IMRN International Mathematics Research Notices
2002, No. 20

A Counterexample to the Hodge Conjecture Extended to Kähler Varieties

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1 Introduction

If X is a smooth projective variety, it is in particular a Kähler variety, and its cohomology groups carry the Hodge decomposition

$$H^k(X, \mathbb{C}) = \oplus_{p+q=k} H^{p,q}(X). \tag{1.1}$$

A class $\alpha \in H^{2p}(X, \mathbb{Q})$ is said to be a rational Hodge class if its image in $H^{2p}(X, \mathbb{C})$ belongs to $H^{p,p}(X)$. As is well known, the classes which are Poincaré dual to irreducible algebraic subvarieties of codimension p of X are Hodge classes of degree $2p$. The Hodge conjecture asserts that any rational Hodge class is a combination with rational coefficients of such classes.

In the case of a general compact Kähler variety X, the conjecture above, where the algebraic subvarieties are replaced with closed analytic subsets, is known to be false (cf. [13]). The simplest example is provided by a complex torus endowed with a holomorphic line bundle of indefinite curvature. If the torus is chosen general enough, it will not contain any analytic hypersurface, while the first Chern class of the line bundle will provide a Hodge class of degree 2.

In fact, another general method to construct Hodge classes is to consider the Chern classes of holomorphic vector bundles. In the projective case, the set of classes generated this way is the same as the set generated by classes of subvarieties. To see this, one introduces a still more general set of classes, which is the set generated by the

Received 22 November 2001. Revision received 21 January 2002.

Chern classes of coherent sheaves on X. Since any coherent sheaf has a finite resolution by locally free sheaves, it follows from Whitney's formula that the set generated by Chern classes of coherent sheaves is not larger than the set generated by Chern classes of locally free sheaves. On the other hand, this latter set obviously contains the classes of subvarieties (we compute for this the Chern class of degree $2p$ of $\mathcal{I}_Z$ for Z irreducible of codimension p, and we show that it is proportional to the class of Z). Finally, to see that the Chern classes of locally free coherent sheaves can be generated by classes of subvarieties, we first reduce, twisting by a very ample line bundle, to the case of globally generated locally free sheaves E. Such an E is the pullback of the tautological quotient bundle $\mathcal{Q}$ on a Grassmannian G by a morphism from X to G. Then we use the fact that the Chern classes of $\mathcal{Q}$ are generated by classes of subvarieties, because the whole cohomology of the Grassmannian is generated by such classes.

In the general Kähler case, none of these equalities between the three sets of Hodge classes introduced above holds. The only obvious result is that the space generated by the Chern classes of analytic coherent sheaves contains both the classes which are Poincaré dual to irreducible closed analytic subspaces and the Chern classes of holomorphic vector bundles (or locally free analytic coherent sheaves). The example above shows that a Hodge class of degree 2 may be the Chern class of a holomorphic line bundle, even if X does not contain any complex analytic subset. On the other hand, we show, in the appendix, that coherent sheaves on a compact Kähler manifold X do not need to admit a resolution by locally free sheaves (although it is true in dimension 2, see [8]), and that more generally, X does not necessarily carry enough vector bundles to generate the Hodge classes of subvarieties or coherent sheaves. Hence, the set $\mathrm{Hdg}(X)_{\mathrm{an}}$ generated by the Chern classes of analytic coherent sheaves is actually larger than the two others.

Notice that using the Grothendieck-Riemann-Roch formula (cf. [3], extended in [7] to the complex analytic case), we can give the following alternative description of the set $\mathrm{Hdg}(X)_{\mathrm{an}}$: it is generated by the classes

$$\phi_* c_i(E), \tag{1.2}$$

where $\phi : Y \to X$ is a morphism from another compact Kähler manifold, E is a holomorphic vector bundle on Y, and i is any integer. Another fact which follows from iterated applications of the Whitney formula, is that the set which is additively generated by the Chern classes of coherent sheaves is equal to the set which is generated as a subring of the cohomology ring by the Chern classes of coherent sheaves. Thus this set is as big and stable as possible.

Now, since we do not know other geometric ways of constructing Hodge classes, the following seems to be a natural extension of the Hodge conjecture to Kähler varieties.

Are the rational Hodge classes of a compact Kähler variety X *generated over* $\mathbb{Q}$ *by rational Chern classes of analytic coherent sheaves on* X*?*

Our goal in this paper is to give a negative answer to this question. We show the following theorem.

Theorem 1.1. There exists a 4-dimensional complex torus X which possesses a nontrivial Hodge class of degree 4, such that any analytic coherent sheaf $\mathcal{F}$ on X satisfies $c_2(\mathcal{F}) = 0$. □

In the appendix, we give a few geometric consequences of a result of Bando and Siu [1], extending Uhlenbeck-Yau's theorem. We show in particular that for a compact Kähler variety X, the analytic coherent sheaves on X do not need to admit finite resolutions by locally free coherent sheaves. This answers a question asked to us by L. Illusie and the referee.

Notation. In this paper, the Chern classes considered are the rational Chern classes

$$c_i(\mathcal{F}) \in H^{2i}(X, \mathbb{Q}), \tag{1.3}$$

for $\mathcal{F}$ a coherent analytic sheaf on a complex manifold X.

2 A criterion for the vanishing of Chern classes of coherent sheaves

We consider in this section a compact Kähler variety X of dimension $n \geq 3$ satisfying the following assumptions:

(1) the Néron-Severi group $NS(X)$ generated by the first Chern classes of holomorphic line bundles on X is equal to 0;

(2) X does not contain any proper closed analytic subset of positive dimension;

(3) for some Kähler class $[\omega] \in H^2(X, \mathbb{R}) \cap H^{1,1}(X)$, the set of Hodge classes $Hdg^4(X, \mathbb{Q})$ is perpendicular to $[\omega]^{n-2}$ for the intersection pairing

$$H^4(X, \mathbb{R}) \otimes H^{2n-4}(X, \mathbb{R}) \longrightarrow \mathbb{R}. \tag{2.1}$$

Our aim is to show the following result.

Proposition 2.1. If X is as above, any analytic coherent sheaf $\mathcal{F}$ on X satisfies $c_2(\mathcal{F}) = 0$. □

Proof. As in [2], the proof is by induction on the rank k of $\mathcal{F}$. We note that because $\dim X \geq 3$, the torsion sheaves supported on points on X have trivial c_1 and c_2. On the other hand, assumption (2) implies that torsion sheaves are supported on points. This deals with the case where $\mathrm{rk}\mathcal{F} = 0$. Furthermore, this shows that we can restrict to torsion free coherent sheaves.

If $\mathcal{F}$ contains a nontrivial subsheaf $\mathcal{G}$ of rank $< k$, we have the exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{G} \longrightarrow 0, \tag{2.2}$$

with $\mathrm{rk}\mathcal{G} < k$ and $\mathrm{rk}\mathcal{F}/\mathcal{G} < k$. Assumption (1) and the induction hypothesis then give

$$c_1(\mathcal{G}) = c_2(\mathcal{G}) = 0, \qquad c_1(\mathcal{F}/\mathcal{G}) = c_2(\mathcal{F}/\mathcal{G}) = 0. \tag{2.3}$$

Therefore Whitney's formula implies that $c_2(\mathcal{F}) = 0$.

We are now reduced to study the case where $\mathcal{F}$ does not contain any nontrivial proper subsheaf of smaller rank. By assumption (2), $\mathcal{F}$ is locally free away from finitely many points $\{x_1, \ldots, x_N\}$ of X. We show now that there exists a variety

$$\tau : \tilde{X} \longrightarrow X, \tag{2.4}$$

which is obtained from X by finitely many successive blowups with smooth centers (and in particular is also Kähler), so that τ restricts to an isomorphism

$$\tilde{X} - \cup_i E_i \cong X - \{x_1, \ldots, x_N\}, \tag{2.5}$$

where $E_i := \tau^{-1}(x_i)$ and such that there exists a locally free sheaf $\tilde{\mathcal{F}}$ on $\tilde{X}$ which is isomorphic to $\mathcal{F}$ on the open set $\tilde{X} - \cup_i E_i$: indeed, the problem is local near each x_i. Choosing a finite presentation

$$\mathcal{O}_X^l \longrightarrow \mathcal{O}_X^r \longrightarrow \mathcal{F} \longrightarrow 0 \tag{2.6}$$

of $\mathcal{F}$ near x_i, we get a morphism to the Grassmannian of k-codimensional subspaces of $\mathbb{C}^r$, which is well defined away from x_i, since $\mathcal{F}$ is free away from x_i. Using the finite presentation, this morphism is easily seen to be meromorphic. Hence by the Hironaka desingularization theorem [5], this morphism can be extended after finitely many blowups. (A priori, [5] works in the algebraic context, but since our meromorphic map takes value in a projective variety and has an isolated point as indeterminacy locus, we can reduce easily to the situation considered in [5].) Then the pullback of the tautological quotient bundle on the Grassmannian will provide the desired extension.

Note that because $\mathcal{F}$ does not contain any nonzero subsheaf of smaller rank, the same is true for $\tilde{\mathcal{F}}$: indeed, assuming there exists a nonzero coherent subsheaf of smaller rank $\mathcal{G} \subset \tilde{\mathcal{F}}$, consider

$$R^0\tau_*\mathcal{G} \subset R^0\tau_*\tilde{\mathcal{F}}. \tag{2.7}$$

This is a nontrivial coherent subsheaf of smaller rank. Now note that $R^0\tau_*\tilde{\mathcal{F}}$ is isomorphic to $\mathcal{F}$ away from the x_i's, hence it is contained in the bidual $\mathcal{F}^{**}$ of $\mathcal{F}$. But the natural inclusion

$$\mathcal{F} \hookrightarrow \mathcal{F}^{**} \tag{2.8}$$

is an isomorphism away from the x_i's, so that its cokernel $\mathcal{T}$ is a coherent sheaf supported on the x_i's. The sheaf

$$\mathcal{G}' := \operatorname{Ker}\left(R^0\tau_*\mathcal{G} \longrightarrow \mathcal{T}\right) \tag{2.9}$$

is then nontrivial of rank equal to the rank of $\mathcal{G}$, hence smaller than k, and it is contained in $\mathcal{F}$, which is a contradiction.

This fact implies that $\tilde{\mathcal{F}}$ is $\tilde{h}$-stable for any Kähler metric $\tilde{h}$ on $\tilde{X}$. The theorem of existence of Hermitian-Yang-Mills connections [10] then provides $\tilde{\mathcal{F}}$ with a Hermitian-Einstein metric $\tilde{k}$ for any Kähler metric $\tilde{h}$ on $\tilde{X}$. This means that the curvature $\tilde{R} \in \Gamma(\mathcal{H}om(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}) \otimes \Omega^2_{\tilde{X},\mathbb{R}})$ of the metric connection on $\tilde{\mathcal{F}}$ associated to $\tilde{k}$ is the sum of a diagonal matrix with all coefficients equal to $\tilde{\mu}\tilde{\omega}$ and of a matrix $\tilde{R}^0$ whose coefficients are $(1,1)$-forms annihilated by the Λ operator relative to the metric $\tilde{h}$. (The connection is then said to be Hermitian-Yang-Mills.) Here $\tilde{\omega}$ is the Kähler form of the metric $\tilde{h}$ and $\tilde{\mu}$ is a constant coefficient, equal to

$$2i\pi\frac{c_1(\tilde{\mathcal{F}})[\tilde{\omega}]^{n-1}}{k[\tilde{\omega}]^n}, \tag{2.10}$$

where $[\tilde{\omega}] \in H^2(\tilde{X}, \mathbb{R})$ denotes the de Rham class of the form $\tilde{\omega}$.

We assume chosen small neighbourhoods V_i of x_i in X, and closed $(1,1)$-forms ω_i on X which vanish outside $\tau^{-1}(V_i)$, and restrict to a Kähler form on the divisor $E_i = \tau^{-1}(x_i)$. That such forms exist follows from the fact that for each i, some combination $\sum_j -n_{ij}E_{ij}$, $n_{ij} > 0$, of the components E_{ij} of the divisor E_i is ample when restricted to E_i. It then follows that there exists a Hermitian metric on the line bundle $\mathcal{O}_{\tilde{X}}(-\sum_j n_{ij}E_{ij})$ whose Chern form restricts to a Kähler form on E_i. We may obviously assume that this metric is the constant metric away from $\tau^{-1}(V_i)$, where we use the natural isomorphism

$$\mathcal{O}_{\tilde{X}}\left(-\sum_j n_{ij}E_{ij}\right) \cong \mathcal{O}_{\tilde{X}} \tag{2.11}$$

away from E_i, and then take for ω_i the Chern form of this metric.

Then we will choose for Kähler form $\tilde{\omega}$ the form

$$\omega_\lambda = \tau^*\omega + \lambda\left(\sum_i \omega_i\right) \tag{2.12}$$

which depends on $\lambda > 0$ and is easily seen to be Kähler for sufficiently small λ.

Let $\tilde{h} = h_\lambda$, $\tilde{k} = k_\lambda$, $\tilde{R} = R_\lambda$. We denote by η^0_λ the closed 4-form

$$\eta^0_\lambda = \operatorname{tr}\left(\frac{R^0_\lambda}{2i\pi}\right)^2. \tag{2.13}$$

We now prove that R_λ tends to 0 with λ in the L^2-sense away from the V_i's. The argument is an extension of Lübke's inequality [6] which proves that a Hermitian-Yang-Mills connection on a vector bundle E with $c_1(E)[\omega]^{n-1} = c_1(E)^2[\omega]^{n-2} = c_2(E)[\omega]^{n-2} = 0$, where $[\omega]$ is the class of the Kähler form on the basis, is in fact flat. We claim the following proposition.

Proposition 2.2. For any differential $(2n-4)$-form α on X, the integral

$$\int_{X-\cup_i V_i} \eta^0_\lambda \wedge \alpha \tag{2.14}$$

tends to 0 with λ. □

Before proving this proposition, we show how it implies that $c_2(\mathcal{F}) = 0$, thus completing the induction step.

Poincaré duality will provide an isomorphism

$$H^4(X - \cup_i V_i, \mathbb{R}) \cong H^{2n-4}(X, \cup_i V_i, \mathbb{R})^*, \tag{2.15}$$

which is realized by integrating closed 4-forms defined over $X - \cup_i V_i$ against closed $(2n-4)$-forms vanishing on the V_i's. Next because $\dim X \geq 3$ we have the isomorphisms

$$\begin{aligned} &H^4(X, \mathbb{R}) \cong H^4(X - \cup_i V_i, \mathbb{R}), \\ &H^{2n-4}(X, \cup_i V_i, \mathbb{R}) \cong H^{2n-4}(X, \mathbb{R}), \end{aligned} \tag{2.16}$$

which are compatible with Poincaré duality. Now the cohomology class of the closed 4-form η^0_λ is easily computed to be

$$[\eta^0_\lambda] = -2c_2(\tilde{\mathcal{F}}) + c_1(\tilde{\mathcal{F}})^2 - \frac{\mu_\lambda}{i\pi}[\omega_\lambda] \cup c_1(\tilde{\mathcal{F}}) + k\left(\frac{\mu_\lambda}{2i\pi}\right)^2 [\omega_\lambda]^2. \tag{2.17}$$

Hence its restriction to $\tilde{X} - \cup_i \tau^{-1}(V_i) \cong X - \cup_i V_i$ satisfies

$$[\eta^0_\lambda]_{|X-\cup_i V_i} = \left(-2c_2(\mathcal{F}) + k\left(\frac{\mu_\lambda}{2i\pi}\right)^2 [\omega]^2 \right)_{|X-\cup_i V_i}, \tag{2.18}$$

In order to show that $c_2(\mathcal{F}) = 0$, it then suffices by (2.15), (2.16), and (2.18) to show that for any closed $(2n-4)$-form α on X, vanishing on $\cup_i V_i$, we have

$$\int_X \left(\eta^0_\lambda - k\left(\frac{\mu_\lambda}{2i\pi}\right)^2 \omega^2\right) \wedge \alpha = 0. \tag{2.19}$$

But this integral is independent of λ and so it suffices to show that

$$\lim_{\lambda \to 0} \int_X \left(\eta^0_\lambda - k\left(\frac{\mu_\lambda}{2i\pi}\right)^2 \omega^2\right) \wedge \alpha = 0. \tag{2.20}$$

Now we have the following lemma, which will be also used in the proof of Proposition 2.2.

Lemma 2.3. The equality

$$\lim_{\lambda \to 0} \mu_\lambda = 0 \tag{2.21}$$

holds. □

Proof. Indeed this follows from formula (2.10), and from the fact that the class $c_1(\tilde{\mathcal{F}})$ pushes forward to 0 in $H^2(X, \mathbb{Z})$ because $NS(X) = 0$. Then the intersection pairing

$$\left\langle c_1(\tilde{\mathcal{F}}), \tau^*[\omega]^{n-1} \right\rangle_{\tilde{X}} = \left\langle \tau_* c_1(\tilde{\mathcal{F}}), [\omega]^{n-1} \right\rangle_X \tag{2.22}$$

is equal to 0, and we conclude using the fact that $\lim_{\lambda \to 0} \omega_\lambda = \tau^*\omega$. ■

Then formula (2.20), and hence Proposition 2.1 follows from (2.21) and Proposition 2.2. ■

Proof of Proposition 2.2. We first claim that

$$\lim_{\lambda \to 0} \int_{\tilde{X}} \eta^0_\lambda \wedge \omega^{n-2}_\lambda = 0. \tag{2.23}$$

Indeed, we know that the space $Hdg^4(X)$ is perpendicular to $[\omega]^{n-2}$ for the intersection pairing. Hence we have

$$\left\langle c_2(\tilde{\mathcal{F}}), \tau^*[\omega]^{n-2}\right\rangle_{\tilde{X}} = \left\langle \tau_* c_2(\tilde{\mathcal{F}}), [\omega]^{n-2}\right\rangle_X = 0. \tag{2.24}$$

On the other hand, this is equal to

$$\lim_{\lambda\to 0}\left\langle c_2(\tilde{\mathcal{F}}), [\omega_\lambda]^{n-2}\right\rangle_{\tilde{X}} \tag{2.25}$$

since $\lim_{\lambda\to 0}\omega_\lambda = \tau^*\omega$. Exactly by the same argument, we show that

$$\lim_{\lambda\to 0}\left\langle c_1^2(\tilde{\mathcal{F}}), [\omega_\lambda]^{n-2}\right\rangle_{\tilde{X}} = 0. \tag{2.26}$$

Then the result follows from formula (2.17) and from Lemma 2.3.

Next we recall that the endomorphism R_λ^0 of $\tilde{\mathcal{F}}$, with forms coefficients is anti-selfadjoint with respect to the metric k_λ. This follows from the fact that R_λ is the curvature of the metric connection with respect to k_λ. In a local orthonormal basis of $\tilde{\mathcal{F}}$, this will be translated into the fact that R_λ^0 is represented by a matrix, whose coefficients are differential 2-forms, which satisfies

$$\overline{{}^tR_\lambda^0} = -R_\lambda^0. \tag{2.27}$$

The second crucial property of R_λ^0 is the fact that its coefficients are primitive differential $(1,1)$-forms on $\tilde{X}$, with respect to the metric h_λ. It is well known that this implies the following equality:

$$*_\lambda\overline{\gamma} = -\overline{\gamma}\wedge\frac{\omega_\lambda^{n-2}}{(n-2)!}, \tag{2.28}$$

where $*_\lambda$ is the Hodge $*$-operator for the metric h_λ. Since h_λ restricts to h on $X - \cup_i V_i$, these forms satisfy as well

$$*\overline{\gamma} = -\overline{\gamma}\wedge\frac{\omega^{n-2}}{(n-2)!} \tag{2.29}$$

on $X - \cup_i V_i$.

Now let α be a differential $(2n-4)$-form on X. Then it follows from (2.29) that there exists a positive constant c_α such that for any primitive $(1,1)$-form γ on X, we have the following pointwise inequality of pseudo-volume forms on X:

$$|\gamma \wedge \overline{\gamma} \wedge \alpha| \leq c_\alpha \gamma \wedge *\overline{\gamma} = -c_\alpha \gamma \wedge \overline{\gamma} \wedge \frac{\omega^{n-2}}{(n-2)!}. \tag{2.30}$$

Working locally in an orthonormal basis of $\tilde{\mathcal{F}}$ and using the fact that the matrix R^0_λ is anti-selfadjoint and with primitive coefficients of $(1,1)$-type, we now get the pointwise inequality of pseudo-volume forms on $X - \cup_i V_i$

$$\left| \operatorname{tr} \left(R^0_\lambda\right)^2 \wedge \alpha \right| \leq c_\alpha \operatorname{tr} \left(R^0_\lambda\right)^2 \wedge \frac{\omega^{n-2}}{(n-2)!}. \tag{2.31}$$

Therefore, we get the inequality

$$\left| \int_{X - \cup_i V_i} \operatorname{tr} \left(R^0_\lambda\right)^2 \wedge \alpha \right| \leq c_\alpha \int_{X - \cup_i V_i} \operatorname{tr} \left(R^0_\lambda\right)^2 \wedge \frac{\omega^{n-2}}{(n-2)!}. \tag{2.32}$$

But by (2.23), and because $\eta^0_\lambda = \operatorname{Tr}(R^0_\lambda/2i\pi)^2$, we know that

$$\lim_{\lambda \to 0} \int_{\tilde{X}} \operatorname{Tr} \left(R^0_\lambda\right)^2 \wedge \omega_\lambda^{n-2} = 0. \tag{2.33}$$

Because the integrand is positive and $\omega = \omega_\lambda$ on $X - \cup_i V_i$, this implies that

$$\lim_{\lambda \to 0} \int_{X - \cup_i V_i} \operatorname{Tr} \left(R^0_\lambda\right)^2 \wedge \omega^{n-2} = 0. \tag{2.34}$$

Hence

$$\lim_{\lambda \to 0} \int_{X - \cup_i V_i} \operatorname{tr} \left(R^0_\lambda\right)^2 \wedge \alpha = 0 = \lim_{\lambda \to 0} \int_{X - \cup_i V_i} \eta^0_\lambda \wedge \alpha. \tag{2.35}$$

Proposition 2.2 is proven. ■

Remark 2.4. A. Teleman mentioned to me the possibility of using the result of [1] (see the appendix) to give a shorter proof of the equality $c_2(\mathcal{F}) = 0$ for stable $\mathcal{F}$. In this paper, the results of Uhlenbeck and Yau [10] are extended to reflexive coherent stable sheaves, and Lübke's inequality, together with the fact that equality implies projective flatness, is proven.

Since in our case we have a much stronger assumption than stability, namely stability of any desingularization of $\mathcal{F}$ with respect to any Kähler metric, we could avoid the reference to the technically hard result of [1] and content ourselves with an argument which appeals only to [10] and elementary computations.

3 Constructing an example

Our example will be of Weil type [12]. The Hodge classes described below have been constructed by Weil in the case of algebraic tori, as a potential counterexample to the Hodge conjecture for algebraic varieties. In the case of a general complex torus, the construction is even simpler. These complex tori have been also considered in [13] by Zucker, who proves there some of the results stated below. (I thank P. Deligne and C. Peters for pointing out this reference to me.)

A complex 4-dimensional torus X with underlying lattice Γ (a lattice of rank 8), is determined by a 4-dimensional complex subspace

$$W \subset \Gamma_{\mathbb{C}} := \Gamma \otimes \mathbb{C} \tag{3.1}$$

satisfying the condition

$$W \cap \Gamma_{\mathbb{R}} = \{0\}, \tag{3.2}$$

where $\Gamma_{\mathbb{R}} = \Gamma \otimes \mathbb{R}$, by the formula

$$X = \frac{\Gamma_{\mathbb{C}}}{W \oplus \Gamma}. \tag{3.3}$$

We start now with a $\mathbb{Z}[I]$-module structure, where $I^2 = -1$, on such a lattice Γ, which makes Γ isomorphic to $\mathbb{Z}[I]^4$. Then

$$\Gamma_{\mathbb{Q}} := \Gamma \otimes \mathbb{Q} \tag{3.4}$$

has an induced structure of K-vector space, where K is the quadratic field $\mathbb{Q}[I]$.

Let

$$\Gamma_{\mathbb{C}} = \mathbb{C}_i^4 \oplus \mathbb{C}_{-i}^4 \tag{3.5}$$

be the associated decomposition into eigenspaces for I. A 4-dimensional complex torus X with underlying lattice Γ and inheriting the I-action is determined as above by a 4-dimensional complex subspace W of $\Gamma_{\mathbb{C}}$, which has to be the direct sum

$$W = W_i \oplus W_{-i} \tag{3.6}$$

of its intersections with $\mathbb{C}_i^4$ and $\mathbb{C}_{-i}^4$.

We will choose W so that

$$\dim W_i = \dim W_{-i} = 2. \tag{3.7}$$

Then W, hence X is determined by the choice of the 2-dimensional subspaces

$$W_i \subset \mathbb{C}^4_i, \qquad W_{-i} \subset \mathbb{C}^4_{-i}, \tag{3.8}$$

which have to be general enough so that condition (3.2) is satisfied.

We have the isomorphisms

$$H^4(X,\mathbb{Q}) \cong H_4(X,\mathbb{Q}) \cong \bigwedge^4 \Gamma_{\mathbb{Q}}. \tag{3.9}$$

Consider the subspace

$$\bigwedge_K^4 \Gamma_{\mathbb{Q}} \subset \bigwedge^4 \Gamma_{\mathbb{Q}}. \tag{3.10}$$

It is defined as follows: note that we have the decomposition

$$\Gamma_K := \Gamma_{\mathbb{Q}} \otimes K = \Gamma_{K,i} \oplus \Gamma_{K,-i} \tag{3.11}$$

into eigenspaces for the I action. Then (3.10) is defined as the image of $\bigwedge_K^4 \Gamma_{K,i} \subset \bigwedge_K^4 \Gamma_K$ under the trace map

$$\bigwedge_K^4 \Gamma_K = \bigwedge_{\mathbb{Q}}^4 \Gamma_{\mathbb{Q}} \otimes K \longrightarrow \bigwedge^4 \Gamma_{\mathbb{Q}}. \tag{3.12}$$

Since $\Gamma_{\mathbb{Q}}$ is a 4-dimensional K-vector space, $\bigwedge_K^4 \Gamma_{\mathbb{Q}}$ is a 1-dimensional K-vector space, and its image is a 2-dimensional $\mathbb{Q}$-vector space. We include for the convenience of the reader a proof of the following lemma.

Lemma 3.1 (Weil [12]). The subspace $\bigwedge_K^4 \Gamma_{\mathbb{Q}} \subset H^4(X,\mathbb{Q})$ consists of Hodge classes, that is, it is contained in the subspace $H^{2,2}(X)$ for the Hodge decomposition. □

Proof. Notice that under the isomorphisms (3.9), tensorized by $\mathbb{C}$, $H^{2,2}(X)$ is equal to the image of

$$\bigwedge^2 W \otimes \bigwedge^2 \overline{W} \tag{3.13}$$

in $\bigwedge^4 \Gamma_{\mathbb{C}}$.

To prove the lemma, we note that we have the natural inclusion

$$\Gamma_K \subset \Gamma_{\mathbb{C}} \tag{3.14}$$

and the equality

$$\Gamma_{K,i} = \Gamma_K \cap \mathbb{C}^4_i. \tag{3.15}$$

The space $\Gamma_{K,i}$ is a 4-dimensional K-vector space which generates over $\mathbb{R}$ the space $\mathbb{C}^4_i$. It follows that the image of $\bigwedge^4_K \Gamma_{K,i}$ in $\bigwedge^4 \Gamma_{\mathbb{C}}$ generates over $\mathbb{C}$ the line $\bigwedge^4 \mathbb{C}^4_i$.

But we know that $\mathbb{C}^4_i$ is the direct sum of the two spaces W_i and $\overline{W_{-i}}$ which are 2-dimensional. Hence

$$\bigwedge^4 \mathbb{C}^4_i = \bigwedge^2 W_i \otimes \bigwedge^2 \overline{W_{-i}} \tag{3.16}$$

is contained in $\bigwedge^2 W \otimes \bigwedge^2 \overline{W}$, that is in $H^{2,2}(X)$. ∎

To conclude the construction of an example satisfying the conclusion of Proposition 2.1, and hence the proof of Theorem 1.1, it remains now only to prove that a general X as above satisfies the assumptions stated at the beginning of Section 2.

We show first of all that the Hodge classes in $\bigwedge^4_K \Gamma_{\mathbb{Q}}$ constructed above are perpendicular to $[\omega]^2$ for a Kähler class $[\omega] \in H^{1,1}(X)$. To see this, note that with the notations as above, these classes lie in $\bigwedge^2 W_i \otimes \bigwedge^2 \overline{W_{-i}}$, with

$$W_i \subset W, \overline{W_{-i}} \subset \overline{W}. \tag{3.17}$$

Via the natural isomorphism $H^1(X, \mathbb{C}) \cong \Gamma_{\mathbb{C}}$, the space $H^{1,0}(X)$ identifies to $\overline{W}^*$ and accordingly the space $H^{1,1}(X)$ identifies to $\overline{W}^* \otimes W^*$. For $[\omega] \in \overline{W}^* \otimes W^*$, the pairing $\langle [\omega]^2, H^{2,2}(X) \rangle$, restricted to $\bigwedge^2 W_i \otimes \bigwedge^2 \overline{W_{-i}}$, is obtained by squaring $[\omega]$ to get an element of

$$\bigwedge^2 \overline{W}^* \otimes \bigwedge^2 W^* \cong \bigwedge^2 W^* \otimes \bigwedge^2 \overline{W}^* \tag{3.18}$$

and by projecting to $\bigwedge^2 W^*_i \otimes \bigwedge^2 \overline{W_{-i}}^*$.

Now choose

$$[\omega] \in \overline{W_i}^* \otimes W^*_i \oplus \overline{W_{-i}}^* \otimes W^*_{-i}. \tag{3.19}$$

Since $\overline{W}^* = \overline{W_i}^* \oplus \overline{W_{-i}}^*$, we can find a Kähler class $[\omega]$ in this space. On the other hand, we see that $[\omega]^2$ belongs to the space

$$\bigwedge^2 \overline{W_i}^* \otimes \bigwedge^2 W_i^* \oplus \overline{W_i}^* \otimes \overline{W_{-i}}^* \otimes W_i^* \otimes W_{-i}^* \oplus \bigwedge^2 \overline{W_{-i}}^* \otimes \bigwedge^2 W_{-i}^*. \tag{3.20}$$

Hence its projection (after switching the factors in the tensor product) to $\bigwedge^2 W_i^* \otimes \bigwedge^2 \overline{W_{-i}}^*$ is equal to 0.

Concerning assumption (2), namely the fact that X does not contain any proper positive dimensional analytic subset, we note that, since X is a complex torus, this property will be satisfied if $NS(X) = 0$ and X is simple. Indeed, it is known that if $Y \subset X$ is a proper positive dimensional subvariety of a simple complex torus, then Y has positive canonical bundle. But X being simple, Y must generate X as a group, and then X must be algebraic, contradicting the fact that $NS(X) = 0$.

In conclusion, the assumptions at the beginning of Section 2 will be a consequence of the following facts.

Proposition 3.2. For a general X as above, the following hold.

(1) $NS(X) = 0$;

(2) X is simple;

(3) the space $Hdg^4(X)$ is equal to the space

$$\bigwedge_K^4 \Gamma_{\mathbb{Q}} \subset \bigwedge^4 \Gamma_{\mathbb{Q}} = H_4(X, \mathbb{Q}) \cong H^4(X, \mathbb{Q}). \tag{3.21}$$

□

Proof. The analogues of these statements have been proven in the algebraic case in [12] (see also [11]). The result is that for a general abelian 4-fold of Weil type, the Néron-Severi group is of rank 1, generated by a class ω, and the space $Hdg^4(X)$ is of rank 3, generated over $\mathbb{Q}$ by the space $\bigwedge_K^4 \Gamma_{\mathbb{Q}}$ and by the class ω^2. Furthermore, property (2) is true for the generic abelian variety X of Weil type.

Property (2) for the general complex torus of Weil type follows immediately, since this is a property satisfied away from the countable union of closed analytic subsets of the moduli space of complex tori of Weil type.

As for properties (1) and (3), we prove them by an infinitesimal argument, starting from an abelian 4-fold of Weil type X satisfying the properties stated above. Assume we can show that for some first order deformation $u \in H^1(T_X)$, tangent to the moduli space of complex tori of Weil type (which is smooth), we have

$$\text{int}(u)(\omega) \neq 0 \quad \text{in } H^2(\mathcal{O}_X), \tag{3.22}$$

where the interior product here is composed of the cup product

$$H^1(T_X) \otimes H^1(\Omega_X) \longrightarrow H^2(T_X \otimes \Omega_X) \tag{3.23}$$

and of the map induced by the contraction

$$H^2(T_X \otimes \Omega_X) \longrightarrow H^2(\mathcal{O}_X). \tag{3.24}$$

Then from the general theory of Hodge loci [4], it will follow that for a general complex torus of Weil type, we have $NS(X) = 0$. Furthermore, it will also follow that

$$\mathrm{int}(u)(\omega^2) \neq 0 \quad \text{in } H^3(\Omega_X), \tag{3.25}$$

because it is equal to $2\omega \cup \mathrm{int}(u)(\omega)$ and the cup product with ω from $H^2(\mathcal{O}_X)$ to $H^3(\Omega_X)$ is injective because ω is a Kähler class on X. But as before this will imply by the theory of Hodge loci that for a general complex torus of Weil type we have $\mathrm{rk}\,\mathrm{Hdg}^4(X) = 2$ so that $\mathrm{Hdg}^4(X) = \bigwedge_K^4 \Gamma_{\mathbb{Q}}$.

Hence it remains only to find such u, which is equivalent to prove that if for any u tangent to the moduli space of complex tori of Weil type, $\mathrm{int}(u)(\omega) = 0$ in $H^2(\mathcal{O}_X)$, then $\omega = 0$ in $H^1(\Omega_X)$. Here the notations are as in the beginning of this section. The tangent space to the deformations of the complex torus X is equal to

$$\mathrm{Hom}\,(W, \Gamma_{\mathbb{C}}/W) = \mathrm{Hom}\,(W, \overline{W}) = W^* \otimes \overline{W}. \tag{3.26}$$

The tangent space to the deformations of X as a complex torus of Weil type is then the subspace

$$\mathrm{Hom}\,(W_i, \mathbb{C}_i^4/W_i) \oplus \mathrm{Hom}\,(W_{-i}, \mathbb{C}_{-i}^4/W_{-i}) = W_i^* \otimes \overline{W_{-i}} \oplus W_{-i}^* \otimes \overline{W_i}. \tag{3.27}$$

Via the identification (3.26), the interior product

$$H^1(X, T_X) \otimes H^1(X, \Omega_X) \longrightarrow H^2(X, \mathcal{O}_X) = \bigwedge^2 W^* \tag{3.28}$$

is equal to the contraction followed by the wedge product

$$W^* \otimes \overline{W} \otimes \overline{W}^* \otimes W^* \longrightarrow \bigwedge^2 W^*. \tag{3.29}$$

We now write

$$\omega = \omega_1 + \omega_2 + \omega_3 + \omega_4, \tag{3.30}$$

where

$$\begin{aligned} &\omega_1 \in \overline{W_i}^* \otimes W_i^*, \qquad \omega_2 \in \overline{W_i}^* \otimes W_{-i}^*, \\ &\omega_3 \in \overline{W_{-i}}^* \otimes W_i^*, \qquad \omega_4 \in \overline{W_{-i}}^* \otimes W_{-i}^*. \end{aligned} \tag{3.31}$$

Then clearly for $u_1 \in W_i^* \otimes \overline{W_{-i}}$ we have

$$\begin{aligned} &\operatorname{int}(u_1)(\omega_1) = \operatorname{int}(u_1)(\omega_2) = 0, \\ &\operatorname{int}(u_1)(\omega_3) \in \bigwedge^2 W_i^*, \qquad \operatorname{int}(u_1)(\omega_4) \in W_i^* \otimes W_{-i}^*. \end{aligned} \tag{3.32}$$

Similarly, for $u_2 \in W_{-i}^* \otimes \overline{W_i}$ we have

$$\begin{aligned} &\operatorname{int}(u_2)(\omega_3) = \operatorname{int}(u_1)(\omega_4) = 0, \\ &\operatorname{int}(u_2)(\omega_2) \in \bigwedge^2 W_{-i}^*, \qquad \operatorname{int}(u_2)(\omega_1) \in W_i^* \otimes W_{-i}^*. \end{aligned} \tag{3.33}$$

The condition

$$\operatorname{int}(u_1)(\omega) = 0 = \operatorname{int}(u_2)(\omega) = 0 \tag{3.34}$$

for any u_1, u_2 then implies that

$$\begin{aligned} &\operatorname{int}(u_1)(\omega_3) = 0 \quad \text{in } \bigwedge^2 W_i^*, \qquad \operatorname{int}(u_1)(\omega_4) = 0 \quad \text{in } W_i^* \otimes W_{-i}^*, \\ &\operatorname{int}(u_2)(\omega_1) = 0 \quad \text{in } W_i^* \otimes W_{-i}^*, \qquad \operatorname{int}(u_2)(\omega_2) = 0 \quad \text{in } \bigwedge^2 W_{-i}^*, \end{aligned} \tag{3.35}$$

for any u_1, u_2. But it is obvious that it implies $\omega_1 = \omega_2 = \omega_3 = \omega_4 = 0$. ■

Hence Proposition 3.2 is proven, which together with Proposition 2.1 completes the proof of Theorem 1.1.

Appendix

Our goal in this appendix is to give a few geometric consequences of the following result, due to Bando and Siu (the second statement given below is only a particular case of [1, Corollary 3], namely the case where the considered sheaf has trivial rational first Chern class).

Theorem A.1. Let X be a compact Kähler variety, endowed with a Kähler metric h and let $\mathcal{F}$ be a reflexive h-stable sheaf on X. Then there exists a Hermite-Einstein metric on $\mathcal{F}$ relative to h. It follows that if we have $\langle c_2(\mathcal{F}), [\omega]^{n-2}\rangle_X = 0$ and $c_1(\mathcal{F}) = 0$, $\mathcal{F}$ is locally free and the associated metric connection is flat. □

Here $[\omega]$ is the Kähler class of the metric h.

Remark A.2. Once we know that the metric connection is flat away from the singular locus Z of $\mathcal{F}$, the fact that $\mathcal{F}$ is locally free is immediate. Indeed, the flat connection is associated to a local system on $X - Z$. But since $\operatorname{codim} Z \geq 2$, this local system extends to X. Hence there exists a holomorphic vector bundle E on X, which admits a unitary flat connection and is isomorphic to $\mathcal{F}$ away from Z. But since $\mathcal{F}$ is reflexive, the isomorphism $E \cong \mathcal{F}$ on $X - Z$ extends to X.

We assume now that X is compact Kähler and satisfies the condition that the group $NS(X) \otimes \mathbb{Q} = Hdg^2(X)$ of rational Hodge classes of degree 2 vanishes, and that the group $Hdg^4(X)$ is perpendicular for the intersection pairing to $[\omega]^{n-2}$ for some Kähler class ω on X. Under these assumptions, X does not contain any proper analytic subset of codimension less than or equal to 2, and any coherent sheaf $\mathcal{F}$ satisfies the conditions

$$c_1(\mathcal{F}) = 0, \qquad \langle c_2(\mathcal{F}), [\omega]^{n-2}\rangle_X = 0. \tag{A.1}$$

We now prove the following proposition.

Proposition A.3. If X is as above, for any torsion free coherent sheaf $\mathcal{F}$ on X, there exist a holomorphic vector bundle E on X, whose all rational Chern classes $c_i(E)$, $i > 0$ vanish, and an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow E \longrightarrow \mathcal{T} \longrightarrow 0 \tag{A.2}$$

where $\mathcal{T}$ is a torsion sheaf on X. □

Before proving the proposition, we state the following corollaries.

Corollary A.4. If E is a holomorphic vector bundle on X, then all rational Chern classes $c_i(E), i > 0$, vanish. □

Proof. Indeed, we know that there exists an inclusion

$$E \hookrightarrow E', \tag{A.3}$$

where E' is a vector bundle of the same rank as E and satisfies the property that all rational Chern classes $c_i(E'), i > 0$, vanish. Now since $Hdg^2(X) = 0$, X does not contain any hypersurface, and it follows that the inclusion above is an isomorphism. ■

Corollary A.5. If X is as above and $Z \subset X$ is a nonempty proper analytic subset, the ideal sheaf $\mathcal{I}_Z$ does not admit a finite free resolution. □

Proof. Indeed, if such a resolution

$$0 \longrightarrow E^n \longrightarrow \cdots \longrightarrow E^i \longrightarrow E^{i-1} \longrightarrow E^0 \longrightarrow \mathcal{I}_Z \longrightarrow 0 \tag{A.4}$$

would exist, then we would get the equality

$$c(\mathcal{I}_Z) = \Pi_i c(E_i)^{\epsilon_i} \tag{A.5}$$

with $\epsilon_i = (-1)^i$. But the left-hand side does not vanish identically in positive degrees since its term of degree $r = \operatorname{codim} Z$ is a nonzero multiple of the class of Z. On the other hand, the right-hand side vanishes in positive degrees by Corollary A.4. ■

Note that the assumptions are satisfied by a general complex torus of dimension at least 3. Taking for Z a point, we get an explicit example of a coherent sheaf which does not admit a finite locally free resolution.

Proof of Proposition A.3. We use again induction on the rank. Let $\mathcal{F}$ be a torsion free coherent sheaf of rank k on X, and assume first that $\mathcal{F}$ does not contain any nonzero subsheaf of smaller rank. There is an inclusion

$$\mathcal{F} \hookrightarrow \mathcal{F}^{**} \tag{A.6}$$

whose cokernel is a torsion sheaf, where the bidual $\mathcal{F}^{**}$ of $\mathcal{F}$ is reflexive. Then $\mathcal{F}^{**}$ does not contain any nonzero subsheaf of smaller rank and hence is stable with respect to the given Kähler metric h on X. The theorem of Bando and Siu [1] together with the fact that

$$c_1(\mathcal{F}^{**}) = 0, \qquad \langle c_2(\mathcal{F}^{**}), [\omega]^{n-2} \rangle_X = 0 \tag{A.7}$$

implies that $\mathcal{F}^{**}$ is a holomorphic vector bundle which is endowed with a flat connection, hence has trivial rational Chern classes and the result is proved in this case.

Assume, otherwise, that there is an exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{H} \longrightarrow 0, \tag{A.8}$$

where the ranks of $\mathcal{G}$ and $\mathcal{H}$ are smaller than the rank of $\mathcal{F}$, and $\mathcal{G}$ and $\mathcal{H}$ are without torsion. This exact sequence determines (and is determined by) an extension class

$$e \in \mathrm{Ext}^1(\mathcal{H}, \mathcal{G}). \tag{A.9}$$

(Here and in the following, all the extension groups refer to extensions as $\mathcal{O}_X$-modules.) Now, by induction on the rank we may assume that we have inclusions

$$\mathcal{G} \hookrightarrow E_1, \qquad \mathcal{H} \hookrightarrow E_2, \tag{A.10}$$

whose cokernels $\mathcal{T}_i$ are of torsion and where the E_i's are holomorphic vector bundles with vanishing Chern classes. The extension class e gives first an extension class $f \in \mathrm{Ext}^1(\mathcal{H}, E_1)$, which provides a sheaf E' containing $\mathcal{F}$ in such way that $E'/\mathcal{F}$ is of torsion, and fitting into an exact sequence

$$0 \longrightarrow E_1 \longrightarrow E' \longrightarrow \mathcal{H} \longrightarrow 0. \tag{A.11}$$

Next, because the torsion sheaf $\mathcal{T}_2$ is supported in codimension ≥ 3, the restriction map provides an isomorphism

$$\mathrm{Ext}^1(E_2, E_1) \cong \mathrm{Ext}^1(\mathcal{H}, E_1). \tag{A.12}$$

Indeed, this is implied using the long exact sequence of Ext's by the vanishing

$$\mathrm{Ext}^1(\mathcal{T}_2, E_1) = \mathrm{Ext}^2(\mathcal{T}_2, E_1) = 0. \tag{A.13}$$

By the local-to-global spectral sequence for Ext's, this in turn is implied by the vanishing

$$\mathcal{E}xt^i(\mathcal{T}_2, E_1) = 0, \quad i \leq 2. \tag{A.14}$$

Since E_1 is locally free, and the statement is local, this follows now from the vanishing

$$\mathcal{E}xt^i(\mathcal{T}, \mathcal{O}_X) = 0, \quad i \leq j, \tag{A.15}$$

for any torsion sheaf $\mathcal{T}$ supported in codimension $\geq j+1$.

The surjectivity of (A.12) implies that f is the image under the restriction map of a class $g \in \mathrm{Ext}^1(E_2, E_1)$. Hence it follows that there is a holomorphic vector bundle E on X, which is an extension of E_2 by E_1, and which contains E' as a subsheaf, such that the quotient E/E' is of torsion. The vector bundle E has vanishing rational Chern classes, because E_i satisfy this property for $i = 1, 2$, and contains $\mathcal{F}$ as a subsheaf such that the quotient $E/\mathcal{F}$ is of torsion. This completes the proof by induction. ■

Acknowledgments

I would like to thank Joseph Le Potier and Andrei Teleman for helpful discussions on this paper. The references [2, 9] have been a starting point for this work. I also thank Pierre Deligne and Luc Illusie for their interest and their questions, and the referee for many constructive comments.

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