On the homotopy types of compact Kähler and complex projective manifolds

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0   Introduction

The celebrated Kodaira theorem [6] says that a compact complex manifold is projective if and only if it admits a Kähler form whose cohomology class is integral. This suggests that Kähler geometry is an extension of projective geometry, obtained by relaxing the integrality condition on a Kähler class. This point of view, together with the many restrictive conditions on the topology of Kähler manifolds provided by Hodge theory (the strongest one being the formality theorem [4]), would indicate that compact Kähler manifolds and complex projective ones cannot be distinguished by topological invariants. This is supported by the results known for Kähler surfaces, for which a much stronger statement is known, as a consequence of Kodaira's classification : recall first that two compact complex manifolds X and X' are said to be deformation equivalent if there exist a family $\pi : \mathcal{X} \to B$, where B is a connected analytic space and π is smooth and proper, and two points b, $b' \in B$ such that

$$X_b \cong X, \ X_{b'} \cong X'.$$

Kodaira shows the following :

Theorem 1 *[7] A compact Kähler surface is deformation equivalent to a complex projective surface.*

Note that since the basis B of the family is path connected, Ehresmann's theorem shows that two deformation equivalent complex manifolds are homeomorphic, and a fortiori have the same homotopy type.

Since Kodaira's result, the situation for higher dimensional manifolds remained unknown. A classical problem, sometimes called the Kodaira problem (see [5]), asks whether theorem 1 still holds in higher dimensions. Many known examples, like hyperkähler manifolds, or more generally K-trivial Kähler manifolds, are deformations of projective ones (see [1]).

In this paper, we show however the following result, which in particular provides a negative answer to Kodaira's problem:

Theorem 2 *In any dimension ≥ 4, there exist compact Kähler manifolds which do not have the homotopy type of a complex projective manifold.*

The case of dimension 3 remains open.

Examples are constructed in section 2. The topological invariant we use here is the integral cohomology ring, so that a stronger version of the theorem above is the fact that there exist compact Kähler manifolds which do not have a cohomology ring isomorphic to the one of a complex projective manifold. In section 3, we explain arguments due to Deligne, (letter to the author,) which show how to get similar statements with the rational cohomology ring and, what is still more surprising, with the complex cohomology algebra.

The examples considered above are birationally equivalent to complex tori, and indeed, the proofs consist in showing that the cohomology algebra prevents the H^1 being endowed with a polarized Hodge structure. In section 3, we construct however a simply connected Kähler manifold satisfying the conclusion of Theorem 2. The proof that its rational cohomology ring is not the one of a projective complex manifold makes a systematic use of Deligne's Lemma 2 (see section 3), combined with the Hodge index theorem.

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1   Complex tori with endomorphisms

Let $\Gamma \cong \mathbb{Z}^{2n}$ be a lattice, and let $\phi : \Gamma \to \Gamma$ be an endomorphism. Let

$$\Gamma_{\mathbb{C}} := \Gamma \otimes \mathbb{C},$$

on which the $\mathbb{C}$-linear extension $\phi_{\mathbb{C}}$ of ϕ acts. We shall assume the following properties :

(P) *The eigenvalues of ϕ all have multiplicity* 1, *and none of them is real.*

We choose, among the $2n$ eigenvalues of ϕ, a set of n elements, say $\lambda_1, \ldots, \lambda_n$, and assume that no two of the λ_i's are conjugate to each other. Let

$$\Gamma' \subset \Gamma_{\mathbb{C}}$$

be the eigenspace associated to these eigenvalues. The last condition then implies that
$$\Gamma_{\mathbb{C}} = \Gamma' \oplus \overline{\Gamma'},$$
so that we have a complex torus T defined as

$$T = \Gamma_{\mathbb{C}}/(\Gamma' \oplus \Gamma).$$

Note that $\phi_{\mathbb{C}}$ leaves stable Γ' and Γ, hence induces an endomorphism, which we will call ϕ_T, of T. The endomorphism of $H_1(T, \mathbb{Z}) \cong \Gamma$ induced by ϕ_T is equal to ϕ.

The characteristic polynomial f of ϕ has $\mathbb{Z}$-coefficients, and can be chosen to be any normalized degree 2n polynomial with integer coefficients, subject to the condition (P).

Remark 1 *Assume conversely that T is a n-dimensional complex torus admitting an endomorphism ϕ_T, whose induced morphism*

$$\phi_{T*} : H_1(T, \mathbb{Z}) \to H_1(T, \mathbb{Z})$$

identifies to ϕ, via an isomorphism $\Gamma \cong H_1(T, \mathbb{Z})$. Then T is one of the tori constructed as above, for a choice of λ_i's.

We have now the following lemma:

Lemma 1 *If $n \geq 2$ and the polynomial f is Galois, that is the Galois group of its splitting field acts as the symmetric group on the roots of f (the eigenvalues of ϕ), the torus T is not an abelian variety.*

Remark 2 *This could be proved using the Albert classification of endomorphism algebras of abelian varieties (see [3], [2]). However, the proof is so easy that it is better to do it directly.*

Proof. Consider the rational Néron-Severi group

$$NS_{\mathbb{Q}}(T) \subset H^2(T, \mathbb{Q}),$$

generated over $\mathbb{Q}$ by Chern classes of holomorphic line bundles on T. This subgroup is contained in the subspace $H^{1,1}(T) \subset H^2(T, \mathbb{C})$ of classes representable by a form of type $(1,1)$, and is stable under the action of ϕ_T^* on $H^2(T, \mathbb{Q})$.

Now the eigenvalues of ϕ_T^* on $H^{1,0}(T) \cong \overline{\Gamma'}^*$ are the $\overline{\lambda_i}$, so that the eigenvalues of ϕ_T^* acting on

$$H^{1,1}(T) \cong \Gamma'^* \otimes \overline{\Gamma'}^*$$

are exactly the $\lambda_i \overline{\lambda_j}$ for any i, j.

The subspace $NS_{\mathbb{Q}}(T) \otimes \mathbb{C}$ of $H^{1,1}(T)$ is stable under ϕ_T^*, hence must be the eigenspace associated to a set S of eigenvalues of the form $\lambda_i \overline{\lambda_j}$. Since it is defined over $\mathbb{Q}$, the Galois group of the splitting field of f has to leave stable the set S. But by assumption, the Galois group acts on the set $\{\lambda_1, \ldots, \lambda_n, \overline{\lambda_1}, \ldots, \overline{\lambda_n}\}$ as the symmetric group. Hence it acts transitively on the products of two distinct elements of this set. Since $n \geq 2$, we can choose $i \neq j$, and then $\lambda_i \lambda_j$ does not belong to the set S. Since S is stable under the Galois group, it follows that S must be empty, so that $NS(T) = 0$ and, a fortiori T is not an abelian variety.

∎

Remark 3 *What we actually proved is the equality $NS(T) = 0$, under the same assumptions. This is to be compared with [8], where Shioda proves that certain automorphism groups acting on surfaces in projective space $\mathbb{P}^3$ force the surface to have Picard group to be as small as possible, namely generated by the class of the line bundle $\mathcal{O}(1)$.*

2 Construction of examples

Let (T, ϕ) be a complex torus with endomorphism as in the previous section.

Our complex manifold X will be constructed as follows : the diagonal of $T \times T$ and the graph of ϕ meet transversally at isolated points $x_1 = (0, 0), \ldots, x_N$. Similarly the graph of ϕ_T meets $T \times 0$ transversally at finitely many points $x_1 = (0, 0), y_1, \ldots, y_M$. We first blow-up the points $x_1, \ldots, x_N, y_1, \ldots, y_M$ in $T \times T$. The proper transforms of the four subtori

$$T \times 0,\ 0 \times T,\ T_{diag},\ T_{graph},$$

of $T \times T$, where the two last ones are respectively the diagonal and the graph of ϕ_T, are smooth and do not meet anymore. So we can blow-up their union, to get a smooth compact manifold, which is Kähler since $T \times T$ is. The following shows that under the assumptions of Lemma 1, X satisfies the conclusion of Theorem 2.

Theorem 3 *Assume $n \geq 2$ and the characteristic polynomial of ϕ is Galois. Let X' be a compact Kähler manifold. Assume that there exists a ring isomorphism*

$$\gamma : H^*(X', \mathbb{Z}) \cong H^*(X, \mathbb{Z}).$$

Then X' is not projective.

Proof. The isomorphism γ shows us that $dim\, X' = dim X = 2n$. Introduce the complex tori

$$Alb\, X,\ Alb\, X'.$$

They are complex tori of dimension

$$b_1(X)/2 = 2n,\ resp.\ b_1(X')/2 = 2n$$

defined respectively as

$$H^0(X, \Omega_X)^*/H_1(X, \mathbb{Z}),\ H^0(X', \Omega_{X'})^*/H_1(X', \mathbb{Z}).$$

We have the Albanese map, which is holomorphic :

$$alb_X : X \to Alb\, X,\ alb_{X'} : X' \to Alb\, X',$$

defined by integrating holomorphic forms along paths, well defined up to closed paths :

$$x \mapsto \int_{x_0}^{x} \in H^0(X, \Omega_X)^*,$$

$x_0 \in X$ being a given base-point, and similarly for X'. This map induces by pull-back an isomorphism

$$alb_X^* : H^1(Alb\, X, \mathbb{Z}) \to H^1(X, \mathbb{Z}),$$

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and similarly for X'. Since $H^*(Alb\,X, \mathbb{Z}) \cong \bigwedge^* H^1(Alb\,X, \mathbb{Z})$, and alb_X^* is compatible with the cup-product, we can identify

$$alb_X^* : H^*(Alb\,X, \mathbb{Z}) \to H^*(X, \mathbb{Z}) \tag{2.1}$$

with the natural map induced by cup-product :

$$\bigwedge^* H^1(X, \mathbb{Z}) \to H^*(X, \mathbb{Z}),$$

and similarly for X'.

In the case of X, the Albanese map is (up to translation) the natural blow-down map, hence it is birational. This implies that in top degree $4n$, the map

$$alb_X^* : H^{4n}(Alb\,X, \mathbb{Z}) \to H^{4n}(X, \mathbb{Z})$$

is an isomorphism. As noted above, this map identifies to the cup-product map in top degree. Via the isomorphism γ, we conclude now that the map

$$alb_{X'}^* : H^{4n}(Alb\,X', \mathbb{Z}) \to H^{4n}(X', \mathbb{Z})$$

is also an isomorphism, which means that the map $alb_{X'}^*$ is of degree 1, that is birational.

Consider now the Gysin maps alb_{X*}, $alb_{X'*}$ on the cohomology of degree 2. They are morphisms of Hodge structures, which can be defined here as the duals with respect to Poincaré duality of the pull-back maps alb_X^*, $alb_{X'}^*$ in degree $4n-2$, because the considered manifolds have torsion free cohomology. Because the Albanese maps are of degree 1 in our case, the Gysin maps satisfy

$$alb_{X*} \circ alb_X^* = Id, \; alb_{X'*} \circ alb_{X'}^* = Id. \tag{2.2}$$

Since $alb_{X'}$ is birational, it induces an isomorphism

$$alb_{X'}^* : H^0(Alb\,X', \Omega^2_{Alb\,X'}) \cong H^0(X', \Omega^2_{X'}).$$

This means that the morphism of Hodge structures

$$alb_{X'}^* : H^2(Alb\,X', \mathbb{Z}) \to H^2(X', \mathbb{Z})$$

induces an isomorphism on $H^{2,0}$. Equation (2.2) shows then that the Hodge structure on $Ker\,alb_{X'*}$ has no $(2,0)$-part, that is, it is made of cohomology classes of type $(1,1)$ in the Hodge decomposition of X'.

Remark 4 *It is only to get this point that we needed the integral cohomology ring. The remainder of the proof will work as well with rational cohomology, replacing in the following argument (polarized) integral Hodge structures of weight 1, (or equivalently (projective) complex tori), with (polarized) rational Hodge structures of weight 1, (or equivalently isogeny classes of (projective) complex tori).*

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We claim now that $Ker\,alb_{X'*}$ is the image under γ^{-1} of the subgroup $Ker\,alb_{X*}$. Indeed, Poincaré duality is given by cup-product followed by the identification given by the orientation
$$H^{4n}(\cdot, \mathbb{Z}) \cong \mathbb{Z}$$
for all the considered manifolds. Hence, since γ is compatible with the cup-product, it is compatible up to sign with Poincaré duality. Now we have seen that γ identifies the images of the pull-back maps (2.1) for X and X'. Since $Ker\,alb_{X'*}$ in degree 2 is the orthogonal complement with respect to Poincaré duality of the image of the pull-back map in degree $4n-2$, and similarly for X, the claim follows.

In conclusion, we have a set of integral Hodge classes of degree 2, provided by the $\gamma^{-1}(\alpha)$, $\alpha \in Ker\,alb_{X*}$. Recalling the construction of X by a sequence of blow-ups, we see that this last group is freely generated by the classes $[\Delta.]$ of the exceptional divisors

$$\Delta_{x_i}, \Delta_{y_j}, \Delta_{T\times 0}, \Delta_{0\times T}, \Delta_{diag}, \Delta_{graph},$$

over the corresponding centers of blow-up x_i, y_j, $T \times 0$, $0 \times T$, T_{diag}, T_{graph}. Let us denote $\delta. := \gamma^{-1}([\Delta.])$.

Each of these classes acts via cup-product on the cohomology of X', and since we proved they are Hodge classes, their action is via morphisms of Hodge structures, of bidegree $(1,1)$. Hence the kernels of their restrictions to $H^1(X', \mathbb{Z})$ provide sub-Hodge structures of $H^1(X', \mathbb{Z})$. On the other hand, since γ is compatible with the cup-product, these groups are the images under γ^{-1} of the corresponding subgroups

$$Ker\,\cup[\Delta.] : H^1(X, \mathbb{Z}) \to H^3(X, \mathbb{Z}).$$

We look now at the maps $\cup[\Delta.]$. Consider the general situation of a blow-up

$$\tau : \widetilde{Y} \to Y$$

of a compact complex manifold Y along a complex submanifold $Z \subset Y$ of codimension ≥ 2. Let $j : E \hookrightarrow \widetilde{Y}$ be the inclusion of the exceptional divisor, and let $\tau_E : E \to Z$ be the restriction of τ to E. Then the map

$$\cup[E] \circ \tau^* : H^*(Y, \mathbb{Z}) \to H^{*+2}(\widetilde{Y}, \mathbb{Z})$$

can be written as

$$j_{E*} \circ \tau_E^* \circ j_Z^*,$$

where j_Z is the inclusion of Z in Y. It is known that the composite $j_{E*} \circ \tau_E^*$ is injective on the cohomology of Z (cf [9], 7.3.3), hence we conclude that

$$Ker\,\cup[E] \circ \tau^* = Ker\,j_Z^*.$$

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We apply this to X and $T \times T$. (We ignore here the initial blow-up of points, since it does not affect H^1 or H^3, hence does not enter in this computation.) For the complex subtorus $T \times 0$ we conclude that the kernel of the morphism

$$\cup[\Delta_{T \times 0}] : H^1(X, \mathbb{Z}) \cong H^1(T \times T, \mathbb{Z}) \to H^3(X, \mathbb{Z})$$

is equal to the kernel of the restriction map :

$$H^1(T \times T, \mathbb{Z}) \to H^1(T \times 0, \mathbb{Z}).$$

Now the left hand side is isomorphic to $H^1(T, \mathbb{Z}) \oplus H^1(T, \mathbb{Z})$ and the restriction map is the first projection pr_1. Similarly, the kernel of the map $\cup[\Delta_{0 \times T}]$ is equal to the kernel of the second projection pr_2. Finally, the kernel of $\cup[\Delta_{diag}]$ is equal to the kernel of $pr_1 + pr_2$, while the kernel of $\cup[\Delta_{graph}]$ is equal to the kernel of $pr_1 + \phi_T^* \circ pr_2$, since the restriction map from $H^1(T \times T, \mathbb{Z})$ to $H^1(T_{graph}, \mathbb{Z}) = H^1(T, \mathbb{Z})$ identifies to $pr_1 + \phi_T^* \circ pr_2$.

Let us now come back to X'. The above shows that $H^1(X', \mathbb{Z})$ contains four integral sub-Hodge structures L_1, L_2, L_3, L_4, images via γ^{-1} of the 4 subgroups

$$Ker\, pr_1,\ Ker\, pr_2,\ Ker\, pr_1 + pr_2,\ Ker\, pr_1 + \phi_T^* \circ pr_2. \tag{2.3}$$

Consider the complex torus (dual to the Albanese torus)

$$Pic^0(X') = H^1(X', \mathbb{C})/(H^{1,0}(X') \oplus H^1(X', \mathbb{Z})).$$

Any integral sub-Hodge structure of $H^1(X', \mathbb{Z})$ provides a corresponding complex subtorus of $Pic^0(X')$ in an obvious way, namely, if

$$L \subset H^1(X', \mathbb{Z})$$

is a primitive sublattice such that

$$L_{\mathbb{C}} = L^{1,0} \oplus L^{0,1},$$

where $L^{1,0} = L_{\mathbb{C}} \cap H^{1,0}(X')$, and $L^{0,1}$ is its complex conjugate, then

$$T_L = L_{\mathbb{C}}/(L^{1,0} \oplus L)$$

is a complex subtorus of $Pic^0 X'$.

Hence we get four subtori T_{L_i} of $Pic^0(X')$, which satisfy the following conditions, (because they are satisfied by the corresponding sublattices) :

1. $T_{L_1} \oplus T_{L_2} = Pic^0(X')$.

2. T_{L_3} is isomorphic to T_{L_1} and T_{L_2} via the two projections induced by the previous isomorphism. In particular T_{L_1} and T_{L_2} are both isomorphic to some torus T', and $Pic^0(X') \cong T' \oplus T'$.

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3. T_{L_4} is isomorphic to T' via the second projection p_2 induced by the last isomorphism. Hence it is the (transpose of the) graph of an endomorphism $\phi'_{T'}$ of T'.

We finally observe that the action of $\phi'_{T'}$ on the homology $H_1(T', \mathbb{Z})$ is determined by the position of the four sublattices L_i. Using the description (2.3) of the $\gamma^{-1}(L_i)$'s, we find that it identifies via γ to ϕ^*_T, that is to the dual of our initial endomorphism ϕ.

So we proved that the variety X' satisfies the property that its Picard torus is a product $T' \times T'$ where T' is a complex torus admitting an automorphism which acts on $H_1(T', \mathbb{Z})$ as the dual of ϕ. We are now in position to apply Lemma 1, combined with Remark 1, which says that T' cannot be projective. On the other hand, if X' were a projective variety, its Picard variety would be also projective (cf [9], 7.2.2). So X' is not projective. ∎

Since the X's above have any possible even dimension ≥ 4 (because they are birational to products $T \times T$, with $dim\, T \geq 2$), Theorem 3 concludes the proof of Theorem 2 for even dimensions. In order to deal with odd dimensions, we prove with very similar arguments the following result :

Proposition 1 *Let X be a variety constructed as above, and let F be an elliptic curve. Let X' be a Kähler compact manifold, such that there exists a ring isomorphism*

$$\gamma : H^*(X', \mathbb{Z}) \to H^*(X \times F, \mathbb{Z}).$$

Then X' is not projective.

Proof. Exactly as in the previous proof, we show first that the Albanese map $alb_{X'}$ of X' is birational, that the kernel of the map

$$alb_{X'*} : H^2(X', \mathbb{Z}) \to H^2(Alb\, X', \mathbb{Z})$$

is made of Hodge classes, and is equal to

$$\gamma^{-1}(Ker\,(alb_{X \times F*} : H^2(X \times F, \mathbb{Z}) \to H^2(Alb\,(X \times F), \mathbb{Z}))).$$

The group $Ker\,(alb_{X \times F*} : H^2(X \times F, \mathbb{Z}) \to H^2(Alb\,(X \times F), \mathbb{Z}))$ is generated by the classes of the exceptional divisors of the blowing down map

$$\tau : X \times F \to T \times T \times F.$$

The exceptional divisors are over centers which either are of the form $point \times F$ or are proper transforms of subtori isomorphic to $T \times F$. For any exceptional divisor Δ over $point \times F$, its class $[\Delta]$ induces the morphism of Hodge structure

$$\cup [\Delta] : H^1(X \times F, \mathbb{Z}) \to H^3(X \times F, \mathbb{Z})$$

which has for kernel (see the previous proof)

$$H^1(T \times T, \mathbb{Z}) \subset H^1(T \times T \times F, \mathbb{Z}).$$

Indeed, this is also

$$Ker \, H^1(T \times T \times F, \mathbb{Z}) \to H^1(point \times F, \mathbb{Z}).$$

Let $L \subset H^1(X', \mathbb{Z})$ be the image of this last kernel via γ^{-1}. Then $L = Ker \cup \delta$, where $\delta = \gamma^{-1}([\Delta])$. Since δ is a Hodge class, L is a sub Hodge structure of $H^1(X', \mathbb{Z})$. It follows that $Pic^0(X')$ contains a complex subtorus T_L corresponding to the sub Hodge structure L.

Now we consider the four other exceptional divisors Δ_i over the proper transforms of

$$0 \times T \times F, \, T \times 0 \times F, \, T_{diag} \times F, \, T_{graph} \times F.$$

There are, as in the previous proof, 4 corresponding sub Hodge structures L_i of $H^1(X', \mathbb{Z})$ which are the kernels of $\cup \delta_i$, $\delta_i = \gamma^{-1}([D_i])$. It is immediate to see that the L_i are contained in L, thus provide subtori T_{L_i} of T_L. As in the previous proof, one shows that :

1. $T_{L_1} \oplus T_{L_2} = T_L$.

2. T_{L_3} is isomorphic to T_{L_1} and T_{L_2} via the two projections induced by the previous isomorphism. In particular T_{L_1} and T_{L_2} are both isomorphic to some torus T', and $T_L \cong T' \oplus T'$.

3. T_{L_4} is isomorphic to T' via the second projection p_2 induced by the last isomorphism. Hence it is the (transpose of the) graph of an endomorphism $\phi'_{T'}$ of T'.

Since the action of the endomorphism $\phi'_{T'}$ on its homology group H_1 is determined by the position of the lattices L_i, we see as in the previous proof that it has to identify to the dual of ϕ. In conclusion T_L has to be a product $T' \times T'$, where T' admits an endomorphism which acts on its homology as the dual of ϕ. Hence the subtorus T_L of $Pic^0(X')$ cannot be projective, so $Pic^0 X'$ is not projective and X' is not projective. ∎

3 Variants

3.1 Other coefficients

This subsection is due to P. Deligne (letter to the author).

Theorem 4 *Let X be as in Theorem 3. If X' is such that there exists a graded isomorphism of rational cohomology rings*

$$\gamma : H^*(X', \mathbb{Q}) \cong H^*(X, \mathbb{Q}),$$

then X' is not a complex projective manifold.

Remark 5 *Here, and in the previous section, we could conclude more generally that X' cannot be smooth complete algebraic or, in a more analytic language, Moïshezon. This is indeed sufficient to imply the existence of Hodge structures on the cohomology groups of X', and of a polarized Hodge structure on $H^1(X')$, which is equivalent to the fact that $Pic^0(X')$ is a projective complex torus.*

The proof of Theorem 4 is based on the following Lemma 2: let $A^* = \oplus_k A^k$ be the rational cohomology ring of a Kähler compact manifold (or a smooth complex complete algebraic variety) and let $A_{\mathbb{C}}^* := A^* \otimes \mathbb{C}$. Let $Z \subset A_{\mathbb{C}}^k$ be an algebraic subset which is defined by homogeneous equations expressed only using the ring structure on A^*. The examples we shall consider here and in the next subsection are :

1. $Z = \{\alpha \in A_{\mathbb{C}}^k / \alpha^l = 0$ in $A_{\mathbb{C}}^{kl}\}$, where l is a given integer.

2. $Z = \{\alpha \in A_{\mathbb{C}}^k / \cup \alpha : A_{\mathbb{C}}^l \to A_{\mathbb{C}}^{k+l}$ is not injective$\}$, where l is a given integer.

Lemma 2 *Let Z be as above, and let Z_1 be an irreducible component of Z. Assume the $\mathbb{C}$-vector space $< Z_1 >$ generated by Z_1 is defined over $\mathbb{Q}$, that is $< Z_1 >= B_{\mathbb{Q}}^k \otimes \mathbb{C}$ for some $B_{\mathbb{Q}}^k \subset A_{\mathbb{Q}}^k$. Then $B_{\mathbb{Q}}^k$ is a rational sub Hodge structure of $A_{\mathbb{Q}}^k$.*

Proof. It suffices to show that $B_{\mathbb{C}}^k = Z_1$ is stable under the Hodge decomposition of $A_{\mathbb{C}}^k$. The Hodge decomposition can be seen as the character decomposition of the action of $\mathbb{C}^*$ on $A_{\mathbb{C}}^*$ given by

$$z \cdot \alpha = z^p \overline{z}^q \alpha, \ \alpha \in A^{p,q}.$$

So it suffices to show that $< Z_1 >$ is stable under this $\mathbb{C}^*$-action. But Z is stable under this action, by its definition and because the action is compatible with the cup-product :

$$z \cdot (\alpha \cup \beta) = z \cdot \alpha \cup z \cdot \beta.$$

Being an irreducible component of Z, Z_1 is also stable under this $\mathbb{C}^*$-action, and so is $< Z_1 >$. ∎

Proof of Theorem 4. Let P be the subspace of $H^2(X', \mathbb{Q})$ defined as the orthogonal complement with respect to Poincaré duality of $\bigwedge^{4n-2} H^1(X', \mathbb{Q}) \subset H^{4n-2}(X', \mathbb{Q})$. As we mentioned in Remark 4, we used integral coefficients in the proof of Theorem 3 only in order to conclude that P consisted of Hodge classes. Let now $P_0 \subset P$ be defined as

$$P_0 = \{\alpha \in P / \cup \alpha : H^1(X', \mathbb{Q}) \to H^3(X', \mathbb{Q}) \text{ is zero}\}.$$

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By Lemma 2, this is a sub Hodge structure of P. It is the image via γ^{-1} of the space generated by classes of exceptional divisors over points. Furthermore, we have for $\alpha \in P/P_0$ an induced cup-product map

$$\cup \alpha : H^1(X', \mathbb{Q}) \to H^3(X', \mathbb{Q}).$$

Introduce now the algebraic subset of $(P/P_0) \otimes \mathbb{C}$:

$$Z = \{\alpha \in (P/P_0) \otimes \mathbb{C}, \ \cup \alpha : H^1(X', \mathbb{Q}) \to H^3(X', \mathbb{Q}) \text{ is not injective}\}.$$

This algebraic subset is the image via γ^{-1} of the union of four 1-dimensional $\mathbb{C}$-vector spaces, in fact defined over $\mathbb{Q}$, generated respectively by the classes of the four exceptional divisors over the blown-up complex subtori of $T \times T$. We can thus apply Deligne's Lemma to each of these lines, thus concluding that the four classes that we have denoted $\delta.$ in the proof of Theorem 3, projected in P/P_0, are Hodge classes for the Hodge structure on P/P_0.

Having this, we conclude as in the proof of Theorem 3.

∎

In the above proof, we needed rational coefficients, in order to make sure that the four lines above were defined over $\mathbb{Q}$. Deligne constructs now an example where the complex cohomology algebra suffices to imply this. The construction works as follows : start with the previous X. The four exceptional divisors $\Delta.$ dominating the four complex subtori $T \times 0$, $0 \times T$, T_{diag}, T_{graph} are of the form $\widetilde{T}. \times \mathbb{P}^{n-1}$, where $\widetilde{T}.$ is obtained by blowing-up finitely many (depending on which subtorus we consider) points of T. Indeed, the normal bundle N of each of these subtori $T.$ in $T \times T$ is trivial, and if $\tau : \widetilde{T}. \to T.$ is the proper transform of $T.$ under the initial blowing-up $\widetilde{T \times T} \to T \times T$ of points, the normal bundle of $\widetilde{T}.$ in $\widetilde{T \times T}$ is isomorphic to $\tau^* N(E)$, where E is the exceptional divisor of τ.

Let us blow-up one subvariety of the form $\widetilde{T}_{0 \times T} \times \alpha_1$ in $\Delta_{0 \times T}$, then two subvarieties of the form $\widetilde{T}_{diag} \times \beta_1$, $\widetilde{T}_{diag} \times \beta_2$ in Δ_{diag}, and three subvarieties of the form $\widetilde{T}_{graph} \times \gamma_1$, $\widetilde{T}_{graph} \times \gamma_2$, $\widetilde{T}_{graph} \times \gamma_3$ in Δ_{graph}. We have constructed this way a smooth Kähler compact manifold X_1.

Theorem 5 *If X_1' is such that there exists a graded isomorphism of complex cohomology algebras*
$$\gamma : H^*(X_1', \mathbb{C}) \cong H^*(X_1, \mathbb{C}),$$
then X_1' is not a complex projective manifold.

Proof. We use the same notations as before and consider the multiplication map
$$\cup \alpha : H^1(X_1', \mathbb{Q}) \to H^3(X_1', \mathbb{Q}),$$
for $\alpha \in P/P_0$. Let us introduce as in the previous proof the algebraic subset Z of $(P/P_0) \otimes \mathbb{C}$ consisting of those α for which $\cup \alpha$ is not injective. On one hand it is defined over $\mathbb{Q}$, and on the other hand, using the isomorphism γ, it

consists of the union of four complex spaces, which are of respective dimensions 1, 2, 3, 4. (The reason is that, for X_1, the class of the exceptional divisor $\Delta_{T \times 0}$ generates the first of these spaces, while the second one is generated by the classes of the two exceptional divisors over $0 \times T$ and so on.) It follows that each of these subspaces is in fact defined over $\mathbb{Q}$. Hence we can apply Lemma 2, to conclude that they are sub Hodge structures of P/P_0. Now for each of these spaces V, there is only one common kernel of the map $\cup \alpha$, $\alpha \in V$, which has to be a sub Hodge structure of $H^1(X_1', \mathbb{Q})$. So we have produced again the four sub Hodge structures of $H^1(X_1', \mathbb{Q})$, and the respective positions of the associated subspaces of $H^1(X_1', \mathbb{C})$ are the same as for X_1 via the isomorphism γ. We then concludes that $H^1(X_1', \mathbb{Q})$ splits into the sum of two copies of a rational Hodge structure which admits an automorphism which over $\mathbb{C}$ is conjugate to ${}^t\phi$. But then this automorphism is conjugate to ${}^t\phi$ also over $\mathbb{Q}$ and we can conclude as in the proof of Theorem 3.

■

Note finally that if Z is any reasonable simply connected compact topological space, Theorem 4 (resp. 5) remains true when X is replaced with $X \times Z$, resp. X_1 is replaced with $X_1 \times Z$: the rational cohomology ring, resp. complex cohomology algebra of $X \times Z$, resp. $X_1 \times Z$, is not the one of a complex projective or Moïshezon variety.

Indeed, the H^1 is not modified by taking product with Z, and we recover the space P introduced in the above proofs as the annihilator of $\bigwedge^{4n-2} H^1$ with respect to the cup-product. The rest of the proof goes as before, leading to the conclusion that $H^1(Y, \mathbb{Q})$ cannot be endowed with a polarized Hodge structure, for any Y having the same rational (resp. complex) cohomology algebra as $X \times Z$, resp. $X_1 \times Z$.

3.2 A simply connected example

In this subsection, we construct *simply connected* compact Kähler manifolds which do not have the homotopy type (in fact the rational cohomology ring) of a projective complex manifold.

We start again with a torus T as in section 1, endowed with an endomorphism ϕ_T, but assume now that its dimension n is ≥ 3. Let us introduce the generalized Kummer variety

$$K = \widetilde{T/\pm 1},$$

that is the desingularization of the quotient of T by the -1 involution, obtained by blowing-up the points of order 2. This is a simply connected compact Kähler manifold. Its cohomology $H^2(K, \mathbb{Q})$ is the direct sum of $\bigwedge^2 H^1(T, \mathbb{Q})$ and of the space generated by the exceptional divisors.

In $K \times K$, let us blow-up the diagonal, and then the proper transform of the graph of ϕ_K. (ϕ_T descends to a rational self-map of K, which we denote ϕ_K. Note that ϕ_K is not holomorphic, because one point of T which

is not of 2-torsion can be sent by ϕ_T to a 2-torsion point of T, producing an indeterminacy point of ϕ_K. However, it is easy to see that the graph of ϕ_K is smooth, isomorphic to the blow-up of K at these points.)

This will be our variety X_2. We shall denote by $\tau : X_2 \to K \times K$ the blowing-down map.

Theorem 6 *If X_2' is such that there exists a graded isomorphism of rational cohomology rings*

$$\gamma : H^*(X_2', \mathbb{Q}) \cong H^*(X_2, \mathbb{Q}),$$

then X_2' is not a complex projective manifold.

The proof will proceed in several lemmas.

Lemma 3 *Consider the subspaces*

$$A_i^2 := \gamma^{-1}(\tau^* \circ pr_i^*(\bigwedge^2 H^1(T, \mathbb{Q}))), \ i = 1, \ 2$$

of $H^2(X_2', \mathbb{Q})$. Then A_i^2 are rational sub Hodge structures of $H^2(X_2', \mathbb{Q})$.

Proof. Let Z' be the algebraic subset of $H^2(X_2', \mathbb{C})$ defined as

$$Z' = \{\alpha \in H^2(X_2', \mathbb{C})/\alpha^2 = 0\}.$$

This Z' is the image under γ^{-1} of the corresponding subset Z of $H^2(X_2, \mathbb{C})$. Using the fact that X_2 has been deduced from $(T/\pm 1) \times (T/\pm 1)$ by a sequence of blow-ups with centers of codimension ≥ 3, one sees easily that $Z = Z_1 \cup Z_2$, where

$$Z_i = \{\tau^* \circ pr_i^*(\alpha)/\alpha \in \bigwedge^2 H^1(T, \mathbb{C}), \ \alpha^2 = 0 \operatorname{in} \bigwedge^4 H^1(T, \mathbb{C})\}.$$

But $\bigwedge^2 H^1(T, \mathbb{C})$ is generated by those α such that $\alpha^2 = 0$. It follows that $< Z_i >= \tau^* \circ pr_i^*(\bigwedge^2 H^1(T, \mathbb{C}))$.

Applying Lemma 2 to $Z_i' := \gamma^{-1}(Z_i)$ gives the result.

∎

Let now A^* be the subalgebra of $H^*(X_2', \mathbb{Q})$ generated by $A_1^2 \oplus A_2^2$ and let $P \subset H^2(X_2', \mathbb{Q})$ be the orthogonal complement with respect to Poincaré duality of A^{4n-2}. This space P is the image under γ^{-1} of the subspace E of $H^2(X_2, \mathbb{Q})$ generated by the classes of exceptional divisors of X_2 over $(T/\pm 1) \times (T/\pm 1)$. This last space contains the two divisors classes $[\Delta_{diag}]$ and $[\Delta_{graph}]$, and the divisors classes $[\Delta_{x_i} \times K]$, $[K \times \Delta_{x_i}]$ where the x_i's are the 2-torsion points of T. As in the previous section, consider the algebraic subset

$$Z' = \{\alpha \in P/ \cup \alpha : A^2 \to H^4(X_2', \mathbb{C}) \text{ is not injective}\}.$$

Z' is the image under γ^{-1} of the corresponding subset Z of E. Now it is easy to check that Z is the union of four vector spaces defined over $\mathbb{Q}$, namely

$$< [\Delta_{diag}] >, \ < [\Delta_{graph}] >, \ < [\Delta_{x_i} \times K], \ i = 1, \ldots, 2^{2n} >, \ < [K \times \Delta_{x_i}], \ i = 1, \ldots, 2^{2n} > .$$

Applying Lemma 2, we conclude that the classes

$$\delta_{diag} := \gamma^{-1}([\Delta_{diag}]), \ \delta_{graph} = \gamma^{-1}([\Delta_{graph}])$$

are Hodge classes in $H^2(X_2', \mathbb{Q})$. The kernels of the cup-product maps

$$\cup \delta. : A^2 \rightarrow H^4(X_2', \mathbb{Q})$$

are thus rational sub Hodge structures of $A_1^2 \oplus A_2^2$. Examining via γ their position in $H^2(X_2, \mathbb{Q})$, we conclude that A_1^2 and A_2^2 are isomorphic as rational Hodge structures and that this rational Hodge structure carries an automorphism which acts as $\phi_T^* = \bigwedge^{2t} \phi$ on $\bigwedge^2 H^1(T, \mathbb{Q})$.

Next, considering the proof of Lemma 1, we see that it actually shows the following :

Lemma 4 *Assume the $\mathbb{Q}$-vector space $\bigwedge^2 H^1(T, \mathbb{Q})$ is endowed with a Hodge structure which is preserved by $\bigwedge^{2t} \phi$. Then either this Hodge structure is trivial, that is contains only Hodge classes, or it has no non zero Hodge classes.*

We have now

Lemma 5 *If the Hodge structure on A_i^2 is trivial, then X_2' cannot be projective.*

Proof. Recall that $A_i^2 = \gamma^{-1}(\tau^* \circ pr_i^*(\bigwedge^2 H^1(T, \mathbb{Q})))$. It follows that it contains a $\mathbb{Q}$-vector subspace V of dimension ≥ 2 such that, for any $\alpha \in V$, $\alpha^2 = 0$ in $H^4(X_2', \mathbb{Q})$. (Take for V a space of the form $\gamma^{-1}(\tau^* \circ pr_i^*(\alpha \wedge H^1(T, \mathbb{Q})))$ for a non zero $\alpha \in H^1(T, \mathbb{Q})$.)

This V is then isotropic for any intersection form on $H^2(X_2', \mathbb{Q})$ of the form

$$q_c(\alpha) = c^{2n-2}\alpha^2,$$

where $c \in H^2(X_2', \mathbb{Q})$. If X_2' were projective, for an ample class c, this intersection form would have, by the Hodge index theorem ([9], 6.3.2), only one positive sign, when restricted to the space of Hodge classes of degree 2. That would contradict the existence of this 2-dimensional isotropic V contained in A_i^2 if the Hodge structure on A_i^2 were trivial.

∎

Remark 6 *The proof even shows that under the same assumptions, X_2' could not be Kähler.*

The two last lemmas together show that the sub Hodge structure A^2 does not contain any Hodge class if X_2' is projective. But then any degree 2 Hodge class on X_2' must be contained in P. On the other hand, we have the following:

Lemma 6 *For any class $c \in P$, the intersection form $q_c(\alpha) = c^{2n-2}\alpha^2$ on $H^2(X_2', \mathbb{Q})$ vanishes on A^2.*

This is proved using the fact that P is the image under γ^{-1} of the space generated by the (exceptional) divisor classes of X_2, and by proving the analogous result on X_2. It is obviously essential here to assume that $n \geq 3$. ∎

The proof of Theorem 6 is now concluded by contradiction. If X_2' were projective, any degree 2 Hodge class on X_2' should be contained in P, and P should contain an ample class c. But then, again by the Hodge index theorem, the form q_c would not vanish on A^2, contradicting Lemma 6.

■

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