Flat pushforwards of Chern classes and the smoothability of cycles in the Whitney range

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Abstract

We prove in this paper the smoothability of cycles modulo rational equivalence in the Whitney range, that is, when the dimension is strictly smaller than the codimension. We introduce and study the class of cycles obtained as "flat pushforwards of Chern classes" (or equivalently, flat pushforwards of products of divisors) and prove that they are smoothable in the Whitney range. Our main result is that all cycles (of any dimension) on a smooth projective variety are flat pushforwards of Chern classes. In the case of abelian varieties, one can even restrict to smooth pushforwards of Chern classes.

1 Introduction

Let X be a smooth projective variety of dimension n. Following [8], we will say that a cycle class $z \in \mathrm{CH}_d(X)$ is smoothable if it belongs to the subgroup of $\mathrm{CH_d(X)}$ generated by the classes of d-dimensional smooth subvarieties of X. A number of nonsmoothability results have been proved outside of the Whitney range, that is, when $2d \geq n$, since the question of smoothability was first raised by Borel and Haefliger [2] for cohomology classes. Hartshorne, Rees and Thomas [8] proved the nonsmoothability of the second Chern class $c_2(\mathcal{E})$ of the tautological subbundle $\mathcal{E}$ on a Grassmannian $G(3, n)$, $n \geq 6$. Debarre [5] proved the nonssmoothability of the minimal class $\frac{\theta^2}{2}$ on a very general Jacobian of curve of genus ≥ 7, where θ is the class of a Theta-divisor. This class of examples has been greatly expanded in [3]. Benoist [1] exhibits examples of nonsmoothable d-cycles on varieties of dimension n for many possible pairs (d, n) outside of the Whitney range, including the case where $2d = n$, under some arithmetic condition on the codimension $c = n-d$. When $2d \geq n$, a big question which remains completely open despite these counterexamples concerns the smoothability of cycles with $\mathbb{Q}$-coefficients.

When $2d - 1 \leq n$, Kleiman proves in [11] that for any cycle $z \in \mathrm{CH}_d(X)$ of codimension c, the cycle $(c-1)!z$ is smoothable. (A similar result in the range $2d < n$ already appeared in [9], but Hironaka acknowledges there the help of Kleiman.) For the cycles themselves (as opposed to a multiple), Hironaka proved in 1968 the following result.

Theorem 1.1. *(Hironaka [9]) Cycles of dimension $d \leq 3$ are smoothable on smooth varieties of dimension $n > 2d$.*

We study in this paper the smoothability problem for cycles modulo rational equivalence in the Whitney range, which is mentioned in the introduction of [1], and our first main result is

Theorem 1.2. *Let X be a smooth projective variety of dimension n, defined over a field of characteristic 0. Then for any integer d such that $2d < n$, any cycle $z \in \mathrm{CH_d(X)}$ is smoothable.*

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Note that this result is related to the easy case of Whitney's embedding theorem in differential topology, thanks to Hironaka resolution theorem [10]. Indeed, for any d-dimensional subvariety $Z \subset X$, we can resolve the singularities of Z and get a proper morphism $\tilde{j} : \widetilde{Z} \to X$ such that $\tilde{j}_*[\widetilde{Z}] = [Z]$ in $\mathrm{CH}_d(X)$. Our statement is that, if $2d < n$, we can replace modulo rational equivalence $\tilde{j} : \widetilde{Z} \to X$ by an integral combination of embeddings $j_i : Z_i \hookrightarrow X$ of smooth subvarieties. The difficulty is the following : We can of course embed $\widetilde{Z}$ in $X \times \mathbb{P}^n$ over X, and then, by easy Whitney type criteria, we see that it suffices to construct a cycle $Z' = \sum_i n_i Z'_i$ in $X \times \mathbb{P}^n$, which is rationally equivalent to $\widetilde{Z}$, and such that the Z'_i are both smooth and in general position. The Chow moving lemma provides such a cycle in general position but unfortunately the Z'_i are not smooth starting from dimension 4.

Remark 1.3. In [1, Theorem 0.3], Benoist produces examples of cycles of dimension d and codimension d that are not smoothable for infinitely many values of d. Theorem 1.2 shows that, at least for these values of d, his examples are optimal and the Whitney condition is necessary for smoothability.

Theorem 1.2 is obtained as a consequence of a more general structure result for algebraic cycles of any dimension that we now describe. For any smooth variety X, we denote by $\mathrm{CH}^*(X)_{\mathrm{Ch}} \subset \mathrm{CH}^*(X)$ the subring generated by Chern classes of vector bundles on X. The standard formula (7) combined with locally free resolutions shows that

$$(c-1)!\mathrm{CH}^c(X) \subset \mathrm{CH}^c(X)_{\mathrm{Ch}}.$$

However, it is well-known that $\mathrm{CH}^*(X)_{\mathrm{Ch}}$ can be a proper subring of $\mathrm{CH}^*(X)$. We refer to [4] for an explicit example and to Lemma 3.5 for another example. As discussed in Section 4, further examples are provided by homogeneous varieties G/H, where G is a semi-simple, simply connected group, H is a Borel subgroup and the torsion order of G is not 1 (we are grateful to Burt Totaro for explaining this to us). Cycles in $\mathrm{CH}_d(X)_{\mathrm{Ch}}$ are relevant for our subject, since we know that they are smoothable under the Whitney condition $2d < \dim X$. (We will give in Section 2 an argument which involves Segre classes and seems slightly different from Kleiman's and Hironaka's arguments.)

Let us now introduce further definitions.

Definition 1.4. *Let X be smooth. We will denote by $\mathrm{CH}(X)_{\mathrm{flpshCh}}$ (for "flat pushforward of Chern classes"), resp. $\mathrm{CH}(X)_{\mathrm{smpshCh}}$ (for "smooth pushforward of Chern classes") the subgroup generated by cycles of the form $\pi_* z'$ for a flat, resp. smooth, proper morphism $\pi : P \to X$, with P smooth, and for some cycle $z' \in \mathrm{CH}(P)_{\mathrm{Ch}}$.*

Lemma 2.4 proved in Section 3.1 says that $\mathrm{CH}^*(X)_{\mathrm{Ch}} \subset \mathrm{CH}^*(X)$ is generated by smooth pushforwards of classes of complete intersections of divisors, so that, in Definition 1.4, we could replace "cycle in $\mathrm{CH}(P)_{\mathrm{Ch}}$" by "cycle in the subring of $\mathrm{CH}^*(P)$ generated by divisors". Further easy properties are discussed in Section 3.1.

We prove in Section 2 the following basic

Proposition 1.5. *(Cf. Proposition 2.9) Cycles in $\mathrm{CH}_d(X)_{\mathrm{flpshCh}}$ are smoothable in the Whitney range $2d < \dim X$.*

The analogous result for cycles in $\mathrm{CH}_d(X)_{\mathrm{smpshCh}}$ is a standard Whitney type statement. Proposition 1.5 is our motivation to introduce Definition 1.4. Our second main theorem is the following

Theorem 1.6. *Let X be a smooth projective variety. Then*

$$\mathrm{CH}(X) = \mathrm{CH}(X)_{\mathrm{flpshCh}}. \tag{1}$$

In other words, any cycle on a smooth projective variety X over a field of characteristic 0 is obtained as a flat pushforward of intersections of divisors on a smooth projective variety Y.

Theorem 1.2 follows from Theorem 1.6 and Proposition 1.5.

The first step in the proof of Theorem 1.6 is the following Theorem 1.7 proved in Section 3.3, which establishes stability properties of $\mathrm{CH}(\cdot)_{\mathrm{flpshCh}}$ under certain non-flat pushforwards. We will use here and in the rest of the paper the terminology of *complete bundle-section in X*, for "closed algebraic subset of codimension c which is the zero-set of a section of a vector bundle of rank c on X".

Theorem 1.7. *(i) (Cf. Proposition 3.7.) Given a finite morphism $j : Y \to X$, where Y and X are smooth projective and $\dim Y = \dim X - 1$, one has*

$$j_*(\mathrm{CH}(Y)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}.$$

(ii) (Cf. Proposition 3.9.) Let X be a smooth projective variety and let $j : Y \hookrightarrow X$ be the inclusion of a smooth subvariety which is a complete bundle-section in X. Then $j_*(\mathrm{CH}(Y)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}$.

(iii) (Cf. Proposition 3.11.) Let X be smooth projective and let $Y \subset X$ be a smooth complete bundle-section in X. Let $\tau : \widetilde{X} = B_Y X \to X$ be the blow-up of X along Y. Then

$$\tau_*(\mathrm{CH}(\widetilde{X})_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}.$$

We will also show in Section 3.3 how Theorem 1.7 easily implies Theorem 1.6 for cycles of dimension ≤ 3 (cf. Theorem 3.12).

Theorem 1.6 is then obtained as a consequence of the following "cbs resolution theorem", which will be proved in Section 4.

Theorem 1.8. *Let $Z \subset X$ be a smooth subvariety of dimension d, with X smooth projective and $\dim X > 4d$. Then, after successive blow-ups of smooth complete bundle-sections $W_i \subset X_i \stackrel{\tau_i}{\to} X_{i-1}$, $i = 0, \ldots, r$, $X_0 = X$, the proper transform $Z_r \subset X_r$ is a smooth connected component of a smooth complete bundle-section Z_r^* on X_r.*

Corollary 1.9. *Let $Z \subset X$ be smooth, projective varieties such that $\dim Z < \frac{1}{4}\dim X$. Then after successive blow-ups $X_{r+1} \stackrel{\tau_{r+1}}{\to} X_r \stackrel{\tau_r}{\to} \cdots \to X_0 := X$ along smooth complete bundle-sections $W_i \subset X_i$, there exists a complete intersection subvariety $Z_{r+1} \subset X_{r+1}$ such that $\Pi_*(Z_{r+1}) = Z$ as effective cycles in X, where $\Pi : X_{r+1} \to X$ is the composition of the τ_i's.*

Proof. Let $X_r \to \cdots \to X_0 := X$ be as in Theorem 1.8, and let $\pi_r : X_{r+1} \to X_r$ be the blow-up of Z_r^*. Let E_{r+1} be the π_r-exceptional divisor lying over Z_r. Choose H_r sufficiently ample on X_r such that $|\pi_r^* H_r - E_{r+1}|$ restricts to a very ample divisor on E_{r+1}. Then we can choose general members $D^i \in |\pi_r^* H_r - E_{r+1}|$ to obtain a complete intersection subvariety $Z_{r+1} := \big(E_{r+1} \cap D^1 \cap \cdots \cap D^c\big)$ satisfying the desired property, where $c = \dim X - \dim Z - 1$. ☐

Theorem 1.7 and Corollary 1.9 immediately imply Theorem 1.6. Indeed, let $Z \subset X$ be a subvariety of dimension d. By desingularizing Z, we get a smooth subvariety $Z' \subset X \times \mathbb{P}^N$, with N large, projecting to $Z \subset X$. This way, we reduced to proving that $[Z] \in \mathrm{CH}_d(X)_{\mathrm{flpshCh}}$ when Z is smooth and $\dim X \geq 4\dim Z$. We apply Corollary 1.9 to $Z \subset X$, and get $Z_{r+1} \subset X_{r+1} \stackrel{\Pi}{\to} X$ such that $[Z] = \Pi_*[Z_{r+1}]$ in $\mathrm{CH}_d(X)$, where Π is a composition of blow-ups along smooth complete bundle-section centers and Z_{r+1} is a complete intersection of divisors in X_{r+1}. By Theorem 1.7(iii), it follows that

$$\Pi_*[Z_{r+1}] = [Z] \in \mathrm{CH}_d(X)_{\mathrm{flpshCh}}.$$

The methods used to prove Theorems 1.6 and 1.7 do not allow us to prove the stronger result that $\mathrm{CH}_d(X) = \mathrm{CH}_d(X)_{\mathrm{smpshCh}}$. In particular, the proof of the main Proposition 3.7 (Theorem 1.7(i) above) does not work if we replace the groups $\mathrm{CH}_d(Y)_{\mathrm{flpshCh}}$ and $\mathrm{CH}_d(X)_{\mathrm{flpshCh}}$ respectively by $\mathrm{CH}_d(Y)_{\mathrm{smpshCh}}$ and $\mathrm{CH}_d(X)_{\mathrm{smpshCh}}$. This leaves open the following

Question 1.10. *Are there smooth projective varieties X such that* $\mathrm{CH}(X) \neq \mathrm{CH}(X)_{\mathrm{smpshCh}}$*?*

As follows from Theorem 1.11 below, the equality $\mathrm{CH}(X) = \mathrm{CH}(X)_{\mathrm{smpshCh}}$ is satisfied by abelian varieties but it could be that for the example treated in Lemma 3.5, or for some homogeneous varieties, we have $\mathrm{CH}_0(X) \neq \mathrm{CH}_0(X)_{\mathrm{smpshCh}}$.

In section 5, we will give an alternative proof of Theorem 1.6 for homogeneous varieties, which does not use the cbs resolution Theorem 1.8, and which, in the case of abelian varieties, even gives a stronger result.

Theorem 1.11. *Let X be a homogeneous projective variety.*

(i) If there exists a smooth projective G-equivariant completion $\overline{G}$ of G which satisfies $\mathrm{CH}_0(\overline{G}) = \mathrm{CH}_0(\overline{G})_{\mathrm{smpshCh}}$*, then*

$$\mathrm{CH}_d(X)_{\mathrm{smpshCh}} = \mathrm{CH}_d(X) \tag{2}$$

for all d.

(ii) If A is an abelian variety, then $\mathrm{CH}_d(A) = \mathrm{CH}_d(A)_{\mathrm{smpshCh}}$ *for any d.*

We do not know if the assumption in Theorem 1.11(i) is always satisfied. The stronger version that there always exists a smooth projective G-equivariant completion $\overline{G}$ of G which satisfies

$$\mathrm{CH}_0(\overline{G}) = \mathrm{CH}_0(\overline{G})_{\mathrm{Ch}} \tag{3}$$

is wrong for abelian varieties. For simply connected groups, the condition (3) seems to be closely related to the torsion order of G being 1 (see [6] and Section 4), but the precise relation is not obvious to us.

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2 Whitney type statements

We first prove the following basic Whitney type result

Proposition 2.1. *Let $\phi : Y \to X$ be a flat morphism between smooth varieties. Let $n = \dim X$. Then for a smooth subvariety $Z \subset Y$ of dimension $d < \frac{n}{2}$ which is in general position and such that the restriction $\phi_{|Z}$ is proper, $\phi_{|Z} : Z \to \phi(Z)$ is an isomorphism, so the closed algebraic subset $\phi(Z) \subset X$ is smooth. Furthermore, if $n = 2d$, $\phi_{|Z}$ is an immersion and the image $\phi(Z)$ has finitely many singular points.*

Although this will be clear from the proof, let us first make precise what we mean by "in general position". This is a transversality condition with respect to ϕ and its infinitesimal properties. For the proposition above, it is enough that Z is the general fiber $\mathcal{Z}_b$ of a family of embeddings

$$\begin{array}{ccc} \mathcal{Z} & \stackrel{f}{\to} & Y \\ p\downarrow & & \\ B & & \end{array} \tag{4}$$

where $\mathcal{Z}$ is smooth and p is smooth, which is very mobile at any point (x, y) of $Z \times Z \cong \mathcal{Z}_b \times \mathcal{Z}_b$, $x \neq y$, (that is, $(f, f) : \mathcal{Z} \times_B \mathcal{Z} \to Y \times Y$ is a submersion at any point (x, y) of $Z \times Z \setminus \Delta_Z$) and whose tangent space is mobile at any point of Z, that is, the morphism

$$F : \mathbb{P}(T_{\mathcal{Z}/B}) \to \mathbb{P}(T_Y)$$

given by the differential of the inclusions $f_b : \mathcal{Z}_b \to Y$, is submersive at any point of Z. The important fact for us is the following

Remark 2.2. Assuming $Y \subset \mathbb{P}^N$ is projective of dimension m, these general position properties will be satisfied by a general complete intersection of $m-d$ very ample divisors.

For the proof of Proposition 2.1, we will use the following consequence of the "general position" assumption.

Lemma 2.3. *Let $Z \subset Y$ be a smooth subvariety of dimension d which is in general position in the above sense. Then,*

(i) If $W \subset Y$ is a closed algebraic subvariety of codimension $> d$, Z does not intersect W.

(ii) If $W \subset Y \times Y$ is a subvariety of codimension $> 2d$, $Z \times Z$ does not intersect W away from the diagonal of Z.

(iii) If $W \subset \mathbb{P}(T_Y)$ is a subvariety of codimension $\geq 2d$, $\mathbb{P}(T_Z)$ does not intersect W.

Proof. (i) We use the notations of (4), with $Z = \mathcal{Z}_b$, $b \in B$ being a general point of B. As f is a submersion along $\mathcal{Z}_b$, there exists a Zariski neighborhood U of $\mathcal{Z}_b$ in $\mathcal{Z}$ such that any component of $f^{-1}(W) \cap U$ has codimension $> d$ in U. As $\dim B = \dim \mathcal{Z} - d$, it follows that

$$p_{|f^{-1}(W)\cap U} : f^{-1}(W) \cap U \to B$$

cannot be dominant, hence for a general $b \in B$, $\mathcal{Z}_b$ does not intersect $f^{-1}(W)$, that is, $Z = f(\mathcal{Z}_b)$ does not intersect W.

(ii) The argument is the same as above with f replaced by $(f,f) : \mathcal{Z} \times_B \mathcal{Z} \to Y \times Y$. The fibers of $(p,p) : \mathcal{Z} \times_B \mathcal{Z} \to B$ are now of dimension $2d$ and (f,f) is a submersion away from the diagonal of $\mathcal{Z}_b$, so if $W \subset Y \times Y$ has codimension $> 2d$, $(f,f)^{-1}(W)$ will have codimension $> 2d$ in $\mathcal{Z} \times_B \mathcal{Z} \setminus \Delta_{\mathcal{Z}}$, and will not dominate B, since $\dim B = \dim(\mathcal{Z} \times_B \mathcal{Z}) - 2d$.

(iii) The argument is the same as above except that we work now with $F : \mathbb{P}(T_{\mathcal{Z}/B}) \to \mathbb{P}(T_Y)$. We simply observe that the fibers of the natural map $\mathbb{P}(T_{\mathcal{Z}/B}) \to B$ are of dimension $2d-1$. $\square$

Proof of Proposition 2.1. Let $\Delta_Y \subset Y \times Y$ be the diagonal and let $Y' \subset Y \times Y \setminus \Delta_Y$ be the closed algebraic subset $Y \times_X Y \setminus \Delta_Y$. By flatness of ϕ, we have $\operatorname{codim}(Y' \subset Y \times Y) = n$. As $\dim Z \times Z = 2d < n$ and Z is in general position, $Z \times Z$ does not intersect Y' away from the diagonal by Lemma 2.3(ii), so $\phi_{|Z}$ is injective. When $n = 2d$, $Z \times Z$ intersects Y' away from the diagonal in at most finitely many points. It remains to prove the infinitesimal statement, for which we only assume that $2d \leq n$. Let $Y_k \subset Y$ be the locally closed subset of Y where the rank of ϕ is equal to k. Then $\dim \phi(Y_k) \leq k$, hence by flatness, $\operatorname{codim}(Y_k \subset Y) \geq n-k$, or equivalently $\dim Y_k \leq m+k$, where $m := \dim Y - n$. Along Y_k, we have the rank k morphism

$$\phi_k := (\phi_*)_{|Y_k} : T_{Y|Y_k} \to (\phi^* T_X)_{|Y_k}$$

with kernel a subbundle $\mathcal{K}_k \subset T_{Y|Y_k}$ of corank k. Let $W \subset \mathbb{P}(T_Y)$ be the set of pairs (y,u), $y \in Y$, $u \in \operatorname{Ker} \phi_{*,y}$. The stratification of Y by the Y_k's describes W as a union

$$W = \sqcup_k \mathbb{P}(\mathcal{K}_k).$$

As $\dim Y_k \leq k+m$ and $\operatorname{rk} \mathcal{K}_k = m+n-k$, we get

$$\dim \mathbb{P}(\mathcal{K}_k) \leq 2m+n-1$$

for any k, and thus $\dim W \leq 2m+n-1$. As $\dim \mathbb{P}(T_Y) = 2(m+n)-1$, it follows that

$$\operatorname{codim}(W \subset \mathbb{P}(T_Y)) \geq n.$$

By Lemma 2.3(iii), Z being of dimension d and in general position with $n \geq 2d$, $\mathbb{P}(T_Z)$ does not intersect W. This means that $\phi_{|Z}$ is an immersion, which concludes the proof. $\square$

We will combine Proposition 2.1 with the following easy result.

Lemma 2.4. *Let X be smooth of dimension n and let $z \in \mathrm{CH}_d(X)$ be a cycle. Assume that z belongs to the subring $\mathrm{CH}^*(X)_{\mathrm{Ch}}$ of $\mathrm{CH}^*(X)$ generated by Chern classes $c_i(E)$ for any coherent sheaf E on X. Then there exist a smooth projective variety Y and a smooth morphism $f : Y \to X$ such that $z = f_*z'$ in $\mathrm{CH}(X)$, where $z' \in \mathrm{CH}(Y)$ belongs to the subring generated by divisors on Y.*

Proof. First of all, using finite locally free resolutions and the Whitney formula, we know that z belongs to the subring of $\mathrm{CH}^*(X)$ generated by the Chern classes $c_i(E)$ for any locally free coherent sheaf E on X. Secondly, we can replace in this statement the Chern classes by the Segre classes, since the total Segre and Chern classes $s(E)$ and $c(E)$ satisfy the relation

$$s(E) = c(E)^{-1},\ c(E) = s(E)^{-1},$$

so any polynomial with integral coefficients in the Segre classes is a polynomial with integral coefficients in the Chern classes and vice-versa.

It thus suffices to prove that any monomial $s_{i_1}(E_1)\ldots s_{i_k}(E_k) \in \mathrm{CH}(X)$, where the E_i's are locally free sheaves on X of rank r_i, satisfies the conclusion of Lemma 2.4. This statement follows from the definition of Segre classes (see [7]). Indeed, let $\pi_i : \mathbb{P}(E_i) \to X$ be the projectivization of E_i and let $H_i \in \mathrm{Pic}(\mathbb{P}(E_i))$ with first Chern class $c_1(H_i) \in \mathrm{CH}^1(\mathbb{P}(E_i))$ be the dual of its Hopf line bundle (so that $R^0\pi_{i*}H_i = E_i^*$). Then

$$s_j(E_i) = \pi_{i*}(c_1(H_i)^{j+r_i-1}) \text{ in } \mathrm{CH}(X). \tag{5}$$

It follows from (5) and the projection formula that

$$s_{i_1}(E_1)\ldots s_{i_k}(E_k) = \pi_*(\mathrm{pr}_1^*c_1(H_1)^{i_1+r_1-1}\ldots \mathrm{pr}_k^*c_1(H_k)^{i_k+r_k-1}) \text{ in } \mathrm{CH}(X), \tag{6}$$

where $\pi : \mathbb{P}(E_1) \times_X \ldots \times_X \mathbb{P}(E_k) \to X$ is the fibred product of the $\pi_i : \mathbb{P}(E_i) \to X$ and pr_i is the projection from $\mathbb{P}(E_1) \times_X \ldots \times_X \mathbb{P}(E_k)$ to its i-th factor. □

Corollary 2.5. *Let X be smooth of dimension n and let $z \in \mathrm{CH}_d(X)_{\mathrm{Ch}}$, with $2d < n$. Then z is smoothable, that is, z is rationally equivalent to a cycle $Z' = \sum_i n_i Z_i'$, where $Z_i' \subset X$ is smooth.*

Proof. Using Lemma 2.4, the result follows from Proposition 2.1 by Remark 2.2. □

Corollary 2.6. *(Hironaka [9], Kleiman [11]) If X is smooth and $2d < \dim X$, any cycle $z \in \mathrm{CH}_d(X)_{\mathbb{Q}}$ is rationally equivalent to a smooth cycle with $\mathbb{Q}$-coefficients. More precisely $(c-1)!z$ is smoothable, where $c := n - d$ is the codimension of z.*

Proof. Indeed, it suffices to prove the result when $z = [Z]$ is the class of a subvariety Z of X of dimension d. Let $\mathcal{O}_Z$ be the structural sheaf of Z, seen as a coherent sheaf on X. It follows from the Grothendieck-Riemann-Roch formula (see [7, Example 15.3.1]) that

$$c_{n-d}(\mathcal{O}_Z) = (-1)^{n-d-1}(n-d-1)![Z] \ \in\ \mathrm{CH}_d(X)_{\mathrm{Ch}} \subset \mathrm{CH}_d(X), \tag{7}$$

so Corollary 2.5 applies. □

Remark 2.7. In [9], which does not use Segre classes but the splitting principle to reduce Chern classes to products of divisors, the coefficient $(c-1)!$ appears multiplied by a constant, which is possibly 1.

Remark 2.8. Kleiman in [11] argues differently by studying singularities of Schubert varieties and proves a result which is of a different nature, as it also includes the cases where $n = 2d$ or $2d - 1$, which are not in the Whitney range.

Combining Lemma 2.4 and Proposition 2.1, we get the following criterion

Proposition 2.9. *Let $\phi : Y \to X$ be a proper flat morphism between smooth varieties. Then for any cycle $z \in \mathrm{CH}_d(Y)_{\mathrm{Ch}}$ with $2d < n = \dim X$, the class $z' = \phi_*z \in \mathrm{CH}_d(X)$ is smoothable on X.*

Proof. By Lemma 2.4, the cycles z as above are of the form $\pi_* z'$ for a smooth proper morphism $\pi : P \to Y$ and for some cycle $z' \in \mathrm{CH}(P)$ which is a combination with integral coefficients of intersections of divisors on P. By Remark 2.2, Proposition 2.9 applies to z' and the flat morphism $\phi \circ \pi : P \to X$, proving the statement. $\square$

3 Flat pushforwards of Chern classes

3.1 Comments on Definition 1.4

We start with the following remarks on Definition 1.4.

Remark 3.1. By Lemma 2.4, in the definition of $\mathrm{CH}(X)_{\mathrm{flpshCh}}$ and $\mathrm{CH}(X)_{\mathrm{smpshCh}}$, we can replace "intersections of divisors on P" by "elements of $\mathrm{CH}(P)_{\mathrm{Ch}}$". Indeed, if Y is smooth, $p : Y \to X$ is a proper flat morphism, and $z \in \mathrm{CH}_d(Y)_{\mathrm{Ch}}$, there exist by Lemma 2.4 a smooth variety P and a smooth morphism $p' : P \to Y$ such that

$$z = p'_*(w) \text{ in } \mathrm{CH}_d(Y),$$

where w belongs to the subring of $\mathrm{CH}(P)$ generated by divisors. The morphism $p \circ p' : P \to X$ is flat and projective, and $p_* z = (p \circ p')_* w$.

Remark 3.2. *If $f : Y \to X$ is a flat (resp. smooth) morphism between smooth projective varieties, one has $f_*(\mathrm{CH}(Y)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}$, resp. $f_*(\mathrm{CH}(Y)_{\mathrm{smpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{smpshCh}}$.*

Let us now establish a few elementary facts.

Lemma 3.3. *(a) One has $\mathrm{CH}(X)_{\mathrm{Ch}} \subset \mathrm{CH}(X)_{\mathrm{smpshCh}} \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}$.*
(b) The subgroup $\mathrm{CH}(X)_{\mathrm{smpshCh}}$ is a subring of $\mathrm{CH}(X)$.
(c) The subgroup $\mathrm{CH}(X)_{\mathrm{flpshCh}}$ is a module over the ring $\mathrm{CH}(X)_{\mathrm{smpshCh}}$.

Proof. (a) The second inclusion is obvious since smoothness implies flatness.

(b) and (c) Let $p_1 : P_1 \to X$, $p_2 : P_2 \to X$ be proper morphisms with P_1, P_2, X smooth and assume p_1 is smooth, p_2 is flat. Then $P_{12} := P_1 \times_X P_2$ is smooth. If Z_1, resp. Z_2 are intersections of divisors on P_1, resp. P_2, their pull-backs Z'_1, resp. Z'_2 to P_{12} via the projections

$$p'_1 : P_{12} \to P_2,\ p'_2 : P_{12} \to P_1$$

are also intersections of divisors, and the projection formula gives

$$p_{1*}Z_1 \cdot p_{2*}Z_2 = p_{12*}(Z'_1 \cdot Z'_2) \text{ in } \mathrm{CH}(X),$$

where $p_{12} : P_1 \times_X P_2 \to X$ is the natural morphism. This proves (b) and (c) since p_{12} is flat and it is smooth if p_2 is smooth. $\square$

Another useful lemma is the following

Lemma 3.4. *Let $\phi : Y_1 \to Y_2$ be a morphism, with Y_1, Y_2 smooth. Then*
(i) One has

$$\phi^*(\mathrm{CH}(Y_2)_{\mathrm{smpshCh}}) \subset \mathrm{CH}(Y_1)_{\mathrm{smpshCh}}, \tag{8}$$

(ii) If ϕ is smooth, then

$$\phi^*(\mathrm{CH}(Y_2)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(Y_1)_{\mathrm{flpshCh}}. \tag{9}$$

Proof. Let $\psi : W \to Y_2$ be a flat (resp. smooth) projective morphism. Then

$$\psi_1 : W_1 := W \times_{Y_2} Y_1 \to Y_1$$

is flat (resp. smooth). Furthermore, if either ψ is smooth (Case (i)) or ϕ is smooth (Case (ii)), W_1 is smooth.

Let $\phi_1 : W_1 \to W$ be the first projection. If $\gamma \in \mathrm{CH}(W)_{\mathrm{Ch}}$, we have $\phi_1^*\gamma \in \mathrm{CH}(W_1)_{\mathrm{Ch}}$, and furthermore we have by [7, Proposition 1.7]

$$\psi_{1*}(\phi_1^*\gamma) = \phi^*(\psi_*\gamma) \text{ in } \mathrm{CH}(Y_1).$$

This proves (8) and (9). □

3.2 Examples of cycles not in $\mathrm{CH}(X)_{\mathrm{Ch}}$

Theorem 1.6 is interesting when the considered cycles do not belong to $\mathrm{CH}(X)_{\mathrm{Ch}}$. Besides the case of 0-cycles on very general abelian varieties with high degree polarization (see [4]), some hypersurfaces in projective space provide such examples. For example, we have

Lemma 3.5. *Let $X \subset \mathbb{P}^4$ be a very general hypersurface of degree 64. Then the class of a point $x \in X$ does not belong to $\mathrm{CH}_0(X)_{\mathrm{Ch}}$. More precisely, for any vector bundle E on X, the degree $\deg c_3(E)$ is divisible by 2.*

Proof. Let E be a vector bundle of rank r on X. By the Hirzebruch-Riemann-Roch formula, the holomorphic Euler-Poincaré characteristic of E is given by the formula

$$\chi(X, E) = \int_X \mathrm{ch}(E)\mathrm{td}(X) = \alpha c_3(E) + \beta c_2(E)c_1(E) + \gamma c_2(E)c_1(X) + q(E), \tag{10}$$

where the constants α, β, γ, δ, ζ independent of E are rational and the quantity $q(E)$ is the part of the Riemann-Roch polynomial (in the Chern classes of E) which involves only $c_1(E)$ and the rank of E, and is an integer since it is equal to

$$\int_X (r-1)\mathrm{td}_3(X) + \mathrm{ch}(\det E)\mathrm{td}(X) = (r-1)\chi(X, \mathcal{O}_X) + \chi(X, \det E).$$

The constants α and β are obtained by expressing $\mathrm{ch}_3(E)$ as a polynomial in the Chern classes $c_i(E)$. One gets

$$\alpha = \tfrac{1}{2},\ \beta = -\tfrac{1}{2}. \tag{11}$$

Finally, the constant γ is obtained by expressing $\mathrm{ch}_2(E)\mathrm{td}_1(X)$ using the Chern classes of E. One gets

$$\gamma = -\tfrac{1}{2}. \tag{12}$$

It is proved in [13] that for X as above, any curve $C \subset X$ has degree divisible by 2. It follows that the numbers $\int_X c_2(E)c_1(E)$ and $\int_X c_2(E)c_1(X)$ are even. We thus deduce from (11) and (12) that $\chi(X, E) = \frac{1}{2}\int_X c_3(E) + C$, where C is an integer. Thus the degree of $c_3(E)$ has to be an even integer. □

Remark 3.6. We see from the proof above that the obstruction to the existence of a vector bundle E on X (or more generally an element of $K_0(X)$) with $\deg c_3(E) = 1$ comes from the defect of the integral Hodge conjecture for degree 4 Hodge classes on X. Conversely, if the integral Hodge conjecture for degree 4 Hodge classes on X holds true, then the generator $\alpha \in H^2(X, \mathbb{Z})$ is algebraic, $\alpha = [Z]$, for some 1-cycle $Z \in \mathrm{CH}_1(X)$. As we have

$$\mathrm{CH}_1(X) = \mathrm{CH}^2(X) = \mathrm{CH}^2(X)_{\mathrm{Ch}}$$

by formula (7), the cycle Z belongs to $\mathrm{CH}^2(X)_{\mathrm{Ch}}$, so the cycle $H \cdot Z$ belongs to $\mathrm{CH}^3(X)_{\mathrm{Ch}}$. Hence there exists a degree 1 element in $\mathrm{CH}^3(X)_{\mathrm{Ch}}$ in this case.

Other examples of smooth projective varieties X for which $\mathrm{CH}(X) \neq \mathrm{CH}(X)_{\mathrm{Ch}}$ are given by generalized flag manifolds for certain affine algebraic groups with torsion index > 1 (see [6] and [18], [19] where this notion is discussed and computed for many groups). Merkurjev proved in [15] that for a simply connected semisimple algebraic group G, and for a closed subgroup H, the K_0-ring of G/H is generated by classes of homogeneous vector bundles on G/H that come from representations of H. If furthermore H is a Borel subgroup of G, then homogeneous vector bundles on G/H coming from representations of H are direct sums of line bundles. In the last case, it follows that the subgroup

$$\mathrm{CH}_0(G/H)_{\mathrm{Ch}} \subset \mathrm{CH}_0(G/H)$$

is also the subgroup generated by products of divisor classes. By definition of the torsion index of G, the index of the later subgroup is a multiple of the torsion index of G. The computations in [18], [19] thus give plenty of examples where $\mathrm{CH}_0(G/H)_{\mathrm{Ch}} \subset \mathrm{CH}_0(G/H)$ is a proper subgroup.

3.3 Some stability results for $\mathrm{CH}(X)_{\mathrm{flpshCh}}$

We will give in this section the proof of Theorem 1.7. It will rely on the following three propositions.

Proposition 3.7. *Let X, Y be smooth projective varieties with $\dim Y = \dim X - 1$, and let $j : Y \to X$ be a finite morphism. Then*

$$j_*(\mathrm{CH}(Y)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}.$$

Proof. Let $T = B_{\Gamma_j}(Y \times X)$ be the smooth projective variety obtained by blowing-up the graph Γ_j of j in $Y \times X$. Let

$$\tau : T \to Y \times X$$

be the blow-up map and let pr_Y, pr_X be the two projections from $Y \times X$ to Y and X. We denote

$$p := \mathrm{pr}_Y \circ \tau : T \to Y,\ q := \mathrm{pr}_X \circ \tau : T \to X$$

the two natural morphisms.

Lemma 3.8. *The morphism p is smooth and the morphism q is flat.*

Proof. Indeed, the fiber of p over $y \in Y$ is the blow-up of X along y, which is smooth. The fiber of q over $x \in X$ is isomorphic to Y when $x \notin j(Y)$, hence it has dimension $n-1$, $n = \dim X$. We claim that all the fibers of q have dimension $\leq n-1$. To see this, we observe that

$$q^{-1}(x) = \tau^{-1}(Y \times \{x\})$$

is the set-theoretic union of several components, some being contained in the exceptional divisor E over Γ_j and mapping via τ to $(Y \times \{x\}) \cap \Gamma_j$, the other being birational to Y. The component which is birational to Y has dimension $n-1$. The other components are also of dimension $\leq n-1$, since the morphism $\tau_{|E} : E \to \Gamma_j \cong Y$ has fibers of dimension $n-1$, and

$$(Y \times \{x\}) \cap \Gamma_j \cong j^{-1}(x) \subset Y \cong \Gamma_j$$

has dimension 0 because j is finite. This proves the claim. The fibers are thus equidimensional, hence q is flat since both T and X are smooth. $\square$

For any class $w \in \mathrm{CH}_d(Y)_{\mathrm{flpshCh}}$, there exist by definition a (not necessarily connected but equidimensional) smooth projective variety W, a flat morphism $\phi : W \to Y$, and divisors $D_1, \ldots, D_{N-d} \in \mathrm{CH}^1(W)$, $N := \dim W$, such that

$$w = \phi_*(D_1 \ldots D_{N-d}) \text{ in } \mathrm{CH}_d(Y). \tag{13}$$

We now observe that

$$j_*w = \Gamma_{j*}(w) = \mathrm{pr}_{X*}(\mathrm{pr}_Y^* z' \cdot \Gamma_j) \tag{14}$$

in $\mathrm{CH}_d(X)$. Furthermore, we have as usual

$$\Gamma_j = \pm \tau_* E^n \text{ in } \mathrm{CH}^n(Y \times X),\ n = \dim X. \tag{15}$$

Let now

$$W_T := W_{\times Y} T$$

with first projection p_W to W, second projection p_T to T and morphism

$$\psi := q \circ p_T : W_T \to X.$$

We first observe that W_T is smooth by Lemma 3.8. Next, combining (13), (14), and (15), and applying the projection formula, we get

$$j_*w = \pm \psi_*(p_W^*(D_1 \dots D_{N-d}) \cdot p_T^* E^n) \text{ in } \mathrm{CH}_d(X), \tag{16}$$

which proves that j_*w belongs to $\mathrm{CH}_d(X)_{\mathrm{flpshCh}}$, since the morphism ψ is flat, being the composition of the two flat morphisms q and p_T. The proof of Proposition 3.7 is finished. □

Proposition 3.7 has the following consequences.

Proposition 3.9. *Let X be smooth projective and let $j : Y \hookrightarrow X$ be the inclusion of a smooth projective subvariety which is the 0-set of a transverse section σ of a vector bundle E on X. Then*

$$j_*(\mathrm{CH}(Y)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}. \tag{17}$$

Proof. We prove the result by induction on the rank of E, the case of rank 1 being a particular case of Proposition 3.7. Let E be a rank r vector bundle on X and let $\pi : \mathbb{P}(E^*) = \mathrm{Proj}\,(\mathrm{Sym}^* E) \to X$ be the projectivization of E^*. Let $\pi^* E \to \mathcal{H}$ be the quotient line bundle on $\mathbb{P}(E^*)$. The section $\pi^*\sigma \in H^0(\mathbb{P}(E^*), \pi^* E)$ projects to a section σ' of $\mathcal{H}$ and we have

Lemma 3.10. *(i) The 0-locus of σ' is a smooth hypersurface X' of $\mathbb{P}(E^*)$.*

(ii) Furthermore, the induced section σ'' of $F := \mathrm{Ker}\,(\pi^* E \to \mathcal{H})$ on X' is transverse with 0-locus $\pi^{-1}(Y) = \mathbb{P}(E^*_{|Y}) \subset X' \subset \mathbb{P}(E^*)$.

Proof. (i) The vanishing locus of σ' is a $\mathbb{P}^{r-2}$-bundle over the open subset $X \setminus Y$ of X where $\sigma \neq 0$, hence it is smooth over $X \setminus Y$. It obviously contains $\pi^{-1}(Y)$ and it remains to show that it is smooth there, which is easy.

(ii) The vanishing locus of σ'' on X' equals scheme-theoretically the vanishing locus of the section $\pi^*\sigma$ of $\pi^* E$ on $\mathbb{P}(E^*)$, hence equals $\pi^{-1}(Y)$. It is thus smooth of codimension $r-1$ in X'. □

Denoting by $\pi_Y : \mathbb{P}(E^*_{|Y}) \to Y$ the restriction of π over Y, we know by Lemma 3.4 that $\pi_Y^* : \mathrm{CH}(Y) \to \mathrm{CH}(\mathbb{P}(E^*_{|Y}))$ maps $\mathrm{CH}(Y)_{\mathrm{flpshCh}}$ to $\mathrm{CH}(\mathbb{P}(E^*_{|Y}))_{\mathrm{flpshCh}}$. Denoting by

$$j' : \mathbb{P}(E^*_{|Y}) \hookrightarrow X',\ j'' : X' \hookrightarrow \mathbb{P}(E^*)$$

the inclusion maps, we get, first by the induction hypothesis on the rank and lemma 3.10, and secondly by Proposition 3.7, that

$$j'_*(\mathrm{CH}(\mathbb{P}(E^*_{|Y}))_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X')_{\mathrm{flpshCh}},\ j''_*(\mathrm{CH}(X')_{\mathrm{flpshCh}}) \subset \mathrm{CH}(\mathbb{P}(E^*))_{\mathrm{flpshCh}}.$$

We conclude that the map $\gamma := j''_* \circ j'_* \circ \pi_Y^* : \mathrm{CH}(Y) \to \mathrm{CH}(\mathbb{P}(E^*))$ has the property that

$$\gamma(\mathrm{CH}(Y)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(\mathbb{P}(E^*))_{\mathrm{flpshCh}}.$$

Recalling from [7, Proposition 1.7] that $\gamma = \pi^* \circ j_*$, we thus proved that

$$\pi^* \circ j_*(\mathrm{CH}(Y)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(\mathbb{P}(E^*))_{\mathrm{flpshCh}}.$$

Let $h = c_1(\mathcal{H}) \in \mathrm{CH}^1(\mathbb{P}(E^*))$. By Lemma 3.3, we have

$$h^{r-1}\mathrm{CH}(\mathbb{P}(E^*))_{\mathrm{flpshCh}} \subset \mathrm{CH}(\mathbb{P}(E^*))_{\mathrm{flpshCh}}$$

and by Remark 3.2, $\pi_*(\mathrm{CH}(\mathbb{P}(E^*))_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}$. As $\pi_*(h^{r-1}\pi^* z) = z$ for any $z \in \mathrm{CH}(X)$, we conclude that for any $z \in \mathrm{CH}(Y)_{\mathrm{flpshCh}}$,

$$j_* z = \pi_*(h^{r-1}\pi^*(j_* z)) \in \mathrm{CH}(X)_{\mathrm{flpshCh}}.$$

$\square$

Another consequence of proposition 3.7 is the following

Proposition 3.11. *Let X be smooth projective and let $Y \subset X$ be a smooth projective subvariety which is the 0-set of a transverse section σ of a vector bundle E on X. Let $\tau : \widetilde{X} = B_Y X \to X$ be the blow-up of X along Y. Then*

$$\tau_*(\mathrm{CH}(\widetilde{X})_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}. \tag{18}$$

Proof. Let $\pi : \mathbb{P}(E) = \mathrm{Proj}(\mathrm{Sym}^* E^*) \to X$ be the projectivization of E. Then the section σ gives a rational section $X \dashrightarrow \mathbb{P}(E)$ of π, whose image is isomorphic to $\widetilde{X}$. Furthermore, as shows a local computation, $\widetilde{X} \subset \mathbb{P}(E)$ is the zero-set of a transverse section $\overline{\sigma}$ of the quotient vector bundle $F := \pi^* E/\mathcal{S}$ on $\mathbb{P}(E)$, namely, $\overline{\sigma}$ is the projection of $\pi^*\sigma$ in $\pi^* E/\mathcal{S}$, where $\mathcal{S} \subset \pi^* E$ is the tautological subbundle. We have

$$\tau_* = \pi_* \circ j_* : \mathrm{CH}(\widetilde{X}) \to \mathrm{CH}(X), \tag{19}$$

where $j : \widetilde{X} \to \mathbb{P}(E)$ is the inclusion map. By Proposition 3.9, we have

$$j_*(\mathrm{CH}(\widetilde{X})_{\mathrm{flpshCh}}) \subset \mathrm{CH}(\mathbb{P}(E))_{\mathrm{flpshCh}}$$

and $\pi_*(\mathrm{CH}(\mathbb{P}(E))_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X)_{\mathrm{flpshCh}}$ by Remark 3.2. Hence (19) implies (18). $\square$

As a consequence of these propositions, we give the easy proof of Theorem 1.6 for cycles of dimension at most 3, and more generally of the following result.

Theorem 3.12. *For any smooth projective variety X, we have*

$$(d-2)!\mathrm{CH}_d(X) \subset \mathrm{CH}_d(X)_{\mathrm{flpshCh}}, \tag{20}$$

with the convention that $(d-2)! = 1$ if $d \le 2$. In particular, for $d \le 3$, we have $\mathrm{CH}_d(X) = \mathrm{CH}_d(X)_{\mathrm{flpshCh}}$.

Proof. Let X be smooth projective of dimension n and let $Z \subset X$ be a subvariety of dimension d. We choose a desingularization $\widetilde{Z} \to Z$ of Z and an embedding $\widetilde{Z} \subset X \times \mathbb{P}^m$ over X for some m. As the projection $p_X : X \times \mathbb{P}^m \to X$ to X is flat, it suffices to prove that

$$(d-2)!\widetilde{Z} \in \mathrm{CH}(X \times \mathbb{P}^m)_{\mathrm{flpshCh}}, \tag{21}$$

as it implies by Remark 3.2 that $(d-2)!Z \in \mathrm{CH}(X)_{\mathrm{flpshCh}}$, which is the contents of (20). In other words, letting

$$Z' = \widetilde{Z},\ X' = X \times \mathbb{P}^m,$$

we reduced to the case of the class of a smooth subvariety $Z' \subset X'$, which we treat now. If $\dim X' \le 2d-1$, this is finished by formula (7) since then $\mathrm{codim}\,(Z' \subset X') \le d-1$. If not, let Y be a smooth general complete intersection of sufficiently ample hypersurfaces in X' containing Z', with Y of dimension $2d$. Such Y exists since Z' is smooth of dimension d and $\dim X' \geq 2d$. Let $j : Y \hookrightarrow X'$ be the inclusion of Y. As Y is a smooth complete bundle-section in X', we have by Proposition 3.9

$$j_*(\mathrm{CH}(Y)_{\mathrm{flpshCh}}) \subset \mathrm{CH}(X')_{\mathrm{flpshCh}}.$$

In order to prove Theorem 3.12, it thus suffices to prove that the class z' of Z' in Y satisfies

$$(d-2)!z' \in \mathrm{CH}_d(Y)_{\mathrm{flpshCh}}. \tag{22}$$

Let $Y' \subset Y$ be a general sufficiently ample hypersurface containing Z'.

Lemma 3.13. *Let N be a smooth variety and $M \subset N$ be a smooth subvariety of dimension d and codimension c. Then a general sufficiently ample hypersurface $H \subset N$ containing M has ordinary quadratic singularities along a smooth subvariety $D \subset M$ of dimension $d - c$. Thus, if $d = c$, H has isolated ordinary quadratic singularities.*

Proof. By Bertini, the singularities of H are on M, and they correspond to the zeroes of the differential

$$d\sigma \in H^0(M, N^*_{M/N}(H)) \tag{23}$$

along M of the defining equation σ of H. When H is sufficiently ample, the section (23) is a general section of the bundle $N^*_{M/N}(H)$ and this bundle is globally generated, so the zero locus of $d\sigma$ is transverse, hence the singular locus of Y' is smooth of dimension $m - c$. In fact, the transversality of the section $d\sigma$ also implies that the singularities are ordinary quadratic as a local computation shows. □

By Lemma 3.13, the hypersurface Y' above is desingularized by a single blow-up along the finite set $W \subset Y'$ of its singular points. We choose now a general 0-dimensional complete intersection $W' \subset Y$ containing W. We have $W' = W \cup W''$, where the set W'' is disjoint from Y'. It follows that the blow-up $\widetilde{Y}$ of Y along W' contains the blow-up $\widetilde{Y'}$ of Y' along W as a smooth hypersurface. Let $\widetilde{Z'} \subset \widetilde{Y'}$ be the proper transform of Z' and denote by $\tilde{z}'$ its class in $\mathrm{CH}_d(\widetilde{Y'})$. As the codimension of $\widetilde{Z'}$ in $\widetilde{Y'}$ is $d-1$, we have $(d-2)!\tilde{z}' \in \mathrm{CH}_d(\widetilde{Y'})_{\mathrm{flpshCh}}$. We now apply Proposition 3.7 to the inclusion i of $\widetilde{Y'}$ in $\widetilde{Y}$ and conclude that $(d-2)!i_*\tilde{z}' \in \mathrm{CH}_d(\widetilde{Y})_{\mathrm{flpshCh}}$. As the morphism $\tau : \widetilde{Y} \to Y$ blows-up the smooth complete bundle-section W in Y, we finally get

$$(d-2)!z' = \tau_*((d-2)!i_*\tilde{z}') \in \mathrm{CH}_d(Y)_{\mathrm{flpshCh}}$$

by Proposition 3.11. □

4 The cbs resolution Theorem

The main result of this section is Theorem 4.2, which says that a smooth subvariety of a smooth variety becomes an irreducible component of a smooth complete bundle-section after a suitable sequence of blow-ups, whose centers are also smooth complete bundle-sections.

In this section we work over an infinite perfect field. All varieties are allowed to be reducible, but assumed pure dimensional.

Blow-up sequences. A *blow-up sequence* is a sequence of morphisms

$$Y_r \xrightarrow{\pi_{r-1}} Y_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_0} Y_0, \tag{24}$$

where each $\pi_i : Y_{i+1} \to Y_i$ is the blow up of a subscheme $C_i \subset Y_i$, called the *center* of the blow-up.

Let $W_0 \subset Y_0$ be a subscheme. If the images of the centers C_i are nowhere dense in W_0, then we let $W_i \subset Y_i$ denote the *birational transform* of W_0. (Also called *proper transform.*)

Here we only deal with blow-up sequences where Y_0 is smooth, and the C_i are smooth and pure dimensional. In this case all the Y_i are smooth.

We say that a blow-up sequence as in (24) is a *complete bundle-section blow-up sequence* (abbreviated as *cbs blow-up sequence*), if the $C_i \subset Y_i$ are all complete bundle-sections.

We consider the following.

Question 4.1. *Let $Z \subset Y$ be smooth, projective varieties. Is there a cbs blow-up sequence $Y_r \to \cdots \to Y_0 := Y$ with centers $C_i \subset Y_i$ such that $\dim C_i < \dim Z$ for every i, and $Z_r \subset Y_r$ is an irreducible component of a smooth, complete bundle-section?*

Most likely the answer is yes, but we prove this only when $\dim Z < \frac{1}{4} \dim Y$; this is sufficient for our purposes, as explained in the introduction.

Theorem 4.2. *Let $Z \subset Y$ be smooth, projective varieties such that $\dim Z < \frac{1}{4} \dim Y$. Then there is cbs blow-up sequence $Y_r \to \cdots \to Y_0 := Y$ with centers $C_i \subset Y_i$, such that $\dim C_i < \dim Z$ for every i, and $Z_r \subset Y_r$ is a union of irreducible components of a smooth complete bundle-section $Z_r^* \subset Y_r$.*

We will prove this theorem as an almost immediate consequence of Property CBS_d stated in 4.6. In the inductive proof of Theorem 4.2 we need a stronger version, where the centers C_i are in 'general position' with respect to some other subvarieties. To understand what we need, consider the blow-up of $H := (xy + z^2) \subset \mathbb{A}^4$ along the line $L := (x = z = t = 0)$. In one chart we get the equation $H' = (x_1 y + z_1^2 t_1 = 0)$. Thus H' does not have ordinary double points. Here $L \subset H$, and it is transversal to the singular set of H, which is $(x = y = z = 0)$.

This leads to the following definition.

Definition 4.3. (Full intersection property) Let $Z \subset Y$ be schemes. A closed subset $U \subset Y$ has *full intersection* with Z, if $Z \cap U$ is a union of connected components of U. A blow-up sequence $Y_r \to \cdots \to Y_0 = Y$ has *full intersection* with Z if the birational transforms $Z_i \subset Y_i$ are defined, and each blow-up center C_i has full intersection with Z_i.

Let Y be a smooth variety and Z, C smooth subvarieties. We will say that Z has normal crossings with C, if the intersection $Z \cap C$ is smooth.

Lemma 4.4. *(Elementary blow-up lemmas) Let $\pi : Y' := B_C Y \to Y$ be the blow-up and $Z' \subset Y'$ the birational transform.*

(i) If Z has normal crossings with C, then Z' is smooth.

(ii) If Z is a complete bundle-section in Y and C has full intersection with Z, then Z' is a complete bundle-section in Y'.

In addition, let $H \subset Y$ be a hypersurface that has only ordinary double points along some smooth $D \subsetneq H$.

(iii) If $C = D$, then H' is smooth.

(iv) If C has full intersection with D and normal crossings with $H \setminus D$, then H' has only ordinary double points, necessarily along D'.

We also need the following subtler variant. This is the main point in the proof where going from complete intersections to complete bundle-sections becomes necessary.

Lemma 4.5. *Let Y be smooth projective and let $M \subset Y$ be a complete bundle-section. Assume that M has only ordinary double points along some smooth subvariety $D \subsetneq M$. Let $\pi : Y' := B_D Y \to Y$ be the blow-up of Y along D and M' the birational transform of M in Y'. Then $M' \subset Y'$ is a smooth complete bundle-section.*

Proof. We know that $M \subset Y$ is the zero-set of a transverse section s of a vector bundle $\mathcal{F}$ on Y. Let E be the exceptional divisor of the blow-up map $\pi : Y' \to Y$. We construct a vector bundle $\mathcal{F}'$ on Y' by modifying $\pi^*\mathcal{F}(-E)$ along E. As $D \subset M$, the section s vanishes along D and has a differential

$$ds : N_{D/Y} \to \mathcal{F}_{|D}.$$

This differential has corank 1, as follows from the fact that M is singular with hypersurface singularities along D. We thus have a quotient line bundle $\mathcal{L}$ of $\mathcal{F}_{|D}$, and denoting $\pi_E : E \to D$ the restriction of π to E, we get a quotient map constructed as the composition

$$q : \pi^*\mathcal{F}(-E) \to \pi_E^*\mathcal{F}_{|D}(-E) \to \pi_E^*\mathcal{L}(-E).$$

We set

$$\mathcal{F}' := \operatorname{Ker} q.$$

This is a vector bundle on Y'. Furthermore, we observe that, by construction, the section π^*s of $\pi^*\mathcal{F}$ provides a section s' of $\mathcal{F}' \subset \pi^*\mathcal{F}$. One then checks that the vanishing locus of s' is exactly the proper transform M'. $\square$

We can now state the inductive forms of Theorem 4.2. Consider the following statements 4.6 and 4.8 depending on dimension d.

4.6. Property CBS_d. *Let $Z \subsetneq X \subset Y$ be smooth, projective varieties, $\dim Z \leq d$ and $\dim Y > 4d$. Assume that $X \subset Y$ is a smooth complete bundle-section, and $\dim X < \frac{1}{2}\dim Y$. Let $Z \subset W^j \subset Y$ be smooth subvarieties such that $\dim W^j < \frac{1}{2}\dim Y$.*

Then there is a cbs blow-up sequence $\Pi : Y_r \to \cdots \to Y_0 := Y$ with centers $C_i \subset Y_i$, such that

1. *$\dim C_i < \dim Z$ for every i,*
2. *each C_i has full intersection with Z_i, X_i and the W_i^j, and*
3. *Z_r is a union of irreducible components of a smooth complete bundle-section $Z_r^* \subset X_r \subset Y_r$.*

Remark 4.7. Note that we do not claim that Z_r^* has full intersection with the W_r^j. In our construction, Z_r^* is essentially the birational transform of a d-dimensional complete intersection Z^* that contains Z. We can thus guarantee that Z_r^* is in general position away from $\Pi^{-1}(Z)$. Note, however, that the blow-up sequence depends on Z^* in a complicated way, so it is unlikely that we can guarantee that Z_r^* is also in general position along $\Pi^{-1}(Z) \setminus Z_r$. This will cause some difficulties in the proof below.

4.8. Property CBS'_d. *Let $Z \subset X \subset Y$ be smooth, projective varieties, with $\dim Z \leq d$ and $\dim Y > 4d$. Assume that $X \subset Y$ is a smooth complete bundle-section and that $\dim X < \frac{1}{2}\dim Y$, $\dim Z < \dim X$. Let $Z \subset W^j \subset Y$ be smooth subvarieties such that $\dim W^j < \frac{1}{2}\dim Y$.*

Then there is a cbs blow-up sequence $Y_r \to \cdots \to Y_0 := Y$ with centers $C_i \subset Y_i$, such that

1. *$\dim C_i < \dim Z$ for every i,*
2. *each C_i has full intersection with Z_i, X_i and the W_i^j, and*
3. *there is a smooth complete bundle-section $X_r^{(1)} \subset Y_r$, such that $Z_r \subset X_r^{(1)} \subset X_r$ and $\dim X_r^{(1)} < \dim X_r$.*

Remark 4.9. In the construction below, $X_r^{(1)}$ is a subset of X_r of codimension 1. If $\dim X \geq \dim Z + 2$, then every irreducible component of X_r contains a unique irreducible component of $X_r^{(1)}$. If $\dim X = \dim Z + 1$, then Z_r is a union of irreducible components of $X_r^{(1)}$, but usually there are other irreducible components as well.

Theorem 4.10. CBS_d *and* CBS'_d *hold for every* d.

For the proof, we use induction on d, and show that $\mathrm{CBS}_{d-1} \Rightarrow \mathrm{CBS}'_d \Rightarrow \mathrm{CBS}_d$. Note that CBS_0 is clear.

Proof of $\mathrm{CBS}'_d \Rightarrow \mathrm{CBS}_d$. Since $\dim Z < \frac{1}{4}\dim Y$, there is a smooth complete intersection $Z \subset X \subset Y$ such that $\dim X < \frac{1}{2}\dim Y$.

We are done if $\dim Z = \dim X$. Otherwise, using Property CBS'_d for $d = \dim Z$, there is a smooth cbs blow-up sequence $Y_{r_1} \to \cdots \to Y_0 = Y$, whose centers have full intersections with Z, X, and a smooth complete bundle-section $X_{r_1}^{(1)} \subset Y_{r_1}$ such that $Z_{r_1} \subset X_{r_1}^{(1)} \subset X_{r_1} \subset Y_{r_1}$ and $\dim X_r^{(1)} < \dim X_r$.

We now replace $Z \subset X \subset Y$ by $Z_r \subset X_r^{(1)} \subset Y_r$ and repeat the argument to get

$$Z_{r_i} \subset X_{r_i}^{(i)} \subset X_{r_i}^{(i-1)} \subset Y_{r_i}, \quad \text{for} \quad i = 2, \ldots$$

With each step we lower the dimension of the smooth complete bundle-section $X_{r_i}^{(i)} \subset Y_{r_i}$, until we reach

$$Z_{r_m} \subset X_{r_m}^{(m)} \subset Y_{r_m},$$

such that $\dim Z_{r_m} = \dim X_{r_m}^{(m)}$. $\square$

Proof of $\mathrm{CBS}_{d-1} \Rightarrow \mathrm{CBS}'_d$. Take a general hypersurface $Z \subset H \subset Y$. Then $H \cap X$ has ordinary double points along some smooth $D \subsetneq Z$ by Lemma 3.13. We apply CBS_{d-1} to D to get D_r which is a union of irreducible component of a smooth cbs D_r^*. Next we should blow up D_r^*. Using Lemma 4.4(iii) we get that the birational transform of $(H \cap X)_r$ is now smooth over D_r, and it is a complete bundle-section by Lemma 4.5. However, we also need to guarantee that the other components $D_r^* \setminus D_r$ have full intersection with H_r and X_r. As we noted in Remark 4.7, this is not clear.

We go around this problem by creating an auxiliary general complete intersection $\bar{X} \subset Y$ that contains D and has dimension $< \frac{1}{2}\dim Y$. Using $\dim W^j < \frac{1}{2}\dim Y$, we can achieve that $\bar{X} \cap X = D$ and $\bar{X} \cap W^j = D$ for every j, scheme theoretically.

Now we apply CBS_{d-1} to $\bar{D} := D \subset \bar{X} \subset Y$ and $\bar{W}^j := W^j$, with the original X playing the role of a new $\bar{W}^0$. We then get $\bar{D}_r^* \supset \bar{D}_r$, which is contained in $\bar{X}_r$. In particular,

$$\bar{D}_r^* \cap \bar{W}_r^j \subset \bar{X}_r \cap \bar{W}_r^j \subset \bar{D}_r = D_r.$$

For $j = 0$ this gives that $\bar{D}_r^* \cap X_r \subset D_r$. Thus $\bar{D}_r^* \setminus \bar{D}_r$ is disjoint from X_r and the W_r^j, hence $\bar{D}_r^*$ has full intersections with Z_r, X_r and W_r^j, as needed. $\square$

Proof of Theorem 4.2. Since $\dim Z < \frac{1}{4}\dim Y$, there is a smooth complete intersection $Z \subset X \subset Y$ such that $\dim X < \frac{1}{2}\dim Y$. We can now apply Property CBS_d stated in 4.6 with $W^j = \emptyset$; the latter is shown to hold in Theorem 4.10. $\square$

5 The case of homogeneous varieties

This section is devoted to the case of cycles on homogeneous varieties, for which stronger results are available.

5.1 Cycles on abelian varieties

We start with the proof of Theorem 1.11(ii). The result in this case is stronger than Theorem 1.6 since it states that $\mathrm{CH}_d(A) = \mathrm{CH}_d(A)_{\mathrm{smpshCh}}$ for any abelian variety A and any integer d.

Proof of Theorem 1.11(ii). Let $z \in \mathrm{CH}_d(A)$. We want to prove that

$$z \in \mathrm{CH}_d(A)_{\mathrm{smpshCh}}. \tag{25}$$

We can assume $z = [Z]$ for some subvariety $Z \subset A$ of dimension d. We denote by $\tau : \widetilde{Z} \to A$ a desingularization of Z. Consider the morphism

$$\phi : A \times \widetilde{Z} \to A$$

$$(x, \tilde{z}) \mapsto x + \tau(\tilde{z}).$$

Obviously ϕ is smooth, since it is A-equivariant. Furthermore, we have, denoting $0_A \in A$ the origin

$$z = \phi_*([0_A \times \widetilde{Z}]). \tag{26}$$

If $[0_A] \in \mathrm{CH}_0(A)$ belongs to the subring of $\mathrm{CH}^*(A)$ generated by Chern classes of coherent sheaves on A, so does $\mathrm{pr}_1^*([0_A]) = [0_A \times \widetilde{Z}] \in \mathrm{CH}(A \times \widetilde{Z})$, hence (26) implies (25) in this case. It is proved however in [4] that for a very general abelian variety A with sufficiently divisible polarization degree and high dimension, the class of a point does not belong to this subring of $\mathrm{CH}^*(A)$, so we cannot apply the argument directly to A. Nevertheless, Debarre also proves in *loc. cit.* that, if J is the Jacobian of a curve C of genus g, then for any point x of J, there exists a rank g vector bundle on J with a section whose zero locus is $\{x\}$ (with its reduced structure). In particular, the class $[x]$ belongs to $\mathrm{CH}^*(J)_{\mathrm{Ch}}$. Let now $j : C \hookrightarrow A$ be the inclusion of a smooth curve of genus g which is a complete intersection of ample hypersurfaces in A. Then by Lefschetz theorem on hyperplane sections, we have a surjective (hence smooth) morphism $\psi = j_* : J = JC \to A$ of abelian varieties, and $\psi_*([0_J]) = [0_A]$. Let

$$\phi_J : J \times \widetilde{Z} \to A$$

be the composite $\phi \circ (\psi, Id)$. Then ϕ_J is smooth and we have

$$z = \phi_{J*}([0_J \times \widetilde{Z}]). \tag{27}$$

As $[0_J]$ belongs to $\mathrm{CH}^*(J)_{\mathrm{Ch}}$, $[0_J \times \widetilde{Z}]$ belongs to $\mathrm{CH}^*(J \times \widetilde{Z})_{\mathrm{Ch}}$, so (25) follows from formula (27). □

5.2 More general homogeneous varieties

We prove in this section Theorem 1.11(i), which is the following statement

Theorem 5.1. *Let X be a homogeneous variety under a group G. If there exists a smooth projective G-equivariant completion $\overline{G}$ of G which satisfies $\mathrm{CH}_0(\overline{G}) = \mathrm{CH}_0(\overline{G})_{\mathrm{smpshCh}}$, then*

$$\mathrm{CH}_d(X)_{\mathrm{smpshCh}} = \mathrm{CH}_d(X) \tag{28}$$

for all d.

Proof of Theorem 5.1. Let $z \in \mathrm{CH}_d(X)$ be the class of a subvariety $Z \subset X$ and let $\tau : \widetilde{Z} \to X$ be a desingularization of Z. We consider the morphism

$$f : G \times \widetilde{Z} \to X,$$

$$(g, \tilde{z}) \mapsto g \cdot \tau(\tilde{z}).$$

The morphism f is obviously smooth since it is G-equivariant. It is however not proper, but it provides a G-equivariant rational map

$$F : \overline{G} \times \widetilde{Z} \dashrightarrow X, \tag{29}$$

where $\overline{G}$ is any smooth projective completion of G on which G acts (which exists by G-equivariant resolution of singularities [14]). The action of G on the left hand side of (29) is via its action on $\overline{G}$. By G-equivariant resolution of indeterminacies [17], there exist a smooth projective variety Y on which G acts, a G-equivariant birational morphism $\eta : Y \to \overline{G} \times \widetilde{Z}$, and a morphism

$$\widetilde{F} : Y \to X,$$

such that $F \circ \eta = \widetilde{F}$ as rational maps to X. The morphism $\widetilde{F}$ is proper since Y is projective, and it is again smooth because it is G-equivariant. Furthermore we have

$$z = \widetilde{F}_*(\eta^*([e \times \widetilde{Z}])) \text{ in } \mathrm{CH}_d(X), \tag{30}$$

where $e \in G \subset \overline{G}$ is the neutral element. We choose now $\overline{G}$ as in Theorem 5.1. Then there exist a smooth projective (nonnecessarily connected but that we can assume equidimensional) variety W and a smooth proper morphism $\phi : W \to \overline{G}$ such that e can be written as

$$e = \phi_*(D_1 \dots D_N) \text{ in } \mathrm{CH}_0(\overline{G}),\ N = \dim W, \tag{31}$$

for some divisors $D_l \in \mathrm{CH}^1(W)$. Let $Y' := W \times_{\overline{G}} Y$, with first projection $q : Y' \to W$ and second projection $p : Y' \to Y$. Then Y' is smooth projective and $p : Y' \to Y$ is smooth. Denoting $\widetilde{F}' := \widetilde{F} \circ p : Y' \to X$, $\widetilde{F}'$ is also smooth. Moreover, by (30) and (31), z can be written as

$$z = \widetilde{F}'_*(q^* D_1 \dots q^* D_N) \text{ in } \mathrm{CH}_d(X). \tag{32}$$

Formula (32) shows that $z \in \mathrm{CH}_d(X)_{\mathrm{smpshCh}}$. $\square$

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