Algebraic geometry versus Kähler geometry

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0 Introduction

This text is divided in two parts, which both illustrate an important gap between complex projective geometry and compact Kähler geometry. These two parts correspond to the two lectures I delivered during the meeting "ADVANCES IN NUMBER THEORY AND GEOMETRY" organized by the Riemann International School of Mathematics in Verbania.

The first gap is of a topological nature. Namely we will show that complex projective manifolds are more topologically restricted than compact Kähler manifolds. One curious aspect of the criterion used here, which is the notion of polarized Hodge structure on a cohomology algebra, is that it can be used as well to provide new topological restrictions on compact Kähler manifolds, which leads to the construction of extremely simple compact symplectic manifolds, built as complex projective bundles over complex tori, but topologically non Kähler. More precisely, the projective manifolds will have a Hodge structure on their cohomology algebra with a *rational* polarization, while in the general Kähler case, the polarization can only be taken to be real. But as we will show, even the existence of a Hodge structure on the cohomology algebra is very restrictive, not to speak of the existence of such a structure with a real polarization.

The second gap is of an analytic nature and concerns Hodge classes. Hodge conjecture could be formulated in the compact Kähler context as well. However we show that some Hodge classes on certain compact Kähler manifolds may exist while there are no non zero Chern classes of coherent sheaves in the corresponding degree (they are the most general Hodge classes built starting from an holomorphic object). This shows that Hodge theory, or more precisely the theory of Hodge structures does not have the same power of information in the general Kähler case as in the projective case. Something is missing in the former situation which prevents to read on the Hodge structure geometric properties of the varieties. In the last section, we will describe some of the supplementary structure the cohomology of a complex projective manifold has (which is related to motives), which is one very fascinating aspect of Hodge theory in the projective case, and which makes the Hodge conjecture more meaningful in the algebro-geometric situation. Namely the Hodge filtration on cohomology with complex coefficients can be computed using algebraic differentials, and more precisely "algebraic de Rham cohomology" and is thus defined over the same field as the variety itself. We will describe the incidence of this on the Hodge conjecture and the best known evidence for the Hodge conjecture.

This text is organized as follows: section 1 introduces the tools from Hodge theory we will be using later on. Section 2 is devoted to our work on the Kodaira problem and further applications of the notion of polarized Hodge structures on cohomology algebras. Finally, section 3 is devoted to a a discussion of the necessity of the projectivity condition in the Hodge conjecture and the particular flavour of the Hodge conjecture in the algebro-geometric context. We refer to [39] for a more extended discussion of the status of the Hodge conjecture.

Thanks. I thank the organizers of this extremely beautiful and interesting meeting around the work of Riemann for inviting me to give lectures there.

1 Hodge theory

1.1 The Hodge decomposition

On any complex manifold, we have the bigradation given by decomposition of $\mathcal{C}^\infty$ complex differential forms into forms of type (p,q). Here $(1,0)$-forms are those which are $\mathbb{C}$-linear and $(0,1)$-forms are the $\mathbb{C}$-antilinear ones. The operator d splits as

$$d = \partial + \overline{\partial},$$

where the operator ∂ sends forms of type (p,q) to forms of type $(p+1,q)$ and the operator $\overline{\partial}$ sends forms of type (p,q) to forms of type $(p,q+1)$

Define $H^{p,q}(X) \subset H^{p+q}(X,\mathbb{C})$ as the set of cohomology classes which can be represented by closed forms of type (p,q). This is not very meaningful topologically for general compact complex manifolds.

Recall now that a complex manifolds is Kähler if it admits a Hermitian metric $h = g - \iota\omega$, where $g = Re\,h$ is a Riemannian metric and ω is closed. For compact Kähler manifolds, Hodge theory of harmonic forms applied to a Kähler metric gives:

Theorem 1.1 *(Hodge decomposition theorem) If X is compact Kähler,*

$$H^k(X,\mathbb{C}) = \oplus_{p+q=k} H^{p,q}(X).$$

A consequence of Theorem 1.1 and the definition of $H^{p,q}(X)$ is known as Hodge symmetry:

Lemma 1.2 *If X is a compact Kähler manifold, one has*

$$\overline{H^{p,q}(X)} = H^{q,p}(X),$$

where we use the complex conjugation acting on $H^{p+q}(X,\mathbb{C}) = H^{p+q}(X,\mathbb{R})\otimes\mathbb{C}$.

Indeed, the map $\eta^{p,q} \mapsto \overline{\eta^{p,q}}$ clearly induces a $\mathbb{C}$-antilinear isomorphism between the space $H^{p,q}(X)$ of de Rham cohomology classes of closed forms of type (p,q) and the space $H^{q,p}(X)$ of de Rham cohomology classes of closed forms of type (q,p).

An obvious but crucial observation for all the results presented in section 2 is the following compatibility result between the Hodge decomposition and the cup-product:

Lemma 1.3 *One has*

$$H^{p,q}\cup H^{p',q'}(X)\subset H^{p+p',q+q'}(X).$$

This follows indeed from the definition of the $H^{q,p}(X)$'s and the fact that the wedge product of a closed form of type (p,q) and a closed form of type (p',q') is a closed form of type $(p+p',q+q')$. ■

1.2 The hard Lefschetz theorem

Another very deep application of Hodge theory is the hard Lefschetz theorem, which says the following: let X be a compact Kähler manifold of dimension n and $\omega\in H^2(X,\mathbb{R})$ the class of a Kähler form on X. Cup-product with ω gives an operator usually denoted by $L: H^*(X,\mathbb{R})\to H^{*+2}(X,\mathbb{R})$.

Theorem 1.4 *For any $k\leq n$,*

$$L^{n-k}: H^k(X,\mathbb{R})\to H^{2n-k}(X,\mathbb{R})$$

is an isomorphism.

So far, the theory sketched above works for general compact Kähler manifolds, and indeed, applications of Hodge theory are the fact that there are strong topological restrictions for a complex compact manifold to be Kähler. In section 2.6, we will describe some of these applications, concerning restrictions on the structure of the cohomology algebra of a compact Kähler manifold. The only difference between Kähler geometry and projective geometry from the above point of view is the fact that in the second case we can choose the class ω to be rational. Our main result in section 2.3 is the fact that one can extract from this further restrictions on the structure of the cohomology algebra of a projective complex manifold.

1.3 Hodge structures

The complex cohomology of a compact Kähler manifold carries the Hodge decomposition. On the other hand, it is not only a complex vector space, since it has a canonical integral structure, namely we have the change of coefficients theorem:

$$H^k(X,\mathbb{C}) = H^k(X,\mathbb{Z})\otimes\mathbb{C}.$$

In the sequel we will denote by $H^k(X,\mathbb{Z})$ the integral cohomology of X modulo torsion. Thus $H^k(X,\mathbb{Z})$ is a lattice, and Hodge theory provides us with an interesting continuous invariant attached to a Kähler complex structure on X, namely the position of the complex spaces $H^{p,q}$ with respect to the lattice $H^k(X,\mathbb{Z})$. This leads to the notion of Hodge structure.

Definition 1.5 *A weight k (integral) Hodge structure is a lattice V, with a decomposition*

$$V_{\mathbb{C}} = \oplus_{p+q=k} V^{p,q},\ V^{q,p} = \overline{V^{p,q}},$$

where $V_{\mathbb{C}} := V \otimes \mathbb{C}$.

The Hodge structure is said to be effective, if $V^{p,q} = 0$ for $p < 0$ or $q < 0$.

Remark 1.6 The Hodge decomposition on $V_{\mathbb{C}}$ satisfying Hodge symmetry gives rise to an action of $\mathbb{C}^*$, seen as a real algebraic group, on $V_{\mathbb{R}}$. Namely $z \in \mathbb{C}^*$ acts by $z^p \overline{z}^q Id$ on $V^{p,q}$.

If X is a compact Kähler manifold, each cohomology group (modulo torsion) $H^k(X,\mathbb{Z})$ carries a canonical effective Hodge structure of weight k.

Example 1.7 The simplest Hodge structures are trivial Hodge structures of even weight $2k$. Namely, one defines $V_{\mathbb{C}} = V^{k,k}$, $V^{p,q} = 0$, $(p,q) \neq (k,k)$.

Example 1.8 The next simplest Hodge structures are weight 1 (effective) Hodge structures. Such a Hodge structure is given by a lattice V (necessary of even rank $2n$), and a decomposition

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1},\ V^{0,1} = \overline{V^{1,0}}.$$

Given V, weight 1 Hodge decompositions as above on $V_{\mathbb{C}}$ are determined by the subspace $V^{1,0}$ of $V_{\mathbb{C}}$ which belongs to the dense open set of the Grassmannian $Grass(n, 2n)$ of rank n complex vector subspaces W of $V_{\mathbb{C}}$ satisfying the property $W \cap \overline{W} = \{0\}$.

If $(V, V_{\mathbb{C}} = V^{1,0} \oplus \overline{V^{1,0}})$ is such a Hodge structure, we have $V_{\mathbb{R}} \cap V^{1,0} = \{0\}$, and thus via the natural projection $V_{\mathbb{C}} \to V_{\mathbb{C}}/V^{1,0}$, $V_{\mathbb{R}}$ projects isomorphically to the right hand side. As $V \subset V_{\mathbb{R}}$ is a lattice, the projection above sends V to a lattice in the complex vector space $V_{\mathbb{C}}/V^{1,0}$. It follows that the quotient

$$T = V_{\mathbb{C}}/(V^{1,0} \oplus V)$$

is a complex torus, the complex structure being given by the complex structure on $V_{\mathbb{C}}/V^{1,0}$.

Conversely, a n-dimensional complex torus T is a quotient of a complex vector space K of rank n by a lattice V of rank $2n$. The inclusion $i : V \hookrightarrow K$ extends by $\mathbb{C}$-linearity to a map of complex vector spaces $i_{\mathbb{C}} : V_{\mathbb{C}} \to K$, which is surjective, as V generates K over $\mathbb{R}$. Thus, denoting $V^{1,0} = Ker\, i_{\mathbb{C}}$, we find that $T = V_{\mathbb{C}}/(V^{1,0} \oplus V)$. As V is a lattice in the quotient $V_{\mathbb{C}}/V^{1,0} = K$, it follows that $V_{\mathbb{R}} \cap V^{1,0} = \{0\}$, or equivalently $V^{1,0} \cap \overline{V^{1,0}} = \{0\}$.

This way we have an equivalence of categories between effective Hodge structures of weight 1 and complex tori (see next section for the notion of morphism of Hodge structures).

1.4 Hodge classes and morphisms of Hodge structures

Definition 1.9 *A morphism of Hodge structures $(V, V^{p,q})$, $(W, W^{p',q'})$ of respective weights k, $k+2r$ is a morphism of lattices*

$$\phi : V \to W,$$

such that the $\mathbb{C}$-linear extension $\phi_{\mathbb{C}}$ of ϕ sends $V^{p,q}$ to $W^{p+r,q+r}$.

Such a morphism is said to be of bidegree (r,r), as it shifts by (r,r) the bigraduation given by the Hodge decomposition. Natural examples of morphisms of Hodge structures are induced by holomorphic maps $f : X \to Y$ between compact Kähler manifolds. The pull-back on cohomology

$$f^* : H^k(Y,\mathbb{Z}) \to H^k(X,\mathbb{Z})$$

is a morphism of Hodge structures of weight k, because the pull-back by f of a closed form of type (p,q) is again a closed form of type (p,q).

We also have the Gysin map

$$f_* : H^k(X,\mathbb{Z}) \to H^{k+2r}(Y,\mathbb{Z}),$$

where $r := dim\, Y - dim\, X$. It is defined on integral cohomology as the composition $PD_Y^{-1} \circ f_* \circ PD_X$, where PD_X is the Poincaré duality isomorphism

$$PD_X : H^l(X, \mathbb{Z}) \cong H_{2n-l}(X, \mathbb{Z}),\ n = dim\, X$$

and similarly for PD_Y, and f_* at the middle is the natural push-forward map induced on homology by f. The map f_* being the transpose with respect to Poincaré duality of f^*, one shows easily by a duality argument that f_* is a morphism of Hodge structures of bidegree (r, r).

Definition 1.10 *Let $(V, V_{\mathbb{C}} = \oplus_{p+q=2k} V^{p,q})$ be a Hodge structure of even weight $2k$. Then the set of rational Hodge classes $Hdg(V)$ of V is defined as the set of classes $\alpha \in V_{\mathbb{Q}} \cap V^{k,k}$, where the intersection is taken inside $V_{\mathbb{C}}$.*

If $V = H^{2k}(X, \mathbb{Z})$ where X is compact Kähler, we will use the notation $Hdg^{2k}(X) := Hdg(V)$.

If X, Y are compact Kähler and $\alpha \in H^l(X, \mathbb{Q}) \otimes H^k(Y, \mathbb{Q}) \subset H^{k+l}(X \times Y, \mathbb{Q})$, $k + l = 2r$, we can see by Poincaré duality α as an element of $Hom\,(H^{2n-l}(X, \mathbb{Q}), H^k(Y, \mathbb{Q}))$, where $n = dim\, X$. Then we have:

Lemma 1.11 *The class α is a Hodge class on $X \times Y$ if and only if the associated morphism is a morphism of Hodge structures (of bidegree (s, s), $s = r - n$).*

Proof. This indeed follows from the fact that by Lemma 1.3, the Hodge structure on the cohomology of $X \times Y$ is the tensor product of the Hodge structures on the cohomology of X and Y, and that the Hodge decomposition is compatible with Poincaré duality in the sense that

$$H^{p,q}(X) = (\oplus_{(p',q') \neq (n-p,n-q)} H^{p',q'}(X))^{\perp} = H^{n-p,n-q}(X)^*.$$

■

This interpretation of Hodge classes on products is crucial for the formulation of the standard conjectures (cf. [24] and section 3.5), which are all particular instances of the Hodge conjecture or consequences of them.

An important fact that we will use is the following:

Lemma 1.12 *If $\alpha \in H^{2k}(X, \mathbb{Q})$ is a Hodge class, then the cup-product by α:*

$$\cup\alpha : H^l(X, \mathbb{Q}) \to H^{l+2k}(X, \mathbb{Q})$$

is a morphism of Hodge structures (of bidegree (k, k)).

This follows indeed immediately from lemma 1.3. ■

1.5 Polarizations

A first formal consequence of the hard Lefschetz theorem 1.4 is the so-called Lefschetz decomposition. With the same notations as before, define for $k \leq n$ the primitive degree k cohomology of X by

$$H^k(X, \mathbb{R})_{prim} := Ker\,(L^{n-k+1} : H^k(X, \mathbb{R}) \to H^{2n+2-k}(X, \mathbb{R})).$$

For example, if $k = 1$, the whole cohomology is primitive, and if $k = 2$, primitive cohomology is the same as the orthogonal subspace, with respect to Poincaré duality, of $\omega^{n-1} \in H^{2n-2}(X, \mathbb{R})$.

The Lefschetz decomposition is the following (it can also be extended to $k > n$ using the hard Lefschetz isomorphism).

Theorem 1.13 *The cohomology groups $H^k(X, \mathbb{R})$ for $k \leq n$ decompose as*

$$H^k(X, \mathbb{R}) = \oplus_{2r \leq k} L^r H^{k-2r}(X, \mathbb{R})_{prim}.$$

1.5.1 Hodge-Riemann bilinear relations

We consider a Kähler compact manifold X with Kähler class ω. We can define an intersection form q_ω on each $H^k(X,\mathbb{R})$ by the formula

$$q_\omega(\alpha,\beta)=\int_X \omega^{n-k}\cup\alpha\cup\beta.$$

By hard Lefschetz theorem and Poincaré duality, q_ω is a non-degenerate bilinear form. It is skew-symmetric if k is odd and symmetric if k is even. Furthermore, the extension of q_ω to $H^k(X,\mathbb{C})$ satisfies the property that

$$q_\omega(\alpha,\beta)=0,\ \alpha\in H^{p,q},\ \beta\in H^{p',q'},\ (p',q')\neq(q,p).$$

This property is indeed an immediate consequence of lemma 1.3 and the fact that $H^{2n}(X,\mathbb{C})=H^{n,n}(X)$, $n=dim_{\mathbb{C}}X$.

Another way to rephrase this is to say that the Hermitian pairing h_ω on $H^k(X,\mathbb{C})$ defined by

$$h_\omega(\alpha,\beta)=\iota^k q_\omega(\alpha,\overline{\beta})$$

has the property that the Hodge decomposition is orthogonal with respect to h_ω.

This property is summarized under the name of first Hodge-Riemann bilinear relations.

Coming back to q_ω, note also that the Lefschetz decomposition is orthogonal with respect to q_ω. Indeed, if $\alpha=L^r\alpha'$, $\beta=L^s\beta'$, with $r<s$, and α', β' primitive, then

$$L^{n-k}\alpha\cup\beta=L^{n-k+r+s}\alpha'\cup\beta',$$

with $L^{n-k+r+s}\alpha'=0$ because $L^{n-k+2r+1}\alpha'=0$.

The second Hodge-Riemann bilinear relations given in Theorem 1.14 below play a crucial role, especially in the study of the period maps. Note first that, because the operator L shifts the Hodge decomposition by $(1,1)$, the primitive cohomology has an induced Hodge decomposition:

$$H^k(X,\mathbb{C})_{prim}=\oplus_{p+q=k}H^{p,q}(X)_{prim},$$

with $H^{p,q}(X)_{prim}:=H^{p,q}(X)\cap H^{p+q}(X,\mathbb{C})_{prim}$. We have now

Theorem 1.14 *The Hermitian form h_ω is definite of sign $(-1)^{\frac{k(k-1)}{2}}\iota^{p-q-k}$ on the component $L^rH^{p,q}(X)_{prim}$, $2r+p+q=k$, of $H^k(X,\mathbb{C})$.*

1.5.2 Rational polarizations and polarized Hodge structures

The Lefschetz decomposition is particularly useful if the Kähler class can be chosen to be rational, or equivalently if the manifold X is projective (see section 2). Indeed, in this case, the Lefschetz decomposition is a decomposition into rational vector subspaces, and as each of these subspaces is stable under the Hodge decomposition, it is a decomposition into Hodge substructures. This is very important for using Hodge theory to study moduli spaces of projective complex manifolds (cf. [18]). Indeed, the period map, which roughly speaking associates to a (Kähler or projective) complex structure the Hodge decomposition on the complex cohomology groups regarded as a varying decomposition on a fixed complex vector space, splits in the projective case into a product of period map for each primitive component (considering deformations of the complex structure with fixed integral Kähler class). Thus it takes values in a product of polarized period domains, which satisfy very strong curvature properties, (at least in the so-called horizontal directions satisfying Griffiths transversality, see [19]).

Let us formalize the notion which emerges from the Hodge-Lefschetz decomposition and the Hodge-Riemann bilinear relations.

Definition 1.15 *A rational polarized Hodge structure of weight k is a Hodge structure $(V, V^{p,q})$ of weight k, together with a rational intersection form q on V, symmetric if k is even, skew-symmetric if k is odd, such that the associated Hermitian bilinear form h on $V_{\mathbb{C}}$, defined by $h(v, w) = \iota^k q(v, \overline{w})$ satisfies the Hodge-Riemann bilinear relations:*

1. *The Hodge decomposition is orthogonal with respect to h.*
2. *The restriction of h to each $V^{p,q}$ is definite of sign $\epsilon_k(-1)^p$, $\epsilon_k = \pm 1$.*

This is (up to a sign) the structure we get on the primitive components of the cohomology of a compact Kähler manifold endowed with a rational Kähler class.

Example 1.16 We have seen that a weight 1 integral Hodge structure is the same thing as a complex torus. In this correspondence, a weight 1 integral polarized Hodge structure is the same thing as a projective complex torus (an abelian variety) with a given integral Kähler cohomology class. Indeed, let $(V, V_{\mathbb{C}} = V^{1,0} \oplus \overline{V^{1,0}})$ be a weight 1 Hodge structure and $q : \bigwedge^2 V \to \mathbb{Z}$ be a polarization. Then, recalling that the corresponding complex torus T is given by

$$T = V_{\mathbb{C}}/(V^{1,0} \oplus V),$$

we find that $\bigwedge^2 V^* \cong \bigwedge^2 H^1(T, \mathbb{Z}) = H^2(T, \mathbb{Z})$, and thus q can be seen as an integral cohomology class on T. Furthermore, q is represented in de Rham cohomology by the $\mathbb{R}$-linear extension $q_{\mathbb{R}}$ of q to $V_{\mathbb{R}}$, which is a 2-form on T, and $V_{\mathbb{R}}$ is isomorphic to $V^{0,1}$ by the projection, which gives the complex structure on the real tangent space $V_{\mathbb{R}}$ of T. Now one verifies ([41], I, 7.2.2) that the first Hodge-Riemann bilinear relation says that $q_{\mathbb{R}}$ is of type $(1, 1)$ and the second Hodge-Riemann bilinear relations say that $q_{\mathbb{R}}$ is a positive real $(1, 1)$ form, that is a Kähler form. Thus q is an integral Kähler class on T and T is projective by the Kodaira criterion (Theorem 2.1), or directly by Riemann's theory of Theta functions (cf. [28]).

2 The Kodaira problem

2.1 The Kodaira criterion

The Kodaira criterion [25] characterizes projective complex manifolds inside the class of compact Kähler manifolds.

Theorem 2.1 *(Kodaira's embedding theorem) A compact complex manifold X is projective if and only if X admits a Kähler class ω which is rational, that is belongs to $H^2(X, \mathbb{Q}) \subset H^2(X, \mathbb{R})$.*

The "only if" is easy. It comes from the fact that if X is projective, one gets a Kähler form on X by restricting the Fubini-Study Kähler form on some projective space $\mathbb{P}^N$ in which X is imbedded as a complex submanifold. But the Fubini-Study Kähler form has integral cohomology class, as its class is the first Chern class of the holomorphic line bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ on $\mathbb{P}^N$.

The converse is a beautiful application of the Kodaira vanishing theorem for line bundles endowed with metrics of positive associated Chern forms.

Definition 2.2 *A polarization on a projective manifold X is the datum of a rational Kähler cohomology class.*

As explained in the previous section, a polarization on X induces an operator L of cup-product with the given Kähler class, a Lefschetz decomposition on each cohomology group $H^k(X, \mathbb{Q})$, and a polarization on each component $L^r H^{k-2r}(X, \mathbb{Q})_{prim}$ of the Lefschetz decomposition.

2.2 Kodaira's theorem on surfaces

Kodaira's embedding theorem 2.1 can also be used to show that certain compact Kähler manifolds X become projective after a small deformations of their complex structure. The point is that the Kähler classes belong to $H^{1,1}(X)_{\mathbb{R}}$, the set of degree 2 cohomology classes which can be represented by a real closed $(1,1)$-form. They even form an open cone, the Kähler cone, in this real vector subspace of $H^2(X, \mathbb{R})$. This subspace deforms differentiably with the complex structure, and by Kodaira's criterion we are reduced to see whether one can arrange that after a small deformation of the complex structure on X, the deformed Kähler cone contains a rational cohomology class.

Example 2.3 Complex tori admit arbitrarily small deformations which are projective.

Let us state the beautiful theorem of Kodaira which was at the origin of the work [36].

Theorem 2.4 *[26] Let S be a compact Kähler surface. Then there is an arbitrarily small deformation of S which is projective.*

Kodaira proved this theorem using his classification of surfaces. Buchdahl [6], [7] gives a proof of Kodaira theorem which does not use the classification. His proof is infinitesimal and shows for example that a rigid compact Kähler surface is projective.

2.3 Various forms of the Kodaira problem

Kodaira's theorem 2.4 immediately leads to formulate a number of questions in higher dimensions:

Question 1: *(The Kodaira problem) Does any compact Kähler manifold admit an arbitrarily small deformation which is projective?*

In order to disprove this, it suffices to find rigid Kähler manifolds which are not projective. However, the paper [13] shows that it is not so easy: if a complex torus T carries three holomorphic line bundles L_1, L_2, L_3 such that the deformations of T preserving the L_i are trivial, then T is projective. The relation with the previous problem is the fact that from (T, L_1, L_2, L_3), one can construct a compact Kähler manifold whose deformations identify to the deformations of the 4-uple (T, L_1, L_2, L_3).

A weaker question concerns global deformations.

Question 2: *(The global Kodaira problem) Does any compact Kähler manifold X admit a deformation which is projective?*

Here we consider any deformation parameterized by a connected analytic space B, that is any smooth proper map $\pi : \mathcal{X} \to B$ between connected analytic spaces, with $X_0 = X$ for some $0 \in B$. Then any fiber X_t will be said to be a deformation of X_0. In that case, even the existence of rigid Kähler manifolds which are not projective would not suffice to provide a negative answer, as there exist complex manifolds which are locally rigid but not globally (consider for example the case of $\mathbb{P}^1 \times \mathbb{P}^1$ which is rigid but deforms to a different Hirzebruch surface). This means that we may have a family of smooth compact complex manifolds $\pi : \mathcal{X} \to B$ whose all fibers X_t for $t \neq 0$ are isomorphic but are not isomorphic to the central fiber X_0.

Note that if X is a deformation of Y, then X and Y are diffeomorphic, because the base B is path connected, and the family of deformations can be trivialized in the $\mathcal{C}^\infty$-category over any path in B.

In particular, X and Y should be homeomorphic, hence have the same homotopy type, hence also the same cohomology ring. Thus Question 2 can be weakened as follows :

Question 3: *(The topological Kodaira problem) Is any compact Kähler manifold X diffeomorphic or homeomorphic to a projective complex manifold?*
Does any compact Kähler manifold X have the homotopy type of a projective complex manifold?

The following theorem answers negatively the questions above.

Theorem 2.5 *[36] There exist, in any dimension ≥ 4, examples of compact Kähler manifolds which do not have the integral cohomology ring of a projective complex manifold.*

Our first proof used integral coefficients and worked only in the non simply connected case. Deligne provided then us with lemma 2.12 due to Deligne, which allowed him to extend the result to cohomology with rational coefficients, and even, after modification of our original example, complex coefficients, [36]). Using his Lemma, we also produce in [36] simply connected examples satisfying the above conclusion.

The examples built in [36] were built by blowing-up in an adequate way compact Kähler manifolds which had themselves the property of deforming to projective ones, namely self-products of complex tori, or self-products of Kummer varieties. This left open the possibility suggested by by Buchdahl, Campana and Yau, that under bimeromorphic transformations, the topological obstructions we obtained above for a Kähler manifold to admit a projective complex structure would disappear.

Question 4: *(The birational Kodaira problem) Is any compact Kähler manifold X bimeromorphic to a smooth compact complex manifold which deforms to a projective complex manifold?*

However we proved in [37] the following result.

Theorem 2.6 *In dimensions ≥ 10, there exist compact Kähler manifolds, no smooth bimeromorphic model of which has the rational cohomology algebra of a projective complex manifold.*

Note however that the compact Kähler manifolds constructed there do not have non-negative Kodaira dimension, as they are bimeromorphic to $\mathbb{P}^1 \times \mathbb{P}^1$-bundles on a product of Kummer manifolds. Thus the following remains open:

Conjecture 2.7 *(Campana 04, Tsunoda 86) Any compact Kähler manifold X of nonnegative Kodaira dimension is bimeromorphic to a smooth compact complex manifold which deforms to a projective complex manifold.*

We will give the main example and the detail of the argument of Theorem 2.5 in the next sections. Let us say that the topological obstruction that we exhibit comes from the notion of *polarized Hodge structure on a cohomology algebra*, where the polarization here is *rational.* The key point is the elementary lemma 1.3 which says that the Hodge decomposition on the cohomology groups of a compact Kähler manifold is compatible with the ring structure: If the ring structure is rich enough, this may force the Hodge structures to admit endomorphisms of Hodge structures. But certain endomorphisms of Hodge structures prevent the existence of a rational polarization.

2.4 Construction of examples

The simplest example of a compact Kähler manifold which cannot admit a projective complex structure for topological reasons is based on the existence of endomorphisms of complex tori which prevent the complex tori in question to be algebraic. Let Γ be a rank $2n$ lattice, and let ϕ be an endomorphism of Γ. Assume that the eigenvalues of ϕ are all distinct and none is real. Choosing n of these eigenvalues $\lambda_1, \ldots, \lambda_n$, so that no two of them are complex conjugate, one can then define

$$\Gamma^{1,0} := \text{eigenspace associated to the } \lambda_i' s \subset \Gamma_{\mathbb{C}}.$$

Clearly $\Gamma_{\mathbb{C}} = \Gamma^{1,0} \oplus \overline{\Gamma^{1,0}}$, so that we get a complex torus

$$T = \Gamma_{\mathbb{C}}/(\Gamma^{1,0} \oplus \Gamma).$$

The extended endomorphism $\phi_{\mathbb{C}}$ of $\Gamma_{\mathbb{C}}$ preserves both $\Gamma^{1,0}$ and Γ, and thus descends to an endomorphism ϕ_T of T. We have then the following :

Proposition 2.8 *[36] If $n \geq 2$ and the Galois group of the field $\mathbb{Q}(\lambda_1, \ldots, \lambda_n, \overline{\lambda}_1, \ldots, \overline{\lambda}_n)$, (that is the splitting field of $\mathbb{Q}(\phi)$,) acts as the full symmetric group $\mathfrak{S}_{2n}$ on the eigenvalues of ϕ, then T has $Hdg^2(T) = 0$ and thus T is not projective.*

Remark 2.9 In fact it would suffice here to know that the Galois group acts bitransitively on the eigenvalues. However, for the purpose of [37], which needs also the absence of Hodge classes of higher degree on $T \times \hat{T}$, except for the obvious ones, this stronger condition on the Galois group is needed.

Proof of proposition 2.8. Indeed, one looks at the action ϕ_T^* of ϕ_T on $H^2(T, \mathbb{Q}) = \bigwedge^2 \Gamma_{\mathbb{Q}}^*$. ϕ_T^* identifies to $\wedge^2\, {}^t\phi$. The assumption on the Galois group then shows that this action is irreducible. On the other hand, this action preserves the subspace $Hdg^2(T)$, which must then be either $\{0\}$ or the whole of $H^2(T, \mathbb{Q})$. As $n \geq 2$, we have $H^{1,1}(T) \neq H^2(T, \mathbb{C})$ and thus $Hdg^2(T) = \{0\}$. ∎

Our first example was the following. Let (T, ϕ_T) be as before, satisfying the assumptions of Proposition 2.8. Inside $T \times T$ we have the four subtori

$$T_1 = T \times 0,\ T_2 = 0 \times T,\ T_3 = Diag,\ T_4 = Graph(\phi_T),$$

which are all isomorphic to T.

These tori meet pairwise transversally in finitely many points $x_1, \ldots, x_N$. Blowing-up these points, the proper transforms $\widetilde{T_i}$ are smooth and do not meet anymore. We can thus blow-up them all to get a compact Kähler manifold X. This is our example.

Theorem 2.10 *X does not have the cohomology ring of a projective complex manifold. In other words, if Y is a compact Kähler manifold such that there exists an isomorphism*

$$\gamma : H^*(Y, \mathbb{Z}) \cong H^*(X, \mathbb{Z})$$

of graded rings, then Y is not projective.

We shall explain later on a simple proof of that. With the help of Lemma 2.12, to be explained later on, Deligne improved the result above, replacing integral coefficients by rational ones. Deligne also modified our example X in such a way that in the statement above, rational cohomology can be replaced with complex cohomology (cf [36], section 3.1).

2.4.1 Proof of theorem 2.10

We want to sketch here the proof of Theorem 2.10. Thus let X be constructed as in section 2.4, and Y be compact Kähler with an isomorphism $\gamma : H^*(Y, \mathbb{Z}) \cong H^*(X, \mathbb{Z})$ of cohomology rings. Our goal is to show that the Hodge structure on $H^1(Y, \mathbb{Z})$ cannot be polarized, thus proving that Y is not projective.

The cohomology group $H^2(X, \mathbb{Z})$ contains the classes e_i of the exceptional divisors E_i over the $\widetilde{T_i}$. We claim the following:

Lemma 2.11 *The classes $a_i := \gamma^{-1}(e_i)$ are Hodge classes on Y.*

Assuming this, it follows that the morphisms of Hodge structures

$$\cup a_i : H^1(Y,\mathbb{Z}) \to H^3(Y,\mathbb{Z})$$

have for kernels Hodge substructures L_i of $H^1(Y,\mathbb{Z})$. Of course $L_i = \gamma^{-1}(Ker \cup e_i)$.

Recall now that X is obtained from $T\times T$ by blow-ups. Thus $H^1(X,\mathbb{Z}) = H^1(T,\mathbb{Z}) \oplus H^1(T,\mathbb{Z})$. Furthermore, an easy computation involving the cohomology ring of a blow-up (cf [41], I, 7.3.3) shows that $Ker \cup e_i$, $i = 1,\ldots,4$, are equal respectively to

$$pr_2^* H^1(T,\mathbb{Z}),\ \ pr_1^* H^1(T,\mathbb{Z}),\ \ \Delta^-,\ \ Graph(\phi_T^*)^-,$$

where

$$\Delta^- := \{(\alpha,-\alpha),\, \alpha\in H^1(T,\mathbb{Z})\},$$

$$Graph(\phi_T^*)^- = \{(\phi_T^*\alpha,-\alpha),\, \alpha\in H^1(T,\mathbb{Z})\}.$$

But it follows that the 4 Hodge substructures L_i of $H^1(Y,\mathbb{Z})$ satisfy

$$L_1 \oplus L_2 = H^1(Y,\mathbb{Z})$$

as Hodge structures, and furthermore

$$L_3 \subset L_1\oplus L_2, \text{ resp. } L_4 \subset L_1 \oplus L_2$$

can be seen as the graphs of an isomorphism (resp. a homomorphism) of Hodge structures between L_1 and L_2. Thus using the isomorphism given by L_3, we can set $L_1 = L_2 =: L$, and the homomorphism given by L_4 gives an endomorphism ψ of L. It is immediate to see that ψ identifies to ϕ_T^*.

Thus we proved that the Hodge structure on $H^1(Y,\mathbb{Z})$ is a direct sum $L\oplus L$ of two copies of a certain weight 1 Hodge structure L carrying an automorphism wich is conjugate to $\phi_T^* = {}^t\phi$. By proposition 2.8 and Example 1.16, L is not polarizable, and neither is the Hodge structure on $H^1(Y,\mathbb{Z})$. ■

It remains to prove lemma 2.11. We will be brief here, as the use of Deligne's lemma 2.12 gives a much better approach to this statement, working for the rational cohomology algebra as well. The point is that looking at the isomorphism γ, we can conclude that the Albanese map a_Y of Y must be birational to its image, as it is the case for X. Indeed this property can be seen on the cohomology ring of a m-dimensional Kähler compact manifold, because it is equivalent to the fact that the natural map given by cup-product:

$$\bigwedge^{2m} H^1(Y,\mathbb{Z}) \to H^{2m}(Y,\mathbb{Z})$$

is an isomorphism.

Having this, one checks that the $\gamma^{-1}(e_i)$ must be in the kernel of the Gysin map

$$(a_Y)_* : H^2(Y,\mathbb{Z}) \to H^2(Alb(Y),\mathbb{Z}),$$

(because this kernel can be described using only the cohomology ring of Y, which is isomorphic to that of X). As a_Y is birational, this kernel is generated by the classes of the exceptional divisors of a_Y, hence consists of Hodge classes. ■

2.4.2 Deligne's lemma and applications

As we have seen in the previous proof, the key point was to show that under the assumptions of Theorem 2.10, certain classes in $H^2(Y,\mathbb{Z})$ must be Hodge classes, and then use them to show the existence of automorphisms of Hodge structures which prevent the existence of a polarization on the Hodge structure of $H^1(Y,\mathbb{Z})$.

The following provides an alternative proof for Lemma 2.11, namely the fact that the classes $\gamma^{-1}(e_i)$ must be Hodge classes, even if γ is only an isomorphism of rational cohomology rings, which leads to the proof (due to Deligne) of theorem 2.10 with rational coefficients.

Let $A^* = \oplus A^k$ be a graded $\mathbb{Q}$-algebra, and assume that each A^k carries a weight k Hodge structure, compatible with the product. (Recall that this means that the product map

$$A^k \otimes A^l \to A^{k+l} \tag{2.1}$$

is a morphism of weight $k+l$ Hodge structures.)

Lemma 2.12 *(Deligne) Let $Z \subset A^k_{\mathbb{C}}$ be a closed algebraic subset defined by homogeneous equations expressed only in terms of the product map on A^*, and let $Z' \subset Z$ be an irreducible component of Z. Assume the vector space $< Z' >$ generated by Z' is defined over $\mathbb{Q}$, that is*

$$< Z' > = B \otimes \mathbb{C},$$

for some $\mathbb{Q}$-vector subspace B of A^k. Then B is a rational Hodge substructure of A^k.

Here, by "defined by homogeneous equations expressed only in terms of the product map on A^*", we mean eg the following kind of algebraic subsets:

1. $Z = \{\alpha \in A^k_{\mathbb{C}},\, \alpha^l = 0 \text{ in } A^{kl}\}$, where l is a fixed integer.
2. $Z = \{\alpha \in A^k_{\mathbb{C}},\, rk\,(\alpha\cdot : A^l \to A^{k+l}) \leq m\}$ where l, m are fixed integers.

Proof of lemma 2.12. Indeed, as B is rational, to say that it is a Hodge substructure of A^k is equivalent to say that $B_{\mathbb{C}}$ is stable under the Hodge decomposition, or equivalently, that $B_{\mathbb{C}} =< Z' >$ is stable under the action of (the real algebraic group) $\mathbb{C}^*$ (see remark 1.6) defining the Hodge decomposition.

But this is immediate because the compatibility of the Hodge structures with the product is equivalent to the fact that the product map (2.1) is equivariant with respect to the $\mathbb{C}^*$-actions on both sides. Hence Z, and also Z', are stable under the $\mathbb{C}^*$-action, and thus, so is $< Z' >$. ∎

A first application of this is the following proof of Lemma 2.11. (In fact we will prove a slightly weaker statement, but which is enough for our purpose.) First of all, consider the subset $P \subset H^2(X,\mathbb{Q})$ generated by classes of exceptional divisors over $T \times T$. This set P is characterized intrinsically by the rational cohomology algebra of X, as being the subspace annihilating (under cup-product) the image of $\bigwedge^{4n-2} H^1(X,\mathbb{Q})$ in $H^{4n-2}(X,\mathbb{Q})$.

Inside P, there is the subspace P' generated by the classes of the total transforms of exceptional divisors over points. This P' has the property that for any $a \in P'$, the cup-product map $a\cup : H^1(X,\mathbb{Q}) \to H^3(X,\mathbb{Q})$ vanishes, and in fact P' is the subspace characterized by this property. Thus by Deligne's lemma, we find that both $\gamma^{-1}(P')$ and $\gamma^{-1}(P)$ are Hodge substructures of $H^2(Y,\mathbb{Q})$.

Finally, we look at the natural map induced by cup-product on Y:

$$\mu : \gamma^{-1}(P)/\gamma^{-1}(P') \to Hom\,(H^1(Y,\mathbb{Q}), H^3(Y,\mathbb{Q})).$$

Looking at the structure of the cohomology ring of X, we find that the set of elements p of $(\gamma^{-1}(P)/\gamma^{-1}(P')) \otimes \mathbb{C}$ for which $\mu(p) : H^1(Y,\mathbb{Q}) \to H^3(Y,\mathbb{Q})$ is not injective is the union of the four lines generated by $a_i = \gamma^{-1}(e_i)$ (or more precisely their projections modulo P'). Hence Deligne's lemma shows that the projection of each a_i in $\gamma^{-1}(P)/\gamma^{-1}(P')$ is a Hodge class. The rest of the proof then goes as before, because we conclude that the $\mu(a_i)$ are morphisms of Hodge structures, which is the only thing we need to conclude the proof as before. ∎

2.5 (Polarized) Hodge structures on cohomology algebras

Here *a cohomology algebra* A^* is the rational cohomology algebra of an orientable compact manifold. However, everything is true for any graded commutative finite dimensional $\mathbb{Q}$-algebra whose top degree term A^d has rank 1 and satisfying Poincaré duality which says that the pairings $A^k \times A^{d-k} \to A^d$, given by the product, are perfect.

Definition 2.13 *A Hodge structure on A^* is a Hodge structure of weight k on each A^k (i.e. a Hodge decomp. on $A^k_{\mathbb{C}}$, satisfying Hodge symmetry), such that:*

$$A^{p,q}_{\mathbb{C}} \cup A^{p',q'}_{\mathbb{C}} \subset A^{p+p',q+q'}_{\mathbb{C}}.$$

Remark 2.14 If there is a Hodge structure on A^*, the top degree d of A^* must be even, $d = 2n$. Indeed, there is a weight d Hodge structure on A^d, which has rank 1.

Definition 2.15 *Such a Hodge structure admits a real polarization if some $\alpha \in A^{1,1}_{\mathbb{R}}$ satisfies the hard Lefschetz property and the Hodge-Riemann bilinear relations. The polarization is said to be rational if α can be chosen in $A^2_{\mathbb{Q}} \cap A^{1,1}$.*

This makes sense abstractly: the hard Lefschetz property implies the Lefschetz decomposition. This is a decomposition into real Hodge substructures, thus giving a Hodge-Lefschetz decomposition. The class α^n trivializes A^{2n}, which gives a Poincaré duality on A^*, and allows to construct the intersection pairings q_α as in section 1.5.1. One can summarize the topological obstruction for our X to be projective as follows:

Theorem 2.16 *The compact Kähler manifolds constructed in the previous sections have the property that their cohomology algebra does not admit a Hodge structure with* rational *polarization.*

Remark 2.17 As they are Kähler compact, their cohomology algebra admits a Hodge structure with *real* polarization.

2.6 Further applications: topological restrictions on compact Kähler manifolds

There is a close geometric relation between symplectic geometry and Kähler geometry. If X is compact Kähler, forgetting the complex structure on X and keeping a Kähler form provides a pair (X, ω) which is a symplectic manifold. In the other direction, given a symplectic manifold (X, ω), the set of compatible almost complex structures J, $J^2 = -Id$ on T_X, i. e. satisfying the conditions

$$\omega(Ju, Jv),\ u, v \in T_{X,x},\ \omega(u, Ju) > 0,\ 0 \neq u \in T_{X,x}$$

is connected and non-empty by Gromov [21].

On the other hand, numerous topological restrictions are satisfied by compact Kähler manifolds, and not by general symplectic manifolds (cf [32]). For example, very strong restrictions on fundamental groups of compact Kähler manifolds have been found (see [1]) while Gompf proves in [16] that fundamental groups of compact symplectic manifolds are unrestricted.

Hodge theory provides two classical restrictions which come directly from what we discussed in section 1.

1. The odd degree Betti numbers $b_{2i+1}(X)$ are even for X compact Kähler. This follows either from the Hodge decomposition combined with Hodge symmetry or from the hard Lefschetz property.

2. The hard Lefschetz property

$$[\omega]^{n-k}\cup : H^k(X,\mathbb{R}) \cong H^{2n-k}(X,\mathbb{R}),\ 2n = dim_{\mathbb{R}}X$$

is satisfied.

Another topological restriction on compact Kähler manifolds is the so-called formality property [10]. A number of methods to produce examples of symplectic topologically non Kähler manifolds were found by Thurston [31], McDuff [27], Gompf [16]. On these examples, one of the properties above was not satified.

We exhibit below numerous other topological restrictions, coming from the notion of Hodge structure on a cohomology algebra. That is we want to exploit the following criterion.

Criterion 2.18 *The cohomology algebra $H^*(X,\mathbb{Q})$ of a Kähler manifold admits a Hodge structure (with real polarization).*

The difficulty one meets here is the fact that none of the data used to define the (polarized) Hodge structure, namely the Hodge numbers $h^{p,q} := dim\, H^{p,q}$ or the class (set of classes) used for the polarization, are topological invariants.

Thus we have to analyse abstractly the constraints without knowing the $h^{p,q}$-numbers or the set of polarization classes. Let us give the simplest example of how one can use this criterion. The proof, which is purely algebraic, is based on Deligne's lemma 2.12.

We start with an orientable compact manifold X and consider a complex vector bundle E on X. We denote by $p : \mathbb{P}(E) \to X$ the corresponding projective bundle. We make the following assumptions on (X, E) :

$$H^*(X) \text{ generated in degree} \leq 2 \text{ and } c_1(E) = 0.$$

By Leray-Hirsch theorem, one has an injection (of algebras) $p^* : H^*(X,\mathbb{Q}) \hookrightarrow H^*(\mathbb{P}(E),\mathbb{Q})$.

Theorem 2.19 *[40] If $H^*(\mathbb{P}(E),\mathbb{Q})$ admits a Hodge structure, then each $H^k(X,\mathbb{C}) \subset H^k(\mathbb{P}(E),\mathbb{C})$ has an induced Hodge decomposition (and thus $H^*(X,\mathbb{Q})$ also admits a Hodge structure).*

Furthermore each $c_i(E) \in H^{2i}(X,\mathbb{Q})$, $i \geq 2$, is of type (i,i) for this Hodge structure on $H^{2i}(X,\mathbb{Q})$.

This allows the construction of symplectic manifolds with abelian fundamental group satisfying formality (cf [10]) and the hard Lefschetz property, but not having the cohomology algebra of a compact Kähler manifold. These manifolds are produced as complex projective bundles over simply connected compact Kähler manifolds (eg complex tori), which easily implies that all the properties above are satisfied. We start with a compact Kähler X having a given class $\alpha \in H^4(X,\mathbb{Q})$ such that for any compatible Hodge decomposition on $H^*(X)$, α is not of type $(2,2)$. Then if E is any complex vector bundle on X satisfying $c_1(E) = 0$, $c_2(E) = \alpha$, $\mathbb{P}(E)$ is topologically non Kähler by theorem 2.19, using the criterion 2.18.

The simplest example of such a pair (X,α) is obtained by choosing for X a complex torus of dimension at least 4 and for α a class satisfying the property that the cup-product map $\alpha\cup : H^1(X,\mathbb{Q}) \to H^5(X,\mathbb{Q})$ has odd rank. Indeed, if α was Hodge for some Hodge structure on the cohomology algebra of X, this morphism would be a morphism of Hodge structure, hence its kernel would be a Hodge substructure of $H^1(X,\mathbb{Q})$, hence of even rank.

3 Hodge classes on projective and Kähler manifolds

This second lecture is devoted to describing the failure of any possible extension of the Hodge conjecture to the Kähler context, and to a discussion of the supplementary structures the cohomology with complex coefficients of a complex projective manifold has; these extra data are deduced from the fact that their cohomology can then be computed algebraically.

3.1 Betti cycle class and the Hodge conjecture

The simplest way of defining the Hodge class $[Z] \in Hdg^{2k}(X)$ of a codimension k closed algebraic subset $Z \stackrel{j}{\hookrightarrow} X$ of a complex projective manifold, (resp. closed analytic subset of a compact Kähler manifold), is to introduce a desingularization $\tilde{j} : \widetilde{Z} \to X$ of Z, and to consider the functional $\alpha \mapsto \int_{\tilde{Z}} \tilde{j}^*\alpha$ defined on $H^{2n-2k}(X, \mathbb{Q})$, $n = dim\, X$.

One then observes that this functional vanishes for type reasons on $\oplus_{(p',q')\neq(n-k,n-k)} H^{p',q'}(X)$. By Poincaré duality, it thus provides a class $[Z] \in H^{2k}(X, \mathbb{Q})$ which is in $H^{k,k}(X)$, that is a Hodge class.

The Hodge conjecture states the following:

Conjecture 3.1 *Assume X is projective. Then any Hodge class can be written as a combination with rational coefficients of classes $[Z]$ constructed above.*

3.2 Chern classes and Hodge classes

In this section, we want to discuss the possibility of enlarging the Hodge conjecture to the Kähler context. In this more general situation cycle classes do not provide in this case enough Hodge classes to generate Hodge classes. The simplest example is provided by a complex torus T admitting a holomorphic line bundle $\mathcal{L}$ of indefinite curvature. This means that a de Rham representative of $c_1(\mathcal{L})$ is given by a real $(1,1)$-form with constant coefficients on T, such that the corresponding Hermitian form is indefinite. If the torus T satisfying this condition is chosen general enough, its space $Hdg^2(T)$ will be generated by $c_1(\mathcal{L})$, as one shows by a deformation argument. It follows that T will not contain any analytic hypersurface, hence no non zero degree 2 Hodge class can be constructed as the Hodge class of a codimension 1 closed analytic subset. Thus we are in a situation where a Hodge classe can be constructed as the first Chern class of a holomorphic line bundle, but not as a combination with rational coefficients of classes of closed analytic subsets.

- Chern classes of holomorphic vector bundles. If E is a complex vector bundle on a topological manifold X, we have the rational Chern classes $c_i(E) \in H^{2i}(X, \mathbb{Q})$. (Note that the Chern classes are usually defined as integral cohomology classes, $c_i \in H^{2i}(X, \mathbb{Z})$, but in this text, the notation c_i will be used for the rational ones.)

These Chern classes of E can be defined in an axiomatic way, starting from the construction of $c_1(L)$ for any complex line bundle (see [15], [22]). Here $c_1(L)$ can be defined by the exponential exact sequence, which provides

$$H^1(X, \mathcal{C}_0^*) \to H^2(X, \mathbb{Z}),$$

where $\mathcal{C}_0^*$ is the sheaf of invertible complex functions on X. It can be also defined as the Euler class of the corresponding oriented real vector bundle of rank 2. One then uses the following decomposition formula for the cohomology of the projective bundle $p : \mathbb{P}(E) \to X$ (cf. [41], 7.3.3):

$$H^*(\mathbb{P}(E), \mathbb{Q}) \cong \oplus_{0 \leq i \leq r-1} h^i p^* H^{*-2i}(X, \mathbb{Q}), \tag{3.2}$$

where $h := c_1(\mathcal{H})$, and $\mathcal{H}$ is the dual of the Hopf (or tautological) sub-line bundle on $\mathbb{P}(E)$. This provides a relation in $H^*(\mathbb{P}(E), \mathbb{Q})$, uniquely determining the $c_i(E) \in H^{2i}(X, \mathbb{Q})$:

$$\sum_{0 \leq i \leq r} (-1)^i h^i c_{r-i}(E) = 0.$$

If E is now a holomorphic vector bundle on a complex manifold X, the Chern classes of E are Hodge classes. This follows from the fact that the classes $c_1(L)$ are Hodge classes, for any holomorphic line bundle on a complex manifold, so in particular $c_1(\mathcal{H})$ is a Hodge class on $\mathbb{P}(E)$, and from the fact that the isomorphism given in (3.2) is an isomorphism of Hodge structures.

One can alternatively use Chern-Weil theory which provides explicit de Rham representatives for the Chern classes, using a complex connection on E.

- Chern classes of coherent sheaves. Coherent sheaves $\mathcal{F}$ on a complex manifold X are sheaves of $\mathcal{O}_X$-modules which are locally presented as quotients

$$\mathcal{O}_X^r \stackrel{\phi}{\to} \mathcal{O}_X^s \to \mathcal{F} \to 0,$$

where ϕ is a matrix of holomorphic functions.

If X is a smooth projective complex manifold of dimension n, it is known (cf [29], [5]) that coherent sheaves on X admit finite locally free resolutions

$$0 \to \mathcal{F}_n \to \ldots \to \mathcal{F}_0 \to \mathcal{F} \to 0,$$

where the $\mathcal{F}_i$ are locally free. Identifying locally free coherent sheaves with holomorphic vector bundles, we can thus define the Chern classes of $\mathcal{F}$ by the Whitney formula :

$$c(\mathcal{F}) := \Pi_l c(\mathcal{F}_l)^{\epsilon_l}. \tag{3.3}$$

In this formula, $\epsilon_l := (-1)^l$, the total Chern class $c(\mathcal{F})$ is given by the formula

$$c(\mathcal{F}) = 1 + c_1(\mathcal{F}) + \ldots + c_n(\mathcal{F}),$$

and the series can be inverted because the cohomology ring is nilpotent in degree > 0 (see [15]). The Whitney formula and the case of holomorphic vector bundles imply that the right hand side of (3.3) is independent of the choice of locally free resolution and that the Chern classes $c_i(\mathcal{F})$ so defined are Hodge classes.

On a general compact Kähler manifold, such a finite locally free resolution does not exist in general (cf [35]). In order to define the $c_i(\mathcal{F})$, one can however use a finite locally free resolution as above of the sheaf $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{H}_X$ by sheaves of locally free $\mathcal{H}_X$-modules, where $\mathcal{H}_X$ is the sheaf of real analytic complex functions. One can then define the Chern classes of $\mathcal{F}$ by a Whitney formula as in (3.3) and some further work is needed to prove that these classes are Hodge classes. An alternative construction avoiding the use of real analytic resolutions can be found in [20].

The construction of rational Hodge classes as rational combinations of Chern classes of coherent sheaves is more general than the two previously given constructions. Namely, it is obvious that it generalizes Chern classes of holomorphic vector bundles, as a coherent sheaf is a more general object than a holomorphic vector bundle, but it is also true that it also generalizes the construction of classes of analytic subsets, for the following reason:

If $Z \subset X$ is a closed analytic subset of codimension k, then its ideal sheaf $\mathcal{I}_Z$ is a coherent sheaf, and one has:

$$c_k(\mathcal{I}_Z) = (-1)^k (k-1)![Z]. \tag{3.4}$$

In the projective case, it is known that the three constructions generate over $\mathbb{Q}$ the same space of Hodge classes (cf. [29] and [5]). It was already mentioned that in the general Kähler case, Chern classes of holomorphic vector bundles or coherent sheaves may provide more Hodge classes than cycle classes. The fact that Chern classes of coherent sheaves allow to construct strictly more Hodge classes than Chern classes of holomorphic vector bundles was proved in [35]. This is something which cannot be detected in degree 2, as in degree 2, Chern classes of holomorphic line bundles generate all integral Hodge classes, a fact which is known as the Lefschetz theorem on $(1,1)$-classes.

If we want to extend the Hodge conjecture to the Kähler case, we therefore are led to consider the following question:

Question 3.2 *Are Hodge classes on compact Kähler manifolds generated over $\mathbb{Q}$ by Chern classes of coherent sheaves?*

The answer to this question is negative, as we proved in [35]. The next subsection will be devoted to describing the counterexample.

3.3 Weil tori and Weil classes

The Hodge classes described below have been constructed by Weil in the case of algebraic tori, as a potential counterexample to the Hodge conjecture for algebraic varieties. In the case of a general complex torus, the construction is still simpler. These complex tori have been also considered in [42] by Zucker. In the application, it will suffice to consider 4-dimensional Weil tori, but the general construction is not more complicated. We will show that they provide a counterexample to Question 3.2.

Recall from Example 1.8 that there is an equivalence of categories between complex tori and weight 1 Hodge structures. In particular, a complex torus admitting an endomorphism I such that $I^2 = -Id$ is the same as a weight 1 Hodge structure Γ, $\Gamma_{\mathbb{C}} = W \oplus \overline{W}$, endowed with a endomorphism I of the lattice Γ, such that $I^2 = -Id$ and I leaves W stable.

We thus start with a $\mathbb{Z}[I]$-action on $\Gamma := \mathbb{Z}^{4n}$, where $I^2 = -Id$, which makes $\Gamma_{\mathbb{Q}} := \Gamma \otimes \mathbb{Q}$ into a K-vector space, where K is the quadratic field $\mathbb{Q}[I]$.

Let $\Gamma_{\mathbb{C}} = \Gamma \otimes \mathbb{C} = \mathbb{C}^{2n}_{\iota} \oplus \mathbb{C}^{2n}_{-\iota}$ be the associated decomposition into eigenspaces for I. A $2n$ dimensional complex subspace W of $\Gamma_{\mathbb{C}}$ which is stable under I has to be the direct sum

$$W = W_{\iota} \oplus W_{-\iota}$$

of its intersections with $\mathbb{C}^{2n}_{\iota}$ and $\mathbb{C}^{2n}_{-\iota}$. It has furthermore to satisfy the condition that

$$W \cap \Gamma_{\mathbb{R}} = \{0\} \tag{3.5}$$

which is equivalent to the fact that W and $\overline{W}$ are complementary subspaces.

Given W, the complex torus X is given by the formula : $X = \Gamma_{\mathbb{C}}/(W \oplus \Gamma)$.

We will choose W so that the following (crucial) condition holds:

$$dim\, W_{\iota} = dim\, W_{-\iota} = n. \tag{3.6}$$

Hence X is determined in this case by the choice of the n-dimensional subspaces

$$W_{\iota} \subset \mathbb{C}^{2n}_{\iota},\ W_{-\iota} \subset \mathbb{C}^{2n}_{-\iota},$$

which have to be general enough so that condition (3.5) is satisfied.

We have isomorphisms

$$H^{2n}(X, \mathbb{Q}) \cong H_{2n}(X, \mathbb{Q}) \cong \bigwedge^{2n} \Gamma_{\mathbb{Q}}. \tag{3.7}$$

Consider the subspace

$$\bigwedge^{2n}_{K} \Gamma_{\mathbb{Q}} \subset \bigwedge^{2n} \Gamma_{\mathbb{Q}}. \tag{3.8}$$

To better understand how this subspace is defined, let us (for ease of notation) identify $K = \mathbb{Q}[I]$ with $\mathbb{Q}[\iota]$. Then as $\Gamma_{\mathbb{Q}}$ is a K-vector space, there is a splitting

$$\Gamma_{\mathbb{Q}} \otimes K = \Gamma_{K,\iota} \oplus \Gamma_{K,-\iota}$$

into eigenspaces for the I-action. Then we have the following maps of $\mathbb{Q}$-vector spaces whose composition gives our desired inclusion:

$$\bigwedge_K^{2n}\Gamma_{\mathbb{Q}}\cong\bigwedge_K^{2n}\Gamma_{K,\iota}\to\bigwedge_K^{2n}\Gamma_K\cong\bigwedge^{2n}\Gamma_{\mathbb{Q}}\otimes K\stackrel{Tr}{\to}\bigwedge^{2n}\Gamma_{\mathbb{Q}}. \tag{3.9}$$

Since $\Gamma_{\mathbb{Q}}$ is a $2n$-dimensional K-vector space, $\bigwedge_K^{2n}\Gamma_{\mathbb{Q}}$ is a one dimensional K-vector space, and its image under this inclusion is thus a 2 dimensional $\mathbb{Q}$-vector space. The claim is that, under the assumption (3.6), $\bigwedge_K^{2n}\Gamma_{\mathbb{Q}}$ is made of Hodge classes, that is, is contained in the subspace $H^{n,n}(X)$ of the Hodge decomposition. Notice that under the isomorphisms (3.7), tensored by $\mathbb{C}$, $H^{n,n}(X)$ identifies with the image of $\bigwedge^n W\otimes\bigwedge^n\overline{W}$ in $\bigwedge^{2n}\Gamma_{\mathbb{C}}$.

The claim follows immediately from the description in (3.9) of the subspace (3.8). Indeed, $\bigwedge_K^{2n}\Gamma_{\mathbb{Q}}\subset\bigwedge^{2n}\Gamma_{\mathbb{Q}}$ is defined as the image of $\bigwedge_K^{2n}\Gamma_{K,\iota}\subset\bigwedge_K^{2n}\Gamma_K$ via the trace map

$$\bigwedge_K^{2n}\Gamma_K=\bigwedge_{\mathbb{Q}}^{2n}\Gamma_{\mathbb{Q}}\otimes K\to\bigwedge^{2n}\Gamma_{\mathbb{Q}}.$$

Now we have the inclusion

$$\Gamma_K\subset\Gamma_{\mathbb{C}},$$

with $\Gamma_{\mathbb{C}}=\Gamma_K\otimes_{\mathbb{Q}}\mathbb{R}$, and the equality

$$\Gamma_{K,\iota}=\Gamma_K\cap\mathbb{C}_\iota^{2n}.$$

The space $\Gamma_{K,\iota}$ is a $2n$ dimensional K-vector space which generates over $\mathbb{R}$ the space $\mathbb{C}_\iota^{2n}$. It follows that the image of $\bigwedge_K^{2n}\Gamma_{K,\iota}$ in $\bigwedge^{2n}\Gamma_{\mathbb{C}}$ generates over $\mathbb{R}$ the complex line $\bigwedge^{2n}\mathbb{C}_\iota^{2n}$. But we know that $\mathbb{C}_\iota^{2n}$ is the direct sum of the two spaces W_ι and $\overline{W}_{-\iota}$ which are n-dimensional. Hence

$$\bigwedge^{2n}\mathbb{C}_\iota^{2n}=\bigwedge^n W_\iota\otimes\bigwedge^n\overline{W}_{-\iota}\subset\bigwedge^n W\otimes\bigwedge^n\overline{W}=H^{n,n}(X).$$

■

We proved in [35] that the Weil Hodge classes on general Weil tori provide a counterexample to question 3.2, thus showing that the projectivity assumption is crucial in the statement of the Hodge conjecture.

Theorem 3.3 *[35] Let T be a general Weil torus of dimension 4. Then any coherent sheaf $\mathcal{F}$ on T satisfies $c_2(\mathcal{F})=0$.*

Thus the Weil Hodge classes constructed in the previous sections are not in the space generated by Chern classes of coherent sheaves.

The proof uses the Uhlenbeck-Yau theorem [34], and can be even shortened by using the Bando-Siu theorem [2], which extends Uhlenbeck-Yau theorem to the case of reflexive coherent sheaves. This proof shows that complex analysis and differential geometry produce in fact restrictions on Hodge classes coming from global holomorphic objects. The main point here is the fact that, as a consequence of the Uhlenbeck-Yau theorem, for a compact Kähler manifold X with a trivial space $Hdg^2(X)$ of Hodge classes of degree 2, Chern classes $c_2(E)$ of stable vector bundles E on X have to satisfy a certain positivity condition. Bando and Siu extend this to coherent reflexive stable sheaves.

3.4 Algebraic de Rham cohomology and algebraic cycle class

Let X be a smooth projective variety defined over a field K of characteristic 0. One has the sheaf of Kähler differentials $\Omega_{X/K}$ which is a locally free algebraic coherent sheaf on X, locally generated near $x\in X(K)$ by df_i, where $f_i(x)=0$ and the f_i generate $\mathcal{M}_x/\mathcal{M}_x^2$ as a K-vector space.

We can form the locally free sheaves $\Omega_{X/K}^l:=\bigwedge^l\Omega_{X/K}$ and by the definition of $\Omega_{X/K}$ we get the differentials $d:\mathcal{O}_X\to\Omega_{X/K},\ d:\Omega_{X/K}^l\to\Omega_{X/K}^{l+1}$ satisfying $d\circ d=0$.

Definition 3.4 *The algebraic de Rham cohomology of X is defined as*

$$H^k_{dR}(X/K) := \mathbb{H}^k(X, \Omega^*_{X/K}).$$

Note that this finite dimensional K-vector space depends on K. However, when $K \subset L$ (field extension), one has

$$H^k_{dR}(X_L/L) = H^k_{dR}(X/K) \otimes_K L.$$

We want now to use this notion to show the following remarkable fact discovered by Grothendieck:

The Betti cohomology with complex coefficients of a smooth complex projective manifold X^{an} corresponding to a smooth projective variety X defined over $\mathbb{C}$, can be computed as an algebraic invariant of X seen as an algebraic variety.

Note that this is not at all true if we change the field of coefficients. Even with $\mathbb{R}$ instead of $\mathbb{C}$, and even if the variety X is defined over $\mathbb{R}$, the cohomology of X^{an} with real coefficients cannot be computed by algebraic means. It is furthermore known by work of Serre (see also [9], [30] for further refined versions of this phenomenon) that the homotopy types indeed does not depend only on the abstract algebraic variety X. In fact, a field automorphism of $\mathbb{C}$ will provide another complex algebraic variety, thus another complex manifold, which is usually not homeomorphic or homotopically equivalent to the original one.

The precise form of the statement above is the following:

Theorem 3.5 *Let X be a smooth algebraic variety defined over $\mathbb{C}$. Then there is a canonical isomorphism*

$$H^k_{dR}(X/\mathbb{C}) = H^k(X^{an}, \mathbb{C}),$$

where X^{an} is the corresponding complex submanifold of $\mathbb{CP}^n$.

Proof. We first recall the GAGA principle: Let $X \subset \mathbb{P}^N(\mathbb{C})$ be a closed algebraic subset, or more precisely a closed subscheme, and let $\mathcal{F}$ be an algebraic coherent sheaf on X. The sheaf $\mathcal{F}$ has an analytic counterpart $\mathcal{F}^{an}$, which is an analytic coherent sheaf on X^{an}. The space of sections of the sheaf $\mathcal{F}^{an}$ over an (usual) open set U of X^{an} is essentially the space of sections of the sheaf $\mathcal{F}$ defined over a Zariski neighbourhood of U, tensored by the space of holomorphic functions on U. Thus morally, it is the sheaf of holomorphic sections of $\mathcal{F}$, considered in the usual topology.

The GAGA comparison theorem [29] says the following:

Theorem 3.6 *(Serre) For any algebraic coherent sheaf $\mathcal{F}$ on X, one has a canonical isomorphism $H^l(X, \mathcal{F}) \to H^l(X^{an}, \mathcal{F}^{an})$, induced by the morphism of ringed spaces $\phi : (X^{an}, \mathcal{O}_{X^{an}}) \to (X, \mathcal{O}_X)$, which satisfies : $\phi^*\mathcal{F} = \mathcal{F}^{an}$.*

We now apply this to the sheaves $\Omega^l_{X/\mathbb{C}}$ of Kähler differentials. It is easy to prove that the corresponding analytic coherent sheaves are nothing but the sheaves of holomorphic differentials $\Omega^l_{X^{an}}$. We thus conclude that for X smooth and algebraic, and for any p, q, we have

$$H^q(X, \Omega^p_{X/\mathbb{C}}) \cong H^q(X^{an}, \Omega^p_{X^{an}}),$$

where on the right we have the Dolbeault cohomology of X^{an}. A spectral sequence argument then allows to conclude that we have a canonical isomorphism of hypercohomology groups:

$$\mathbb{H}^k(X, \Omega^*_{X/\mathbb{C}}) \cong \mathbb{H}^k(X^{an}, \Omega^*_{X^{an}}). \tag{3.10}$$

But we have the holomorphic de Rham resolution

$$0 \to \mathbb{C} \to \mathcal{O}_{X^{an}} \xrightarrow{d} \Omega_{X^{an}} \to \dots \to \Omega^n_{X^{an}} \to 0,\ n = dim\, X$$

of the constant sheaf $\mathbb{C}$ on X^{an}, which makes the constant sheaf $\mathbb{C}$ on X^{an} quasiisomorphic to the holomorphic de Rham complex $\Omega^*_{X^{an}}$. This implies that (3.10) can be written as

$$H^k_{dR}(X/\mathbb{C}) := \mathbb{H}^k(X, \Omega^*_{X/\mathbb{C}}) \cong H^k(X^{an}, \mathbb{C}). \tag{3.11}$$

■

Remark 3.7 What makes striking Theorem 3.5 is the fact that the algebraic de Rham complex is not at all acyclic in positive degree in the Zariski topology, so that the proof above is completely indirect. In fact, by the affine version of Theorem 3.5, its degree i cohomology sheaf is the complexified version of the sheaf $\mathcal{H}^i$ studied by Bloch and Ogus [3].

3.4.1 Algebraic cycle class

Let X be a smooth projective variety defined over K and $Z \subset X$ be a local complete intersection closed algebraic subset of X, also defined over K. Following Bloch [4], we construct a cycle class

$$[Z]_{alg} \in H^{2k}_{dR}(X/K)$$

which by construction lies in fact in

$$F^k H^{2k}_{dR}(X/K) := Im\,(\mathbb{H}^{2k}(X, \Omega^{*\geq k}_{X/K}) \to \mathbb{H}^{2k}(X, \Omega^*_{X/K})).$$

Being a local complete intersection, Z can be locally in the Zariski topology defined by k equations $f_1, \ldots, f_k$. On a Zariski open set U where these k equations define $U \cap Z$, we have a covering of U by k open sets

$$U_i := \{x \in U,\ f_i(x) \neq 0\}.$$

On the intersection $U_1 \cap \ldots \cap U_k$, the closed degree k differential $\frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_k}{f_k}$ has no poles hence defines a section of $\Omega^k_{U/K}$ which is in fact a closed form. We can see it as a Čech cocycle on U relative to the open cover above, with value in $\Omega^{k,c}_{U/K}$, where the superscript c stands for "closed". We thus get an element of $H^{k-1}(U, \Omega^{k,c}_{U/K})$. Observe now that there is an obvious map of complexes

$$\Omega^{k,c}_{X/K} \to \Omega^{*\geq k}_{X/K},$$

where the left hand side should be put in degree k. We thus get a class in

$$\mathbb{H}^{2k-1}(U, \Omega^{*\geq k}_{U/K}) \cong \mathbb{H}^{2k}_{Z\cap U}(U, \Omega^{*\geq k}_{U/K}).$$

This class can be shown to be independent of the choice of equations f_i. These locally defined classes thus provide a global section of the sheaf of hypercohomology with support $\mathcal{H}^{2k}_Z(\Omega^{*\geq k}_{X/K})$. Examining the local to global spectral sequence for $\mathbb{H}^{2k}_Z(X, \Omega^{*\geq k}_{X/K})$ one finds now that it is very degenerate, so that

$$H^0(Z, \mathcal{H}^{2k}_Z(\Omega^{*\geq k}_{X/K})) = \mathbb{H}^{2k}_Z(X, \Omega^{*\geq k}_{X/K})$$

which provides us with a class in $\mathbb{H}^{2k}_Z(X, \Omega^{*\geq k}_{X/K})$. Using the natural map

$$\mathbb{H}^{2k}_Z(X, \Omega^{*\geq k}_{X/K}) \to \mathbb{H}^{2k}(X, \Omega^{*\geq k}_{X/K}),$$

we finally get the desired cycle class $[Z]_{alg}$.

Let now X be a smooth projective variety defined over $\mathbb{C}$ and $Z \subset X$ be as above. The following comparison result can be verified to hold as a consequence of Cauchy formula or rather multiple residue formula:

Theorem 3.8 *Via the isomorphism (3.11) in degree $2k$, one has*

$$[Z]_{alg} = (2\iota\pi)^k [Z].$$

3.5 Absolute Hodge classes

Here we enter one of the most fascinating aspects of the Hodge conjecture, which seriously involves the fact that the complex manifolds we are considering are algebraic.

Let us first introduce the notion of (de Rham) absolute Hodge class (cf. [12]). First of all, let us make a change of definition: a Hodge class of degree $2k$ on X will be in this section a class $\alpha \in (2i\pi)^k H^{2k}(X,\mathbb{Q}) \cap H^{k,k}(X)$. The reason for this shift is the fact that we want to use the algebraic cycle class $[Z]_{alg}$ introduced in section 3.4.1, which takes value in algebraic de Rham cohomology, and which equals (via the isomorphism (3.11)) $(2i\pi)^k[Z]$ by Theorem 3.8.

Let X^{an} be a complex projective manifold and $\alpha \in Hdg^{2k}(X^{an})$ be a Hodge class. Thus $\alpha \in (2i\pi)^k H^{2k}(X,\mathbb{Q})$ and

$$\alpha \in F^k H^{2k}(X^{an},\mathbb{C}) \cong \mathbb{H}^{2k}(X^{an},\Omega^{*\geq k}_{X^{an}}). \tag{3.12}$$

Here, the left hand side can be shown to be given by the Hodge filtration

$$F^k H^{2k}(X^{an},\mathbb{C}) = \oplus_{p\geq k} H^{p,2k-p}(X^{an})$$

on the Betti cohomology of the complex manifold X^{an} and the isomorphism of (3.12) is induced by the holomorphic de Rham resolution.

As before, the right hand side in (3.12) can be computed as in Theorem 3.5 as the hypercohomology of the algebraic variety X with value in the complex of algebraic differentials:

$$\mathbb{H}^{2k}(X^{an},\Omega^{*\geq k}_{X^{an}}) \cong \mathbb{H}^{2k}(X,\Omega^{*\geq k}_{X/\mathbb{C}}). \tag{3.13}$$

Let us denote by $\mathcal{E}$ the set of fields embeddings of $\mathbb{C}$ in $\mathbb{C}$. For each element σ of $\mathcal{E}$, we get a new algebraic variety X_σ defined over $\mathbb{C}$, obtained from X by applying σ to the coefficients of the defining equations of X, and we have a similar isomorphism for X_σ. Note that σ acts on complex points of $\mathbb{P}^n$, which induces a natural map from $X(\mathbb{C})$ (which as a set is X^{an}) to $X_\sigma(\mathbb{C})$ (which as a set is X_σ^{an}), but that this map $X^{an} \to X_\sigma^{an}$ is not continuous in general (the only non trivial continuous automorphism of $\mathbb{C}$ being complex conjugation).

But as an algebraic variety, X_σ is deduced from X by applying σ, and it follows that there is a natural (only $\sigma(\mathbb{C})$-linear) map between algebraic de Rham cohomology spaces:

$$\mathbb{H}^{2k}(X,\Omega^{*\geq k}_{X/\mathbb{C}}) \to \mathbb{H}^{2k}(X_\sigma,\Omega^{*\geq k}_{X_\sigma/\mathbb{C}}).$$

Applying the comparison isomorphism (3.13) in the reverse way, the class α provides a (de Rham or Betti) complex cohomology class

$$\alpha_\sigma \in \mathbb{H}^{2k}(X_\sigma,\Omega^{*\geq k}_{X_\sigma}) = F^k H^{2k}(X_\sigma^{an},\mathbb{C})$$

for each $\sigma \in \mathcal{E}$.

Definition 3.9 *(cf [12])* The class α is said to be (de Rham) absolute Hodge if α_σ is a Hodge class for each σ, that is $\alpha_\sigma = (2i\pi)^k\beta_\sigma$, for some rational cohomology class $\beta_\sigma \in H^{2k}(X_\sigma^{an},\mathbb{Q})$.

The main reason for introducing this definition is the following:

Proposition 3.10 *If $Z \subset X$ is a complex subvariety of codimension k, then $(2i\pi)^k[Z] \in (2i\pi)^k H^{2k}(X,\mathbb{Q})$ is an absolute Hodge class.*

Proof. This indeed follows from the comparison theorem 3.8 which tells us that $(2i\pi)^k[Z] = [Z]_{alg}$, and from the fact that

$$[Z]_{alg,\sigma} = [Z_\sigma]_{alg} = (2i\pi)^k[Z_\sigma],$$

as shows the explicit construction described in section 3.4.1. Here $Z_\sigma \subset X_\sigma$ is deduced from Z by applying the field embedding σ to the defining equations of Z. ■

Proposition 3.10 shows that the Hodge conjecture contains naturally the following sub-conjectures:

Conjecture 3.11 *Hodge classes on smooth complex projective varieties are absolute Hodge.*

This conjecture is solved affirmatively by Deligne for Hodge classes on abelian varieties (cf. [12]). An important but easy point in this proof is the fact that Weil classes (cf. section 3.3) on Weil abelian varieties are absolute Hodge.

To conclude this section, let us mention two crucial examples of absolute Hodge classes. They play an important role in the theory of algebraic cycles (cf. [24]) and are not known to be algebraic (that is to satisfy the Hodge conjecture).

Example 3.12 *Let X be smooth projective of dimension n. The class $[\Delta_X]$ of the diagonal in $X \times X$ is an algebraic class in $H^{2n}(X \times X, \mathbb{Q})$. This spaces identifies under Künneth decomposition and Poincaré duality to $\oplus_i End\, H^i(X, \mathbb{Q})$. The (algebraic) class of the diagonal identifies to $(2\iota\pi)^n \sum_i Id_{H^i(X,\mathbb{Q})}$. By lemma 1.11, for each $i \leq 2n$, the class corresponding to $(2\iota\pi)^n Id_{H^i(X,\mathbb{Q})}$ gives a Hodge class on $X \times X$. This Hodge class is absolute.*

Example 3.13 *X being as above, recall from Theorem 1.4 that if $h = c_1(H)$ where H is an ample line bundle on X, there is for each $k \leq n$ an isomorphism of Hodge structures*

$$h^{n-k} : H^k(X, \mathbb{Q}) \cong H^{2n-k}(X, \mathbb{Q}).$$

The corresponding Hodge class of degree $4n - 2k$ on $X \times X$ is algebraic. In fact it is the class of the cycle H^{n-k} supported on the diagonal Δ_X.

Consider now the inverse of the Lefschetz isomorphism:

$$(h^{n-k})^{-1} : H^{2n-k}(X, \mathbb{Q}) \cong H^k(X, \mathbb{Q}).$$

It provides a Hodge class of degree $2k$ on $X \times X$, which is absolute Hodge, and is not known to be algebraic.

3.6 Absolute Hodge classes and the structure of Hodge loci

3.6.1 Locus of Hodge classes

The key point in which algebraic geometry differs from Kähler geometry is the fact that a smooth complex projective variety X does not come alone, but accompanied by a full family of deformations $\pi : \mathcal{X} \to T$, where π is smooth and projective (that is $\mathcal{X} \subset T \times \mathbb{P}^N$ over T, for some integer N), and where the basis T is quasi-projective smooth and defined over $\mathbb{Q}$. (Here T is not supposed to be geometrically connected). Indeed, one can take for T a desingularization of a Zariski open set of the reduced Hilbert scheme parameterizing subschemes of $\mathbb{P}^n$ with same Hilbert polynomial as X. The existence of this family of deformations is reflected in the transformations $X \mapsto X_\sigma$ considered above. Namely, the variety T being defined over $\mathbb{Q}$, σ acts on its complex points, and if X is the fiber over some complex point $0 \in T(\mathbb{C})$, then X_σ is the fiber over the complex point $\sigma(0)$ of $T(\mathbb{C})$.

The total space $\mathcal{X}$ is thus an algebraic variety defined over $\mathbb{Q}$ (and in fact we may even complete it to a smooth projective variety defined over $\mathbb{Q}$), but for the moment, let us consider it as a family of smooth complex varieties, that is, let us work with $\pi : \mathcal{X}^{an} \to T^{an}$.

Associated to this family are the Hodge bundles

$$F^l\mathcal{H}^k := R^k\pi_*(\Omega^{*\geq l}_{\mathcal{X}^{an}/T^{an}}) \subset \mathcal{H}^k := R^k\pi_*(\Omega^*_{\mathcal{X}^{an}/T^{an}}) = R^k\pi_*\mathbb{C} \otimes \mathcal{O}_{T^{an}}, \qquad (3.14)$$

which are coherent analytic locally free sheaves with respective fibers over $t \in T$

$$F^l H^k(X_t, \mathbb{C}) \subset H^k(X_t, \mathbb{C}).$$

In (3.14), the last isomorphism $R^k\pi_*(\Omega^*_{\mathcal{X}^{an}/T^{an}}) = R^k\pi_*\mathbb{C} \otimes \mathcal{O}_{T^{an}}$ is induced by the resolution of the sheaf $\pi^{-1}\mathcal{O}_{T^{an}}$ by the relative holomorphic de Rham complex.

We shall denote by $F^l H^k$ the total space of the corresponding vector bundles.

Definition 3.14 *(cf [8]) The locus of Hodge classes for the family $\mathcal{X} \to T$ and in degree $2k$ is the subset $Z \subset F^kH^{2k}$ consisting of pairs (t, α) where $t \in T(\mathbb{C})$ and α_t is a Hodge class on X_t, $\alpha_t \in F^kH^{2k}(X_t, \mathbb{C}) \cap (2i\pi)^k H^{2k}(X_t, \mathbb{Q})$.*

This locus is thus the set of all Hodge classes in fibers of π.

For $\alpha \in Z$ we shall denote by Z_α the connected component of Z passing through α and by T_α the projection of Z_α to T. T_α is the Hodge locus of α, that is the locus of deformations of X where α deforms as a Hodge class.

Let us give a local analytic description of the locus of Hodge classes : we want to describe all the pairs (t, α), $t \in T$, $\alpha \in (2i\pi)^k H^{2k}(X_t, \mathbb{Q})$, such that $\alpha \in F^kH^{2k}(X_t, \mathbb{C})$. Let us choose a connected and simply connected neighbourhood U of 0 in T. Then all the fibers X_t, $t \in U$, are canonically homeomorphic to X_0, so that we have a canonical identification

$$H^{2k}(X_t, \mathbb{Q}) \cong H^{2k}(X_0, \mathbb{Q})$$

and more precisely a holomorphic trivialization of the vector bundle H^{2k} over U:

$$H^{2k} \cong U \times H^{2k}(X_0, \mathbb{C}). \tag{3.15}$$

This identification preserves the rational structures on both sides.

Let us consider the following composite holomorphic map:

$$\Phi : F^kH^{2k} \hookrightarrow H^{2k} \cong U \times H^{2k}(X_0, \mathbb{C}) \stackrel{pr_2}{\to} H^{2k}(X_0, \mathbb{C}).$$

By definition, a Hodge class on X_t is a class in $F^kH^{2k}(X_t, \mathbb{C})$ which via Φ is sent to a $(2\iota\pi)^k$ times rational cohomology class in $H^{2k}(X_0, \mathbb{C})$. It follows that the locus of Hodge classes is locally defined as $\Phi^{-1}((2\iota\pi)^k H^{2k}(X_0, \mathbb{Q}))$, which is a countable union a closed analytic subsets of F^kH^{2k}.

The above description is highly transcendental, as it makes use of the trivialization (3.15). Note now the fact that the complex (maybe reducible) manifold F^kH^{2k} is in fact algebraic and defined over $\mathbb{Q}$. Indeed, using GAGA principle, the coherent sheaf $F^k\mathcal{H}^{2k}$ is simply the analytic coherent sheaf associated to the algebraic sheaf $R^{2k}\pi_*(\Omega^{*\geq k}_{\mathcal{X}/T})$ which is defined over $\mathbb{Q}$ on T. (Here the $R^{2k}\pi_*$ is the algebraic derived functor.)

Thinking a little more, we see that if $\sigma : \mathbb{C} \to \mathbb{C}$ is a field embedding, then σ acts on the complex points of the complex manifold F^kH^{2k} (because it is defined over $\mathbb{Q}$), and that if $(t, \alpha_t) \in F^kH^{2k}$ is a complex point of this complex manifold, then $\sigma(t, \alpha_t)$ is nothing but the class $\alpha_{t,\sigma} \in F^kH^{2k}(X^{an}_{t,\sigma})$ we considered in the previous section. We deduce from this the following interpretation of the notion of absolute Hodge class.

Lemma 3.15 *(cf. [38]) i) To say that Hodge classes of degree $2k$ on fibers of the family $\mathcal{X} \to T$ are absolute Hodge is equivalent to say that the locus Z is a countable union of closed algebraic subsets of F^kH^{2k} defined over $\mathbb{Q}$.*

ii) To say that α is an absolute Hodge class is equivalent to say that Z_α is a closed algebraic subset of F^kH^{2k} defined over $\overline{\mathbb{Q}}$ and that its images under Gal $(\overline{\mathbb{Q}} : \mathbb{Q})$ are again components of the locus of Hodge classes.

Proof. i) The implication $\Leftarrow$ is obvious once one observes that $\alpha \mapsto \alpha_\sigma$ is nothing but the action of $\sigma \in Aut\,\mathbb{C}$ on the complex points of F^kH^{2k}; indeed, if Z is of the stated form, it is then stable under the action of $Aut\,\mathbb{C}$, and Hodge classes are absolute Hodge. In the other direction, we have to prove that if Z is stable under $Aut\,\mathbb{C}$, it is a countable union of closed algebraic subsets defined over $\mathbb{Q}$. Observe now that if $\alpha \in Z$, the orbit of α under $Aut\,\mathbb{C}$ fills in the complement of a countable union of proper Zariski closed subsets of the $\mathbb{Q}$-Zariski closure of α in F^kH^{2k}. Taking for α a sufficiently general point in a local connected component of Z, and using the local structure of Z as a countable union of closed analytic subsets of F^kH^{2k}, we then conclude that Z equals the countable union of $\mathbb{Q}$-Zariski closure of adequately chosen $\alpha \in Z$.

ii) One direction is again obvious. In the other direction, we will use the algebraicity theorem 3.16 below. It thus only remains to prove that Z_α is defined over $\overline{\mathbb{Q}}$. For this, one observes as in [12] that if α is absolute Hodge, then any $\alpha' \in Z_\alpha$ is also absolute Hodge (for which one needs in fact Theorem 3.16 below). But then, if α is absolute Hodge, for any $\sigma \in Aut\,\mathbb{C}$ the algebraic subset $Z_{\alpha,\sigma} \subset F^kH^{2k}$ is also contained in the locus of Hodge classes, and as an easy argument shows, $Z_{\alpha,\sigma}$ is locally a component of this locus. But the locus of Hodge classes has only countably many components (as shows its local description) and thus we conclude that there only countably many translates $Z_{\alpha,\sigma} \subset F^kH^{2k}$, $\sigma \in Aut\,\mathbb{C}$. We then use the fact that a closed algebraic subset of a complex variety defined over $\mathbb{Q}$ which has only countably many translates under $Aut\,\mathbb{C}$ is defined over $\overline{\mathbb{Q}}$. ■

The following result, which anwers partially conjecture 3.11, establishes part of the expected structure of the locus of Hodge classes.

Theorem 3.16 *[8] The connected components Z_α of Z are closed algebraic subsets of F^kH^{2k}. As a consequence, the Hodge loci T_α are closed algebraic subsets of T.*

This theorem is very deep. It is very much expected if one believes in the Hodge conjecture, because then the Z_α will be the images by the relative algebraic cycle class of universal relative cycles parameterized by components of the relative Hilbert scheme of $\mathcal{X} \to T$.

Let us now investigate the arithmetic aspect of the notion of absolute Hodge class, exploiting its relation with the definition field of the component of the Hodge loci.

Proposition 3.17 *[38] i) Let $\alpha \in F^kH^{2k}$ be an absolute Hodge class. Then the Hodge conjecture is true for α if it is true for absolute Hodge classes on varieties defined over $\overline{\mathbb{Q}}$.*

ii) Let $\alpha \in F^kH^{2k}$ be a Hodge class, such that the Hodge locus T_α is defined over $\overline{\mathbb{Q}}$. Then the Hodge conjecture is true for α if it is true for Hodge classes on varieties defined over $\overline{\mathbb{Q}}$.

Proof. i) The key ingredient is the global invariant cycle theorem or more classically "Theorem of the fixed part" (Theorem 3.18 below) due to Deligne [11]. Let Y be a smooth complex projective variety, and $U \subset Y$ a Zariski open set. Let $\phi : U \to B$ be a smooth proper algebraic morphism, where B is quasi-projective. Thus the fibers of ϕ are smooth complex projective varieties and there is a monodromy action:

$$\rho : \pi_1(B, 0) \to Aut\,H^l(Y_0, \mathbb{Q}),\ 0 \in B.$$

Theorem 3.18 *The space of invariant classes*

$$H^l(Y_0, \mathbb{Q})^\rho := \{\alpha \in H^l(Y_0, \mathbb{Q}),\ \rho(\gamma)(\alpha) = \alpha, \forall \gamma \in \pi_1(B, 0)\}$$

is equal to the image of the restriction map (which is a morphism of Hodge structures) :

$$H^l(Y, \mathbb{Q}) \to H^l(Y_0, \mathbb{Q}).$$

In particular it is a Hodge substructure of $H^l(Y_0, \mathbb{Q})$.

Let us now put everything together: let X be complex projective and $\alpha \in (2i\pi)^kH^{2k}(X, \mathbb{Q})$ be an absolute Hodge class. There is a smooth projective map $\pi : \mathcal{X} \to T$ defined over $\overline{\mathbb{Q}}$, with $\mathcal{X}$ and T smooth quasi-projective, and such that X is the fiber of π over a complex point of T (the smooth locus of π).

As we explained above, the fact that α is absolute Hodge implies that the component of the Hodge locus containing α, say Z_α, is defined over $\overline{\mathbb{Q}}$. We consider the reduced subscheme underlying Z_α, say R_α, which we may assume by shrinking to be smooth and connected. Then we make the base change $R_\alpha \to T$, which gives $\pi_\alpha : \mathcal{X}_\alpha \to R_\alpha$, where both varieties are smooth quasi-projective and defined over $\overline{\mathbb{Q}}$, and π_α is smooth projective.

Tautologically, over R_α we have a holomorphic section $\tilde{\alpha}$ of F^kH^{2k}. By definition of R_α, this holomorphic section has the particularity that at any point $t \in R_\alpha$, $\alpha_t \in$

$(2i\pi)^k H^{2k}(X_t, \mathbb{Q})$. But then, by countability, we conclude that $\tilde{\alpha}$ is a locally constant section of H^{2k}, which is everywhere of type (k,k). Hence our class α extends to a global section of the local system $(2i\pi)^k R^{2k}\pi_{\alpha *}\mathbb{Q}$, which means equivalently that α is invariant under the monodromy action along R_α.

Now we introduce a smooth compactification $\overline{\mathcal{X}_\alpha}$ defined over $\overline{\mathbb{Q}}$. The global invariant cycles theorem tells us that there is a class

$$\beta \in (2i\pi)^k H^{2k}(\overline{\mathcal{X}_\alpha}, \overline{\mathbb{Q}}) \cap F^k H^{2k}(\overline{\mathcal{X}_\alpha})$$

which restricts to α on X. With some more work, using the polarizations introduced in section 1.5.2, one can show that β can be chosen absolute Hodge. Now, if the Hodge conjecture is true for β, it is true for α.

Statement ii) is proved in a similar way. ∎

Proposition 3.17 is one motivation to investigate the question whether the Hodge loci T_α are defined over $\overline{\mathbb{Q}}$, which by Lemma 3.15 is weaker than the question whether Hodge class are absolute.

Concerning this last problem, we conclude with the following criterion, also proved in [38]:

Theorem 3.19 *Let $\alpha \in F^k H^{2k}(X, \mathbb{C})$ be a Hodge class. Suppose that any locally constant Hodge substructures $L \subset H^{2k}(X_t, \mathbb{Q})$, $t \in T_\alpha$, is purely of type (k,k). Then T_α is defined over $\overline{\mathbb{Q}}$, and its translates under $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ are again of the form T_β.*

The assumptions in the theorem are reasonably easy to check in practice, for example by infinitesimal methods. On the other hand, they are clearly not satisfied in a case where the component T_α of the Hodge locus consists of one isolated point, if the Hodge structure on $H^{2k}(X)$ is not trivial. In this case, what predicts the Hodge conjecture is that this point should be defined over $\overline{\mathbb{Q}}$. But our criterion does not give this: in fact our criterion applies only when we actually have a non trivial variation of Hodge structure along T_α.

Note that Theorem 3.19 also addresses partially Conjecture 3.11, in view of Lemma 3.15. Indeed, we know by Theorem 3.16 that the Z_α's are algebraic, and thus by Lemma 3.15, Conjecture 3.11 is a question about the definition field of the Z_α's and their translates under $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Theorem 3.19 addresses the same question for the T_α's instead of Z_α.

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