

MONOIDAL JANTZEN FILTRATIONS

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ABSTRACT. We introduce a monoidal analogue of Jantzen filtrations in the framework of monoidal abelian categories with generic braidings. It leads to a deformation of the multiplication of the Grothendieck ring. We conjecture, and we prove in many remarkable situations, that this deformation is associative so that our construction yields a quantization of the Grothendieck ring as well as analogs of Kazhdan-Lusztig polynomials. As a first main example, for finite-dimensional representations of simply-laced quantum loop algebras, we prove the associativity and we establish that the resulting quantization coincides with the quantum Grothendieck ring constructed by Nakajima and Varagnolo-Vasserot in a geometric manner. Hence, it yields a unified representation-theoretic interpretation of the quantum Grothendieck ring. As a second main example, we establish an analogous result for a monoidal category of finite-dimensional modules over symmetric quiver Hecke algebras categorifying the coordinate ring of a unipotent group associated with a Weyl group element.

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1. INTRODUCTION

Jantzen filtrations are at the origin of fundamental developments of representation theory. For instance, the celebrated Jantzen conjecture [31] (and its reformulation by Gabber-Joseph [23]), originally proved by Beilinson-Bernstein [4], implies that the (original) Kazhdan-Lusztig polynomials [43] are interpreted in terms of Jantzen filtrations of Verma modules in the category \mathcal{O} of a simple Lie algebra. This explains remarkable properties of these polynomials: their coefficients are positive and their evaluation at 1 are the multiplicities of simple modules in certain distinguished representations. This gives rise to the Kazhdan-Lusztig algorithm to compute characters of simple modules in certain important categories by using geometric representation theory.

The definition of Jantzen filtrations relies on an isomorphism of \mathbb{K} -vector spaces

$$\phi: V \otimes_{\mathbb{0}} \mathbb{K} \simeq W \otimes_{\mathbb{0}} \mathbb{K}$$

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where \mathbb{K} is the fraction field of an integral domain \mathbb{O} , and V, W are \mathbb{O} -modules. For \mathfrak{p} a maximal ideal of \mathbb{O} , one has the respective filtrations $\mathfrak{p}^i V$ and $\mathfrak{p}^j W$ of V and W , $i, j \geq 0$. The Jantzen filtrations are obtained from their interplay via the isomorphism ϕ (see [32, II.8] for the precise definition).

We introduce a monoidal analogue of Jantzen filtrations in the framework of monoidal categories with generic braidings, which we call R -matrices, with the following salient points in comparison to ordinary Jantzen filtrations:

(1) Instead of one isomorphism ϕ , our definition of the filtration of W is obtained from two remarkable isomorphisms

$$V \otimes_{\mathbb{O}} \mathbb{K} \simeq W \otimes_{\mathbb{O}} \mathbb{K} \simeq V' \otimes_{\mathbb{O}} \mathbb{K},$$

by an interplay of the images of three relevant filtrations.

(2) Our filtrations lead to the deformation not only of certain multiplicities, but also of the structure constants of the Grothendieck ring of the monoidal category.

The precise formula for the monoidal Jantzen filtrations is given in (2.8).

Our general construction depends on the choice of a PBW-theory in the monoidal category, that is a choice of a family of simple objects (the cuspidal objects) whose monoidal products (the mixed products) satisfy certain remarkable properties. Then the construction involves a deformation of this PBW-theory along a formal parameter together with R -matrices, crucial isomorphisms between deformations of the mixed products. These are isomorphisms in (1) where W is a mixed product and V, V' are distinguished mixed products, called respectively standard and costandard.

Our monoidal Jantzen filtrations are filtrations by subobjects

$$F_{\bullet} M: \quad M \supset \cdots \supset F_{-1} M \supset F_0 M \supset F_1 M \supset \cdots \supset \{0\}.$$

We establish that, under mild conditions, the filtrations are compatible with specializations of R -matrices and satisfy certain duality properties.

The decategorification version of the filtration $F_{\bullet} M$ is defined as

$$[M]_t := \sum_{n \in \mathbb{Z}} [\mathrm{Gr}_n^F M] t^n.$$

It belongs to the Grothendieck group of the category, with the coefficients extended to $\mathbb{Z}[t^{\pm 1}]$ for a formal variable t .

Some of these coefficients are defined to be the analogues of Kazhdan-Lusztig polynomials. We establish the existence of a corresponding canonical basis under reasonable conditions.

Moreover, this decategorification defines a $\mathbb{Z}[t^{\pm 1/2}]$ -bilinear operation $*$ (after a slight twist) that deforms the multiplication of the Grothendieck ring. We conjecture that in a general setting, this deformation defines a ring, that is the operation $*$ is associative. This is one of the new salient points in comparison to the original theory of Jantzen filtrations.

Our first main examples for monoidal Jantzen filtrations are realized in categories of finite-dimensional representations of the quantum loop algebra $U_q(L\mathfrak{g})$ associated with a complex simple Lie algebra \mathfrak{g} and a generic quantum parameter $q \in \mathbb{C}^{\times}$. This is a Hopf algebra whose finite-dimensional modules form an interesting abelian monoidal category \mathcal{C} , which is neither semisimple nor braided. In particular, the tensor product $V \otimes W$ is not isomorphic to its opposite $W \otimes V$ for general simple modules $V, W \in \mathcal{C}$. Nevertheless, their Jordan-Hölder factors coincide up to reordering. In other words, we have $[V \otimes W] = [W \otimes V]$ in the Grothendieck ring $K(\mathcal{C})$, and hence $K(\mathcal{C})$ is commutative. Indeed, this commutativity

follows from the injectivity of the so-called q -character homomorphism $\chi_q: K(\mathcal{C}) \rightarrow \mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1} \mid i \in I, a \in \mathbb{C}^\times]$ due to Frenkel-Reshetikhin [17], where I is an index set of the simple roots of \mathfrak{g} . Thus, one may identify $K(\mathcal{C})$ with a subring of \mathcal{Y} .

By the classification result due to Chari-Pressley [9], the set of classes of simple modules in \mathcal{C} is in bijection with the set $\mathcal{M}^+ \subset \mathcal{Y}$ of monomials in the variables $Y_{i,a}$. For each $m \in \mathcal{M}^+$, the corresponding simple module $L(m)$ is of highest weight m , namely $\chi_q(L(m))$ has m as its highest term. The problem to compute $\chi_q(L(m))$ for all $m \in \mathcal{M}^+$ is of fundamental importance. At the present moment, a general closed formula (like the Weyl character formula) is not known.

One possible strategy is to find an algorithm to compute $\chi_q(L(m))$ recursively, analogous to the Kazhdan-Lusztig algorithm. For each $x \in I \times \mathbb{C}^\times$, the q -character of the simple module $V_x := L(Y_x)$ (called a fundamental module) can be computed by an algorithm due to Frenkel-Mukhin [15]. For each monomial $m = Y_{x_1} \cdots Y_{x_d} \in \mathcal{M}^+$, if (x_1, \dots, x_d) is ordered suitably, the corresponding tensor product $M(m) := V_{x_1} \otimes \cdots \otimes V_{x_d}$ has a simple head isomorphic to $L(m)$. Moreover, there exists a partial ordering of \mathcal{M} (called the Nakajima partial ordering) such that we have

$$[M(m)] = [L(m)] + \sum_{m' < m} P_{m,m'} [L(m')]$$

in $K(\mathcal{C})$. The module $M(m)$ is called a standard module. Since we know $\chi_q(M(m))$, it is enough to compute the multiplicities $P_{m,m'}$. For this purpose, we consider a one-parameter (non-commutative) deformation of $K(\mathcal{C})$, called the quantum Grothendieck ring. It was introduced by Nakajima [53] and by Varagnolo-Vasserot [57] for \mathfrak{g} of simply-laced type, and by the second author [27] for general \mathfrak{g} . The quantum Grothendieck ring $K_t(\mathcal{C})$ is a $\mathbb{Z}[t^{\pm 1/2}]$ -subalgebra of a quantum torus \mathcal{Y}_t deforming \mathcal{Y} , stable under a natural anti-involution $y \mapsto \bar{y}$ of \mathcal{Y}_t , and comes with a standard $\mathbb{Z}[t^{\pm 1/2}]$ -basis $\{M_t(m)\}_{m \in \mathcal{M}^+}$. Under the specialization $t \rightarrow 1$, $M_t(m)$ goes to $[M(m)]$. We can prove (see [27, 53]) that there exists the canonical basis $\{L_t(m)\}_{m \in \mathcal{M}^+}$ satisfying $\overline{L_t(m)} = L_t(m)$ and

$$M_t(m) = L_t(m) + \sum_{m' < m} P_{m,m'}(t) L_t(m')$$

for some $P_{m,m'}(t) \in t\mathbb{Z}[t]$. This characterization enables us to compute the polynomials $P_{m,m'}(t)$ recursively. When \mathfrak{g} is of simply-laced type, the following result was obtained by using perverse sheaves on quiver varieties.

Theorem 1.1 ([53, 57]). *When \mathfrak{g} is of simply-laced type, the following properties hold:*

- (KL) *Analogue of Kazhdan-Lusztig conjecture: under the specialization $t \rightarrow 1$, $L_t(m)$ goes to $[L(m)]$, or equivalently, we have $P_{m,m'}(1) = P_{m,m'}$.*
- (P) *Positivity: for any $m' < m$, we have $P_{m,m'}(t) \in \mathbb{Z}_{\geq 0}[t]$.*

The second author [27] conjectured that these properties hold for general \mathfrak{g} . Very recently, with Oh and Oya, we obtained some pieces of evidence of this conjecture.

Theorem 1.2 ([20, 21]). *The property (KL) also holds when \mathfrak{g} is of type B. For general \mathfrak{g} , the property (KL) also holds for all simple modules that are reachable (in the sense of cluster algebras). The property (P) holds for general \mathfrak{g} .*

Having these results, we ask: what is representation-theoretic meaning of $K_t(\mathcal{C})$ or $P_{m,m'}(t)$?

Here we propose an answer to this question by introducing monoidal Jantzen filtrations for any tensor products of fundamental modules. For any sequence $\epsilon = (\epsilon_1, \dots, \epsilon_d)$ of elements of $I \times \mathbb{C}^\times$, let $M(\epsilon) := V_{\epsilon_1} \otimes \cdots \otimes V_{\epsilon_d}$ be the corresponding tensor product, which is not

necessarily a standard module (we call it is a mixed product). By using R -matrices, we define a monoidal Jantzen filtration $F_\bullet M(\epsilon)$ by $U_q(L\mathfrak{g})$ -submodules (in the paper, we will also handle more general PBW-theories in \mathcal{C}). The decategorification gives a corresponding element $[M(\epsilon)]_t$ of the t -deformed Grothendieck group $K(\mathcal{C})_t := K(\mathcal{C}) \otimes \mathbb{Z}[t^{\pm 1/2}]$.

Then we define a $\mathbb{Z}[t^{\pm 1/2}]$ -bilinear map $*$: $K(\mathcal{C})_t \times K(\mathcal{C})_t \rightarrow K(\mathcal{C})_t$ by

$$[M(m)]_t * [M(m')]_t := t^{\gamma(m, m')} [M(m) \otimes M(m')]_t,$$

where γ is a certain skew-symmetric bilinear form on \mathcal{M}^+ . Also, $K(\mathcal{C})_t$ is endowed with a natural involution $\overline{X \otimes f(t)} = X \otimes f(t^{-1})$. Now we propose the following:

Conjecture 1.3 (= Conjecture 3.20). *The pair $(K(\mathcal{C})_t, *)$ defines a $\mathbb{Z}[t^{\pm 1/2}]$ -algebra with anti-involution, and it is isomorphic to the quantum Grothendieck ring $K_t(\mathcal{C})$ identifying the standard basis $\{M_t(m)\}_{m \in \mathcal{M}^+}$ with the basis $\{[M(m)]_t\}_{m \in \mathcal{M}^+}$.*

Note that the associativity of the map $*$ is unclear from the definition. Besides, Conjecture 1.3 implies the above properties (KL) and (P). We prove the Conjecture 1.3 for \mathfrak{g} of simply-laced type. This is one of the main results of this paper.

Theorem 1.4 (= Theorem 3.26). *Conjecture 1.3 is true when \mathfrak{g} is of simply-laced type.*

As a consequence, we obtain a categorification of the quantum Grothendieck ring in terms of finite-dimensional representations enhanced with their monoidal Jantzen filtrations. Note also that it was established in [30] that, when \mathfrak{g} is of simply-laced type, the quantum Grothendieck ring contains a copy of the positive part $U_q(\mathfrak{n})$ of the finite-type quantum group $U_q(\mathfrak{g})$ (it corresponds to the quantum Grothendieck of a monoidal subcategory of finite-dimensional representations). Hence we obtain as well a new categorification of $U_q(\mathfrak{n})$ in terms of our monoidal Jantzen filtrations.

Our proof of Theorem 1.4 uses geometric method due to Nakajima involving perverse sheaves on quiver varieties. Actually, our strategy is much inspired by Grojnowski's unpublished note [26], which studies filtrations on standard modules over quantum loop algebras and affine Hecke algebras using perverse sheaves.

Recall that the first proof of the original Jantzen conjecture for Verma modules by Beilinson-Bernstein [4] was also geometric, where the Jantzen filtrations are identified with the weight filtrations of some standard \mathcal{D} -modules on flag manifolds through the Beilinson-Bernstein localization. There is another approach due to Soergel [56] and Kübel [47], which is a Koszul dual picture to Beilinson-Bernstein's proof. In this second approach, the Jantzen filtrations are related to the Andersen filtrations on the Hom-space from Verma to tilting modules in the category \mathcal{O} , which is in turn identified with the degree filtrations of the local intersection cohomology of Schubert varieties. A key ingredient here is the hard Lefschetz theorem applied to the setting of the "Fundamental Example" of Bernstein-Lunts [3]. See the introduction of [60] for more details and recent further development.

Our proof of Theorem 1.4 has a similar flavor to this second approach. Based on Nakajima's geometric construction, we identify our monoidal Jantzen filtrations of the mixed products $M(\epsilon)$ with the degree filtrations of certain hyperbolic localizations (in the sense of Braden [6]) of perverse sheaves on graded quiver varieties. Here, key ingredients are again the hard Lefschetz property and the "Fundamental Example" mentioned above. Since the Poincaré polynomials of these hyperbolic localizations serve the structure constants of the quantum Grothendieck ring $K_t(\mathcal{C})$ in its geometric definition [58], we obtain the desired result.

Our second examples of the monoidal Jantzen filtrations are given by the finite-dimensional modules over symmetric quiver Hecke algebras. For any symmetric Kac-Moody algebra \mathfrak{g} and

an element w of its Weyl group, one has a monoidal abelian category \mathcal{C}_w consisting of finite-dimensional ungraded modules over the quiver Hecke algebras (or rather their completions), which categorifies the coordinate ring $\mathbb{C}[N(w)]$ of a unipotent algebraic group $N(w)$. Note that this category \mathcal{C}_w is obtained from its graded version \mathcal{C}_w^\bullet categorifying the quantized coordinate ring $A_t[N(w)]$ by forgetting the grading. To each reduced word $\mathbf{i} = (i_1, i_2, \dots, i_\ell)$ for w , one can associate the dual PBW-basis of $A_t[N(w)]$, which are categorified by ordered products of the so-called cuspidal modules in \mathcal{C}_w^\bullet [38]. Forgetting the grading, we have a basis of standard modules for the Grothendieck ring $K(\mathcal{C}_w)$. Since the category \mathcal{C}_w has generic braidings, one can apply the same construction as above to define the monoidal Jantzen filtrations and hence get a deformation $(K(\mathcal{C}_w)_t, *)$ of $K(\mathcal{C}_w)$. Thus, it makes sense to expect that $(K(\mathcal{C}_w)_t, *)$ defines an associative algebra isomorphic to the quantum coordinate ring $A_t[N(w)]$. In other words, the monoidal Jantzen filtrations in \mathcal{C}_w may recover the forgotten gradings of the Jordan-Hölder multiplicities in \mathcal{C}_w^\bullet . This is an analog of Conjecture 1.3 above. Note also that the same construction applies to the affine Hecke algebras of general linear groups as well, since their central completions are identical to the completions of quiver Hecke algebras of type A . In this paper, we verify the conjecture in the following special case.

Theorem 1.5 (= Theorem 4.14). *The analog of Conjecture 1.3 for \mathcal{C}_w is true when the reduced word \mathbf{i} is adapted to a quiver.*

In fact, when the reduced word \mathbf{i} is adapted to a quiver Q , we have a geometric interpretation of the quiver Hecke algebra due to Varagnolo-Vasserot [59] and the relevant mixed product modules in terms of the equivariant perverse sheaves on the space of representations of the quiver Q , which appear in the construction of the canonical bases of quantized enveloping algebras due to Lusztig [49]. Theorem 1.5 can be proved by applying the aforementioned strategy for Theorem 1.4 to this geometric situation.

We can expect our theory extends to other frameworks, such as to the coherent Satake category [7] or to the representation theory of p -adic groups.

Organization. This paper is organized as follows. In Section 2, we develop a general theory of monoidal Jantzen filtrations in the setting of monoidal abelian category of representations (modules) over an algebra. In Section 3, we discuss the case of quantum loop algebras and state our main Conjecture 1.3. We also provide some concrete examples of monoidal Jantzen filtrations at the end (Section 3.7). In Section 4, we discuss the case of quiver Hecke algebras and state the analogous conjecture. The remaining part of the paper is devoted to the proofs of our main theorems, where we apply some geometric methods including perverse sheaves. Before going into individual discussions, in Section 5, we assemble some relevant facts on equivariant perverse sheaves which we commonly use in the proofs. Finally, we prove our main Theorems 1.4 and 1.5 above in Sections 6 and 7 respectively.

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Overall conventions.

- (1) For a statement P , we set $\delta(P)$ to be 1 or 0 according that P is true or false. We often abbreviate $\delta(i = j)$ as $\delta_{i,j}$.
- (2) For an object X in a category, we denote by id_X the identity morphism on X . We often abbreviate it as id suppressing the subscript X when it is clear from the context.

- (3) We write \mathbb{Z} , \mathbb{N} , \mathbb{Q} , and \mathbb{C} for the sets of integers, non-negative integers, rational numbers, and complex numbers, respectively. Note that we have $0 \in \mathbb{N}$ in our convention.
- (4) For a set J , we define $\mathbb{N}^{\oplus J}$ to be the subset of \mathbb{N}^J consisting of J -tuples $\mathbf{d} = (d_j)_{j \in J}$ with finite support, i.e., $\#\{j \in J \mid d_j > 0\} < \infty$. For each $i \in J$, let $\delta_i := (\delta_{i,j})_{j \in J} \in \mathbb{N}^{\oplus J}$ be the delta function.

2. GENERAL DEFINITIONS FOR MONOIDAL CATEGORIES OF REPRESENTATIONS

In this section we explain our general categorical framework to construct monoidal Jantzen filtrations (Definition 2.14 and Formula (2.8)). They depend on a PBW-theory in a monoidal category (Section 2.1) and on a deformation of this PBW-theory (Section 2.2) together with R -matrices (Section 2.4). We establish in general the compatibility of the monoidal Jantzen filtrations with specialized R -matrices (Propositions 2.16, 2.17). We explain in Section 2.6 the decategorification process and the construction of analogs of Kazhdan-Lusztig polynomials. We conjecture that we obtain a ring through this process (Conjectures 2.19, 2.20). Then we establish a general duality result (Proposition 2.22) between filtrations of standard and costandard objects, a Kazhdan-Lusztig type characterization of a canonical basis and we state a Duality Conjecture 2.25 related to the existence of a bar involution.

2.1. PBW-theory for monoidal categories of representations. Let A be an associative algebra over a field \mathbb{k} . In what follows, we abbreviate $\otimes_{\mathbb{k}}$ as \otimes . We assume that there is a non-trivial \mathbb{k} -algebra homomorphism $\varepsilon: A \rightarrow \mathbb{k}$, through which \mathbb{k} is regarded as an A -module. Let B be an $(A, A^{\otimes 2})$ -bimodule which is free of finite rank as a right $A^{\otimes 2}$ -module and equipped with isomorphisms

$$(2.1) \quad B \otimes_{A^{\otimes 2}} (B \otimes A) \simeq B \otimes_{A^{\otimes 2}} (A \otimes B)$$

of $(A, A^{\otimes 3})$ -bimodules, and

$$(2.2) \quad B \otimes_{A^{\otimes 2}} (A \otimes \mathbb{k}) \simeq B \otimes_{A^{\otimes 2}} (\mathbb{k} \otimes A) \simeq A$$

of (A, A) -bimodules, making the category of (left) A -modules into a \mathbb{k} -linear monoidal category with respect to the product

$$(2.3) \quad M \star N := B \otimes_{A^{\otimes 2}} (M \otimes N).$$

Note that the category $A\text{-mod}$ of finite-dimensional left A -modules is stable under this monoidal structure, and the Grothendieck group $K(A\text{-mod})$ becomes a ring with a canonical \mathbb{Z} -basis formed by the classes of finite-dimensional simple A -modules.

Example 2.1. We mainly consider the following case. Let A be a bialgebra over \mathbb{k} with coproduct $\Delta: A \rightarrow A^{\otimes 2}$ and counit $\varepsilon: A \rightarrow \mathbb{k}$. We regard $B := A^{\otimes 2}$ as an $(A, A^{\otimes 2})$ -bimodule with the structure map (Δ, id) . Then the product \star is the ordinary tensor product of left A -modules. The case of quantum loop algebras will be of particular interest in the following (see Section 3).

We can also consider a slight generalization of the above situation. Now, we may not assume that A is unital, but we assume that there is a collection of mutually orthogonal central idempotents $\{1_\gamma\}_{\gamma \in \Gamma} \subset A$ labelled by a monoid $\Gamma = (\Gamma, +)$ such that $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$, where $A_\gamma := 1_\gamma A$. An A -module M is always supposed to satisfy $M = \bigoplus_{\gamma \in \Gamma} 1_\gamma M$. Let $\varepsilon: A \rightarrow \mathbb{k}$ be a non-trivial \mathbb{k} -algebra homomorphism satisfying $\varepsilon(1_\gamma) = \delta_{\gamma,0}$. Let B be an $(A, A^{\otimes 2})$ -bimodule, which is Γ -graded (that is, $B = \bigoplus_{\gamma \in \Gamma} 1_\gamma B$ and $1_\gamma B = \bigoplus_{\gamma' + \gamma'' = \gamma} B(1_{\gamma'} \otimes 1_{\gamma''})$) for all $\gamma \in \Gamma$) and locally free of finite rank as a right $A^{\otimes 2}$ -module (that is, $B(1_\gamma \otimes 1_{\gamma'})$ is free of finite rank as a right $A_\gamma \otimes A_{\gamma'}$ -module for each $\gamma, \gamma' \in \Gamma$). We assume that these are

equipped with isomorphisms as in (2.1), (2.2) making the category of (left) A -modules into a \mathbb{k} -linear monoidal category with respect to the product \star in (2.3). Note that the category $A\text{-mod}$ in this case is a Γ -graded monoidal category, that is, we have a natural decomposition $A\text{-mod} = \bigoplus_{\gamma \in \Gamma} A_\gamma\text{-mod}$ with $(A_\gamma\text{-mod}) \star (A_{\gamma'}\text{-mod}) \subset A_{\gamma+\gamma'}\text{-mod}$. The situation in the previous paragraph can be thought of a special case where Γ is trivial.

Example 2.2. We mainly consider the following case. Let $A = \bigoplus_{\beta \in \mathbb{Q}_+} \widehat{H}_\beta$, where \widehat{H}_β is a natural completion of the quiver Hecke algebra H_β , and $\varepsilon: A \rightarrow \mathbb{k}$ the projection to $\widehat{H}_0 = \mathbb{k}$. We take $B = \bigoplus_{\beta, \beta' \in \mathbb{Q}_+} \widehat{H}_{\beta+\beta'} e(\beta, \beta')$ with a natural $(A, A^{\otimes 2})$ -bimodule structure. Then the product \star is the usual convolution product (or parabolic induction) of left A -modules (see Section 4.3 below for details).

Remark 2.3. One could develop our theory of monoidal Jantzen filtrations in a more general setting of an abstract monoidal abelian category with an appropriate notion of deformation. For example, one may employ the notion of affinization in an abstract monoidal abelian category recently studied in [37].

Let \mathcal{C} be a monoidal Serre subcategory of $A\text{-mod}$.

Definition 2.4. Let $\{L_j\}_{j \in J}$ be a collection of simple objects of \mathcal{C} parameterized by a subset $J \subset \mathbb{Z}$, and \preceq a partial ordering of the set $\mathbb{N}^{\oplus J}$. We say that such a pair $(\{L_j\}_{j \in J}, \preceq)$ gives a *PBW-theory* of \mathcal{C} if the following conditions are satisfied:

- (1) For each $\mathbf{d} = (d_j)_{j \in J} \in \mathbb{N}^{\oplus J}$, the *oppositely* ordered product (here the ordering of $J \subset \mathbb{Z}$ is induced from the natural ordering of \mathbb{Z})

$$M(\mathbf{d}) := \overset{\leftarrow}{\star}_{j \in J} L_j^{\star d_j}$$

has a simple head $L(\mathbf{d})$.

- (2) The set $\{L(\mathbf{d})\}_{\mathbf{d} \in \mathbb{N}^{\oplus J}}$ gives a complete collection of simple objects of \mathcal{C} up to isomorphisms.
- (3) In the Grothendieck ring $K(\mathcal{C})$, for each $\mathbf{d} \in \mathbb{N}^{\oplus J}$, we have

$$[M(\mathbf{d})] = [L(\mathbf{d})] + \sum_{\mathbf{d}' \prec \mathbf{d}} P_{\mathbf{d}, \mathbf{d}'} [L(\mathbf{d}')],$$

where $P_{\mathbf{d}, \mathbf{d}'} = [M(\mathbf{d}) : L(\mathbf{d}')] \in \mathbb{N}$ is the Jordan-Hölder multiplicity.

We refer to the modules $M(\mathbf{d})$ as the *standard modules*. Note that their classes $\{[M(\mathbf{d})]\}_{\mathbf{d} \in \mathbb{N}^{\oplus J}}$ form a \mathbb{Z} -basis of $K(\mathcal{C})$. On the other hand, we also consider the naturally ordered product

$$M^\vee(\mathbf{d}) := \overset{\rightarrow}{\star}_{j \in J} L_j^{\star d_j},$$

which we refer to as the *constandard modules*.

Remark 2.5. Let $(\{L_j\}_{j \in J}, \preceq)$ be a PBW-theory of \mathcal{C} . For each $i \in J$, let $\delta_i = (\delta_{i,j})_{j \in J} \in \mathbb{N}^{\oplus J}$ denote the delta function. By definition, we have

$$M(\delta_i) = M^\vee(\delta_i) = L(\delta_i) = L_i.$$

Remark 2.6. In all the examples below, we will only encounter the situation where the partial ordering \preceq of $\mathbb{N}^{\oplus J}$ can be taken to be the bi-lexicographic ordering.

2.2. Generically commutative deformations of simple modules. For a commutative \mathbb{k} -algebra R , we write $A_R := A \otimes R$ and $B_R := B \otimes R$. Note that B_R is an $(A_R, A_R \otimes_R A_R)$ -bimodule. Let $A_R\text{-mod}$ denote the category of left A_R -modules which are finitely generated over R . This is an R -linear monoidal category with respect to the product

$$M \star_R N := B_R \otimes_{(A_R \otimes_R A_R)} (M \otimes_R N).$$

Consider an indeterminate z . Let $\mathbb{O} := \mathbb{k}[[z]]$ be the ring of formal power series and $\mathbb{K} := \mathbb{k}((z))$ its fraction field (the ring of Laurent series). For an \mathbb{O} -module M , we write

$$(2.4) \quad M_{\mathbb{K}} := M \otimes_{\mathbb{O}} \mathbb{K} \quad \text{and} \quad M_0 := M \otimes_{\mathbb{O}} \mathbb{k}.$$

These operations give the monoidal functors

$$A\text{-mod} \leftarrow A_{\mathbb{O}}\text{-mod} \rightarrow A_{\mathbb{K}}\text{-mod}.$$

Definition 2.7. Let $\{L_j\}_{j \in J}$ be a collection of simple objects of $A\text{-mod}$ labelled by a subset $J \subset \mathbb{Z}$. We say that a collection $\{\tilde{L}_j\}_{j \in J}$ of objects of $A_{\mathbb{O}}\text{-mod}$ gives a *generically commutative deformation* of $\{L_j\}_{j \in J}$ if the following conditions are satisfied:

- (D1) For each $j \in J$, we have $(\tilde{L}_j)_0 \simeq L_j$ and \tilde{L}_j is free over \mathbb{O} .
- (D2) For any $i, j \in J$, we have an isomorphism of $A_{\mathbb{K}}\text{-modules}$

$$(\tilde{L}_i \star_{\mathbb{O}} \tilde{L}_j)_{\mathbb{K}} \simeq (\tilde{L}_j \star_{\mathbb{O}} \tilde{L}_i)_{\mathbb{K}}$$

and an equality

$$\text{End}_{A_{\mathbb{K}}} \left((\tilde{L}_i \star_{\mathbb{O}} \tilde{L}_j)_{\mathbb{K}} \right) = \mathbb{K}\text{id}.$$

Under the condition (D2), we always find an isomorphism of $A_{\mathbb{K}}\text{-modules}$

$$R_{i,j}: (\tilde{L}_i \star_{\mathbb{O}} \tilde{L}_j)_{\mathbb{K}} \rightarrow (\tilde{L}_j \star_{\mathbb{O}} \tilde{L}_i)_{\mathbb{K}}$$

satisfying $R_{i,j}(\tilde{L}_i \star_{\mathbb{O}} \tilde{L}_j) \subset \tilde{L}_j \star_{\mathbb{O}} \tilde{L}_i$ and $R_{i,j}(\tilde{L}_i \star_{\mathbb{O}} \tilde{L}_j) \not\subset z(\tilde{L}_j \star_{\mathbb{O}} \tilde{L}_i)$. Here we naturally regard $\tilde{L}_i \star_{\mathbb{O}} \tilde{L}_j$ as an \mathbb{O} -lattice of $(\tilde{L}_i \star_{\mathbb{O}} \tilde{L}_j)_{\mathbb{K}}$. Such a morphism $R_{i,j}$ is unique up to a multiple in \mathbb{O}^{\times} and is called a *renormalized R -matrix*.

Lemma 2.8 (cf. [12, 5.5.4]). *Let \mathcal{C} be a monoidal Serre subcategory of $A\text{-mod}$. If there is a PBW-theory $(\{L_j\}_{j \in J}, \prec)$ of \mathcal{C} which admits a generically commutative deformation $\{\tilde{L}_j\}_{j \in J}$, the Grothendieck ring $K(\mathcal{C})$ is isomorphic to a polynomial ring in J -many variables:*

$$K(\mathcal{C}) \simeq \mathbb{Z}[X_j \mid j \in J]; \quad [L_j] \mapsto X_j.$$

In particular, $K(\mathcal{C})$ is a commutative ring.

Proof. For any $M \in A_{\mathbb{K}}\text{-mod}$ and $A_{\mathbb{O}}$ -lattices $N, N' \subset M$, we have $[N_0] = [N'_0]$ in $K(A\text{-mod})$ (cf. [10, Lemma 2.3.4]). Applying this fact to the case when $M = (\tilde{L}_j \star_{\mathbb{O}} \tilde{L}_i)_{\mathbb{K}}$, $N = R_{ij}(\tilde{L}_i \star_{\mathbb{O}} \tilde{L}_j)$ and $N' = \tilde{L}_j \star_{\mathbb{O}} \tilde{L}_i$, we find $[L_i \star L_j] = [L_j \star L_i]$ for any $i, j \in I$. Since $\{[M(\mathbf{d})]\}_{\mathbf{d} \in \mathbb{N}^{\oplus J}}$ forms a \mathbb{Z} -basis of $K(\mathcal{C})$, we obtain the assertion. \square

Let $(\{L_j\}_{j \in J}, \preceq)$ be a PBW-theory of a monoidal Serre subcategory $\mathcal{C} \subset A\text{-mod}$. Assume $\{L_j\}_{j \in J}$ admits a generically commutative deformation $\{\tilde{L}_j\}_{j \in J}$. Then for any $d \in \mathbb{N}$ and any sequence $\epsilon = (\epsilon_1, \dots, \epsilon_d) \in J^d$, we define the *mixed product* $M(\epsilon) \in \mathcal{C}$ and its deformation $\tilde{M}(\epsilon) \in A_{\mathbb{O}}\text{-mod}$ by

$$M(\epsilon) := L_{\epsilon_1} \star \cdots \star L_{\epsilon_d} \quad \text{and} \quad \tilde{M}(\epsilon) := \tilde{L}_{\epsilon_1} \star_{\mathbb{O}} \cdots \star_{\mathbb{O}} \tilde{L}_{\epsilon_d}.$$

By definition, we have $\tilde{M}(\epsilon)_0 = M(\epsilon)$.

Definition 2.9. For $J \subset \mathbb{Z}$ and $\mathbf{d} = (d_j)_{j \in J} \in \mathbb{N}^{\oplus J}$, we set

$$J^{\mathbf{d}} := \{\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_d) \in J^{\mathbf{d}} \mid \#\{k \mid \epsilon_k = j\} = d_j, \forall j \in J\},$$

where $d := \sum_{j \in J} d_j$. A sequence $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_d) \in J^{\mathbf{d}}$ is said to be *standard* (resp. *costandard*) if it satisfies $\epsilon_1 \geq \dots \geq \epsilon_d$ (resp. $\epsilon_1 \leq \dots \leq \epsilon_d$). Given $\mathbf{d} \in \mathbb{N}^{\oplus J}$, there is a unique standard (resp. costandard) sequence in $J^{\mathbf{d}}$, which we often denote by $\boldsymbol{\epsilon}_s = \boldsymbol{\epsilon}_s(\mathbf{d})$ (resp. $\boldsymbol{\epsilon}_c = \boldsymbol{\epsilon}_c(\mathbf{d})$). By definition, we have $M(\boldsymbol{\epsilon}_s) = M(\mathbf{d})$ and $M(\boldsymbol{\epsilon}_c) = M^\vee(\mathbf{d})$.

2.3. Intertwiners arising from R -matrices. Let $\{L_j\}_{j \in J}$ be a collection of simple modules in \mathcal{C} labelled by a set $J \subset \mathbb{Z}$, and $\{\tilde{L}_j\}_{j \in J}$ its generically commutative deformation. For any pair $(i, j) \in J^2$, we have a unique non-negative integer $\alpha(i, j)$ satisfying

$$(2.5) \quad R_{i,j} \circ R_{j,i} \equiv z^{\alpha(i,j)} \text{id} \pmod{\mathbb{O}^\times}$$

by the condition (D2) in Definition 2.7. Note that

$$\alpha(i, j) = \alpha(j, i) \quad \text{and} \quad \alpha(i, i) = 0$$

hold. We have the following three cases:

- (i) $\alpha(i, j) = 0$;
- (ii) $\alpha(i, j) > 0$ and $i > j$;
- (iii) $\alpha(i, j) > 0$ and $i < j$.

We say that the renormalized R -matrix $R_{i,j}$ (or an isomorphism of the form $\text{id} \star R_{i,j} \star \text{id}$) is *neutral* (resp. *positive*, *negative*) when the above condition (i) (resp. (ii), (iii)) is satisfied.

Definition 2.10. Let $\mathbf{d} \in \mathbb{N}^{\oplus J}$. For $\boldsymbol{\epsilon}, \boldsymbol{\epsilon}' \in J^{\mathbf{d}}$, we write $\boldsymbol{\epsilon} \lesssim \boldsymbol{\epsilon}'$ if $\boldsymbol{\epsilon}'$ is obtained from $\boldsymbol{\epsilon}$ by replacing a consecutive pair (i, j) in $\boldsymbol{\epsilon}$ satisfying either (i) or (ii) (see above) with the opposite pair (j, i) . It generates a preorder on the set $J^{\mathbf{d}}$, which we denote by the same symbol \lesssim . Let \sim denote the induced equivalence relation on $J^{\mathbf{d}}$. In other words, for $\boldsymbol{\epsilon}, \boldsymbol{\epsilon}' \in J^{\mathbf{d}}$, we write $\boldsymbol{\epsilon} \sim \boldsymbol{\epsilon}'$ if and only if $\boldsymbol{\epsilon} \lesssim \boldsymbol{\epsilon}'$ and $\boldsymbol{\epsilon}' \lesssim \boldsymbol{\epsilon}$.

Now let us assume our generically commutative deformation is consistent in the following sense.

Definition 2.11. A generically commutative deformation $\{\tilde{L}_j\}_{j \in J}$ of $\{L_j\}_{j \in J}$ is said to be *consistent* if

(D3) For $i < j < k$, we have the *quantum Yang-Baxter relation*:

$$(2.6) \quad (R_{j,k} \star_{\mathbb{O}} \text{id}) \circ (\text{id} \star_{\mathbb{O}} R_{i,k}) \circ (R_{i,j} \star_{\mathbb{O}} \text{id}) \equiv (\text{id} \star_{\mathbb{O}} R_{i,j}) \circ (R_{i,k} \star_{\mathbb{O}} \text{id}) \circ (\text{id} \star_{\mathbb{O}} R_{j,k}) \pmod{\mathbb{O}^\times}$$

as morphisms from $\tilde{L}_i \star_{\mathbb{O}} \tilde{L}_j \star_{\mathbb{O}} \tilde{L}_k$ to $\tilde{L}_k \star_{\mathbb{O}} \tilde{L}_j \star_{\mathbb{O}} \tilde{L}_i$.

Remark 2.12. Thanks to (2.5), the above consistency condition (3) ensures the quantum Yang-Baxter relation (2.6) holds for any triple (i, j, k) in J . For example, if we multiply by $(R_{j,k} \star_{\mathbb{O}} \text{id})^{-1} \equiv z^{-\alpha(j,k)}(R_{k,j} \star_{\mathbb{O}} \text{id})$ from the left and by $(\text{id} \star_{\mathbb{O}} R_{j,k})^{-1} \equiv z^{-\alpha(j,k)}(\text{id} \star_{\mathbb{O}} R_{k,j})$ from the right to the relation (2.6) with $i < j < k$, we obtain the quantum Yang-Baxter relation for the triple (k, i, j) .

Assume that $\{\tilde{L}_j\}_{j \in J}$ is a consistent generically commutative deformation of $\{L_j\}_{j \in J}$. Let $\mathbf{d} \in \mathbb{N}^{\oplus J}$ and $\boldsymbol{\epsilon}, \boldsymbol{\epsilon}' \in J^{\mathbf{d}}$. When $\boldsymbol{\epsilon} \lesssim \boldsymbol{\epsilon}'$ (resp. $\boldsymbol{\epsilon}' \lesssim \boldsymbol{\epsilon}$), we can consider the $A_{\mathbb{K}}$ -isomorphism

$$R_{\boldsymbol{\epsilon}', \boldsymbol{\epsilon}}: \tilde{M}(\boldsymbol{\epsilon})_{\mathbb{K}} \rightarrow \tilde{M}(\boldsymbol{\epsilon}')_{\mathbb{K}}$$

obtained by composing the neutral or positive (resp. negative) renormalized R -matrices. Thanks to the quantum Yang-Baxter relation (2.6) and Remark 2.12, it is well-defined up to multiples in \mathbb{O}^\times . If $\epsilon \lesssim \epsilon' \lesssim \epsilon''$ or $\epsilon'' \lesssim \epsilon' \lesssim \epsilon$, we have

$$R_{\epsilon'', \epsilon'} \circ R_{\epsilon', \epsilon} \equiv R_{\epsilon'', \epsilon} \pmod{\mathbb{O}^\times}.$$

In particular, we obtain the following.

Proposition 2.13. *If $\epsilon \sim \epsilon'$, the homomorphism $R_{\epsilon', \epsilon}$ induces isomorphisms*

$$\tilde{M}(\epsilon) \simeq \tilde{M}(\epsilon') \quad \text{and} \quad M(\epsilon) \simeq M(\epsilon').$$

2.4. Monoidal Jantzen filtrations. In what follows, let $(\{L_j\}_{j \in J}, \preceq)$ is a PBW-theory of a monoidal Serre subcategory $\mathcal{C} \subset A\text{-mod}$ and $\{\tilde{L}_j\}_{j \in J}$ a consistent generically commutative deformation of $\{L_j\}_{j \in J}$. Fix $\mathbf{d} \in \mathbb{N}^{\oplus J}$ and write ϵ_s and ϵ_c for the standard and costandard sequences in $J^{\mathbf{d}}$ respectively. For any $\epsilon \in J^{\mathbf{d}}$, we have $\epsilon_s \lesssim \epsilon \lesssim \epsilon_c$ and hence the $A_{\mathbb{K}}$ -isomorphisms

$$\tilde{M}(\epsilon_s)_{\mathbb{K}} \xrightarrow{R_{\epsilon, \epsilon_s}} \tilde{M}(\epsilon)_{\mathbb{K}} \xrightarrow{R_{\epsilon_c, \epsilon}} \tilde{M}(\epsilon_c)_{\mathbb{K}}$$

constructed in the previous subsection. We regard $\tilde{M}(\epsilon)$ as an \mathbb{O} -lattice of $\tilde{M}(\epsilon)_{\mathbb{K}}$. Now, we define the decreasing filtration of A -submodules

$$(2.7) \quad M(\epsilon) \supset \cdots \supset F_{-1}M(\epsilon) \supset F_0M(\epsilon) \supset F_1M(\epsilon) \supset \cdots$$

by the formula

$$(2.8) \quad F_nM(\epsilon) := \text{ev}_{z=0} \left(\tilde{M}(\epsilon) \cap \sum_{k \in \mathbb{Z}} \left(z^k R_{\epsilon, \epsilon_s} \tilde{M}(\epsilon_s) \cap z^{n-k} R_{\epsilon_c, \epsilon}^{-1} \tilde{M}(\epsilon_c) \right) \right)$$

for each $n \in \mathbb{Z}$, where $\text{ev}_{z=0}: \tilde{M}(\epsilon) \rightarrow \tilde{M}(\epsilon)_0 = M(\epsilon)$ is the natural evaluation map (recall (2.4)). By construction, we have $F_{-n}M(\epsilon) = M(\epsilon)$ and $F_nM(\epsilon) = \{0\}$ for n large enough.

Definition 2.14. We call the filtration $F_\bullet M(\epsilon) = \{F_nM(\epsilon)\}_{n \in \mathbb{Z}}$ in (2.7) the (*monoidal*) *Jantzen filtration* of $M(\epsilon)$.

Example 2.15. When $\epsilon = \epsilon_s$, we have $R_{\epsilon, \epsilon_s} \in \mathbb{O}^\times \text{id}$ and hence

$$F_nM(\epsilon_s) = \text{ev}_{z=0} \left(\tilde{M}(\epsilon_s) \cap z^n R_{\epsilon_c, \epsilon_s}^{-1} \tilde{M}(\epsilon_c) \right)$$

for each $n \in \mathbb{Z}$. The filtration $F_\bullet M(\mathbf{d})$ of the standard module $M(\mathbf{d}) = M(\epsilon_s)$ is given in this way, which is analogous to the usual Jantzen filtration of standard (Verma) modules of Lie algebras.

Dually, when $\epsilon = \epsilon_c$, we have $R_{\epsilon_c, \epsilon} \in \mathbb{O}^\times \text{id}$ and hence

$$F_nM(\epsilon_c) = \text{ev}_{z=0} \left(z^n R_{\epsilon_c, \epsilon_s} \tilde{M}(\epsilon_s) \cap \tilde{M}(\epsilon_c) \right)$$

for each $n \in \mathbb{Z}$. The filtration $F_\bullet M^\vee(\mathbf{d})$ of the costandard module $M^\vee(\mathbf{d}) = M(\epsilon_c)$ is given in this way.

2.5. Specialized R -matrices. We keep the assumption from the previous subsection. Suppose that $\epsilon \lesssim \epsilon'$ or $\epsilon' \lesssim \epsilon$. Then $R_{\epsilon',\epsilon}$ is defined and there is a unique integer $\beta(\epsilon', \epsilon) \geq 0$ so that

$$R_{\epsilon',\epsilon} \tilde{M}(\epsilon) \subset z^{\beta(\epsilon',\epsilon)} \tilde{M}(\epsilon') \quad \text{and} \quad R_{\epsilon',\epsilon} \tilde{M}(\epsilon) \not\subset z^{\beta(\epsilon',\epsilon)+1} \tilde{M}(\epsilon').$$

Note that if $R_{\epsilon',\epsilon}$ is of the form $\text{id} \star R_{i,j} \star \text{id}$, then $\beta(\epsilon', \epsilon) = 0$. Under the same assumption, there is also a unique integer $\alpha(\epsilon', \epsilon) \geq 0$ such that

$$R_{\epsilon',\epsilon} \circ R_{\epsilon,\epsilon'} \equiv z^{\alpha(\epsilon',\epsilon)} \text{id}_{\tilde{M}(\epsilon')} \quad \text{and} \quad R_{\epsilon,\epsilon'} \circ R_{\epsilon',\epsilon} \equiv z^{\alpha(\epsilon',\epsilon)} \text{id}_{\tilde{M}(\epsilon)} \pmod{\mathbb{O}^\times}.$$

These numbers satisfy the following properties:

- (1) By definition, we have $\alpha(\epsilon, \epsilon') = \alpha(\epsilon', \epsilon)$;
- (2) Recall the notation $\alpha(i, j)$ for $i, j \in J$ from the previous section. When $R_{\epsilon',\epsilon}$ is the composition of homomorphisms of the form $\text{id} \star R_{i_k, j_k} \star \text{id}$ for $1 \leq k \leq n$, we have $\alpha(\epsilon', \epsilon) = \sum_{k=1}^n \alpha(i_k, j_k)$. In particular, if $\epsilon \lesssim \epsilon' \lesssim \epsilon''$, we have the additivity $\alpha(\epsilon'', \epsilon) = \alpha(\epsilon'', \epsilon') + \alpha(\epsilon', \epsilon)$;
- (3) We have $\epsilon \sim \epsilon'$ if and only if $\alpha(\epsilon, \epsilon') = 0$ (case of Proposition 2.13);
- (4) If $\epsilon \lesssim \epsilon' \lesssim \epsilon''$, we have $\beta(\epsilon'', \epsilon') + \beta(\epsilon', \epsilon) \leq \beta(\epsilon'', \epsilon)$ and $\beta(\epsilon, \epsilon') + \beta(\epsilon', \epsilon'') \leq \beta(\epsilon, \epsilon'')$;
- (5) For $\epsilon \lesssim \epsilon'$, we have $\alpha(\epsilon, \epsilon') \geq \beta(\epsilon, \epsilon') + \beta(\epsilon', \epsilon)$. In particular, we always have $\alpha(\epsilon, \epsilon') - \beta(\epsilon, \epsilon') \geq 0$.

Now $z^{-\beta(\epsilon',\epsilon)} R_{\epsilon',\epsilon}$ induces a non-zero morphism of A -modules

$$\mathbf{r}_{\epsilon',\epsilon}: M(\epsilon) \rightarrow M(\epsilon')$$

called the *specialized R -matrix*, which is uniquely determined up to an invertible element in \mathbb{k} . The following propositions are useful to compute examples.

Proposition 2.16. *Let $\epsilon, \epsilon' \in J^d$ satisfying $\epsilon \lesssim \epsilon'$. For any $N \in \mathbb{Z}$, we have*

$$\mathbf{r}_{\epsilon',\epsilon}(F_N M(\epsilon)) \subset F_{N-2\beta(\epsilon',\epsilon)} M(\epsilon').$$

Proof. We have the following commutative (up to multiples in \mathbb{O}^\times) diagram

$$\begin{array}{ccc} \tilde{M}(\epsilon_s)_{\mathbb{K}} & \xrightarrow{R_{\epsilon,\epsilon_s}} & \tilde{M}(\epsilon)_{\mathbb{K}} \\ \downarrow R_{\epsilon',\epsilon_s} & \searrow R_{\epsilon',\epsilon} & \downarrow R_{\epsilon_c,\epsilon} \\ \tilde{M}(\epsilon')_{\mathbb{K}} & \xrightarrow{R_{\epsilon_c,\epsilon'}} & \tilde{M}(\epsilon_c)_{\mathbb{K}} \end{array}$$

Let $y(z) \in z^k R_{\epsilon,\epsilon_s} \tilde{M}(\epsilon_s) \cap z^{N-k} R_{\epsilon_c,\epsilon}^{-1} \tilde{M}(\epsilon_c)$ with $k \in \mathbb{Z}$. Then

$$y(z) = R_{\epsilon,\epsilon_s}(z^k x(z)) \quad \text{and} \quad R_{\epsilon_c,\epsilon}(y(z)) = z^{N-k} x'(z),$$

for some $x(z) \in \tilde{M}(\epsilon_s)$ and $x'(z) \in \tilde{M}(\epsilon_c)$. Then $y'(z) = z^{-\beta(\epsilon',\epsilon)} R_{\epsilon',\epsilon}(y(z)) \in \tilde{M}(\epsilon')$ satisfies

$$y'(z) = z^{-\beta(\epsilon',\epsilon)} (R_{\epsilon',\epsilon} \circ R_{\epsilon,\epsilon_s})(z^k x(z)) = R_{\epsilon',\epsilon_s}(z^{k-\beta(\epsilon',\epsilon)} x(z))$$

and

$$R_{\epsilon_c,\epsilon'}(y'(z)) = z^{-\beta(\epsilon',\epsilon)} R_{\epsilon_c,\epsilon}(y(z)) = z^{N-k-\beta(\epsilon',\epsilon)} x'(z)$$

up to multiples in \mathbb{O}^\times . Then the result follows from

$$y'(z) \in z^{k-\beta(\epsilon',\epsilon)} R_{\epsilon',\epsilon_s} \tilde{M}(\epsilon_s) \cap z^{N-k-\beta(\epsilon',\epsilon)} R_{\epsilon_c,\epsilon'}^{-1} \tilde{M}(\epsilon_c). \quad \square$$

Proposition 2.17. *Let $\epsilon, \epsilon' \in J^{\mathbf{d}}$ satisfying $\epsilon' \lesssim \epsilon$. For any $N \in \mathbb{Z}$, we have*

$$\mathbf{r}_{\epsilon', \epsilon}(F_N M(\epsilon)) \subset F_{N+2\alpha(\epsilon', \epsilon)-2\beta(\epsilon', \epsilon)} M(\epsilon').$$

Proof. We have the same diagram as in the proof of Proposition 2.16 and we consider $y(z)$, $x(z)$, $x'(z)$, $y'(z)$ in the same way. But now we have

$$\begin{aligned} y'(z) &= z^{-\beta(\epsilon', \epsilon)} (R_{\epsilon', \epsilon} \circ R_{\epsilon_s, \epsilon_s})(z^k x(z)) = z^{\alpha(\epsilon', \epsilon) - \beta(\epsilon', \epsilon)} R_{\epsilon', \epsilon_s}(z^k x(z)), \\ R_{\epsilon_c, \epsilon'}(y'(z)) &= z^{\alpha(\epsilon', \epsilon) - \beta(\epsilon', \epsilon)} R_{\epsilon_c, \epsilon}(y(z)) = z^{N-k+\alpha(\epsilon', \epsilon) - \beta(\epsilon', \epsilon)} x'(z) \end{aligned}$$

up to multiples in \mathbb{O}^\times . Then the result follows from

$$y'(z) \in z^{k+\alpha(\epsilon', \epsilon) - \beta(\epsilon', \epsilon)} R_{\epsilon', \epsilon_s} \tilde{M}(\epsilon_s) \cap z^{N-k+\alpha(\epsilon', \epsilon) - \beta(\epsilon', \epsilon)} R_{\epsilon_c, \epsilon'}^{-1} \tilde{M}(\epsilon_c). \quad \square$$

Remark 2.18. For $\mathbf{d} \in \mathbb{N}^{\oplus J}$, we set

$$\beta(\mathbf{d}) := \beta(\epsilon_c(\mathbf{d}), \epsilon_s(\mathbf{d})),$$

where $\epsilon_s(\mathbf{d})$ and $\epsilon_c(\mathbf{d})$ are the standard and costandard sequences in $J^{\mathbf{d}}$ respectively. By Example 2.15, we have

$$M(\mathbf{d}) = F_{\beta(\mathbf{d})} M(\mathbf{d}) \supsetneq F_{\beta(\mathbf{d})+1} M(\mathbf{d}).$$

In particular, the simple head $L(\mathbf{d})$ of $M(\mathbf{d})$ contributes to $F_n M(\mathbf{d})/F_{n+1} M(\mathbf{d})$ as a composition factor if and only if $n = \beta(\mathbf{d})$.

2.6. Decategorification. We keep the same assumption from the previous subsections. Let t be another indeterminate with a formal square root $t^{1/2}$. Consider the $\mathbb{Z}[t^{\pm 1/2}]$ -module

$$K(\mathcal{C})_t := K(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1/2}] = \bigoplus_{\mathbf{d} \in \mathbb{N}^{\oplus J}} \mathbb{Z}[t^{\pm 1/2}][L(\mathbf{d})],$$

where we abbreviate $[M] \otimes 1$ as $[M]$. For each $\epsilon \in J^{\mathbf{d}}$ with $\mathbf{d} \in \mathbb{N}$, using the Jantzen filtration (2.7), we define an element $[M(\epsilon)]_t \in K(\mathcal{C})_t$ by

$$(2.9) \quad [M(\epsilon)]_t := \sum_{n \in \mathbb{Z}} [\mathrm{Gr}_n^F M(\epsilon)] t^n,$$

where $\mathrm{Gr}_n^F M(\epsilon) := F_n M(\epsilon)/F_{n+1} M(\epsilon)$. As a special case, for each $\mathbf{d} \in \mathbb{N}^{\oplus J}$, we have defined the element $[M(\mathbf{d})]_t = [M(\epsilon_s(\mathbf{d}))]_t$. By Definition 2.4, it comes with the relation

$$(2.10) \quad [M(\mathbf{d})]_t = t^{\beta(\mathbf{d})} \left([L(\mathbf{d})] + \sum_{\mathbf{d}' \prec \mathbf{d}} P_{\mathbf{d}, \mathbf{d}'}(t) [L(\mathbf{d}')] \right),$$

where $\beta(\mathbf{d}) := \beta(\epsilon_c(\mathbf{d}), \epsilon_s(\mathbf{d})) \in \mathbb{N}$ is as in Remark 2.18, and $P_{\mathbf{d}, \mathbf{d}'}(t) \in \mathbb{N}[t]$ is an analog of Kazhdan-Lusztig polynomial defined by

$$P_{\mathbf{d}, \mathbf{d}'}(t) := t^{-\beta(\mathbf{d})} \sum_{n \in \mathbb{Z}} [\mathrm{Gr}_n^F M(\mathbf{d}) : L(\mathbf{d}')] t^n.$$

Here $[M : L]$ denotes the Jordan-Hölder multiplicity of L in M . Then, by definition, we have

$$P_{\mathbf{d}, \mathbf{d}'}(1) = P_{\mathbf{d}, \mathbf{d}'} = [M(\mathbf{d}) : L(\mathbf{d}')] \quad \text{for any } \mathbf{d}' \prec \mathbf{d}.$$

Note that $\{[M(\mathbf{d})]_t\}_{\mathbf{d} \in \mathbb{N}^{\oplus J}}$ forms a $\mathbb{Z}[t^{\pm 1/2}]$ -basis of $K(\mathcal{C})_t$ by the relation (2.10).

Let $\gamma: \mathbb{N}^{\oplus J} \times \mathbb{N}^{\oplus J} \rightarrow \frac{1}{2}\mathbb{Z}$ be a skew-symmetric bilinear map. With the above notation, we define a $\mathbb{Z}[t^{\pm 1/2}]$ -bilinear operation $* = *_{\gamma}$ on $K(\mathcal{C})_t$ in terms of the basis $\{[M(\mathbf{d})]_t\}_{\mathbf{d} \in \mathbb{N}^{\oplus J}}$ by

$$[M(\mathbf{d})]_t * [M(\mathbf{d}')]_t := t^{\gamma(\mathbf{d}, \mathbf{d}')} [M(\mathbf{d}) \star M(\mathbf{d}')]_t,$$

where the RHS is given by (2.9) with $M(\epsilon) = M(\mathbf{d}) \star M(\mathbf{d}')$ (that is, $\epsilon \in J^{\mathbf{d}+\mathbf{d}'}$ is the concatenation of two standard sequences $\epsilon_s(\mathbf{d})$ and $\epsilon_s(\mathbf{d}')$, which is not necessarily standard). Be aware that the operation \star depends on many choices: a PBW-theory $(\{L_j\}_{j \in J}, \preceq)$, its consistent, generically commutative deformation $\{\tilde{L}_j\}_{j \in J}$, and a bilinear form γ .

We may expect the associativity of \star , but it seems unclear from the construction. We state it as our general conjecture.

Conjecture 2.19 (Associativity Conjecture). *The $\mathbb{Z}[t^{\pm 1/2}]$ -module $K(\mathcal{C})_t$ with this operation \star is a $\mathbb{Z}[t^{\pm 1/2}]$ -algebra, and so it gives a (not necessarily commutative) t -deformation of the Grothendieck ring $K(\mathcal{C})$.*

We also write a stronger version of the above Conjecture. For each integer $n \geq 2$, we consider the $\mathbb{Z}[t^{\pm 1/2}]$ -multilinear operation $m_n: K(\mathcal{C})_t^n \rightarrow K(\mathcal{C})_t$ given by

$$m_n([M(\mathbf{d}_1)]_t, \dots, [M(\mathbf{d}_n)]_t) := t^{\sum_{1 \leq k < l \leq n} \gamma(\mathbf{d}_k, \mathbf{d}_l)} [M(\mathbf{d}_1) \star \dots \star M(\mathbf{d}_n)]_t$$

for $\mathbf{d}_1, \dots, \mathbf{d}_n \in \mathbb{N}^{\oplus J}$. Of course, we have $m_2(x, y) = x \star y$. By convention, we set $m_1 := \text{id}$.

Conjecture 2.20 (Strong Associativity Conjecture). *For any integers $n \geq 3$ and $0 < k < n$, we have*

$$m_n(x_1, \dots, x_n) = m_k(x_1, \dots, x_k) \star m_{n-k}(x_{k+1}, \dots, x_n)$$

for any $x_1, \dots, x_n \in K(\mathcal{C})_t$.

Remark 2.21. If Conjecture 2.20 holds, then Conjecture 2.19 also holds and moreover, for any $d \in \mathbb{N}$ and any sequence $\epsilon = (\epsilon_1, \dots, \epsilon_d) \in J^d$, we have

$$(2.11) \quad [L_{\epsilon_1}] \star \dots \star [L_{\epsilon_d}] = t^{\sum_{1 \leq k < l \leq d} \gamma(\delta_{\epsilon_k}, \delta_{\epsilon_l})} [M(\epsilon)]_t,$$

where $\delta_i \in \mathbb{N}^{\oplus J}$ denotes the delta function. Note that the converse is true. Namely, Conjecture 2.20 holds if and only if Conjecture 2.19 and (2.11) hold for any $d \in \mathbb{N}$ and $\epsilon \in J^d$.

2.7. Bar-involution and normality. Let $\{L_j\}_{j \in J}$ be a PBW-theory for a monoidal Serre subcategory $\mathcal{C} \subset A\text{-mod}$ which admits a consistent, generically commutative deformation $\{\tilde{L}_j\}_{j \in J}$ as above. We have the following general fact.

Proposition 2.22. *For each $\mathbf{d} \in \mathbb{N}^{\oplus J}$ and $n \in \mathbb{Z}$, we have an isomorphism of A -modules:*

$$\text{Gr}_n^F M(\mathbf{d}) \simeq \text{Gr}_{-n}^F M^\vee(\mathbf{d}).$$

Proof. Recall Example 2.15. Let $\epsilon_s, \epsilon_c \in J^d$ be the standard and costandard sequences respectively. For brevity, we write $M := \tilde{M}(\epsilon_s)$, $N := \tilde{M}(\epsilon_c)$ and $R = R_{\epsilon_c, \epsilon_s}: M_{\mathbb{K}} \simeq N_{\mathbb{K}}$. Then, we have isomorphisms of A -modules:

$$\begin{aligned} \text{Gr}_n^F M(\mathbf{d}) &\simeq \frac{M \cap z^n R^{-1} N}{(M \cap z^{n+1} R^{-1} N) + (z M \cap z^n R^{-1} N)} \\ &\simeq \frac{z^{-n} R M \cap N}{(z^{-n} R M \cap z N) + (z^{-n+1} R M \cap N)} \simeq \text{Gr}_{-n}^F M^\vee(\mathbf{d}), \end{aligned}$$

where the second isomorphism is induced by the isomorphism $z^{-n} R$. \square

Let $\overline{(\cdot)}: K(\mathcal{C})_t \rightarrow K(\mathcal{C})_t$ be the involution of abelian group given by

$$\overline{t^n [L(\mathbf{d})]} := t^{-n} [L(\mathbf{d})]$$

for any $n \in \frac{1}{2}\mathbb{Z}$ and $\mathbf{d} \in \mathbb{N}^{\oplus J}$. The following is an immediate consequence of Proposition 2.22.

Corollary 2.23. *For each $\mathbf{d} \in \mathbb{N}^{\oplus J}$, we have*

$$[M^\vee(\mathbf{d})]_t = t^{-\beta(\mathbf{d})} \left([L(\mathbf{d})] + \sum_{\mathbf{d}' \prec \mathbf{d}} P_{\mathbf{d}, \mathbf{d}'}(t^{-1}) [L(\mathbf{d}')] \right) = \overline{[M(\mathbf{d})]_t}.$$

In particular, for any $i, j \in J$, we have

$$\overline{[L_i] * [L_j]} = [L_j] * [L_i].$$

Remark 2.24. Note that we need the map γ to be skew-symmetric for the second assertion of the above Corollary 2.23. This justifies this condition on γ , which will be satisfied in all examples below.

As a generalization, we also conjecture the following.

Conjecture 2.25 (Duality Conjecture). *For any $d \in \mathbb{N}$ and $\epsilon = (\epsilon_1, \dots, \epsilon_d) \in J^d$, we have*

$$\overline{[M(\epsilon)]_t} = [M(\epsilon^{\text{op}})]_t,$$

where $\epsilon^{\text{op}} = (\epsilon_d, \dots, \epsilon_1)$ is the opposite sequence.

Note that, if both the Strong Associativity Conjecture (= Conjecture 2.20) and the Duality Conjecture (= Conjecture 2.25) are true, the involution $\overline{(\cdot)}$ defines an anti-algebra involution of $(K(\mathcal{C})_t, *)$.

We finish this section by introducing the notion of normality, following [36, Definition 2.5].

Definition 2.26 (Normality). We say that our deformation $\{\tilde{L}_j\}_{j \in J}$ as above is *normal* if

- (N1) we have $\beta(\mathbf{d}) = 0$ for each $\mathbf{d} \in \mathbb{N}^{\oplus J}$ (cf. Remark 2.18), and
- (N2) $\text{Gr}_0^F M(\mathbf{d}) \simeq L(\mathbf{d})$ for each $\mathbf{d} \in \mathbb{N}^{\oplus J}$.

If $\{\tilde{L}_j\}_{j \in J}$ satisfies the condition (N1), the non-zero homomorphism

$$\mathbf{r}_\mathbf{d} := \mathbf{r}_{\epsilon_c(\mathbf{d}), \epsilon_s(\mathbf{d})} : M(\mathbf{d}) \rightarrow M^\vee(\mathbf{d})$$

is induced by $R_{\epsilon_c, \epsilon_s}$ (no rescaling here). Therefore, assuming (N1), the condition (N2) is equivalent to the condition

- (N2)' $\text{Im}(\mathbf{r}_\mathbf{d}) \simeq L(\mathbf{d})$ for each $\mathbf{d} \in \mathbb{N}^{\oplus J}$.

Note that this condition (N2)' is automatically satisfied if $M^\vee(\mathbf{d})$ has a simple socle for each $\mathbf{d} \in \mathbb{N}^{\oplus J}$. If $\{\tilde{L}_j\}_{j \in J}$ is normal, we have

$$(2.12) \quad P_{\mathbf{d}, \mathbf{d}'}(t) \in t\mathbb{N}[t]$$

for any $\mathbf{d}' \prec \mathbf{d}$. Therefore, we obtain the following Kazhdan-Lusztig type characterization of $\{[L(\mathbf{d})]\}_{\mathbf{d} \in \mathbb{N}^{\oplus J}}$ that can be seen as a canonical basis of $K(\mathcal{C})_t$.

Proposition 2.27. *Assume that our deformation $\{\tilde{L}_j\}_{j \in J}$ as above is normal. Then, the $\mathbb{Z}[t^{\pm 1/2}]$ -basis $\{[L(\mathbf{d})]\}_{\mathbf{d} \in \mathbb{N}^{\oplus J}}$ of $K(\mathcal{C})_t$ is characterized by the following two properties:*

- (1) $\overline{[L(\mathbf{d})]} = [L(\mathbf{d})]$, and
- (2) $[L(\mathbf{d})] - [M(\mathbf{d})]_t \in \sum_{\mathbf{d}' \prec \mathbf{d}} t\mathbb{Z}[t][M(\mathbf{d}')]_t$.

3. MONOIDAL JANTZEN FILTRATIONS FOR QUANTUM LOOP ALGEBRAS

We study our first main examples for monoidal Jantzen filtrations, realized in categories of finite-dimensional representations of quantum loop algebras. More precisely, we first give general reminders on these representations. Then we introduce the ordinary PBW-theory arising from fundamental modules (Theorem 3.2) and more general PBW-theories of affine cuspidal modules from [38]. We recall the relevant R -matrices, we introduce relevant deformations of the PBW-theory in Section 3.4 and we check it fits into our general framework (Theorem 3.7). Hence we obtain monoidal Jantzen filtrations. Independently, we recall the construction of quantum Grothendieck rings, the corresponding Kazhdan-Lusztig polynomials which are now known to be positive (Theorem 3.17). We conjecture that our decategorified monoidal Jantzen filtrations recover this quantum Grothendieck ring (Conjectures 3.20, 3.23). So this gives an explanation for the positivity of Kazhdan-Lusztig polynomials in this context.

3.1. Quantum loop algebras and their representations. Let \mathfrak{g} be a complex finite-dimensional simple Lie algebra. Let $C = (c_{ij})_{i,j \in I}$ denote the Cartan matrix of \mathfrak{g} , where I is the set of Dynkin nodes. Let $r \in \{1, 2, 3\}$ be the lacing number of \mathfrak{g} , and $(r_i)_{i \in I} \in \{1, r\}^I$ the left symmetrizer of C , i.e., satisfying $r_i c_{ij} = r_j c_{ji}$ for all $i, j \in I$.

Let $U_q(L\mathfrak{g})$ be quantum loop algebra associated to \mathfrak{g} . It is a Hopf algebra defined over an algebraic closed field $\mathbb{k} = \overline{\mathbb{Q}(q)}$, where q is a formal parameter. It has a family of Chevalley generators $e_i, f_i, k_i^{\pm 1}$ where $i \in I \sqcup \{0\}$.

Let \mathcal{C} denote the rigid monoidal category of finite-dimensional $U_q(L\mathfrak{g})$ -modules (with the standard type 1 condition). Recall that the isomorphism classes of simple modules of the category \mathcal{C} are parameterized by the set $(1 + z\mathbb{k}[z])^I$ of I -tuples of monic polynomials (the Drinfeld polynomials). Such a I -tuple encode the eigenvalues of distinguished operators on a highest weight vector of the simple representation [9].

We will focus on the monoidal subcategory $\mathcal{C}_{\mathbb{Z}}$ of \mathcal{C} introduced by Hernandez-Leclerc [29] and so that every prime simple module of \mathcal{C} (that is every simple module which can not be factorized into a non trivial tensor product of modules) is in $\mathcal{C}_{\mathbb{Z}}$ after a suitable spectral parameter shift. Precisely, we fix a parity function $\varepsilon: I \rightarrow \{0, 1\}$ satisfying the condition

$$\varepsilon_i \equiv \varepsilon_j + \min(r_i, r_j) \pmod{2} \quad \text{if } c_{ij} < 0,$$

and let

$$\hat{I} := \{(i, p) \in I \times \mathbb{Z} \mid p \equiv \varepsilon_i \pmod{2}\}.$$

We introduce a formal variable $Y_{i,p}$ for each $(i, p) \in \hat{I}$ and \mathcal{M} be the group of all the Laurent monomials

$$m = \prod_{(i,p) \in \hat{I}} Y_{i,p}^{u_{i,p}(m)}.$$

We say that $m \in \mathcal{M}$ is *dominant* if $u_{i,p}(m) \geq 0$ for all $(i, p) \in \hat{I}$, and denote the set of dominant monomials by \mathcal{M}^+ . For each such dominant monomial, we have a simple module $L(m) \in \mathcal{C}$ corresponding to the Drinfeld polynomials $(\prod_p (1 - q^p z)^{u_{i,p}(m)})_{i \in I}$. The category $\mathcal{C}_{\mathbb{Z}}$ is defined to be the Serre subcategory of \mathcal{C} generated by these simple modules. It is closed under taking tensor products and left/right duals. In other words, $\mathcal{C}_{\mathbb{Z}}$ is a rigid monoidal subcategory of \mathcal{C} .

3.2. Standard modules and PBW-theory. For $(i, p) \in \hat{I}$, consider the fundamental representation defined by

$$V_{i,p} = L(Y_{i,p}).$$

We choose a numbering $I = \{1, \dots, n\}$ where n is the rank of \mathfrak{g} and we define an embedding $e: \hat{I} \rightarrow \mathbb{Z}$ by setting

$$e: (i, p) \mapsto i + np.$$

This induces an ordering on \hat{I} so that $p < q$ implies $(i, p) < (j, q)$. We will just denote $V_{e(i,p)} = V_{i,p}$ so that we have a family of simple modules $\{V_j\}_{j \in J}$ as in Section 2.1, where J is the image $e(\hat{I}) \subset \mathbb{Z}$.

Remark 3.1. More generally, one can work with any embedding $e: \hat{I} \rightarrow \mathbb{Z}$ satisfying the condition

$$e(i, p) < e(j, s) \quad \text{if } \mathfrak{o}(V_{i,p}, V_{j,s}) > 0,$$

where the number $\mathfrak{o}(M, N) \in \mathbb{N}$ is the pole order of the normalized R -matrix defined below. It follows that the resulting deformed product $*$ on $K(\mathcal{C}_{\mathbb{Z}})_t$ does not depend on the choice of such an embedding at least when \mathfrak{g} is of simply-laced type from the proof of Theorem 3.26 given in Section 6 below.

In what follows, we often identify \mathcal{M} with $\mathbb{Z}^{\oplus \hat{I}}$ by the correspondence $m \mapsto (u_{i,p}(m))$. Then, the set \mathcal{M}^+ is identified with $\mathbb{N}^{\oplus \hat{I}}$. We define a partial ordering on $\mathcal{M}^+ \simeq \mathbb{N}^{\oplus \hat{I}}$ in the following way. For each $(i, p) \in I \times \mathbb{Z}$ with $(i, p - r_i) \in \hat{I}$, following [16], we define the loop analog of a simple root

$$A_{i,p} = Y_{i,p-r_i} Y_{i,p+r_i} \prod_{(j,s) \in \hat{I}: c_{i,j} < 0, |s-p| < r_i} Y_{j,s}^{-1} \in \mathcal{M}.$$

For $m, m' \in \mathcal{M}$, we write $m \preceq m'$ if $m'm^{-1}$ is a monomial in various $A_{i,p}$ for $(i, p - r_i) \in \hat{I}$. This defines a partial ordering on \mathcal{M} , called the *Nakajima partial ordering*. As one also can view an element in $\mathcal{M} \simeq \mathbb{Z}^{\oplus \hat{I}}$ as an element in $\mathbb{Z}^{\oplus J}$ through the map e , this induces a partial ordering \preceq on $\mathbb{N}^{\oplus J}$.

The following is a reformulation of well-know results by various authors, in particular [8, 9, 35, 57].

Theorem 3.2. *The pair $(\{V_j\}_{j \in J}, \preceq)$ gives a PBW-theory of $\mathcal{C}_{\mathbb{Z}}$.*

We will call the corresponding standard modules the ordinary standard modules as they were studied by many authors, in particular from the point of view of geometric representation theory for simply-laced quantum loop algebras.

A generalization of this PBW-theory is proposed Kashiwara–Kim–Oh–Park in [38]. Consider a strong complete duality datum in the sense of [38] (such a family can be obtained from a Q -datum in the sense of [22]). Then there is a corresponding collection of simple representations $(S_k)_{k \in \mathbb{Z}}$ in $\mathcal{C}_{\mathbb{Z}}$ called the affine cuspidal modules, see [38, Section 5.2] (in the particular case above, the affine cuspidal modules are fundamental representations, now parameterized by \mathbb{Z} , that is we have fixed an increasing bijection between J and \mathbb{Z}). Then let \preceq be the bi-lexicographic ordering on $\mathbb{N}^{\oplus \mathbb{Z}}$.

Theorem 3.3 ([38]). *The pair $(\{S_k\}_{k \in \mathbb{Z}}, \preceq)$ gives a PBW-theory of $\mathcal{C}_{\mathbb{Z}}$.*

The ordinary PBW-theory given by fundamental representations in Theorem 3.2 is a particular case of this result (see [38, Remark 6.4]), but there are more general PBW-theories arising in this form.

3.3. R -matrices. The algebra $U_q(\mathbf{Lg})$ has a \mathbb{Z} -grading defined on Chevalley generators by $\deg(e_i) = \deg(f_i) = \deg(k_i^{\pm 1}) = 0$ for $i \in I$ and $\deg(e_0) = -\deg(f_0) = 1$. There is a corresponding algebra morphism $\tau_u: U_q(\mathbf{Lg}) \rightarrow U_q(\mathbf{Lg})[u^{\pm 1}]$ such that a homogeneous element g of degree $m \in \mathbb{Z}$ satisfies $\tau_u(g) = u^m g$.

Let V be a representation of $U_q(\mathbf{Lg})$. Consider the ring $\mathbb{O} = \mathbb{k}[[z]]$ as above with the formal variable $z = u - 1$. Then the \mathbb{O} -module $(V)_u = V \otimes \mathbb{O}$ has a structure of $U_q(\mathbf{Lg})_{\mathbb{O}}$ -module obtained as the twist of the module structure of V by τ_u . The morphism τ_u is compatible with the coproduct of $U_q(\mathbf{Lg})$, and so for two $U_q(\mathbf{Lg})$ -modules V and W we have

$$(V \otimes W)_u \simeq (V)_u \otimes (W)_u.$$

We can also consider the tensor product $V_u \otimes W_v$ and its scalar extension

$$((V)_u \otimes (W)_v)_{\mathbb{k}((z,w))} := ((V)_u \otimes (W)_v) \otimes_{\mathbb{k}[[z]] \otimes \mathbb{k}[[w]]} \mathbb{k}((z,w))$$

to the ring of Laurent formal power series with two variables $z = u - 1$ and $w = v - 1$.

Theorem 3.4. *Let M, N, P be simple modules in $\mathcal{C}_{\mathbb{Z}}$. There is a unique isomorphism of $U_q(\mathbf{Lg})$ -modules*

$$T_{M,N}(u, v): ((M)_u \otimes (N)_v)_{\mathbb{k}((z,w))} \rightarrow ((N)_v \otimes (M)_u)_{\mathbb{k}((z,w))},$$

normalized so that for $y \in M, y' \in N$ highest weight vectors, the image of $y \otimes y'$ by $(T_{M,N}(u, v))$ is $y' \otimes y$. Moreover $T_{M,N}(u, v) = T_{M,N}(u/v)$ depends only on u/v and is rational

$$T_{M,N}(u, v)(M \otimes N) \subset (N \otimes M) \otimes \mathbb{k}(u/v).$$

It satisfies the quantum Yang-Baxter equation, that is we have

$$\begin{aligned} & (T_{N,P}(v) \otimes \text{id}) \circ (\text{id} \otimes T_{M,P}(u)) \circ (T_{M,N}(u/v) \otimes \text{id}) \\ &= (T_{M,N}(u/v) \otimes \text{id}) \circ (\text{id} \otimes T_{M,P}(u)) \circ (T_{N,P}(v) \otimes \text{id}). \end{aligned}$$

The isomorphism $T_{M,N}(u, v)$ is obtained by the specialization of the universal R -matrix normalized on tensor products of highest weight vectors (see [18] and [13, Proposition 9.5.3]).

Let us consider the order of 1 as a pole of $T_{M,N}(u)$:

$$\mathfrak{o}(M, N) \in \mathbb{N}.$$

The renormalized R -matrix is defined as

$$R_{M,N}(u) = (u - 1)^{\mathfrak{o}(M,N)} T_{M,N}(u).$$

Its limit at $u \rightarrow 1$ is a non zero morphism of $U_q(\mathbf{Lg})$ -modules (considered in [33]):

$$\mathbf{r}_{M,N}: M \otimes N \rightarrow N \otimes M.$$

Remark 3.5. It is not clear how to define the quantity $\mathfrak{o}(M, N)$ for general categories as considered in Section 2. However, for the ordinary PBW-theory $(\{V_j\}_{j \in J}, \preceq)$ in Theorem 3.2 and its generically commutative deformation introduced in the next subsection, we have

$$\mathfrak{o}(V_i, V_j) = \begin{cases} \alpha(i, j) & \text{if } i < j, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha(i, j)$ is the number defined in Section 2. The operators $\mathbf{r}_{M,N}$ defined as the limits of operators $R_{M,N}$ coincide with the specialized R -matrices in the general framework of Section 2.4. In the situations considered below, these notations will not lead to confusions because, as explained above, they are well-defined up to multiples in \mathbb{k}^{\times} .

Example 3.6. Let $\mathfrak{g} = \mathfrak{sl}_2$ and $M = L(Y_{1,a})$, $N = L(Y_{1,b})$ be fundamental representations. The structure of $M \otimes N$ is well-known. We have $\mathfrak{o}(M, N) = \delta_{b-a,2}$ and $R_{M,N}$ is an isomorphism if $|b-a| \neq 2$. If $b = a - 2$, its image is simple of dimension 3 isomorphic to $L(Y_{1,a}Y_{1,b})$ and its kernel is the trivial module of dimension 1. If $b = a + 2$, its image is simple of dimension 1 and its kernel is isomorphic to $L(Y_{1,a}Y_{1,b})$. All this can be checked by direct computations. Indeed, there are respective bases (v_a^+, v_a^-) and (v_b^+, v_b^-) of weight vectors of M and of N , so that in the basis $(v_a^+ \otimes v_b^+, v_a^+ \otimes v_b^-, v_a^- \otimes v_b^+, v_a^- \otimes v_b^-)$, we see that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{u(1-q^{-2})}{u-q^{b-a-2}} & \frac{q^{-1}(u-q^{b-a})}{u-q^{b-a-2}} & 0 \\ 0 & \frac{q^{-1}(u-q^{b-a})}{u-q^{b-a-2}} & \frac{q^{b-a}(1-q^{-2})}{u-q^{b-a-2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is the matrix of $T_{M,N}(u)$. From the basis

$$(v_a^+ \otimes v_b^+, v_a^+ \otimes v_b^- + q^{-1}v_a^- \otimes v_b^+, q^{-1}v_a^+ \otimes v_b^- - v_a^- \otimes v_b^+, v_a^- \otimes v_b^-)$$

to the basis

$$(v_b^+ \otimes v_a^+, v_b^+ \otimes v_a^- + q^{-1}v_b^- \otimes v_a^+, q^{-1}v_b^- \otimes v_a^+ - q^{-1}v_b^+ \otimes v_a^-, v_b^- \otimes v_a^-)$$

the matrix is diagonal

$$T_{M,N}(u) = \text{diag}(1, 1, \delta(u), 1),$$

where $\delta(u) = \frac{q^{b-a} - uq^{-2}}{u - q^{b-a-2}}$. When $a = b + 2$, at the limit $u \rightarrow 1$ one obtains

$$\mathbf{r}_{M,N} = \text{diag}(1, 1, 0, 1).$$

When $b = a + 2$, multiplying by $u - 1$, we obtain at the limit

$$\mathbf{r}_{M,N} = \text{diag}(0, 0, q^2 - q^{-2}, 0).$$

We note that in these cases $|b-a| = 2$ we have

$$R_{M,N}(u) \circ R_{N,M}(u) = (u-1)\text{id}.$$

3.4. Deformation. We fix a PBW-theory $(\{S_k\}_{k \in J}, \preceq)$ of $\mathcal{C}_{\mathbb{Z}}$ as above ($J = e(\hat{I})$ or \mathbb{Z}). We set

$$\tilde{S}_k := (S_k)_{\exp(kz)}.$$

It is a $U_q(L\mathfrak{g})_{\mathbb{O}}$ -module. For any $k, k' \in J$, we have an isomorphism

$$R_{k,k'} = R_{S_k, S_{k'}}(\exp((k-k')z)): (\tilde{S}_k \star_{\mathbb{O}} \tilde{S}_{k'})_{\mathbb{K}} \simeq (\tilde{S}_{k'} \star_{\mathbb{O}} \tilde{S}_k)_{\mathbb{K}}.$$

Theorem 3.7. *The collection $\{\tilde{S}_k\}_{k \in J}$ is a normal, consistent, generically commutative deformation of $\{S_k\}_{k \in J}$.*

Proof. The statement follows from the results recalled above, and [38, Proposition 5.7(iii)] for the normality. \square

Remark 3.8. Recall $\beta(\epsilon', \epsilon)$ defined in Section 2.5. For the ordinary PBW-theory of the quantum loop algebras in Theorem 3.2, we have $\beta(\epsilon', \epsilon) = 0$ if $\epsilon \lesssim \epsilon'$ by considering highest weight vectors as in the proof above. If $\epsilon' \lesssim \epsilon$, we may have $\beta(\epsilon', \epsilon) > 0$, but we have $\alpha(\epsilon', \epsilon) = \beta(\epsilon', \epsilon)$.

As a consequence of Theorem 3.7, we obtain a generalization of (2.10) and Corollary 2.23 in the situation of this section. The class $[L]$ of the simple quotient L of $M(\epsilon_s)$ occurs with multiplicity 1 in $M(\epsilon)$:

$$[M(\epsilon)]_t = [L] + \sum_{L' \prec L} P_{L',\epsilon}(t)[L']$$

where $P_{L',\epsilon}(t) \in \mathbb{N}[t^{\pm 1}]$ and \prec is the Nakajima partial ordering on simple classes.

Example 3.9. We continue Example 3.6 and we compute the corresponding monoidal Jantzen filtrations. We consider

$$\epsilon = \epsilon_s = (3, 1)$$

with

$$S_3 = L(Y_{1,2}) \text{ and } S_1 = L(Y_{1,0}).$$

Then $M(\epsilon_s) = S_3 \otimes S_1$ has a unique proper submodule S of dimension 1 and $M(\epsilon_c) = S_1 \otimes S_3$ has a unique proper submodule L of dimension 3.

For $M(\epsilon_s)$, we are in the first situation of Example 2.15. We have $R_{\epsilon_c, \epsilon_s} = R_{3,1}$ and

$$z^N R_{3,1}^{-1} \tilde{M}(\epsilon_c) \cap \tilde{M}(\epsilon_s) = \begin{cases} \tilde{M}(\epsilon_s) & \text{if } N \leq 0, \\ z^N \tilde{M}(\epsilon_s) + z^{N-1} \mathbb{O}S & \text{if } N \geq 1, \end{cases}$$

$$F_\bullet M(\epsilon_s): \quad \cdots \supset F_0 M(\epsilon_s) = M(\epsilon_s) \supset F_1 M(\epsilon_s) = S \supset F_2 M(\epsilon_s) = 0 \supset \cdots.$$

For $M(\epsilon_c)$, we are in the second situation of Example 2.15. We have

$$z^N R_{3,1} \tilde{M}(\epsilon_s) \cap \tilde{M}(\epsilon_c) = \begin{cases} \tilde{M}(\epsilon_c) & \text{if } N \leq -1, \\ z^{N+1} \tilde{M}(\epsilon_c) + z^N \mathbb{O}L & \text{if } N \geq 0, \end{cases}$$

$$F_\bullet M(\epsilon_c): \quad \cdots \supset F_{-1} M(\epsilon_c) = M(\epsilon_c) \supset F_0 M(\epsilon_c) = L \supset F_1 M(\epsilon_c) = 0 \supset \cdots.$$

Example 3.10. Let us illustrate Proposition 2.16 with the filtrations computed in Example 3.9 for the morphism

$$\mathbf{r}_{\epsilon_c, \epsilon_s}: M(\epsilon_s) \rightarrow M(\epsilon_c).$$

Then we have:

$$\begin{aligned} \mathbf{r}_{\epsilon_c, \epsilon_s}(F_0 M(\epsilon_s)) &\subset \text{Im}(\mathbf{r}_{\epsilon_c, \epsilon_s}) = L = F_0 M(\epsilon_c), \\ \mathbf{r}_{\epsilon_c, \epsilon_s}(F_1 M(\epsilon_s)) &= \mathbf{r}_{\epsilon_c, \epsilon_s}(S) = 0 = F_1 M(\epsilon_c). \end{aligned}$$

3.5. Quantum Grothendieck ring. We recall the construction of the quantum Grothendieck ring. For a representation M in $\mathcal{C}_{\mathbb{Z}}$ we have its q -character defined in [17]. It can be proved [29] that as M is in $\mathcal{C}_{\mathbb{Z}}$, we have

$$\chi_q(M) \in \mathcal{Y} = \mathbb{Z}[Y_{i,p}^{\pm 1}]_{(i,p) \in \hat{I}}.$$

It defines the q -character morphism on the Grothendieck ring $K(\mathcal{C}_{\mathbb{Z}})$ of $\mathcal{C}_{\mathbb{Z}}$

$$\chi_q: K(\mathcal{C}_{\mathbb{Z}}) \rightarrow \mathcal{Y}.$$

Consider the quantum Cartan matrix $C(z) = (C_{i,j}(z))_{i,j \in I}$ defined by $C_{i,j}(z) = [c_{i,j}]_z$ if $i \neq j$ and $C_{i,i}(z) = [2]_{z^{r_i}}$ for $i \in I$, where $[k]_z := (z^k - z^{-k})/(z - z^{-1})$ is the standard quantum integer. We will denote $\tilde{C}_{i,j}(z) = \sum_{m \geq 1} \tilde{c}_{i,j}(m) z^m$ the expansion of the (i,j) -entry of the inverse $\tilde{C}(z)$ of the quantum Cartan matrix $C(z)$ at $z = 0$. We also extend the definition of $\tilde{c}_{i,j}(m)$ to every $m \in \mathbb{Z}$ by setting $\tilde{c}_{i,j}(m) = 0$ if $m \leq 0$.

For $(i,p), (j,s) \in \hat{I}$, following [27], we set

$$\mathcal{N}(i,p; j,s) := \tilde{c}_{i,j}(p-s-r_i) - \tilde{c}_{i,j}(p-s+r_i) - \tilde{c}_{i,j}(s-p-r_i) + \tilde{c}_{i,j}(s-p+r_i).$$

As $\mathcal{N}(i, p; j, s) = -\mathcal{N}(j, s; i, p)$, this defines a skew-symmetric bilinear form

$$\mathcal{N}: \mathbb{N}^{\oplus \hat{I}} \times \mathbb{N}^{\oplus \hat{I}} \rightarrow \mathbb{Z}.$$

Definition 3.11 ([27]). We define the quantum torus \mathcal{Y}_t as the $\mathbb{Z}[t^{\pm 1/2}]$ -algebra presented by the set of generators $\{\tilde{Y}_{i,p}^{\pm 1} \mid (i, p) \in \hat{I}\}$ and the following relations:

- (1) $\tilde{Y}_{i,p} \tilde{Y}_{i,p}^{-1} = \tilde{Y}_{i,p}^{-1} \tilde{Y}_{i,p} = 1$ for each $(i, p) \in \hat{I}$,
- (2) $\tilde{Y}_{i,p} \tilde{Y}_{j,s} = t^{-\mathcal{N}(i,p;j,s)} \tilde{Y}_{j,s} \tilde{Y}_{i,p}$ for each $(i, p), (j, s) \in \hat{I}$.

Remark 3.12. See [30, Remark 3.1] for comments on the relations with the quantum torus in [58] and [53] for simply-laced quantum loop algebras.

Example 3.13. Let $\mathfrak{g} = \mathfrak{sl}_2$. Then $\tilde{Y}_{1,2} \tilde{Y}_{1,0} = t^{-2} \tilde{Y}_{1,0} \tilde{Y}_{1,2}$.

The evaluation at $t = 1$ is the \mathbb{Z} -algebra homomorphism $\text{ev}_{t=1}: \mathcal{Y}_t \rightarrow \mathcal{Y}$ given by

$$t^{1/2} \mapsto 1, \quad \tilde{Y}_{i,p} \mapsto Y_{i,p}.$$

An element $\tilde{m} \in \mathcal{Y}_t$ is called a *monomial* if it is a product of the generators $\tilde{Y}_{i,p}$ for $(i, p) \in \hat{I}$ and $t^{\pm 1/2}$. For a monomial $\tilde{m} \in \mathcal{Y}_t$, we set $u_{i,p}(\tilde{m})$ to be the power of $Y_{i,p}$ in $\text{ev}_{t=1}(\tilde{m})$. A monomial \tilde{m} in \mathcal{Y}_t is said to be *dominant* if $\text{ev}_{t=1}(\tilde{m}) \in \mathcal{M}^+$. Moreover, for monomials \tilde{m}, \tilde{m}' in \mathcal{Y}_t , set

$$\tilde{m} \preceq \tilde{m}' \text{ if and only if } \text{ev}_{t=1}(\tilde{m}) \preceq \text{ev}_{t=1}(\tilde{m}'),$$

with the ordering on \mathcal{M} defined above. Following [27, Section 6.3], we define the \mathbb{Z} -algebra anti-involution $\overline{(\cdot)}$ on \mathcal{Y}_t by

$$t^{1/2} \mapsto t^{-1/2}, \quad \tilde{Y}_{i,p} \mapsto t \tilde{Y}_{i,p}.$$

For any monomial \tilde{m} in \mathcal{Y}_t , there uniquely exists $a \in \mathbb{Z}$ such that $\underline{\tilde{m}} = t^{a/2} \tilde{m}$ is $\overline{(\cdot)}$ -invariant. As $\underline{\tilde{m}}$ depends only on $\text{ev}_{t=1}(\tilde{m})$, for every monomial $m \in \mathcal{M}$, the element \underline{m} is well-defined in \mathcal{Y}_t . These elements form the free $\mathbb{Z}[t^{\pm 1/2}]$ -basis of \mathcal{Y}_t called the basis of *commutative monomials*. For example, for $(i, p) \in \hat{I}$, we set

$$\tilde{A}_{i,p+r_i} := \underline{A}_{i,p+r_i}.$$

For each $i \in I$, denote by $K_{i,t}$ the $\mathbb{Z}[t^{\pm 1/2}]$ -subalgebra of \mathcal{Y}_t generated by

$$\{\tilde{Y}_{i,p}(1 + t^{-1} \tilde{A}_{i,p+r_i}^{-1}) \mid (i, p) \in \hat{I}\} \cup \{\tilde{Y}_{j,s}^{\pm 1} \mid (j, s) \in \hat{I}, j \neq i\}.$$

Following [27, 53, 58], the quantum Grothendieck ring of $\mathcal{C}_{\mathbb{Z}}$ is defined as

$$K_t(\mathcal{C}_{\mathbb{Z}}) := \bigcap_{i \in I} K_{i,t}.$$

By construction, the quantum Grothendieck is stable by the $\overline{(\cdot)}$ -involution.

Theorem 3.14 ([27, Theorem 5.11]). *For every dominant monomial \tilde{m} in \mathcal{Y}_t , there uniquely exists an element $F_t(\tilde{m})$ of $K_t(\mathcal{C}_{\mathbb{Z}})$ such that \tilde{m} is the unique dominant monomial occurring in $F_t(\tilde{m})$. The monomials \tilde{m}' occurring in $F_t(\tilde{m}) - \tilde{m}$ satisfy $\tilde{m}' \prec \tilde{m}$. In particular, the set $\{F_t(\underline{m}) \mid m \in \mathcal{M}^+\}$ forms a $\mathbb{Z}[t^{\pm 1/2}]$ -basis of $K_t(\mathcal{C}_{\mathbb{Z}})$.*

Note that the $F_t(\underline{m})$ are $\overline{(\cdot)}$ -invariant.

For a dominant monomial \tilde{m} in \mathcal{Y}_t and $u_{i,p}(\tilde{m})$ the power of $Y_{i,p}$ in $\text{ev}_{t=1}(\tilde{m})$, set

$$E_t(\tilde{m}) := \tilde{m} \left(\prod_{p \in \mathbb{Z}}^{\leftarrow} \left(\prod_{i \in I: (i,p) \in \hat{I}} \tilde{Y}_{i,p}^{u_{i,p}(\tilde{m})} \right) \right)^{-1} \prod_{p \in \mathbb{Z}}^{\leftarrow} \left(\prod_{i \in I: (i,p) \in \hat{I}} F_t(\tilde{Y}_{i,p})^{u_{i,p}(\tilde{m})} \right).$$

Note that by [27], the products are well-defined.

The element $E_t(\underline{m})$ is called *the (q,t) -character of the ordinary standard module $M(m)$* associated to m as above. By [15, 28], the image by $\text{ev}_{t=1}$ is $\chi_q(M(m))$.

We consider another kind of elements $L_t(\underline{m})$ in $K_t(\mathcal{C}_{\mathbb{Z}})$ which is conjecturally a t -quantum version of the q -character of simple modules.

Theorem 3.15 ([53, Theorem 8.1], [27, Theorem 6.9]). *For a dominant monomial $m \in \mathcal{M}^+$, there exists a unique element $L_t(\underline{m})$ in $K_t(\mathcal{C}_{\mathbb{Z}})$ such that*

$$(S1) \quad \overline{L_t(\underline{m})} = L_t(\underline{m}), \text{ and}$$

$$(S2) \quad L_t(\underline{m}) = E_t(\underline{m}) + \sum_{m' \in \mathcal{M}^+} Q_{m,m'}(t) E_t(\underline{m}') \text{ with } Q_{m,m'}(t) \in t\mathbb{Z}[t].$$

Moreover, we have $Q_{m,m'}(t) = 0$ unless $m' \prec m$. In particular, the set $\{L_t(\underline{m}) \mid m \in \mathcal{M}^+\}$ forms a $\mathbb{Z}[t^{\pm 1/2}]$ -basis of $K_t(\mathcal{C}_{\mathbb{Z}})$.

The element $L_t(\underline{m})$ is called *the (q,t) -character of the simple module $L(m)$* .

In what follows, for a dominant monomial $m \in \mathcal{M}^+$, we will write for simplicity

$$F_t(m) := F_t(\underline{m}), \quad E_t(m) := E_t(\underline{m}), \quad L_t(m) := L_t(\underline{m}).$$

Conjecture 3.16 ([27, Conjecture 7.3]). *For all $m \in \mathcal{M}^+$, we have*

$$\text{ev}_{t=1}(L_t(m)) = \chi_q(L(m)).$$

A fundamental theorem of Nakajima [53, Theorem 8.1] states that this holds true when \mathfrak{g} is of simply-laced type. The proof used the geometry of quiver varieties. This was the main motivation for this conjecture. This conjecture is now proved for type B in [21] and for all simple modules that are reachable (in the sense of cluster algebras) for general types in [20].

Thanks to the unitriangular property (S2), we can write

$$E_t(m) = L_t(m) + \sum_{m' \in \mathcal{M}^+: m' \prec m} P_{m,m'}(t) L_t(m')$$

with some $P_{m,m'}(t) \in t\mathbb{Z}[t]$ for each $m \in \mathcal{M}^+$. The polynomials $P_{m,m'}(t)$ are analogs of Kazhdan-Lusztig polynomials for finite-dimensional representations of quantum loop algebras. The following was proved by Nakajima [53] for simply-laced types, and by the authors of [21] for general types.

Theorem 3.17 ([21, 53]). *The polynomials $P_{m,m'}(t)$ are positive.*

Example 3.18. Let $\mathfrak{g} = \mathfrak{sl}_2$ and $m = Y_{1,0}Y_{1,2}$. Then $E_t(m)$ equals

$$t(\tilde{Y}_{1,2} + \tilde{Y}_{1,4}^{-1})(\tilde{Y}_{1,0} + \tilde{Y}_{1,2}^{-1}) = \underline{Y_{1,0}Y_{1,2}} + \underline{Y_{1,0}Y_{1,4}^{-1}} + \underline{Y_{1,2}^{-1}Y_{1,4}^{-1}} + t = L_t(m) + tL_t(1).$$

The specialization at $t = 1$ corresponds to $[L(Y_{1,2}) \otimes L(Y_{1,0})] = [L(m)] + 1$ in $K(\mathcal{C}_{\mathbb{Z}})$.

3.6. Quantum Grothendieck ring conjecture. Recall \mathcal{N} defined in the previous Section. By considering the powers of the variable of dominant monomials m, m' in \mathcal{M}^+ , (3.5) also defines $\mathcal{N}(m, m') \in \mathbb{Z}$.

Remark 3.19. There is an interpretation of \mathcal{N} . Let M and N be simple modules in $\mathcal{C}_{\mathbb{Z}}$. Then set

$$\Lambda(M, N) = \mathcal{N}(M, N) + 2\mathfrak{o}(M, N),$$

where $\mathcal{N}(M, N) = \mathcal{N}(m_M, m_N)$ with $m_M, m_N \in \mathcal{M}^+$ dominant monomials parameterizing M and N respectively. As proved in [22], $\Lambda(M, N)$ coincides with the invariant defined in [40].

We continue with a PBW-theory as in Section 3.4. Let us denote $m_k \in \mathcal{M}^+$ the dominant monomial so that $S_k = L(m_k)$. We consider the skew-symmetric bilinear form γ defined on $\mathbb{N}^{\oplus J} \times \mathbb{N}^{\oplus J}$ and so that for any $k, k' \in J$:

$$\gamma(\delta_k, \delta_{k'}) = -\mathcal{N}(m_k, m_{k'})/2.$$

We consider the associated bilinear operation $* = *_{\gamma}$ on $K(\mathcal{C}_{\mathbb{Z}})_t$. Be aware that this operation $*$ also depends on our choice of PBW-theory and its deformation.

Let us define the $\mathbb{Z}[t^{\pm 1/2}]$ -linear isomorphism $\phi: K(\mathcal{C}_{\mathbb{Z}})_t \simeq K_t(\mathcal{C}_{\mathbb{Z}})$ by $\phi([L(m)]) = L_t(m)$ for all $m \in \mathcal{M}^+$. Clearly, we have $\phi \circ \overline{(\cdot)} = \overline{(\cdot)} \circ \phi$.

Conjecture 3.20 (Quantum Grothendieck Ring Conjecture). *With a chosen PBW-theory and its deformation, Associativity Conjectures 2.19 and 2.20 hold for $(K(\mathcal{C}_{\mathbb{Z}})_t, *)$, and the linear isomorphism ϕ gives a $\mathbb{Z}[t^{\pm 1/2}]$ -algebra isomorphism from $(K(\mathcal{C}_{\mathbb{Z}})_t, *)$ to the quantum Grothendieck ring $K_t(\mathcal{C}_{\mathbb{Z}})$.*

Remark 3.21. If Conjecture 3.20 is true for any PBW-theory, it implies that the ring structure $(K(\mathcal{C}_{\mathbb{Z}})_t, *)$, with its canonical basis, does not depend on the choice of PBW-theory.

Example 3.22. We can illustrate first with the filtrations computed in Example 3.9 with $\epsilon_s = (3, 1)$. We have

$$[M(\epsilon_s)]_t = [L] + t \quad \text{and} \quad [M(\epsilon_c)]_t = [L] + t^{-1}.$$

As $\mathcal{N}(1, 3; 1, 1) = 2$, we recover the well-known formulas (see Example 3.18):

$$[S_3] * [S_1] = t^{-1}[L] + 1 \quad \text{and} \quad [S_1] * [S_3] = t[L] + 1.$$

Now, let us consider the ordinary PBW-theory of fundamental modules as in Theorem 3.2, and assume that Conjecture 3.20 is true for this case. By Remark 2.21, it implies that, for any $m \in \mathcal{M}^+$, we have

$$\phi([M(m)]_t) = E_t(m).$$

More generally, for each $d \in \mathbb{N}$ and $\epsilon = (\epsilon_1, \dots, \epsilon_d) \in J^d$, letting

$$E_t(\epsilon) := t^{\sum_{1 \leq k < l \leq d} \mathcal{N}(i_k, p_k; i_l, p_l)/2} F_t(Y_{i_1, p_1}) \cdots F_t(Y_{i_d, p_d}),$$

where (i_k, p_k) denotes the element of \hat{I} such that $e(i_k, p_k) = \epsilon_k$ for each $1 \leq k \leq d$, we obtain

$$(3.1) \quad \phi([M(\epsilon)]_t) = E_t(\epsilon).$$

Compare with (2.11). The converse statement is true. Namely, Conjecture 3.20 for the ordinary PBW-theory is equivalent to the following.

Conjecture 3.23. *The equality (3.1) holds for any $d \in \mathbb{N}$ and $\epsilon \in J^d$.*

Remark 3.24. Note that Conjecture 3.23 also implies Duality Conjecture 2.25 for the ordinary PBW-theory. Indeed, for each $\epsilon \in J^d$, we have $\phi([M(\epsilon)]_t) = E_t(\epsilon) = E_t(\epsilon^{\text{op}}) = \phi([M(\epsilon^{\text{op}})]_t)$ and so $\overline{[M(\epsilon)]_t} = [M(\epsilon^{\text{op}})]_t$.

Remark 3.25. Conjecture 3.23 also implies the analog of Kazhdan-Lusztig conjecture (= Conjecture 3.16). In addition, as (2.12) is a consequence of Conjecture 3.23, the positivity of Kazhdan-Lusztig polynomials of Theorem 3.17 can be seen as an evidence for this conjecture.

When \mathfrak{g} is of simply-laced type, we can actually establish that the conjectures are true for the ordinary PBW-theory with the help of geometry. Namely, we have the following, whose proof will be given in Section 6 below.

Theorem 3.26. *When \mathfrak{g} is of simply-laced type, Conjecture 3.23 holds, or equivalently, Conjecture 3.20 for the ordinary PBW theory holds.*

3.7. Examples.

3.7.1. We consider examples as in Example 3.9 but with more factors:

$$\epsilon_s = (3, 3, 1), \quad \epsilon = (3, 1, 3), \quad \epsilon_c = (1, 3, 3)$$

so that we have

$$M(\epsilon_s) = S_3 \otimes S_3 \otimes S_1, \quad M(\epsilon) = S_3 \otimes S_1 \otimes S_3, \quad M(\epsilon_c) = S_1 \otimes S_3 \otimes S_3.$$

We have the morphisms:

$$M(\epsilon_s) \xrightarrow{\mathbf{r}_{\epsilon, \epsilon_s}} M(\epsilon) \xrightarrow{\mathbf{r}_{\epsilon_c, \epsilon}} M(\epsilon_c).$$

We obtain the monoidal Jantzen filtrations

$$F_0 M(\epsilon_s) = M(\epsilon_s) \supset F_1 M(\epsilon_s) = F_2 M(\epsilon_s) = \text{Ker}(\mathbf{r}_{\epsilon, \epsilon_s}) \supset F_3 M(\epsilon_s) = \{0\},$$

$$F_0 M(\epsilon) = M(\epsilon) \supset F_1 M(\epsilon) = \{0\},$$

$$F_{-2} M(\epsilon_c) = M(\epsilon_c) \supset F_{-1} M(\epsilon_c) = F_0 M(\epsilon_c) = \text{Im}(\mathbf{r}_{\epsilon_c, \epsilon}) \supset F_1(M(\epsilon_c)) = \{0\}.$$

Let L be the simple quotient of $M(\epsilon_s)$. We obtain

$$[M(\epsilon_s)]_t = [L] + t^2[S_3], \quad [M(\epsilon)]_t = [L] + [S_3], \quad [M(\epsilon_c)]_t = [L] + t^{-2}[S_3],$$

$$m_3([S_3], [S_3], [S_1]) = t^{-2}[L] + [S_3], \quad m_3([S_3], [S_1], [S_3]) = [L] + [S_3],$$

$$m_3([S_1], [S_3], [S_3]) = t^2[L] + [S_3].$$

This is completely analogous to the case of $\epsilon_s = (3, 1, 1)$.

3.7.2. Now we set

$$\epsilon_s = (5, 3, 1), \quad \epsilon_c = (1, 3, 5)$$

so that we have

$$M(\epsilon_s) = S_5 \otimes S_3 \otimes S_1, \quad M(\epsilon_c) = S_1 \otimes S_3 \otimes S_3.$$

As $S_1 \otimes S_5 \simeq S_5 \otimes S_1$, we have two intermediate modules:

$$M(\epsilon_1) = M(3, 5, 1) \simeq M(3, 1, 5) \quad \text{and} \quad M(\epsilon_2) = M(5, 1, 3) \simeq M(1, 5, 3).$$

We have the morphisms:

$$\begin{array}{ccc} & M(\epsilon_1) & \\ \mathbf{r}_{\epsilon_1, \epsilon_s} \nearrow & & \searrow \mathbf{r}_{\epsilon_c, \epsilon_1} \\ M(\epsilon_s) & & M(\epsilon_c) \\ \mathbf{r}_{\epsilon_2, \epsilon_s} \searrow & & \nearrow \mathbf{r}_{\epsilon_c, \epsilon_2} \\ & M(\epsilon_2) & \end{array}$$

We obtain the monoidal Jantzen filtrations

$$F_0M(\epsilon_s) = M(\epsilon_s) \supset F_1M(\epsilon_s) = \text{Ker}(\mathbf{r}_{\epsilon_c, \epsilon_s}) = \text{Ker}(\mathbf{r}_{\epsilon_1, \epsilon_s}) + \text{Ker}(\mathbf{r}_{\epsilon_2, \epsilon_s}) \supset F_2M(\epsilon_s) = \{0\},$$

$$\begin{aligned} F_{-1}M(\epsilon_1) &= M(\epsilon_1) \supset F_0M(\epsilon_1) = \text{Im}(\mathbf{r}_{\epsilon_1, \epsilon_s}) \\ &\supset F_1M(\epsilon_1) = \text{Ker}(\mathbf{r}_{\epsilon_c, \epsilon_1}) \cap \text{Im}(\mathbf{r}_{\epsilon_1, \epsilon_s}) \supset F_2M(\epsilon_1) = \{0\}, \\ F_{-1}M(\epsilon_2) &= M(\epsilon_2) \supset F_0M(\epsilon_2) = \text{Im}(\mathbf{r}_{\epsilon_2, \epsilon_s}) \\ &\supset F_1M(\epsilon_2) = \text{Ker}(\mathbf{r}_{\epsilon_c, \epsilon_2}) \cap \text{Im}(\mathbf{r}_{\epsilon_2, \epsilon_s}) \supset F_2M(\epsilon_2) = \{0\}, \end{aligned}$$

$$F_{-1}M(\epsilon_c) = M(\epsilon_c) \supset F_0M(\epsilon_c) = \text{Im}(\mathbf{r}_{\epsilon_c, \epsilon}) = \text{Im}(\mathbf{r}_{\epsilon_c, \epsilon_1}) \cap \text{Im}(\mathbf{r}_{\epsilon_c, \epsilon_2}) \supset F_1(M(\epsilon_c)) = \{0\}.$$

Let L be the simple quotient of $M(\epsilon_s)$. We obtain

$$\begin{aligned} [M(\epsilon_s)]_t &= [L] + t[S_1] + t[S_5], & [M(\epsilon_1)]_t &= [L] + t^{-1}[S_1] + t[S_5] \\ [M(\epsilon_2)]_t &= [L] + t[S_1] + t^{-1}[S_5], & [M(\epsilon_c)]_t &= [L] + t^{-1}[S_1] + t^{-1}[S_5], \\ m_3([S_5], [S_3], [S_1]) &= t^{-1}[L] + [S_1] + [S_5], & m_3([S_1], [S_3], [S_5]) &= t[L] + [S_1] + [S_5], \\ m_3([S_3], [S_5], [S_1]) &= t[L] + [S_1] + t^2[S_5] = t^2m_3([S_3], [S_1], [S_5]), \\ m_3([S_5], [S_1], [S_3]) &= t[L] + t^2[S_1] + t[S_5] = t^2m_3([S_1], [S_5], [S_3]). \end{aligned}$$

3.7.3. Now we set

$$\epsilon_s = (3, 3, 1, 1), \quad \epsilon_c = (1, 1, 3, 3)$$

so that we have

$$M(\epsilon_s) = S_3^{\otimes 2} \otimes S_1^{\otimes 2}, \quad M(\epsilon_c) = S_1^{\otimes 2} \otimes S_3^{\otimes 2}.$$

We have four intermediate modules:

$$\begin{aligned} M(\epsilon_1) &= M(3, 1, 3, 1), & M(\epsilon_2) &= M(3, 1, 1, 3), \\ M(\epsilon_3) &= M(1, 3, 3, 1), & M(\epsilon_4) &= M(1, 3, 1, 3). \end{aligned}$$

We have the specialized R -matrices:

$$\begin{array}{ccccc} & & M(\epsilon_2) & & \\ & \nearrow^{\mathbf{r}_{\epsilon_2, \epsilon_1}} & & \searrow_{\mathbf{r}_{\epsilon_3, \epsilon_2}} & \\ M(\epsilon_s) & \xrightarrow{\mathbf{r}_{\epsilon_1, \epsilon_s}} & M(\epsilon_1) & & M(\epsilon_4) \xrightarrow{\mathbf{r}_{\epsilon_c, \epsilon_4}} M(\epsilon_c) \\ & \searrow_{\mathbf{r}_{\epsilon_3, \epsilon_1}} & & \nearrow_{\mathbf{r}_{\epsilon_4, \epsilon_3}} & \\ & & M(\epsilon_3) & & \end{array}$$

as well as the morphism $\mathbf{r}_{\epsilon_s, \epsilon_c} : M(\epsilon_c) \rightarrow M(\epsilon_s)$ obtained as in Section 3.3.

We obtain the monoidal Jantzen filtrations:

$$\begin{aligned} F_0M(\epsilon_s) &= M(\epsilon_s) \supset F_1M(\epsilon_s) = \text{Ker}(\mathbf{r}_{\epsilon_c, \epsilon_s}) \supset F_2M(\epsilon_s) = F_3M(\epsilon_s) = \text{Ker}(\mathbf{r}_{\epsilon_1, \epsilon_s}) \\ &\supset F_4M(\epsilon_s) = \text{Im}(\mathbf{r}_{\epsilon_s, \epsilon_c}) \supset F_5M(\epsilon_s) = \{0\}, \\ F_0M(\epsilon_1) &= M(\epsilon_1) \supset F_1M(\epsilon_1) = \text{Ker}(\mathbf{r}_{\epsilon_2, \epsilon_1}) + \text{Ker}(\mathbf{r}_{\epsilon_3, \epsilon_1}) \\ &\supset F_2M(\epsilon_1) = \text{Ker}(\mathbf{r}_{\epsilon_2, \epsilon_1}) \cap \text{Ker}(\mathbf{r}_{\epsilon_3, \epsilon_1}) \supset F_3M(\epsilon_1) = \{0\}, \\ F_{-1}M(\epsilon_2) &= M(\epsilon_2) \supset F_0M(\epsilon_2) = \text{Ker}(\mathbf{r}_{\epsilon_4, \epsilon_2}) + \text{Im}(\mathbf{r}_{\epsilon_2, \epsilon_s}) \\ &\supset F_1M(\epsilon_2) = \text{Ker}(\mathbf{r}_{\epsilon_4, \epsilon_2}) \cap \text{Im}(\mathbf{r}_{\epsilon_2, \epsilon_s}) \supset F_2M(\epsilon_2) = \{0\}, \\ &\supset F_1M(\epsilon_3) = \text{Ker}(\mathbf{r}_{\epsilon_4, \epsilon_3}) \cap \text{Im}(\mathbf{r}_{\epsilon_3, \epsilon_s}) \supset F_2M(\epsilon_3) = \{0\}, \\ F_{-2}M(\epsilon_4) &= M(\epsilon_4) \supset F_{-1}M(\epsilon_4) = \text{Im}(\mathbf{r}_{\epsilon_4, \epsilon_2}) + \text{Im}(\mathbf{r}_{\epsilon_4, \epsilon_3}) \\ &\supset F_0M(\epsilon_4) = \text{Im}(\mathbf{r}_{\epsilon_4, \epsilon_2}) \cap \text{Im}(\mathbf{r}_{\epsilon_4, \epsilon_3}) \supset F_1M(\epsilon_4) = \{0\}, \\ F_{-4}M(\epsilon_c) &= M(\epsilon_c) \supset F_{-3}M(\epsilon_c) = \text{Ker}(\mathbf{r}_{\epsilon_s, \epsilon_c}) \end{aligned}$$

$$\supset F_{-2}(M(\epsilon_c)) = F_{-1}M(\epsilon_c) = \text{Im}(\mathbf{r}_{\epsilon_c, \epsilon_4}) \supset F_0M(\epsilon_c) = \text{Im}(\mathbf{r}_{\epsilon_c, \epsilon_s}) \supset F_1M(\epsilon_c) = \{0\}.$$

Let L be the simple quotient of $M(\epsilon_s)$ and $K = L(Y_{1,q^0}Y_{1,q^2})$. We obtain:

$$[M(\epsilon_s)]_t = [L] + (t^3 + t)[K] + t^4, \quad [M(\epsilon_1)]_t = [L] + 2t[K] + t^2,$$

$$[M(\epsilon_2)]_t = [L] + (t + t^{-1})[K] + 1 = [M(\epsilon_3)]_t,$$

$$[M(\epsilon_4)]_t = [L] + 2t^{-1}[K] + t^{-2}, \quad [M(\epsilon_c)]_t = [L] + (t^{-3} + t^{-1})[K] + t^{-4}.$$

This gives for the products:

$$[S_3^{\otimes 2}] * [S_1^{\otimes 2}] = t^{-4}[L] + (t^{-1} + t^{-3})[K] + 1, \quad [S_1^{\otimes 2}] * [S_3^{\otimes 2}] = t^4[L] + (t^3 + t)[K] + 1,$$

$$m_4([S_3], [S_1], [S_3], [S_1]) = t^{-2}[L] + 2t^{-1}[K] + 1,$$

$$m_3([S_3], [S_1^{\otimes 2}], [S_3]) = [L] + (t + t^{-1})[K] + 1 = m_3([S_1], [S_3^{\otimes 2}], [S_1]),$$

$$m_4([S_1], [S_3], [S_1], [S_3]) = t^2[L] + 2t[K] + 1.$$

3.7.4. Let \mathfrak{g} be of simply-laced type, $i \in I$, $r \in \mathbb{Z}$ and set

$$\epsilon_s = (i + (r + 2)n, i + rn), \quad \epsilon_c = (i + rn, i + (r + 2)n)$$

so that we have

$$M(\epsilon_s) = L(Y_{i,r+2}) \otimes L(Y_{i,r}), \quad M(\epsilon_c) = L(Y_{i,r}) \otimes L(Y_{i,r+2}).$$

We have the morphism:

$$\mathbf{r}_{\epsilon_c, \epsilon_s} : M(\epsilon_s) \rightarrow M(\epsilon_c).$$

of simple image L isomorphic to $L(Y_{i,r}Y_{i,r+2})$ and kernel K simple isomorphic to $\bigotimes_{j \in I: c_{i,j} = -1} L(Y_{j,r+1})$. The monoidal Jantzen filtrations are characterized by

$$F_0M(\epsilon_s) = M(\epsilon_s) \supset F_1M(\epsilon_s) = K \supset F_2M(\epsilon_s) = \{0\},$$

$$F_{-1}M(\epsilon_c) = M(\epsilon_c) \supset F_0M(\epsilon_c) = L \supset F_1(M(\epsilon_c)) = \{0\}.$$

We obtain

$$[M(\epsilon_s)]_t = [L] + t[K], \quad [M(\epsilon_c)]_t = [L] + t^{-1}[K],$$

$$[S_{i+(r+2)n}] * [S_{i,r+n}] = t^\alpha[L] + t^{\alpha+1}[K], \quad [S_{i+rn}] * [S_{i+(r+2)n}] = t^{-\alpha}[L] + t^{-\alpha-1}[K],$$

where $\alpha = -1 + (\tilde{c}_{i,i}(1) + \tilde{c}_{i,i}(3))/2$. For $\mathfrak{g} = \mathfrak{sl}_3$, $K \simeq S_4$ and $\alpha = -1/2$.

Remark 3.27. In all the above examples, we find that every filter submodule $F_nM(\epsilon)$ of $M(\epsilon)$ can be expressed only in terms of the specialized R -matrices. In particular, the monoidal Jantzen filtrations do not depend on the choice of deformations in these examples. It would be interesting to study in which situation the monoidal Jantzen filtrations are characterized only by the specialized R -matrices.

4. MONOIDAL JANTZEN FILTRATIONS FOR SYMMETRIC QUIVER HECKE ALGEBRAS

We study our second main examples for monoidal Jantzen filtrations, realized in categories of representations of symmetric quiver Hecke algebras.

We first give reminders on quantum unipotent coordinate rings with their PBW and canonical bases (Section 4.2). In Section 4.3, we recall their categorification in terms of representations of quiver Hecke algebras (Theorem 4.3) which are compatible with specialization (Corollary 4.4). Note that quiver Hecke algebras have a natural grading and categorify the *quantum* unipotent coordinate rings (whereas quantum loop algebras above produce commutative Grothendieck rings), although we work with ungraded modules in this paper. We recall the categories \mathcal{C}_w of (ungraded) representations together with their PBW-theory from [39] (Theorem 4.5). We construct the deformation of such a PBW-theory (Section 4.5). Hence we obtain monoidal Jantzen filtrations. We state the analog of the quantum Grothendieck ring conjecture, which we call the quantum unipotent ring conjecture (Conjectures 4.9, 4.11). It expresses our expectation that our monoidal Jantzen filtrations for ungraded modules should recover the graded Jordan-Hölder multiplicities for their graded counterparts.

4.1. Notation. Let $C = (c_{ij})_{i,j \in I}$ be a *symmetric* generalized Cartan matrix with I being the set of Dynkin nodes. We write $i \sim j$ if $c_{ij} < 0$. We denote the associated Kac-Moody Lie algebra by \mathfrak{g} . Let \mathbf{Q} be a free abelian with a basis $\{\alpha_i\}_{i \in I}$ endowed with the symmetric bilinear form $(-, -)$ given by $(\alpha_i, \alpha_j) = c_{ij}$. We set $\mathbf{Q}^+ := \sum_{i \in I} \mathbb{N}\alpha_i \subset \mathbf{Q}$. For each $i \in I$, the simple reflection s_i is defined by $s_i\alpha_j = \alpha_j - c_{ij}\alpha_i$. The Weyl group W is the subgroup of $\text{Aut}(\mathbf{Q})$ generated by the simple reflections $\{s_i\}_{i \in I}$. The pair $(W, \{s_i\}_{i \in I})$ forms a Coxeter system. The length of an element $w \in W$ is denoted by $\ell(w)$. The set of real roots is defined by $R := \bigcup_{i \in I} W\alpha_i$. We have $R = R^+ \sqcup (-R^+)$ with $R^+ := R \cap \mathbf{Q}^+$.

4.2. Quantum unipotent coordinate rings. Let t be an indeterminate. For $n \in \mathbb{N}$, we set $[n]_t := \frac{t^n - t^{-n}}{t - t^{-1}} \in \mathbb{Z}[t^{\pm 1}]$ and $[n]_t! := \prod_{k=1}^n [k]_t$. Let $U_t^+(\mathfrak{g})$ denote the positive half of the quantized enveloping algebra of \mathfrak{g} . By definition, it is the $\mathbb{Q}(t)$ -algebra presented by the generators $\{e_i\}_{i \in I}$ and the quantum Serre relations $\sum_{k=0}^{1-c_{ij}} e_i^{(k)} e_j e_i^{(1-c_{ij}-k)} = 0$ for any $i, j \in I$ with $i \neq j$, where $e_i^{(n)} := e_i^n / [n]_t!$ is the divided power. The algebra $U_t^+(\mathfrak{g})$ has the natural \mathbf{Q}^+ -grading $U_t^+(\mathfrak{g}) = \bigoplus_{\beta \in \mathbf{Q}^+} U_t^+(\mathfrak{g})_\beta$ with $e_i \in U_t^+(\mathfrak{g})_{\alpha_i}$. In addition, we have an algebra involution ι given by $\iota(t) = t^{-1}$ and $\iota(e_i) = e_i$ for any $i \in I$. Let $U_t^+(\mathfrak{g})_{\mathbb{Z}[t^{\pm 1}]}$ denote the $\mathbb{Z}[t^{\pm 1}]$ -subalgebra generated by all the divided powers $\{e_i^{(n)}\}_{i \in I, n \in \mathbb{N}}$. The algebra $U_t^+(\mathfrak{g})_{\mathbb{Z}[t^{\pm 1}]}$ is free over $\mathbb{Z}[t^{\pm 1}]$ and has the canonical (or lower global) basis \mathbf{B} due to Lusztig and Kashiwara. Each element in \mathbf{B} is fixed by the involution ι . We give a review of Lusztig's construction of \mathbf{B} later in Section 7.1.

We equip the tensor product $U_t^+(\mathfrak{g}) \otimes_{\mathbb{Q}(t)} U_t^+(\mathfrak{g})$ with the structure of $\mathbb{Q}(t)$ -algebra by

$$(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = t^{-(\beta_2, \gamma_1)} (x_1 y_1 \otimes x_2 y_2),$$

where $x_i \in U_t^+(\mathfrak{g})_{\beta_i}, y_i \in U_t^+(\mathfrak{g})_{\gamma_i}$ for $i = 1, 2$. There is a unique $\mathbb{Q}(t)$ -algebra homomorphism

$$r: U_t^+(\mathfrak{g}) \rightarrow U_t^+(\mathfrak{g}) \otimes_{\mathbb{Q}(t)} U_t^+(\mathfrak{g})$$

satisfying $r(e_i) = e_i \otimes 1 + 1 \otimes e_i$ for each $i \in I$. Then, we have a unique non-degenerate symmetric bilinear pairing $\langle -, - \rangle$ on $U_t^+(\mathfrak{g})$ satisfying

$$\langle 1, 1 \rangle = 1, \quad \langle e_i, e_j \rangle = \delta_{i,j} / (1 - t^2), \quad \langle x, yz \rangle = \langle r(x), y \otimes z \rangle$$

for any $x, y, z \in U_t^+(\mathfrak{g})$, where $\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle := \langle x_1, y_1 \rangle \cdot \langle x_2, y_2 \rangle$. Let ι' be the involution of $U_t^+(\mathfrak{g})$ dual to ι with respect to $\langle -, - \rangle$. By definition, it satisfies $\langle \iota'(x), y \rangle = \overline{\langle x, \iota(y) \rangle}$ for any $x, y \in U_t^+(\mathfrak{g})$, where $\overline{f(t)} := f(t^{-1})$ for $f(t) \in \mathbb{Q}(t)$. It is known that

$$(4.1) \quad \iota'(xy) = t^{(\beta, \gamma)} \iota'(y) \iota'(x)$$

holds for any $x \in U_t^+(\mathfrak{g})_\beta$ and $y \in U_t^+(\mathfrak{g})_\gamma$.

Let $A_t[N]_{\mathbb{Z}[t^{\pm 1}]}$ be the dual of $U_t^+(\mathfrak{g})_{\mathbb{Z}[t^{\pm 1}]}$, that is,

$$A_t[N]_{\mathbb{Z}[t^{\pm 1}]} = \{x \in U_t^+(\mathfrak{g}) \mid \langle x, U_t^+(\mathfrak{g})_{\mathbb{Z}[t^{\pm 1}]} \rangle \subset \mathbb{Z}[t^{\pm 1}]\}.$$

This is a $\mathbb{Z}[t^{\pm 1}]$ -subalgebra of $U_t^+(\mathfrak{g})$, endowed with the dual canonical basis \mathbf{B}^* . Each element of \mathbf{B}^* is fixed by the dual involution ι' . The algebra $A_t[N]_{\mathbb{Z}[t^{\pm 1}]}$ is specialized at $t = 1$ to a commutative ring, identical to the coordinate ring of the (pro-)unipotent group N associated with the positive part of \mathfrak{g} . We call $A_t[N]_{\mathbb{Z}[t^{\pm 1}]}$ the quantum unipotent coordinate ring.

Fix $w \in W$. We choose a reduced word $\mathbf{i} = (i_1, \dots, i_\ell) \in I^\ell$ for w , that is, we have $w = s_{i_1} \cdots s_{i_\ell}$ and $\ell = \ell(w)$. In what follows, we set

$$J := \{j \in \mathbb{Z} \mid 1 \leq j \leq \ell\}.$$

For each $j \in J$, we define a real positive root $\alpha_{\mathbf{i}, j} \in \mathbb{R}^+$ by

$$\alpha_{\mathbf{i}, j} := s_{i_1} \cdots s_{i_{j-1}} \alpha_{i_j}.$$

Then, we have $\{\alpha_{\mathbf{i}, j} \mid j \in J\} = \mathbb{R}^+ \cap w(-\mathbb{R}^+)$. Correspondingly, we define the root vector $E_{\mathbf{i}, j}$ and its dual $E_{\mathbf{i}, j}^*$ for each $k \in J$ to be the elements of $U_t^+(\mathfrak{g})_{\alpha_{\mathbf{i}, j}}$ given by

$$E_{\mathbf{i}, j} := T_{i_1} \cdots T_{i_{j-1}}(e_{i_j}), \quad E_{\mathbf{i}, j}^* := (1 - t^2)E_{\mathbf{i}, j},$$

where T_i denotes Lusztig's braid symmetry ($= T'_{i, -1}$ in Lusztig's notation, see [49, 37.1.3] for its precise definition). We have $E_{\mathbf{i}, j} \in U_t^+(\mathfrak{g})_{\mathbb{Z}[t^{\pm 1}]}$ and $E_{\mathbf{i}, j}^* \in A_t[N]_{\mathbb{Z}[t^{\pm 1}]}$.

Let $A_t[N(w)]_{\mathbb{Z}[t^{\pm 1}]}$ denote the $\mathbb{Z}[t^{\pm 1}]$ -subalgebra of $A_t[N]_{\mathbb{Z}[t^{\pm 1}]}$ generated by $\{E_{\mathbf{i}, j}^* \mid j \in J\}$. As the notation suggests, this is independent of the choice of reduced word \mathbf{i} , and can be thought of as the quantum coordinate ring of the unipotent group $N(w)$ corresponding to the finite-dimensional nilpotent Lie subalgebra $\bigoplus_{\alpha \in \mathbb{R}^+ \cap w(-\mathbb{R}^+)} \mathfrak{g}_\alpha$ of \mathfrak{g} .

For each $\mathbf{d} = (d_j)_{j \in J} \in \mathbb{N}^{\oplus J}$, we define

$$(4.2) \quad E_{\mathbf{i}}^*(\mathbf{d}) := t^{\sum_{j \in J} d_j (d_j - 1)/2} \prod_{j \in J}^{\leftarrow} (E_{\mathbf{i}, j}^*)^{d_j}.$$

Then, the set $\{E_{\mathbf{i}}^*(\mathbf{d}) \mid \mathbf{d} \in \mathbb{N}^{\oplus J}\}$ forms a free $\mathbb{Z}[t^{\pm 1}]$ -basis of $A_t[N(w)]_{\mathbb{Z}[t^{\pm 1}]}$, called the dual PBW basis associated to the reduced word \mathbf{i} .

Theorem 4.1 ([46, Theorem 4.29]). *There exists a unique free $\mathbb{Z}[t^{\pm 1}]$ -basis $\{B_{\mathbf{i}}^*(\mathbf{d}) \mid \mathbf{d} \in \mathbb{N}^{\oplus J}\}$ of $A_t[N(w)]_{\mathbb{Z}[t^{\pm 1}]}$ satisfying $\iota' B_{\mathbf{i}}^*(\mathbf{d}) = B_{\mathbf{i}}^*(\mathbf{d})$ and*

$$E_{\mathbf{i}}^*(\mathbf{d}) = B_{\mathbf{i}}^*(\mathbf{d}) + \sum_{\mathbf{d}' \prec \mathbf{d}} c_{\mathbf{i}}[\mathbf{d}; \mathbf{d}'] B_{\mathbf{i}}^*(\mathbf{d}') \quad \text{for some } c_{\mathbf{i}}[\mathbf{d}, \mathbf{d}'] \in t\mathbb{Z}[t],$$

for each $\mathbf{d} \in \mathbb{N}^{\oplus J}$, where \preceq is the bi-lexicographic ordering. Moreover, we have

$$\mathbf{B}^*(w) := \mathbf{B}^* \cap A_t[N(w)]_{\mathbb{Z}[t^{\pm 1}]} = \{B_{\mathbf{i}}^*(\mathbf{d}) \mid \mathbf{d} \in \mathbb{N}^{\oplus J}\}.$$

In particular, we have $E_{\mathbf{i}, j}^* = E_{\mathbf{i}}^*(\boldsymbol{\delta}_j) = B_{\mathbf{i}}^*(\boldsymbol{\delta}_j) \in \mathbf{B}^*(w)$ for each $j \in J$.

4.3. Symmetric quiver Hecke algebra. Fix $\beta = \sum_{i \in I} b_i \alpha_i \in \mathbb{Q}^+$ and set $|\beta| := \sum_{i \in I} b_i$. Consider the finite set

$$I^\beta := \{\nu = (\nu_1, \dots, \nu_{|\beta|}) \in I^\beta \mid \alpha_{\nu_1} + \dots + \alpha_{\nu_{|\beta|}} = \beta\}.$$

The symmetric group $\mathfrak{S}_{|\beta|}$ acts on I^β by place permutations. We write $\sigma_k \in \mathfrak{S}_{|\beta|}$ for the transposition of k and $k+1$ for each $1 \leq k < |\beta|$. Let \mathbb{k} be a field of characteristic 0. For each $i, j \in I$, we define a polynomial $Q_{ij}(u, v) \in \mathbb{k}[u, v]$ by

$$Q_{ij}(u, v) = \delta(i \neq j)(u - v)^{-c_{ij}}.$$

Definition 4.2. Let $\beta \in \mathbb{Q}^+$ as above. The symmetric quiver Hecke algebra H_β is the \mathbb{Z} -graded \mathbb{k} -algebra presented by the three kinds of generators $\{e(\nu) \mid \nu \in I^\beta\}$, $\{x_1, \dots, x_{|\beta|}\}$, $\{\tau_1, \dots, \tau_{|\beta|-1}\}$ and the following relations:

$$\begin{aligned} e(\nu)e(\nu') &= \delta_{\nu, \nu'}e(\nu), \quad \sum_{\nu \in I^\beta} e(\nu) = 1, \quad x_k x_l = x_l x_k, \quad x_k e(\nu) = e(\nu) x_k, \\ \tau_k e(\nu) &= e(\sigma_k \nu) \tau_k, \quad \tau_k \tau_l = \tau_l \tau_k \quad \text{if } |k - l| > 1, \quad \tau_k^2 e(\nu) = Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) e(\nu), \\ &(\tau_k x_l - x_{\sigma_k(l)} \tau_k) e(\nu) = \delta_{\nu_k, \nu_{k+1}} (\delta_{l, k+1} - \delta_{l, k}) e(\nu), \\ &(\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k) e(\nu) = \delta_{\nu_k, \nu_{k+2}} \frac{Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) - Q_{\nu_k, \nu_{k+1}}(x_{k+2}, x_{k+1})}{x_k - x_{k+2}} e(\nu). \end{aligned}$$

We endow H_β with a \mathbb{Z} -grading by

$$\deg(e(\nu)) = 0, \quad \deg(x_k) = 2, \quad \deg(\tau_k e(\nu)) = -c_{\nu_k, \nu_{k+1}}.$$

We denote by \mathcal{M}_β the category of left H_β -modules, and by $\mathcal{M}_\beta^\bullet$ the category of graded left H_β -modules (whose morphisms are homogeneous). We also denote by $\mathcal{M}_{f, \beta} \subset \mathcal{M}_\beta$ and $\mathcal{M}_{f, \beta}^\bullet \subset \mathcal{M}_\beta^\bullet$ the full subcategories of finite-dimensional modules.

There is an anti-involution of H_β given by

$$e(\nu) \mapsto e(\nu^{\text{op}}), \quad x_k \mapsto x_{|\beta|-k+1}, \quad \tau_k \mapsto \tau_{|\beta|-k},$$

where ν^{op} denote the opposite sequence of ν . For a (graded) H_β -module M , we equip the (graded) dual vector space M^\vee with the structure of left H_β -module by twisting the natural right module structure with the above anti-involution. We say that $M \in \mathcal{M}_{f, \beta}^\bullet$ is self-dual if $M \simeq M^\vee$ as graded H_β -modules. Every simple module in $\mathcal{M}_{f, \beta}^\bullet$ is known to be self-dual after a grading shift.

For $\beta, \beta' \in \mathbb{Q}^+$, we consider an idempotent

$$e(\beta, \beta') := \sum_{\nu \in I^\beta, \nu' \in I^{\beta'}} e(\nu * \nu') \in H_{\beta + \beta'},$$

where $\nu * \nu' \in I^{\beta + \beta'}$ is the concatenation of the sequences ν and ν' . Similarly, for $\beta_1, \dots, \beta_n \in \mathbb{Q}^+$, we define $e(\beta_1, \dots, \beta_n) \in H_{\beta_1 + \dots + \beta_n}$.

We regard $H_{\beta + \beta'} e(\beta, \beta')$ as a graded $(H_{\beta + \beta'}, H_\beta \otimes H_{\beta'})$ -bimodule in a natural way. For $M \in \mathcal{M}_\beta$ and $N \in \mathcal{M}_{\beta'}$, we define the convolution product $M \star N \in \mathcal{M}_{\beta + \beta'}$ by

$$M \star N := H_{\beta + \beta'} e(\beta, \beta') \otimes_{H_\beta \otimes H_{\beta'}} (M \otimes N).$$

It endows the category $\mathcal{M} := \bigoplus_{\beta \in \mathbb{Q}^+} \mathcal{M}_\beta$ (resp. $\mathcal{M}^\bullet := \bigoplus_{\beta \in \mathbb{Q}^+} \mathcal{M}_\beta^\bullet$) with a structure of \mathbb{k} -linear monoidal category (resp. graded monoidal category). The subcategories $\mathcal{M}_f :=$

$\bigoplus_{\beta \in \mathbb{Q}^+} \mathcal{M}_{f,\beta}$ and $\mathcal{M}_f^\bullet := \bigoplus_{\beta \in \mathbb{Q}^+} \mathcal{M}_{f,\beta}^\bullet$ are closed under these monoidal structures. For any $M_k \in \mathcal{M}_{\beta_k}$, $k = 1, \dots, n$, we have a natural isomorphism

$$M_1 \star \cdots \star M_n \simeq H_\beta e(\beta_1, \dots, \beta_n) \otimes_{H_{\beta_1, \dots, \beta_n}} (M_1 \otimes \cdots \otimes M_n),$$

where $\beta = \sum_{k=1}^n \beta_k$ and $H_{\beta_1, \dots, \beta_n} := H_{\beta_1} \otimes \cdots \otimes H_{\beta_n}$.

Let $K(\mathcal{M}_f^\bullet)$ denote the Grothendieck ring of the category \mathcal{M}_f^\bullet . The following celebrated result is due to Khovanov-Lauda [45], Rouquier [54, 55] and Varagnolo-Vasserot [59].

Theorem 4.3 ([45, 54, 55, 59]). *There is an isomorphism of algebras*

$$(4.3) \quad K(\mathcal{M}_f^\bullet) \simeq A_t[N]_{\mathbb{Z}[t^{\pm 1}]},$$

where the multiplication of $t^{\pm 1}$ on the right hand side corresponds to the grading shift $\langle \pm 1 \rangle$ on the left hand side. It induces a bijection between the set of the classes of self-dual simple modules and the dual canonical basis \mathbf{B}^* .

Let $\mathcal{M}_f^{\text{nilp}} \subset \mathcal{M}_f$ be the full subcategory of modules on which the element x_k acts nilpotently for all k . This is identical to the monoidal Serre subcategory generated by the image of the forgetful functor $\mathcal{M}_f^\bullet \rightarrow \mathcal{M}_f$. We think of \mathbb{Z} as a $\mathbb{Z}[t^{\pm 1}]$ -algebra through the specialization $\mathbb{Z}[t^{\pm 1}] \rightarrow \mathbb{Z}$ at $t = 1$. Let $A_t[N]|_{t=1} := A_t[N]_{\mathbb{Z}[t^{\pm 1}]} \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}$. This is a commutative ring endowed with the specialized dual canonical basis $\mathbf{B}^*|_{t=1} := \mathbf{B}^* \otimes 1$.

Corollary 4.4. *There is an isomorphism of algebras*

$$K(\mathcal{M}_f^{\text{nilp}}) \simeq A_t[N]|_{t=1}$$

through which the basis formed by the classes of simple modules correspond to the specialized dual canonical basis $\mathbf{B}^*|_{t=1}$.

Let $\widehat{H} := \bigoplus_{\beta \in \mathbb{Q}^+} \widehat{H}_\beta$, where \widehat{H}_β denotes the completion of H_β along the grading. The (non-unital) algebra \widehat{H} and the $(\widehat{H}, \widehat{H} \otimes \widehat{H})$ -bimodule $\bigoplus_{\beta, \beta'} \widehat{H}_{\beta+\beta'} e(\beta, \beta')$ satisfy the assumptions in Section 2.1. Thus the category $\widehat{H}\text{-mod}$ is a monoidal category. In addition, we have a natural isomorphism $\widehat{H}\text{-mod} \simeq \mathcal{M}_f^{\text{nilp}}$ of monoidal categories, and hence

$$(4.4) \quad K(\widehat{H}\text{-mod}) \simeq A_t[N]|_{t=1}$$

through which the basis formed by the classes of simple modules corresponds to the specialized dual canonical basis $\mathbf{B}^*|_{t=1}$. In what follows, we identify $\mathcal{M}_f^{\text{nilp}}$ with $\widehat{H}\text{-mod}$.

4.4. Category \mathcal{C}_w and PBW-theory. Let us fix an element $w \in W$. We define the category \mathcal{C}_w^\bullet (resp. \mathcal{C}_w) to be the Serre subcategory of \mathcal{M}_f^\bullet (resp. $\widehat{H}\text{-mod}$) generated by the simple modules corresponding to the elements of $t^{\mathbb{Z}}\mathbf{B}^*(w)$ (resp. $\mathbf{B}^*(w)|_{t=1}$) under the isomorphism (4.3) (resp. (4.4)). Theorem 4.1 and Theorem 4.3 (resp. Corollary 4.3) tell us that the category \mathcal{C}_w^\bullet (resp. \mathcal{C}_w) is closed under the monoidal structure \star and that we have the isomorphism

$$K(\mathcal{C}_w^\bullet) \simeq A_t[N(w)]_{\mathbb{Z}[t^{\pm 1}]} \quad (\text{resp. } K(\mathcal{C}_w) \simeq A_t[N(w)]|_{t=1}),$$

where $A_t[N(w)]|_{t=1} := A_t[N(w)]_{\mathbb{Z}[t^{\pm 1}]} \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}$ is the specialization at $t = 1$.

Now, let us choose a reduced word $\mathbf{i} = (i_1, \dots, i_\ell)$ for w . For each $j \in J$, let $L_{\mathbf{i},j}^\bullet \in \mathcal{C}_w^\bullet$ (resp. $L_{\mathbf{i},j}$) be a simple module whose isomorphism class corresponds to the dual root vector $E_{\mathbf{i},j}^*$ (resp. $E_{\mathbf{i},j}^*|_{t=1}$) through the above isomorphism (4.3) (resp. (4.4)). The module $L_{\mathbf{i},j}$ is obtained from $L_{\mathbf{i},j}^\bullet$ by forgetting the grading. These modules are called *cuspidal modules*. We recall the following fundamental result due to Kashiwara-Kim-Oh-Park [39].

Theorem 4.5 ([39]). *For any $w \in W$ and any reduced word \mathbf{i} for w , the pair $(\{L_{\mathbf{i},j}\}_{j \in J}, \preceq)$ gives a PBW-theory of the monoidal category \mathcal{C}_w in the sense of Definition 2.4, where \preceq is the bi-lexicographic ordering on the set $\mathbb{N}^{\oplus J}$.*

In what follows, given a reduced word \mathbf{i} for w , we write $M_{\mathbf{i}}(\mathbf{d})$ and $L_{\mathbf{i}}(\mathbf{d})$ respectively for the standard module and its simple head, and write $M_{\mathbf{i}}(\epsilon)$ for the mixed tensor product, associated with the PBW-theory in Theorem 4.5. Note that the class $[L_{\mathbf{i}}(\mathbf{d})]$ corresponds to the specialized dual canonical basis element $B_{\mathbf{i}}^*(\mathbf{d})|_{t=1}$ under the isomorphism (4.4).

4.5. R -matrices and deformed PBW-theory. Let $\beta \in \mathbb{Q}^+$. We define an element φ_k of H_{β} for each $1 \leq k < |\beta|$ by

$$\varphi_k e(\nu) := \delta_{\nu_k, \nu_{k+1}} (\tau_k x_k - x_k \tau_k) e(\nu) + (1 - \delta_{\nu_k, \nu_{k+1}}) \tau_k e(\nu).$$

Since $\{\varphi_k\}_{1 \leq k < |\beta|}$ satisfy the braid relations, we get a well-defined element φ_g for each $g \in \mathfrak{S}_{|\beta|}$ by composing them so that we have $\varphi_g = \varphi_{i_1} \cdots \varphi_{i_n}$ if $g = \sigma_{i_1} \cdots \sigma_{i_n}$ is a reduced expression. For any $M \in \mathcal{M}_{\beta}$ and $M' \in \mathcal{M}_{\beta'}$, we have the unique $H_{\beta+\beta'}$ -homomorphism

$$R_{M, M'} : M \star M' \rightarrow M' \star M$$

extending the $H_{\beta} \otimes H_{\beta'}$ -homomorphism $M \otimes M' \rightarrow M' \star M$ given by $v \otimes v' \mapsto \varphi_{\sigma} e(\beta', \beta)(v' \otimes v)$, where $\sigma \in \mathfrak{S}_{|\beta|+|\beta'|}$ is the permutation defined by $\sigma(k) := k + (-1)^{\delta(k > |\beta'|)} |\beta'|$. Note that $R_{M, M'}$ also yields an \widehat{H} -homomorphism if M, M' are \widehat{H} -modules. By construction, they satisfy the quantum Yang-Baxter equation, that is, we have

$$(4.5) \quad \begin{aligned} & (R_{M', M''} \star \text{id}_M) \circ (\text{id}_{M'} \star R_{M, M''}) \circ (R_{M, M''} \star \text{id}_{M''}) \\ & = (\text{id}_{M''} \star R_{M, M'}) \circ (R_{M, M''} \star \text{id}_{M'}) \circ (\text{id}_M \star R_{M', M''}) \end{aligned}$$

for any \widehat{H} -modules M, M', M'' .

Next, we introduce deformations. Let z be an indeterminate and set $\mathbb{O} = \mathbb{k}[[z]]$, $\mathbb{K} = \mathbb{k}((z))$ as before. For $M \in \widehat{H}\text{-mod}$, we define its deformation $M_{a(z)}$ with $a(z) \in z\mathbb{O}$ to be the \mathbb{O} -module $M \otimes \mathbb{O}$ equipped with the \widehat{H} -action given by

$$(4.6) \quad \begin{aligned} e(\nu) \cdot (v \otimes f(z)) & := e(\nu) v \otimes f(z), \\ x_k \cdot (v \otimes f(z)) & := x_k v \otimes f(z) + v \otimes a(z) f(z), \\ \tau_l \cdot (v \otimes f(z)) & := \tau_l v \otimes f(z) \end{aligned}$$

for any $v \in M$ and $f(z) \in \mathbb{O}$. Therefore, $M_{a(z)}$ is an $\widehat{H}_{\mathbb{O}}$ -module such that $(M_{a(z)})_0 \simeq M$.

Although the following result is essentially due to [41, §2.3], we shall give a proof for completeness. Recall that a simple module $M \in \widehat{H}\text{-mod}$ is said to be *real* if $M \star M$ is simple.

Lemma 4.6. *Let M, N be simple modules in $\widehat{H}\text{-mod}$, and $a(z), b(z) \in z\mathbb{O}$ with $a(z) \neq b(z)$.*

(1) *We have an isomorphism of $\widehat{H}_{\mathbb{K}}$ -modules*

$$(M_{a(z)} \star_{\mathbb{O}} N_{b(z)})_{\mathbb{K}} \simeq (N_{b(z)} \star_{\mathbb{O}} M_{a(z)})_{\mathbb{K}},$$

induced from $R_{M_{a(z)}, N_{b(z)}}$.

(2) *Assuming that at least one of M and N is real, we have an isomorphism*

$$\text{End}_{\widehat{H}_{\mathbb{K}}}((M_{a(z)} \star_{\mathbb{O}} N_{b(z)})_{\mathbb{K}}) \simeq \mathbb{K}\text{id}.$$

Proof. By [33, Proposition 1.10], for any $M' \in H_\beta\text{-Mod}$ and $N' \in H_{\beta'}\text{-Mod}$, the homomorphism $(R_{N',M'} \circ R_{M',N'})|_{e(\beta,\beta')(M' \otimes N')}$ is given by the multiplication by

$$X := \sum_{\nu \in I^\beta, \nu' \in I^{\beta'}} \left(\prod_{1 \leq k \leq d, 1 \leq l \leq d', \nu_k \neq \nu'_l} Q_{\nu_k, \nu'_l}(x_k, x_{d+l}) \right) e(\nu * \nu').$$

This element X is in $Z_+(H_\beta) \otimes H_{\beta'} + H_\beta \otimes Z_+(H_{\beta'})$, where $Z_+(H_\beta)$ denotes the positive degree part of the center of H_β . Consider the case when $M' = M_{a(z)}$ and $N' = N_{b(z)}$. Since $Z_+(H_\beta)$ acts by zero on a simple module, the action of X on $e(\beta, \beta')(M_{a(z)} \otimes_{\mathbb{O}} N_{b(z)})$ becomes the multiplication by

$$\sum_{\nu \in I^\beta, \nu' \in I^{\beta'}} \left(\prod_{1 \leq k \leq d, 1 \leq l \leq d', \nu_k \neq \nu'_l} (a(z) - b(z))^{-c_{\nu_k, \nu'_l}} \right) e(\nu * \nu') = (a(z) - b(z))^N e(\beta, \beta')$$

for some $N \in \mathbb{N}$. It is invertible after the localization as $a(z) \neq b(z)$. Thus, (1) is proved.

To prove (2), it is enough to show $\text{End}_{\widehat{H}_0}(M_{a(z)} \star_{\mathbb{O}} N_{b(z)}) = \mathbb{O}\text{id}$ as we have

$$\text{End}_{\widehat{H}_{\mathbb{K}}}((M_{a(z)} \star_{\mathbb{O}} N_{b(z)})_{\mathbb{K}}) \simeq \text{End}_{\widehat{H}_0}(M_{a(z)} \star_{\mathbb{O}} N_{b(z)}) \otimes_{\mathbb{O}} \mathbb{K}.$$

For simplicity, put $T := M_{a(z)} \star_{\mathbb{O}} N_{b(z)}$. Then, $T_0 = T/zT \simeq M \star N$. By the assumption and [34, Proposition 3.8], we have

$$(4.7) \quad \text{End}_{\widehat{H}}(T_0) \simeq \text{End}_{\widehat{H}}(M \star N) \simeq \mathbb{k}\text{id}.$$

Let $f \in \text{End}_{\widehat{H}_0}(T)$ be a non-zero homomorphism. There exists a unique integer $s \in \mathbb{N}$ such that $f(T) \subset z^s T$ and $f(T) \not\subset z^{s+1} T$. By (4.7), there exists a unique $c_s \in \mathbb{K}^\times$ such that $(f - c_s z^s \text{id})(T) \subset z^{s+1} T$. Repeating the same argument, we inductively find for any integer $l \geq s$ a scalar $c_l \in \mathbb{k}$ such that $(f - \sum_{k=s}^l c_k z^k \text{id})(T) \subset z^{l+1} T$. Then, we get $f = (\sum_{k \geq s} c_k z^k) \text{id}$ as $\bigcap_{k \in \mathbb{N}} z^k T = \{0\}$, which proves $\text{End}_{\widehat{H}_0}(T) = \mathbb{O}\text{id}$ as desired. \square

Let $(\{L_{\mathbf{i},j}\}_{j \in J}, \preceq)$ be the PBW-theory in Theorem 4.5 associated with a reduced word \mathbf{i} of w . We define a collection $\{\tilde{L}_{\mathbf{i},j}\}_{j \in J}$ of \widehat{H}_0 -modules by

$$(4.8) \quad \tilde{L}_{\mathbf{i},j} := (L_{\mathbf{i},j})_{jz}$$

for each $j \in J$.

Corollary 4.7. *The collection $\{\tilde{L}_{\mathbf{i},j}\}_{j \in J}$ defined in (4.8) gives a normal, consistent, generically commutative deformation of $\{L_{\mathbf{i},j}\}_{j \in J}$ in the sense of Section 2.*

Proof. It is known that the simple module $L_{\mathbf{i},j}$ is real (cf. [39, Proposition 4.2]). Then, it is clear from Lemma 4.6 that the collection $\{L_{\mathbf{i},j}\}_{j \in J}$ gives a generically commutative deformation. Since the renormalized R -matrix R_{ij} in this case is induced from the homomorphism $z^{s_{ij}} R_{\tilde{L}_{\mathbf{i},i}, \tilde{L}_{\mathbf{i},j}}$ with s_{ij} being a uniquely defined integer, the consistency follows from the quantum Yang-Baxter equation (4.5). The normality is proved in [36, Proposition 2.11]. \square

4.6. Quantum unipotent coordinate ring conjecture. We state the analog of the quantum Grothendieck ring conjecture for the quiver Hecke algebras, which we call the quantum unipotent ring conjecture. For this purpose, we need to introduce a renormalization of the dual canonical basis.

Let $t^{1/2}$ be a formal square root of the indeterminate t , and let

$$A_t[N(w)]_{\mathbb{Z}[t^{\pm 1/2}]} := A_t[N(w)]_{\mathbb{Z}[t^{\pm 1}]} \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}[t^{\pm 1/2}].$$

Note that we have $(\beta, \beta) \in 2\mathbb{Z}$ for any $\beta \in \mathbb{Q}$. For a homogeneous element $x \in A_t[N(w)]_{\mathbb{Z}[t^{\pm 1/2}]}$ of degree $\beta \in \mathbb{Q}^+$, we write $\tilde{x} := t^{-(\beta, \beta)/4}x$. In particular, for any reduced word \mathbf{i} for w and $\mathbf{d} = (d_j)_{j \in J} \in \mathbb{N}^{\oplus J}$, we write

$$\tilde{B}_{\mathbf{i}}^*(\mathbf{d}) = t^{-(\beta, \beta)/4}B_{\mathbf{i}}^*(\mathbf{d}), \quad \tilde{E}_{\mathbf{i}}^*(\mathbf{d}) = t^{-(\beta, \beta)/4}E_{\mathbf{i}}^*(\mathbf{d}),$$

where $\beta := \sum_{j \in J} d_j \alpha_{\mathbf{i}, j}$. We define the renormalized involution $\overline{(\cdot)}$ of $A_t[N(w)]_{\mathbb{Z}[t^{\pm 1/2}]}$ by $\overline{x} := t^{-(\beta, \beta)/2} \iota'(x)$ if x is homogeneous of degree β , so that it fixes each renormalized dual canonical basis element $\tilde{B}_{\mathbf{i}}^*(\mathbf{d})$. The identity (4.1) implies that $\overline{(\cdot)}$ is an anti-involution, i.e., we have $\overline{x \cdot y} = \overline{y} \cdot \overline{x}$ for any $x, y \in A_t[N(w)]_{\mathbb{Z}[t^{\pm 1/2}]}$.

Remark 4.8. By [24] (and [34]), the algebra $A_t[N(w)]_{\mathbb{Z}[t^{\pm 1/2}]}$ has the structure of quantum cluster algebra in the sense of [2]. The anti-involution $\overline{(\cdot)}$ coincides with the natural bar-involution of the quantum cluster algebra.

In terms of the renormalized elements, the equation (4.2) is rewritten as

$$\tilde{E}_{\mathbf{i}}^*(\mathbf{d}) = t^{-\sum_{1 \leq j < k \leq \ell} d_j d_k (\alpha_{\mathbf{i}, j}, \alpha_{\mathbf{i}, k})/2} \prod_{j \in J}^{\leftarrow} (\tilde{E}_{\mathbf{i}, j}^*)^{d_j}$$

Comparing with (2.11), we define the skew-symmetric bilinear map $\gamma_{\mathbf{i}}: \mathbb{N}^{\oplus J} \times \mathbb{N}^{\oplus J} \rightarrow \frac{1}{2}\mathbb{Z}$ by

$$\gamma_{\mathbf{i}}(\mathbf{d}, \mathbf{d}') := \frac{1}{2} \sum_{1 \leq j < k \leq \ell} (d_j d'_k - d_k d'_j) (\alpha_{\mathbf{i}, j}, \alpha_{\mathbf{i}, k}).$$

With the consistent deformation $\{\tilde{L}_{\mathbf{i}, j}\}_{j \in J}$ constructed in Section 4.5 and $\gamma = \gamma_{\mathbf{i}}$ defined as above, we obtain the associated bilinear operation $* = *_{\gamma_{\mathbf{i}}}$ on $K(\mathcal{C}_w)_t$. Be aware that it depends on the choice of reduced word \mathbf{i} .

We define a $\mathbb{Z}[t^{\pm 1/2}]$ -linear isomorphism $\phi: K(\mathcal{C}_w)_t \simeq A_t[N(w)]_{\mathbb{Z}[t^{\pm 1/2}]}$ by $\phi([L_{\mathbf{i}}(\mathbf{d})]) = \tilde{B}_{\mathbf{i}}^*(\mathbf{d})$ for all $\mathbf{d} \in \mathbb{N}^{\oplus J}$. Note that this isomorphism ϕ does not depend on the choice of reduced word \mathbf{i} . Clearly, we have $\phi \circ \overline{(\cdot)} = \overline{(\cdot)} \circ \phi$.

Conjecture 4.9 (Quantum Unipotent Coordinate Ring Conjecture). *With a chosen reduced word \mathbf{i} for w , Associativity Conjectures 2.19 and 2.20 hold for $(K(\mathcal{C}_w)_t, *)$, and the linear isomorphism ϕ gives a $\mathbb{Z}[t^{\pm 1/2}]$ -algebra isomorphism from $(K(\mathcal{C}_w)_t, *)$ to quantum unipotent coordinate ring $A_t[N(w)]_{\mathbb{Z}[t^{\pm 1/2}]}$.*

Remark 4.10. If Conjecture 4.9 is true for any reduced word \mathbf{i} for w , it implies that the ring structure $(K(\mathcal{C}_w)_t, *)$ does not depend on the choice of \mathbf{i} .

Assume that Conjecture 4.9 is true for a chosen reduced word \mathbf{i} for a while. Then, we have

$$\phi([M_{\mathbf{i}}(\mathbf{d})]_t) = \tilde{E}_{\mathbf{i}}^*(\mathbf{d})$$

for any $\mathbf{d} \in \mathbb{N}^{\oplus J}$. More generally, for any $d \in \mathbb{N}$ and sequence $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_d) \in J^d$, letting

$$(4.9) \quad \tilde{E}_{\mathbf{i}}^*(\boldsymbol{\epsilon}) := t^{\sum_{1 \leq k < l \leq d} \gamma_{\mathbf{i}}(\delta_{\epsilon_k}, \delta_{\epsilon_l})} \tilde{E}_{\mathbf{i}, \epsilon_1}^* \cdots \tilde{E}_{\mathbf{i}, \epsilon_d}^*,$$

we obtain the equality

$$(4.10) \quad \phi([M_{\mathbf{i}}(\boldsymbol{\epsilon})]_t) = \tilde{E}_{\mathbf{i}}^*(\boldsymbol{\epsilon}).$$

The converse statement is true. Namely, Conjecture 4.9 is equivalent to the following.

Conjecture 4.11. *The equality (4.10) holds for any $d \in \mathbb{N}$ and $\boldsymbol{\epsilon} \in J^d$.*

Remark 4.12. Note that Conjecture 4.11 also implies Duality Conjecture 2.25 in this case. Indeed, for each $\epsilon \in J^d$, we have $\overline{\phi([M_{\mathbf{i}}(\epsilon)]_t)} = \overline{\tilde{E}_{\mathbf{i}}^*(\epsilon)} = \tilde{E}_{\mathbf{i}}^*(\epsilon^{\text{op}}) = \phi([M_{\mathbf{i}}(\epsilon^{\text{op}})]_t)$, and hence $\overline{[M_{\mathbf{i}}(\epsilon)]_t} = [M_{\mathbf{i}}(\epsilon^{\text{op}})]_t$.

Definition 4.13. Let Q be a quiver. We understand it as a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is the set of vertices, Q_1 is the set of arrows and s (resp. t) is the map $Q_1 \rightarrow Q_0$ assigning an arrow with its source (resp. target). We say that a quiver Q without edge loops is of type \mathfrak{g} if $Q_0 = I$ and, for any $i, j \in I$ with $i \neq j$, we have

$$-c_{ij} = -c_{ji} = \#\{a \in Q_1 \mid \{s(a), t(a)\} = \{i, j\}\}.$$

A vertex i is called a *source* (resp. *sink*) of the quiver Q if there is no arrow $a \in Q_1$ with $i = t(a)$ (resp. $i = s(a)$). A sequence $\mathbf{i} = (i_1, i_2, \dots, i_\ell)$ in I is said to be *adapted to Q* if the vertex i_k is a source of the quiver $s_{i_{k-1}} \cdots s_{i_2} s_{i_1} Q$ for any $1 \leq k \leq \ell$, where $s_i Q$ denotes the quiver obtained from Q by inverting the orientations of all the arrows incident to i .

When our reduced word \mathbf{i} for w is adapted to a quiver Q of type \mathfrak{g} , we have a geometric realization of the deformed PBW theory $\{\tilde{L}_{\mathbf{i}, j}\}_{j \in J}$ and their mixed products (see Section 7 below). In this case, we establish that our conjectures are true with the help of geometry.

Theorem 4.14. *When our reduced word \mathbf{i} for w is adapted to a quiver Q of type \mathfrak{g} , Conjecture 4.11 holds, and hence Conjecture 4.9 holds.*

A proof will be given in Section 7 below.

5. PRELIMINARIES FOR GEOMETRIC PROOFS

In the remaining part of this paper, we prove our main Theorems 3.26 and 4.14 with the help of geometry. In this section, before going into individual discussions, we recall some preliminary facts commonly used in the proofs. They are based on Grojnowski's unpublished note [26], "Fundamental Example" of Bernstein-Lunts [3] (also outlined in [26]), and the hyperbolic localization theorem due to Braden [6].

5.1. Hard Lefschetz property. Let \mathbb{k} be a field and z an indeterminate. For a $\mathbb{k}[z]$ -module M , we often write $z_M: M \rightarrow M$ for the endomorphism given by the action of z . We endow the polynomial ring $\mathbb{k}[z]$ with a \mathbb{Z} -grading by setting $\deg z := 2$. Let $\mathbb{k}[z]\text{-Mod}_{\mathbb{Z}}$ be the category of \mathbb{Z} -graded $\mathbb{k}[z]$ -modules. For $M \in \mathbb{k}[z]\text{-Mod}_{\mathbb{Z}}$, its n -th graded piece is denoted by M^n . For $k \in \mathbb{Z}$, the grading shift functor $\langle k \rangle$ is defined by $(M \langle k \rangle)^n = M^{n+k}$ for any $n \in \mathbb{Z}$. For each $n \in \mathbb{Z}$, we set $M^{\geq n} := \bigoplus_{k \geq n} M^k$, which is a graded $\mathbb{k}[z]$ -submodule of M .

Definition 5.1. We say that a module $M \in \mathbb{k}[z]\text{-Mod}_{\mathbb{Z}}$ satisfies the *hard Lefschetz property* if the endomorphism z_M^n restricts to a \mathbb{k} -linear isomorphism $M^{-n} \xrightarrow{\sim} M^n$ for any $n \in \mathbb{N}$.

Lemma 5.2. *Let M be a finitely generated \mathbb{Z} -graded $\mathbb{k}[z]$ -module satisfying the hard Lefschetz property. For any $n \in \mathbb{Z}$, we have*

$$M^{\geq n} = \sum_{k, l \in \mathbb{N}: k-l=n} \text{Im}(z_M^k) \cap \text{Ker}(z_M^{l+1}).$$

Proof. Note that a finitely generated \mathbb{Z} -graded $\mathbb{k}[z]$ -module is bounded from below with all its graded pieces being finite-dimensional. Since M satisfies the hard Lefschetz property, it is finite-dimensional and decomposes into a finite direct sum of the modules of the form $M_p := (\mathbb{C}[z]/z^{p+1}\mathbb{C}[z])\langle p \rangle$ for various $p \in \mathbb{N}$. Thus, it suffices to prove the assertion when

$M = M_p$. From the definition of M_p , we have $\text{Im}(z_{M_p}^k) = z^k M_p$ and $\text{Ker}(z_{M_p}^{l+1}) = z^{p-l} M_p$. Therefore, we have

$$\sum_{k-l=n} \text{Im}(z_{M_p}^k) \cap \text{Ker}(z_{M_p}^{l+1}) = \sum_{0 \leq k \leq p+n} z^{\max(k, p+n-k)} M_p = z^{\lceil (p+n)/2 \rceil} M_p.$$

Observe that $z^k M_p = M_p^{\geq -p+2k}$ for any $k \in \mathbb{N}$. If $p+n$ is even, we have $2\lceil (p+n)/2 \rceil = p+n$ and hence $z^{\lceil (p+n)/2 \rceil} M_p = M_p^{\geq n}$, which implies the assertion. If $p+n$ is odd, we have $2\lceil (p+n)/2 \rceil = p+n+1$ and hence $z^{\lceil (p+n)/2 \rceil} M_p = M_p^{\geq n+1}$, which also implies the assertion as $M_p^n = 0$ in this case. \square

Assume that there is a short exact sequence

$$0 \rightarrow M^! \rightarrow M^* \rightarrow N\langle 1 \rangle \rightarrow 0$$

in $\mathbb{k}[z]\text{-Mod}_{\mathbb{Z}}$ satisfying the following three conditions:

- (i) The modules $M^!$ and M^* are free of finite rank over $\mathbb{k}[z]$;
- (ii) Setting $\bar{M}^! := M^!/zM^!$ and $\bar{M}^* := M^*/zM^*$, we have $(\bar{M}^!)^{-n} = 0$ and $(\bar{M}^*)^n = 0$ for any $n > 0$;
- (iii) The module N satisfies the hard Lefschetz property.

In what follows, we regard $M^!$ as a $\mathbb{k}[z]$ -submodule of M^* through the given injection.

Lemma 5.3 ([26]). *With the above assumption, the graded $\mathbb{k}[z]$ -module $L := M^*/zM^!$ satisfies the hard Lefschetz property.*

Proof. From the condition (i), the endomorphism z_{M^*} is injective and hence we have $\bar{M}^! \simeq \text{Ker}(z_L)$. In addition, we have the natural isomorphisms $\bar{M}^* \simeq \text{Coker}(z_L)$ and $N\langle 1 \rangle \simeq L/\bar{M}^!$. These isomorphisms give the exact sequences

$$0 \rightarrow \bar{M}^! \rightarrow L \xrightarrow{a} N\langle 1 \rangle \rightarrow 0 \quad \text{and} \quad 0 \rightarrow N\langle -1 \rangle \xrightarrow{b} L \rightarrow \bar{M}^* \rightarrow 0$$

in $\mathbb{k}[z]\text{-Mod}_{\mathbb{Z}}$ satisfying $b\langle 2 \rangle \circ a = z_L: L \rightarrow L\langle 2 \rangle$. By the condition (ii), for any $n > 0$, the homomorphisms a and b induce the \mathbb{k} -linear isomorphisms $a_{-n}: L^{-n} \xrightarrow{\sim} N^{-n+1}$ and $b_n: N^{n-1} \xrightarrow{\sim} L^n$ respectively. Now, for each $n > 0$, we have the commutative diagram

$$\begin{array}{ccccc} L^{-n} & \xrightarrow{z_L^{n-1}} & L^{n-2} & \xrightarrow{z_L} & L^n \\ a_{-n} \downarrow & & a_{n-2} \downarrow & \nearrow b_n & \\ N^{-n+1} & \xrightarrow{z_N^{n-1}} & N^{n-1} & & \end{array}$$

with the bottom arrow being an isomorphism by the condition (iii). Therefore, the \mathbb{k} -linear map $z_L^n = z_L \circ z_L^{n-1}$ gives an isomorphism $L^{-n} \xrightarrow{\sim} L^n$ for any $n > 0$. \square

5.2. Notation around equivariant sheaves. In this subsection, we assume that \mathbb{k} is a field of characteristic zero. Let G be a complex linear algebraic group. By a G -variety, we mean a complex algebraic variety endowed with an algebraic action of G . For a G -variety X , let $D_G^b(X, \mathbb{k})$ denote the G -equivariant bounded derived category of constructible complexes of sheaves of \mathbb{k} -vector spaces on X in the sense of Bernstein-Lunts [3]. It is a \mathbb{k} -linear triangulated category, whose shift is denoted by $[1]$. It is endowed with the perverse t -structure, whose heart $\text{Perv}_G(X, \mathbb{k})$ is the category of the G -equivariant perverse sheaves. When G is a trivial group $G = \{1\}$, we simply write $D^b(X, \mathbb{k})$ and $\text{Perv}(X, \mathbb{k})$ dropping the symbol G .

For $\mathcal{F}, \mathcal{G} \in D_G^b(X, \mathbb{k})$, we abbreviate $\text{Hom}_{D_G^b(X, \mathbb{k})}(\mathcal{F}, \mathcal{G})$ as $\text{Hom}_G(\mathcal{F}, \mathcal{G})$, and for $n \in \mathbb{Z}$, we set $\text{Hom}_G^n(\mathcal{F}, \mathcal{G}) := \text{Hom}_G(\mathcal{F}, \mathcal{G}[n])$. Letting $\underline{\mathbb{k}}_X$ be the constant \mathbb{k} -sheaf on X , we set

$H_G^n(\mathcal{F}) := \text{Hom}_G^n(\mathbb{k}_X, \mathcal{F})$. The \mathbb{Z} -graded \mathbb{k} -vector spaces $\text{Hom}_G^\bullet(\mathcal{F}, \mathcal{G}) := \bigoplus_{n \in \mathbb{Z}} \text{Hom}_G^n(\mathcal{F}, \mathcal{G})$ and $H_G^\bullet(\mathcal{F}) := \bigoplus_{n \in \mathbb{Z}} H_G^n(\mathcal{F})$ are graded modules over $H_G^\bullet(\text{pt}, \mathbb{k}) = H_G^\bullet(\mathbb{k}_{\text{pt}})$ (the G -equivariant cohomology ring of a point).

The Verdier duality of $D_G^b(X, \mathbb{k})$ is denoted by \mathbb{D}_X , or simply by \mathbb{D} . For an equivariant morphism f of G -varieties, we use the symbols $f^*, f_*, f^!, f_!$ for the associated functors of the G -equivariant derived categories. Given a homomorphism of algebraic groups $\varphi: G' \rightarrow G$, we regard X as an G' -variety through φ . Then, we have a natural functor $\text{Res}_\varphi: D_G^b(X, \mathbb{k}) \rightarrow D_{G'}^b(X, \mathbb{k})$, which commutes with the Verdier duality and all the functors $f^*, f_*, f^!, f_!$ above, see [3, Proposition 7.2]. When φ is the trivial inclusion $\{1\} \hookrightarrow G$, the functor Res_φ is identical to the forgetful functor $\text{For}: D_G^b(X, \mathbb{k}) \rightarrow D^b(X, \mathbb{k})$. When G is connected, For induces a full embedding $\text{Perv}_G(X, \mathbb{k}) \hookrightarrow \text{Perv}(X, \mathbb{k})$, see [1, Proposition 6.2.15], through which we think of $\text{Perv}_G(X, \mathbb{k})$ as a full subcategory of $\text{Perv}(X, \mathbb{k})$.

We denote by $\text{IC}(X, \mathbb{k})$ the intersection cohomology complex of X . This is a simple object of $\text{Perv}_G(X, \mathbb{k})$. We set $\text{IH}_G^\bullet(X, \mathbb{k}) := H_G^\bullet(\text{IC}(X, \mathbb{k}))$.

5.3. “Fundamental Example” of Bernstein-Lunts. In the remainder of this section, we consider the following situation. Let E be a finite dimensional \mathbb{C} -vector space endowed with a linear action of a complex algebraic torus T . Let $X^*(T)$ (resp. $X_*(T)$) denote the character (resp. cocharacter) lattice of T . We have the weight space decomposition $E = \bigoplus_{\lambda \in X^*(T)} E_\lambda$. We assume that the T -action on E is *attractive*, that is,

$$(5.1) \quad \text{there exists } \rho^\vee \in X_*(T) \text{ such that } \langle \rho^\vee, \lambda \rangle > 0 \text{ for any } \lambda \in \text{wt}(E),$$

where $\text{wt}(E) := \{\lambda \in X^*(T) \mid E_\lambda \neq \{0\}\}$ and $\langle -, - \rangle: X_*(T) \times X^*(T) \rightarrow \mathbb{Z}$ denotes the natural pairing. It particularly implies that the T -fixed locus E^T consists of a single point $0 \in E$ and $\lim_{s \rightarrow 0} \rho^\vee(s) \cdot x = 0$ for all $x \in E$.

Let $i: \{0\} \rightarrow E$ and $p: E \rightarrow \{0\}$ be the obvious morphisms. Applying p_* and $p_!$ respectively to the adjunction morphisms $\text{id} \rightarrow i_* i^*$ and $i_! i^! \rightarrow \text{id}$, we get the natural morphisms

$$(5.2) \quad p_* \rightarrow i^* \quad \text{and} \quad i^! \rightarrow p_!$$

of functors from $D_T^b(E, \mathbb{k})$ to $D_T^b(\{0\}, \mathbb{k})$.

Proposition 5.4 ([14, Proposition 2.3]). *The morphisms in (5.2) are isomorphisms.*

In what follows, we fix a cocharacter ρ^\vee satisfying (5.1) and regard E as a \mathbb{C}^\times -variety through $\rho^\vee: \mathbb{C}^\times \rightarrow T$. We make an identification $H_{\mathbb{C}^\times}^\bullet(\text{pt}, \mathbb{k}) = \mathbb{k}[z]$ with $\deg z = 2$. Note that the condition (5.1) particularly implies that the stabilizer in \mathbb{C}^\times of a point $x \in E \setminus \{0\}$ is always finite. For any closed \mathbb{C}^\times -subvariety $X \subset E$, we consider the quotient $\mathbb{P}_{\rho^\vee} X := (X \setminus \{0\})/\mathbb{C}^\times$, which is projective as a closed subvariety of the weighted projective space $\mathbb{P}_{\rho^\vee} E$.

Proposition 5.5 ([3]). *For any \mathbb{C}^\times -stable closed variety X of E , we have an isomorphism*

$$\text{IH}_{\mathbb{C}^\times}^\bullet(X \setminus \{0\}, \mathbb{k}) \simeq \text{IH}^\bullet(\mathbb{P}_{\rho^\vee} X, \mathbb{k}) \langle 1 \rangle$$

of finite-dimensional \mathbb{Z} -graded \mathbb{k} -vector spaces, under which the action of $z \in \mathbb{k}[z] = H_{\mathbb{C}^\times}^\bullet(\text{pt}, \mathbb{k})$ on the LHS corresponds the Lefschetz operator (i.e., multiplication by the first Chern class of an ample line bundle) on the RHS up to multiples in \mathbb{k}^\times .

Proof. The existence of the isomorphism follows from [3, Theorem 9.1] (here, we need the assumption that \mathbb{k} is of characteristic zero). The latter assertion is [3, Lemma 14.5]. \square

Let $j: E \setminus \{0\} \hookrightarrow E$ be the open inclusion of the complement of $\{0\}$. Let $X \subset E$ be a closed T -subvariety. Applying i^* to the standard exact triangle

$$i_! i^! \mathrm{IC}(X, \mathbb{k}) \rightarrow \mathrm{IC}(X, \mathbb{k}) \rightarrow j_* j^* \mathrm{IC}(X, \mathbb{k}) \xrightarrow{+1},$$

we get the exact triangle

$$(5.3) \quad i^! \mathrm{IC}(X, \mathbb{k}) \rightarrow i^* \mathrm{IC}(X, \mathbb{k}) \rightarrow i^* j_* j^* \mathrm{IC}(X, \mathbb{k}) \xrightarrow{+1}$$

in $D_T^b(V, \mathbb{k})$. By applying $\mathrm{H}_{\mathbb{C}^\times}^\bullet(-) \circ \mathrm{Res}_{\rho^\vee}$ to the third term and using Propositions 5.4 and 5.5, we obtain the isomorphisms

$$\mathrm{H}_{\mathbb{C}^\times}^\bullet(i^* j_* j^* \mathrm{IC}(X, \mathbb{k})) \simeq \mathrm{H}_{\mathbb{C}^\times}^\bullet((p \circ j)_* \mathrm{IC}(X \setminus \{0\}, \mathbb{k})) \simeq \mathrm{IH}^\bullet(\mathbb{P}_{\rho^\vee} X, \mathbb{k})\langle 1 \rangle.$$

Theorem 5.6 (“Fundamental Example” [3]). *Applying $\mathrm{H}_{\mathbb{C}^\times}^\bullet(-) \circ \mathrm{Res}_{\rho^\vee}$ to the exact triangle (5.3) yields a short exact sequence*

$$(5.4) \quad 0 \rightarrow \mathrm{H}_{\mathbb{C}^\times}^\bullet(i^! \mathrm{IC}(X, \mathbb{k})) \rightarrow \mathrm{H}_{\mathbb{C}^\times}^\bullet(i^* \mathrm{IC}(X, \mathbb{k})) \rightarrow \mathrm{IH}^\bullet(\mathbb{P}_{\rho^\vee} X, \mathbb{k})\langle 1 \rangle \rightarrow 0$$

in $\mathbb{k}[z]\text{-Mod}_{\mathbb{Z}}$ satisfying the conditions (i), (ii), (iii) in Section 5.1 above.

Proof. When \mathbb{k} is the field of real numbers, the assertion is proved in [3, Section 14]. To deal with the general case, it is enough to consider the case when \mathbb{k} is the field of rational numbers. For this case, we may employ the fact that $\mathrm{IC}(X, \mathbb{k})$ underlies a simple \mathbb{C}^\times -equivariant mixed Hodge module of pure weight 0, and both functors i^* and $i^!$ preserve the purity thanks to Proposition 5.4. Then, it follows that the connecting homomorphisms in the long exact sequence obtained by applying $\mathrm{H}_{\mathbb{C}^\times}^0(-) \circ \mathrm{Res}_{\rho^\vee}$ to (5.3) are all zero. See [5, Proof of Proposition 4.4] for more details. Together with Proposition 5.5, it leads to the desired short exact sequence (5.4). The conditions (i) and (ii) can be verified as a special case of Corollary 5.9 below (see also Example 5.8). The condition (iii) follows from the latter assertion of Proposition 5.5 and the hard Lefschetz theorem for $\mathrm{IH}^\bullet(\mathbb{P}_{\rho^\vee} X, \mathbb{k})$. \square

5.4. Hyperbolic localization. We finish this section by recalling an equivariant version of the *hyperbolic localization theorem* due to Braden [6]. We keep the assumption from the previous subsection. Let $\tau \in X_*(T)$ be a cocharacter of T . We have the associated decomposition

$$(5.5) \quad E = E_\tau^+ \oplus E_\tau^0 \oplus E_\tau^-,$$

where the component E_τ^\pm (resp. E_τ^0) is the sum of weight spaces E_λ satisfying $\pm \langle \tau, \lambda \rangle > 0$ (resp. $\langle \tau, \lambda \rangle = 0$). Let $i_\tau^\pm: E_\tau^\pm \oplus E_\tau^0 \hookrightarrow E$ and $i_{\tau,0}^\pm: E_\tau^0 \hookrightarrow E_\tau^\pm \oplus E_\tau^0$ be the inclusions.

Theorem 5.7 ([6]). *For any cocharacter $\tau \in X_*(T)$, the followings hold.*

- (1) *There is a natural isomorphism $(i_{\tau,0}^+)^*(i_\tau^+)^! \simeq (i_{\tau,0}^-)^!(i_\tau^-)^*$ of functors from $D_T^b(E, \mathbb{k})$ to $D_T^b(E_\tau^0, \mathbb{k})$.*
- (2) *For any simple perverse sheaf $\mathcal{F} \in \mathrm{Perv}_T(E, \mathbb{k})$, its image $(i_{\tau,0}^+)^*(i_\tau^+)^! \mathcal{F}$ is a finite direct sum of shifted simple perverse sheaves on E_τ^0 .*

We call the functor $(i_{\tau,0}^+)^*(i_\tau^+)^!$ the *hyperbolic localization* associated with τ .

Proof. By the similar argument as in [14, §2.6], one can easily lift the main theorems in [6] to the equivariant setting, which proves the assertions. \square

We say that a cocharacter $\tau \in X_*(T)$ is *generic* if $E_\tau^0 = \{0\}$.

Example 5.8. For example, $\tau = \pm\rho^\vee$ is a generic cocharacter. In this case, we have $E_{\pm\rho^\vee}^\pm = E$ and $E_{\pm\rho^\vee}^\mp = \{0\}$. Therefore, we have $i_{\rho^\vee,0}^+ = i_{-\rho^\vee}^+ = i$, $i_{\rho^\vee}^+ = \text{id}_E$ and $i_{-\rho^\vee,0}^+ = \text{id}_{\{0\}}$. Thus, the functors $i^* = (i_{\rho^\vee,0}^+)^*(i_{\rho^\vee}^+)^!$ and $i^! = (i_{-\rho^\vee,0}^+)^*(i_{-\rho^\vee}^+)^!$ are special cases of hyperbolic localization.

Corollary 5.9. *Let $\tau \in X_*(T)$ be generic. For any simple perverse sheaf $\mathcal{F} \in \text{Perv}_T(E)$, we have an isomorphism of \mathbb{Z} -graded $\mathbb{k}[z]$ -modules*

$$\mathbf{H}_{\mathbb{C}^\times}^\bullet((i_{\tau,0}^+)^*(i_\tau^+)^! \text{Res}_{\rho^\vee}(\mathcal{F})) \simeq \mathbf{H}^\bullet((i_{\tau,0}^+)^*(i_\tau^+)^! \text{For}(\mathcal{F})) \otimes \mathbb{k}[z].$$

Proof. Since \mathcal{F} is a simple perverse sheaf, Theorem 5.7 (2) enables us to find an isomorphism

$$(i_{\tau,0}^+)^*(i_\tau^+)^! \mathcal{F} \simeq \bigoplus_{k \in \mathbb{Z}} \mathbf{H}^k((i_{\tau,0}^+)^*(i_\tau^+)^! \text{For}(\mathcal{F})) \otimes \mathbb{k}_{\{0\}}[-k]$$

in $D_T^b(\{0\}, \mathbb{k})$. Applying $\mathbf{H}_{\mathbb{C}^\times}^\bullet(-) \circ \text{Res}_{\rho^\vee}$, we obtain the desired isomorphism. \square

6. PROOF OF THEOREM 3.26

In this section, we give a proof of Theorem 3.26 using the geometric construction of $U_q(L\mathfrak{g})$ -modules due to Nakajima [51, 52]. We retain the notation from Section 3 above. Throughout this section, we assume that our Lie algebra \mathfrak{g} is of simply-laced type.

6.1. Geometric construction of mixed tensor products. Fix $\mathbf{d} = (d_{i,p}) \in \mathbb{N}^{\hat{I}}$. It determines a dominant monomial $Y^\mathbf{d} = \prod_{(i,p) \in \hat{I}} Y_{i,p}^{d_{i,p}} \in \mathcal{M}^+$. Let $\mathfrak{M}^\bullet(\mathbf{d})$ and $\mathfrak{M}_0^\bullet(\mathbf{d})$ be the graded quiver varieties, smooth and affine respectively, associated with a \hat{I} -graded \mathbb{C} -vector space $D = \bigoplus_{(i,p) \in \hat{I}} D_{i,p}$ such that $\dim_{\mathbb{C}} D_{i,p} = d_{i,p}$. See [53, Section 4] (and also [19, Section 4.4]) for the definition. They come with natural actions of the group $G_\mathbf{d} := \prod_{(i,p) \in \hat{I}} GL(D_{i,p})$ and there is a canonical $G_\mathbf{d}$ -equivariant proper morphism of varieties $\pi_\mathbf{d}: \mathfrak{M}^\bullet(\mathbf{d}) \rightarrow \mathfrak{M}_0^\bullet(\mathbf{d})$. Let $Z^\bullet(\mathbf{d}) := \mathfrak{M}^\bullet(\mathbf{d}) \times_{\mathfrak{M}_0^\bullet(\mathbf{d})} \mathfrak{M}^\bullet(\mathbf{d})$ be the Steinberg type variety. The equivariant algebraic K -theory $K^{G_\mathbf{d}}(Z^\bullet(\mathbf{d}))$ is an associative algebra with respect to the convolution. By Nakajima [51], for any group homomorphism $\varphi: G \rightarrow G_\mathbf{d}$, there is a \mathbb{k} -algebra homomorphism

$$(6.1) \quad U_q(L\mathfrak{g}) \rightarrow \widehat{K}^G(Z^\bullet(\mathbf{d}))_{\mathbb{k}},$$

where $\widehat{K}^G(-)_{\mathbb{k}}$ denote the completion of the equivariant K -theory $K^G(-) \otimes_{\mathbb{Z}} \mathbb{k}$ with respect to the ideal of the representation ring $R(G) = K^G(\text{pt})$ formed by virtual G -representations of dimension 0. For the completion, see also [19, Section 4.6]. By the equivariant Riemann-Roch theorem, we have a homomorphism of \mathbb{k} -algebras

$$\widehat{K}^G(Z^\bullet(\mathbf{d}))_{\mathbb{k}} \rightarrow \widehat{\mathbf{H}}_\bullet^G(Z^\bullet(\mathbf{d}), \mathbb{k}),$$

where the RHS is the convolution algebra of the completed G -equivariant Borel-Moore homology. It is an algebra over the completion $\widehat{\mathbf{H}}_\bullet^G(\text{pt}, \mathbb{k})$. Composed with the homomorphism (6.1), we get a \mathbb{k} -algebra homomorphism

$$(6.2) \quad \Psi_{\mathbf{d}, \varphi}: U_q(L\mathfrak{g}) \rightarrow \widehat{\mathbf{H}}_\bullet^G(Z^\bullet(\mathbf{d}), \mathbb{k}).$$

We consider an action of \mathbb{C}^\times on the vector space D such that the (i, p) -component $D_{i,p}$ is of weight $-e(i, p)$ for each $(i, p) \in \hat{I}$. It defines a group homomorphism $\rho^\vee: \mathbb{C}^\times \rightarrow G_\mathbf{d}$. In what follows, we consider the case $G = \mathbb{C}^\times$ and $\varphi = \rho^\vee$ in (6.2). We identify the ring $\mathbf{H}_{\mathbb{C}^\times}^\bullet(\text{pt}, \mathbb{k})$ with the polynomial ring $\mathbb{k}[z]$ so that the indeterminate z corresponds to the *negative* fundamental weight of \mathbb{C}^\times . In particular, we have the identification $\mathbb{O} = \mathbb{k}[[z]] = \widehat{\mathbf{H}}_{\mathbb{C}^\times}^\bullet(\text{pt}, \mathbb{k})$.

From now on, through the bijection $e: \hat{I} \rightarrow J \subset \mathbb{Z}$, we identify \mathbf{d} with an element $\mathbf{d} \in \mathbb{N}^{\oplus J}$. Namely, we set $d_j = d_{i,p}$ if $j = e(i,p)$. In the similar way, we identify D with a J -graded vector space. Consider the action of the symmetric group \mathfrak{S}_d on the set $J^{\mathbf{d}}$ by place permutations, where $d := \sum_{j \in J} d_j$. Let $\epsilon_s = (j_1, \dots, j_d)$ denote the unique standard sequence in $J^{\mathbf{d}}$. We fix a basis $\{v_1, \dots, v_d\}$ of D such that $v_k \in D_{j_k}$. This yields a maximal torus $T_{\mathbf{d}}$ of $G_{\mathbf{d}}$ consisting of diagonal matrices with respect to the basis. Note that the homomorphism $\rho^{\vee}: \mathbb{C}^{\times} \rightarrow G_{\mathbf{d}}$ factors through $T_{\mathbf{d}}$ and hence $\rho^{\vee} \in X_*(T_{\mathbf{d}})$. For each sequence $\epsilon \in J^{\mathbf{d}}$, let $\sigma_{\epsilon} \in \mathfrak{S}_d$ denote the element of the smallest length such that $\epsilon = (j_{\sigma_{\epsilon}(1)}, \dots, j_{\sigma_{\epsilon}(d)})$. Then we define $\tau_{\epsilon} \in X_*(T_{\mathbf{d}})$ by $\tau_{\epsilon}(t) \cdot v_{\sigma_{\epsilon}(k)} = t^k v_{\sigma_{\epsilon}(k)}$ for $1 \leq k \leq d$. Following Nakajima [52], we consider the closed subvariety $\mathfrak{T}(\epsilon)$ of $\mathfrak{M}^{\bullet}(\mathbf{d})$ defined by

$$\mathfrak{T}(\epsilon) := \{x \in \mathfrak{M}^{\bullet}(\mathbf{d}) \mid \lim_{s \rightarrow 0} \tau_{\epsilon}(s) \cdot \pi_{\mathbf{d}}(x) = 0 \in \mathfrak{M}_0^{\bullet}(\mathbf{d})\}.$$

The equivariant Borel-Moore homology $H_{\bullet}^{\mathbb{C}^{\times}}(\mathfrak{T}(\epsilon), \mathbb{k})$ is a (left) module over the algebra $H_{\bullet}^{\mathbb{C}^{\times}}(Z^{\bullet}(\mathbf{d}), \mathbb{k})$ by the convolution, where the \mathbb{C}^{\times} -action is given by ρ^{\vee} . Through the Nakajima homomorphism $\Psi_{\mathbf{d}, \rho^{\vee}}$, we regard the completed \mathbb{C}^{\times} -equivariant Borel-Moore homology $\widehat{H}_{\bullet}^{\mathbb{C}^{\times}}(\mathfrak{T}(\epsilon), \mathbb{k})$ as a $U_q(\mathfrak{Lg})_{\mathbb{Q}}$ -module.

Theorem 6.1 ([52]). *For any $\epsilon \in J^{\mathbf{d}}$, we have an isomorphism of $U_q(\mathfrak{Lg})_{\mathbb{Q}}$ -modules*

$$\widehat{H}_{\bullet}^{\mathbb{C}^{\times}}(\mathfrak{T}(\epsilon), \mathbb{k}) \simeq \widetilde{M}(\epsilon),$$

which specialize to

$$H_{\bullet}^{\mathbb{C}^{\times}}(\mathfrak{T}(\epsilon), \mathbb{k}) \simeq M(\epsilon).$$

6.2. Sheaf theoretic interpretation. Recall that the affine graded quiver variety $\mathfrak{M}_0^{\bullet}(\mathbf{d})$ has a canonical stratification whose stratum $\mathfrak{M}_0^{\bullet \text{reg}}(\mathbf{v}, \mathbf{d})$ is labelled by $\mathbf{v} = (v_{i,p}) \in \mathbb{N}^{\oplus \hat{I}'}$, where $\hat{I}' := (I \times \mathbb{Z}) \setminus \hat{I}$. See [53, Section 4]. Write $A^{-\mathbf{v}} := \prod_{(i,p) \in \hat{I}'} A_{i,p}^{-v_{i,p}} \in \mathcal{M}$. A stratum $\mathfrak{M}_0^{\bullet \text{reg}}(\mathbf{v}, \mathbf{d})$ is non-empty if and only if $Y^{\mathbf{d}} A^{-\mathbf{v}} \in \mathcal{M}^+$ and the simple module $L(Y^{\mathbf{d}} A^{-\mathbf{v}})$ contributes as a composition factor of the standard module $M(\mathbf{d}) = M(Y^{\mathbf{d}})$. In particular, we have only finitely many non-empty strata, including $\mathfrak{M}_0^{\bullet \text{reg}}(0, \mathbf{d}) = \{0\}$.

Let $\mathcal{A}_{\mathbf{d}} := (\pi_{\mathbf{d}})_* \mathbb{k}_{\mathfrak{M}^{\bullet}(\mathbf{d})}$ denote the (derived) push-forward of the constant sheaf on $\mathfrak{M}^{\bullet}(\mathbf{d})$ along the proper morphism $\pi_{\mathbf{d}}$. Then, by an equivariant version of [10, Section 8.6], we have an isomorphism of \mathbb{k} -algebras

$$(6.3) \quad H_{\bullet}^{\mathbb{C}^{\times}}(Z^{\bullet}(\mathbf{d}), \mathbb{k}) \simeq \text{Hom}_{\mathbb{C}^{\times}}^{\bullet}(\mathcal{A}_{\mathbf{d}}, \mathcal{A}_{\mathbf{d}}),$$

where the algebra structure on the RHS is given by the Yoneda product. By [51, Theorem 14.3.2], we have a decomposition in $D_{\mathbb{C}^{\times}}^b(\mathfrak{M}_0^{\bullet}(\mathbf{d}), \mathbb{k})$:

$$(6.4) \quad \mathcal{A}_{\mathbf{d}} \simeq \bigoplus_{\mathbf{v}} \text{IC}(\mathbf{v}, \mathbf{d}) \otimes_{\mathbb{k}} L^{\bullet}(\mathbf{v}, \mathbf{d}),$$

where \mathbf{v} runs over all the elements of $\mathbb{N}^{\oplus \hat{I}'}$ satisfying $\mathfrak{M}_0^{\bullet \text{reg}}(\mathbf{v}, \mathbf{d}) \neq \emptyset$, $\text{IC}(\mathbf{v}, \mathbf{d})$ denotes the intersection cohomology complex of the closure of the stratum $\mathfrak{M}_0^{\bullet \text{reg}}(\mathbf{v}, \mathbf{d})$ (with coefficients in \mathbb{k}), and $L^{\bullet}(\mathbf{v}, \mathbf{d}) \in D^b(\text{pt}, \mathbb{k})$ is a non-zero object, which we regard as a non-zero finite-dimensional \mathbb{Z} -graded \mathbb{k} -vector space. We consider the total perverse cohomology

$$\bar{\mathcal{A}}_{\mathbf{d}} := \bigoplus_{k \in \mathbb{Z}} {}^p \mathcal{H}^k(\mathcal{A}_{\mathbf{d}}) = \bigoplus_{\mathbf{v}} \text{IC}(\mathbf{v}, \mathbf{d}) \otimes_{\mathbb{k}} L(\mathbf{v}, \mathbf{d}),$$

where $L(\mathbf{v}, \mathbf{d})$ denotes the underlying ungraded \mathbb{k} -vector space of $L^\bullet(\mathbf{v}, \mathbf{d})$. Since $\bar{\mathcal{A}}_{\mathbf{d}}$ is a semisimple perverse sheaf, its Yoneda algebra

$$A_{\mathbf{d}} := \mathrm{Hom}_{\mathbb{C}^\times}^\bullet(\bar{\mathcal{A}}_{\mathbf{d}}, \bar{\mathcal{A}}_{\mathbf{d}})$$

is a non-negatively graded \mathbb{k} -algebra whose degree zero component A^0 is isomorphic to the semisimple algebra $\bigoplus_{\mathbf{v}} \mathrm{End}_{\mathbb{k}}(L(\mathbf{v}, \mathbf{d}))$. Let $\widehat{A}_{\mathbf{d}} := \prod_{n \geq 0} A_{\mathbf{d}}^n$ denote the completion of $A_{\mathbf{d}}$ along the grading. The set $\{L(\mathbf{v}, \mathbf{d})\}_{\mathbf{v}}$ gives a complete system of simple $\widehat{A}_{\mathbf{d}}$ -modules.

From (6.3) and (6.4), we obtain an isomorphism of \mathbb{k} -algebras

$$\widehat{H}_{\bullet}^{\mathbb{C}^\times}(Z^\bullet(\mathbf{d}), \mathbb{k}) \simeq \widehat{A}_{\mathbf{d}}.$$

Composed with the Nakajima homomorphism (6.1), we get a \mathbb{k} -algebra homomorphism $U_q(\mathrm{Lg}) \rightarrow \widehat{A}_{\mathbf{d}}$, through which we regard an $\widehat{A}_{\mathbf{d}}$ -module as a $U_q(\mathrm{Lg})$ -module.

Theorem 6.2 ([51, Theorem 14.3.2]). *The simple $\widehat{A}_{\mathbf{d}}$ -module $L(\mathbf{v}, \mathbf{d})$ is isomorphic to the simple $U_q(\mathrm{Lg})$ -module $L(Y^{\mathbf{d}} A^{-\mathbf{v}})$.*

Let Γ be an infinite quiver whose set of vertices is \hat{I} and whose set of arrows Γ_1 is given by the following rule: the number of arrows from (i, p) to (j, r) is equal to the pole order $\mathfrak{o}(V_{j,r}, V_{i,p})$ of the normalized R -matrix (cf. Section 3.3). We define

$$E(\mathbf{d}) := \bigoplus_{x \in \Gamma_1} \mathrm{Hom}_{\mathbb{C}}(D_{s(x)}, D_{t(x)}),$$

where $s(x)$ (resp. $t(x)$) denotes the source (resp. target) of an arrow x . The group $G_{\mathbf{d}}$ acts on $E(\mathbf{d})$ by conjugation. Note that \mathbb{C}^\times acts on $E(\mathbf{d})$ through $\rho^\vee: \mathbb{C}^\times \rightarrow T_{\mathbf{d}}$ with strictly positive weights because $\mathfrak{o}(V_{j,r}, V_{i,p}) > 0$ implies $r < p$. In other words, $\rho^\vee \in X_*(T_{\mathbf{d}})$ satisfies the condition (5.1), and hence the $T_{\mathbf{d}}$ -action on $E(\mathbf{d})$ is attractive.

Theorem 6.3 ([19, 44]). *The affine graded quiver variety $\mathfrak{M}_0^\bullet(\mathbf{d})$ is $G_{\mathbf{d}}$ -equivariantly isomorphic to a $G_{\mathbf{d}}$ -stable closed subvariety of the affine space $E(\mathbf{d})$.*

In what follows, we identify $\mathfrak{M}_0^\bullet(\mathbf{d})$ with the $G_{\mathbf{d}}$ -stable closed subvariety of $E(\mathbf{d})$.

Recall the cocharacter $\tau_\epsilon \in X_*(T)$ for each $\epsilon \in J^{\mathbf{d}}$. Let

$$E(\epsilon) := \{x \in E(\mathbf{d}) \mid \lim_{s \rightarrow 0} \tau_\epsilon(s) \cdot x = 0\} = E(\mathbf{d})_{\tau_\epsilon}^+$$

in the notation of (5.5). By definition, the variety $\mathfrak{T}(\epsilon)$ is identical to the fiber product $\mathfrak{M}^\bullet(\mathbf{d}) \times_{E(\mathbf{d})} E(\epsilon)$ arising from the canonical proper morphism $\pi_{\mathbf{d}}: \mathfrak{M}^\bullet(\mathbf{d}) \rightarrow \mathfrak{M}_0^\bullet(\mathbf{d}) \subset E(\mathbf{d})$ and the inclusion $i_\epsilon: E(\epsilon) \hookrightarrow E(\mathbf{d})$. Therefore, similarly to the isomorphism in (6.3), we have

$$H_{\bullet}^{\mathbb{C}^\times}(\mathfrak{T}(\epsilon), \mathbb{k}) = H_{\bullet}^{\mathbb{C}^\times}(\mathfrak{M}^\bullet(\mathbf{d}) \times_{E(\mathbf{d})} E(\epsilon)) \simeq \mathrm{Hom}_{\mathbb{C}^\times}^\bullet(i_{\epsilon*} \mathbb{k}_{E(\epsilon)}, \mathcal{A}_{\mathbf{d}}) \simeq H_{\mathbb{C}^\times}^\bullet(p_{\epsilon*} i_\epsilon^! \mathcal{A}_{\mathbf{d}})$$

as $A_{\mathbf{d}}$ -modules, where $p_\epsilon: E(\epsilon) \rightarrow \{0\}$ is the obvious morphism. Here the \mathbb{Z} -gradings are disregarded. Let $i_{\epsilon,0}: \{0\} \hookrightarrow E(\epsilon)$ be the inclusion. By Proposition 5.4, we have an isomorphism $p_{\epsilon*} \simeq i_{\epsilon,0}^*$ of functors from $D_{\mathbb{C}^\times}^b(E(\epsilon), \mathbb{k})$ to $D_{\mathbb{C}^\times}^b(\{0\}, \mathbb{k})$. After the completion, we get

$$\widehat{H}_{\bullet}^{\mathbb{C}^\times}(\mathfrak{T}(\epsilon), \mathbb{C}) \simeq \widehat{H}_{\mathbb{C}^\times}^\bullet(p_{\epsilon*} i_\epsilon^! \bar{\mathcal{A}}_{\mathbf{d}}) \simeq \widehat{H}_{\mathbb{C}^\times}^\bullet(i_{\epsilon,0}^* i_\epsilon^! \bar{\mathcal{A}}_{\mathbf{d}})$$

as $\widehat{A}_{\mathbf{d}}$ -modules. Combined with Theorem 6.1, we obtain the following.

Proposition 6.4. *For each $\epsilon \in J^{\mathbf{d}}$, we have an isomorphism of $U_q(\mathrm{Lg})_{\mathbb{O}}$ -modules*

$$\widehat{H}_{\mathbb{C}^\times}^\bullet(i_{\epsilon,0}^* i_\epsilon^! \bar{\mathcal{A}}_{\mathbf{d}}) \simeq \widetilde{M}(\epsilon),$$

which specialize to

$$H_{\mathbb{C}^\times}^\bullet(i_{\epsilon,0}^* i_\epsilon^! \bar{\mathcal{A}}_{\mathbf{d}}) \simeq M(\epsilon).$$

Remark 6.5. By construction, we have $E(\mathbf{d})_{\tau_{\epsilon}^+}^+ = E(\mathbf{d})_{\tau_{\epsilon^{\text{op}}}^-}$, where ϵ^{op} is the opposite sequence of ϵ . By Theorem 5.7, it implies an isomorphism $\mathbb{D}(i_{\epsilon,0}^* i_{\epsilon}^! \bar{\mathcal{A}}_{\mathbf{d}}) \simeq i_{\epsilon^{\text{op}},0}^* i_{\epsilon^{\text{op}}}^! \bar{\mathcal{A}}_{\mathbf{d}}$.

6.3. Geometric interpretation of R -matrices. Recall the preorder \lesssim of the set $J^{\mathbf{d}}$ from Section 2.4. The following lemma is clear from the definition.

Lemma 6.6. *For $\epsilon, \epsilon' \in J^{\mathbf{d}}$, we have $E(\epsilon) \subset E(\epsilon')$ if and only if $\epsilon \lesssim \epsilon'$. For the standard (resp. costandard) sequence ϵ_s (resp. ϵ_c), we have $E(\epsilon_s) = \{0\}$ (resp. $E(\epsilon_c) = E(\mathbf{d})$).*

For $\epsilon, \epsilon' \in J^{\mathbf{d}}$ satisfying $\epsilon \lesssim \epsilon'$, let $i_{\epsilon',\epsilon}: E(\epsilon) \hookrightarrow E(\epsilon')$ denote the inclusion. Note that we have $i_{\epsilon,0} = i_{\epsilon,\epsilon_s}$ and $i_{\epsilon} = i_{\epsilon_c,\epsilon}$ for any $\epsilon \in J^{\mathbf{d}}$. We have the following diagram of inclusions:

$$\begin{array}{ccc}
 & E(\epsilon_c) = E(\mathbf{d}) & \\
 i_{\epsilon} \nearrow & & \nwarrow i_{\epsilon'} \\
 E(\epsilon) & \xrightarrow{i_{\epsilon',\epsilon}} & E(\epsilon') \\
 i_{\epsilon,0} \nwarrow & & \nearrow i_{\epsilon',0} \\
 & E(\epsilon_s) = \{0\} &
 \end{array}$$

The canonical morphism of functors $i_{\epsilon',\epsilon}^! \rightarrow i_{\epsilon',\epsilon}^*$ induces a morphism

$$(6.5) \quad i_{\epsilon,0}^* i_{\epsilon}^! \bar{\mathcal{A}}_{\mathbf{d}} = i_{\epsilon,0}^* i_{\epsilon',\epsilon}^! i_{\epsilon'}^! \bar{\mathcal{A}}_{\mathbf{d}} \rightarrow i_{\epsilon,0}^* i_{\epsilon',\epsilon}^* i_{\epsilon'}^! \bar{\mathcal{A}}_{\mathbf{d}} = i_{\epsilon',0}^* i_{\epsilon'}^! \bar{\mathcal{A}}_{\mathbf{d}}.$$

Taking the cohomology, we obtain a homomorphism

$$\mathcal{R}_{\epsilon',\epsilon}: H_{\mathbb{C}^\times}^\bullet(i_{\epsilon,0}^* i_{\epsilon}^! \bar{\mathcal{A}}_{\mathbf{d}}) \rightarrow H_{\mathbb{C}^\times}^\bullet(i_{\epsilon',0}^* i_{\epsilon'}^! \bar{\mathcal{A}}_{\mathbf{d}})$$

of graded $A_{\mathbf{d}}$ -modules. From the construction, it satisfies

- (1) $\mathcal{R}_{\epsilon'',\epsilon'} \circ \mathcal{R}_{\epsilon',\epsilon} = \mathcal{R}_{\epsilon'',\epsilon}$ if $\epsilon \lesssim \epsilon' \lesssim \epsilon''$;
- (2) $\mathcal{R}_{\epsilon,\epsilon'} \circ \mathcal{R}_{\epsilon',\epsilon} = \text{id}$ if $\epsilon \sim \epsilon'$.

Proposition 6.7. *Let $\epsilon, \epsilon' \in J^{\mathbf{d}}$ satisfying $\epsilon \lesssim \epsilon'$. Through the isomorphism in Proposition 6.4, the completion $\widehat{\mathcal{R}}_{\epsilon',\epsilon}$ of the homomorphism $\mathcal{R}_{\epsilon',\epsilon}$ is identified with the intertwiner $\widehat{\mathcal{R}}_{\epsilon',\epsilon}$, i.e., the following diagram commutes up to multiples in \mathbb{O}^\times :*

$$\begin{array}{ccc}
 \widehat{H}_{\mathbb{C}^\times}^\bullet(i_{\epsilon,0}^* i_{\epsilon}^! \bar{\mathcal{A}}_{\mathbf{d}}) & \xrightarrow{\widehat{\mathcal{R}}_{\epsilon',\epsilon}} & \widehat{H}_{\mathbb{C}^\times}^\bullet(i_{\epsilon',0}^* i_{\epsilon'}^! \bar{\mathcal{A}}_{\mathbf{d}}) \\
 \simeq \downarrow & & \downarrow \simeq \\
 \widetilde{M}(\epsilon) & \xrightarrow{R_{\epsilon',\epsilon}} & \widetilde{M}(\epsilon'),
 \end{array}$$

where the vertical arrows are the isomorphisms in Proposition 6.4.

Proof. Since the $U_q(\mathbf{Lg})_{\mathbb{K}}$ -module $\widetilde{M}(\epsilon)_{\mathbb{K}} \simeq \widetilde{M}(\epsilon')_{\mathbb{K}}$ is simple and we have $\beta(\epsilon', \epsilon) = 0$ by the normality, the intertwiner $R_{\epsilon',\epsilon}$ is characterized as a unique (up to multiples in \mathbb{O}^\times) $U_q(\mathbf{Lg})_{\mathbb{O}}$ -homomorphism $\widetilde{M}(\epsilon) \rightarrow \widetilde{M}(\epsilon')$ whose specialization at $z = 0$ is non-zero. By construction, $\widehat{\mathcal{R}}_{\epsilon',\epsilon}$ is a $U_q(\mathbf{Lg})_{\mathbb{O}}$ -homomorphism. Thus, it suffices to show that the specialization at $z = 0$ of $\mathcal{R}_{\epsilon',\epsilon}$ is non-zero. By Corollary 5.9, the specialization of $\mathcal{R}_{\epsilon',\epsilon}$ is the homomorphism $H^\bullet(i_{\epsilon,0}^* i_{\epsilon}^! \bar{\mathcal{A}}_{\mathbf{d}}) \rightarrow H^\bullet(i_{\epsilon',0}^* i_{\epsilon'}^! \bar{\mathcal{A}}_{\mathbf{d}})$ arising from (6.5). Since $\bar{\mathcal{A}}_{\mathbf{d}}$ contains $\mathbb{k}_{\{0\}}$ as a summand, it is non-zero. \square

Corollary 6.8. *For any $\epsilon, \epsilon' \in J^{\mathbf{d}}$ satisfying $\epsilon \lesssim \epsilon'$, the homomorphism $\mathcal{R}_{\epsilon',\epsilon}$ is injective.*

Proof. Since $H_{\mathbb{C}^\times}^\bullet(i_{\epsilon,0}^*i_\epsilon^!\mathcal{A})$ is a free $\mathbb{k}[z]$ -module of finite rank by Corollary 5.9, it suffices to show that the completion $\widehat{\mathcal{R}}_{\epsilon',\epsilon}$ is injective. This latter assertion follows from Proposition 6.7 and the injectivity of the intertwiner $R_{\epsilon',\epsilon}$. \square

6.4. Proof of Theorem 3.26. Now we shall prove the following crucial result using the facts from Section 5.

Theorem 6.9. *For any $\mathbf{d} \in \mathbb{N}^{\oplus \hat{I}}$, $\epsilon \in J^{\mathbf{d}}$ and $n \in \mathbb{Z}$, the isomorphism $M(\epsilon) \simeq H^\bullet(i_{\epsilon,0}^*i_\epsilon^!\bar{\mathcal{A}}_{\mathbf{d}})$ in Proposition 6.4 induces the isomorphism*

$$F_n M(\epsilon) \simeq H^{\geq n}(i_{\epsilon,0}^*i_\epsilon^!\bar{\mathcal{A}}_{\mathbf{d}}).$$

In particular, we have the following equality in $K(\mathcal{C}_{\mathbb{Z}})_t$:

$$(6.6) \quad [M(\epsilon)]_t = \sum_{\mathbf{v}} \left(\sum_{n \in \mathbb{Z}} t^n \dim_{\mathbb{k}} H^n(i_{\epsilon,0}^*i_\epsilon^!\mathrm{IC}(\mathbf{v}, \mathbf{d})) \right) [L(Y^{\mathbf{d}} A^{-\mathbf{v}})].$$

Proof. In this proof, to lighten the notation, for each $\epsilon \in J^{\mathbf{d}}$, we set

$$H(\epsilon) := H_{\mathbb{C}^\times}^\bullet(i_{\epsilon,0}^*i_\epsilon^!\bar{\mathcal{A}}_{\mathbf{d}})$$

and regard it as a graded $A_{\mathbf{d}}$ -submodule of $H(\epsilon_c)$ through the injective homomorphism $\mathcal{R}_{\epsilon_c,\epsilon}: H(\epsilon) \rightarrow H(\epsilon_c)$ (cf. Corollary 6.8). Thus, we have the inclusions of graded $A_{\mathbf{d}}$ -submodules $H(\epsilon_s) \subset H(\epsilon) \subset H(\epsilon_c)$ for any $\epsilon \in J^{\mathbf{d}}$. By Corollary 5.9, $H(\epsilon)$ is a graded free $\mathbb{k}[z]$ -module of finite rank and the quotient $H(\epsilon)/zH(\epsilon)$ is identical to the non-equivariant cohomology $H^\bullet(i_{\epsilon,0}^*i_\epsilon^!\bar{\mathcal{A}}_{\mathbf{d}})$ as a graded \mathbb{k} -vector space. By Theorem 5.6 and Lemma 5.3, the graded $\mathbb{k}[z]$ -module $L := H(\epsilon_c)/zH(\epsilon_s)$ satisfies the hard Lefschetz property.

By the definition (2.7), the filter submodule $F_n M(\epsilon)$ is the image of

$$F_n \tilde{M}(\epsilon) := \tilde{M}(\epsilon) \cap \sum_{k \in \mathbb{Z}} \left(z^k R_{\epsilon,\epsilon_s} \tilde{M}(\epsilon_s) \cap z^{n-k} R_{\epsilon_c,\epsilon}^{-1} \tilde{M}(\epsilon_c) \right)$$

under the evaluation map $\mathrm{ev}_{z=0}: \tilde{M}(\epsilon) \rightarrow M(\epsilon) = \tilde{M}(\epsilon)/z\tilde{M}(\epsilon)$. Consider a quotient map

$$f: \tilde{M}(\epsilon) \rightarrow \tilde{M}(\epsilon)/zR_{\epsilon,\epsilon_s}\tilde{M}(\epsilon_s) \simeq \widehat{H}(\epsilon)/z\widehat{H}(\epsilon_s) \simeq H(\epsilon)/zH(\epsilon_s) \subset H(\epsilon_c)/zH(\epsilon_s) = L,$$

where the first isomorphism is induced by the one in Proposition 6.4. We have

$$\begin{aligned} f(F_n \tilde{M}(\epsilon)) &= \left(\sum_{k-l=n} \mathrm{Ker}(z_L^{l+1}) \cap \mathrm{Im}(z_L^k) \right) \cap (H(\epsilon)/zH(\epsilon_s)) \\ &= L^{\geq n} \cap (H(\epsilon)/zH(\epsilon_s)) \\ &= (H(\epsilon)/zH(\epsilon_s))^{\geq n}, \end{aligned}$$

where the second equality is due to Lemma 5.2. Letting $g: L = H(\epsilon_c)/zH(\epsilon_s) \rightarrow H(\epsilon_c)/zH(\epsilon)$ be the quotient map, we obtain

$$F_n M(\epsilon) = \mathrm{ev}_{z=0}(F_n \tilde{M}(\epsilon)) \simeq g \left(f(F_n \tilde{M}(\epsilon)) \right) = g \left((H(\epsilon)/zH(\epsilon_s))^{\geq n} \right) = H^\bullet(i_{\epsilon,0}^*i_\epsilon^!\bar{\mathcal{A}}_{\mathbf{d}})^{\geq n},$$

which proves the former assertion. The other assertion (6.6) follows from the former one together with the definition of $\bar{\mathcal{A}}_{\mathbf{d}}$ and Theorem 6.2. \square

Corollary 6.10. *For any ϵ and $n \in \mathbb{Z}$, the filtration layer $\mathrm{Gr}_n^F M(\epsilon) = F_n M(\epsilon)/F_{n+1} M(\epsilon)$ is a semisimple $U_q(\mathrm{Lg})$ -module.*

Proof. This is because the $U_q(L\mathfrak{g})$ -action on $\mathrm{Gr}_n^F M(\epsilon)$ factors through the action of the semisimple algebra A^0 by the above Theorem 6.9. \square

On the other hand, the geometric construction of the quantum Grothendieck ring $K_t(\mathcal{C}_{\mathbb{Z}})$ due to Varagnolo-Vasserot [57] implies the following.

Theorem 6.11 ([58]). *For any $\mathbf{d} \in \mathbb{N}^{\oplus \hat{I}}$ and $\epsilon \in J^{\mathbf{d}}$, we have the following equality in $K_t(\mathcal{C}_{\mathbb{Z}})$:*

$$(6.7) \quad E_t(\epsilon) = \sum_{\mathbf{v}} \left(\sum_{n \in \mathbb{Z}} t^n \dim_{\mathbb{k}} H^n(i_{\epsilon,0}^* i_{\epsilon}^! \mathrm{IC}(\mathbf{v}, \mathbf{d})) \right) L_t(Y^{\mathbf{d}} A^{-\mathbf{v}}).$$

Proof. This is a direct consequence of the geometric definition of $K_t(\mathcal{C}_{\mathbb{Z}})$ in [58]. See [30, Section 5.5] for a comparison with our algebraic definition in Section 3.5. \square

Comparing (6.6) with (6.7), we arrive at the desired equality (3.1). Thus, we have proved Theorem 3.26.

6.5. Examples. We give some explicit computations of the hyperbolic localizations $i_{\epsilon,0}^* i_{\epsilon}^! \bar{\mathcal{A}}_{\mathbf{d}}$ in the simplest examples (and we check that we recover examples that we computed in the previous sections). Let $\mathfrak{g} = \mathfrak{sl}_2$. In this case, $\hat{I} = \{1\} \times 2\mathbb{Z}$, and $J = e(\hat{I})$ is the set of all odd integers. The quiver Γ is depicted as

$$\dots \longleftarrow \circ \longleftarrow \circ \longleftarrow \circ \longleftarrow \circ \longleftarrow \circ \longleftarrow \dots$$

(1, -4) (1, -2) (1, 0) (1, 2) (1, 4)

We have $S_j = L(Y_{1,j-1})$ for any $j \in J$.

6.5.1. We consider the case of Section 3.7.1, where $\mathbf{d} \in \mathbb{N}^{\oplus \hat{I}}$ is given by $d_{1,2} = 2$, $d_{1,0} = 1$ and $d_{1,2k} = 0$ if $k \notin \{0, 1\}$. The affine graded quiver variety $\mathfrak{M}_0^{\bullet}(\mathbf{d})$ coincides with the 2-dimensional linear space $E := E(\mathbf{d}) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^2, \mathbb{C})$ and the stratification is given by $E = (E \setminus \{0\}) \sqcup \{0\}$. We have

$$\bar{\mathcal{A}}_{\mathbf{d}} \simeq (\mathbb{k}_{\{0\}} \otimes L) \oplus (\mathbb{k}_E[2] \otimes S_3),$$

where $L = L(Y_{1,2}^2 Y_{1,0})$. The set $J^{\mathbf{d}}$ consists of 3 elements $\epsilon_s = (3, 3, 1) < \epsilon = (3, 1, 3) < \epsilon_c = (1, 3, 3)$ and $E(\epsilon)$ is a 1-dimensional linear subspace of E . We have

$$\begin{aligned} i_{\epsilon_s,0}^* i_{\epsilon_s}^! \bar{\mathcal{A}}_{\mathbf{d}} &= i_0^! \bar{\mathcal{A}}_{\mathbf{d}} = (\mathbb{k}_{\{0\}} \otimes L) \oplus (\mathbb{k}_{\{0\}}[-2] \otimes S_3), \\ i_{\epsilon,0}^* i_{\epsilon}^! \bar{\mathcal{A}}_{\mathbf{d}} &= (\mathbb{k}_{\{0\}} \otimes L) \oplus (\mathbb{k}_{\{0\}} \otimes S_3), \\ i_{\epsilon_c,0}^* i_{\epsilon_c}^! \bar{\mathcal{A}}_{\mathbf{d}} &= i_0^* \bar{\mathcal{A}}_{\mathbf{d}} = (\mathbb{k}_{\{0\}} \otimes L) \oplus (\mathbb{k}_{\{0\}}[2] \otimes S_3), \end{aligned}$$

where (and hereafter) i_0 denotes the inclusion of the origin.

6.5.2. We consider the case of Section 3.7.2, where $\mathbf{d} \in \mathbb{N}^{\oplus \hat{I}}$ is given by $d_{1,0} = d_{1,2} = d_{1,4} = 1$ and $d_{1,2k} = 0$ if $k \notin \{0, 1, 2\}$. The affine graded quiver variety $\mathfrak{M}_0^{\bullet}(\mathbf{d})$ coincides with the closed subvariety X of the 2-dimensional linear space $E(\mathbf{d}) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})^{\oplus 2} = \{(a, b) \mid a, b \in \mathbb{C}\}$ defined by the equation $ab = 0$. The stratification is given by $\mathfrak{M}_0^{\bullet}(\mathbf{d}) = X = X_a \sqcup X_b \sqcup \{0\}$, where $X_a := \{(a, b) \in \mathbb{C}^2 \mid a \neq 0, b = 0\}$ and $X_b := \{(a, b) \in \mathbb{C}^2 \mid a = 0, b \neq 0\}$. We have

$$\bar{\mathcal{A}}_{\mathbf{d}} \simeq (\mathbb{k}_{\{0\}} \otimes L) \oplus (\mathbb{k}_{\bar{X}_a}[1] \otimes S_5) \oplus (\mathbb{k}_{\bar{X}_b}[1] \otimes S_1),$$

where $L = L(Y_{1,0}Y_{1,2}Y_{1,4})$. The set $J^{\mathbf{d}}/\sim$ consists of 4 equivalence classes represented by $\epsilon_s = (5, 3, 1)$, $\epsilon_1 = (3, 5, 1)$, $\epsilon_2 = (5, 1, 3)$, and $\epsilon_c = (1, 3, 5)$. We have $E(\epsilon_1) = \bar{X}_b$ and $E(\epsilon_2) = \bar{X}_a$. Therefore, we can compute as:

$$\begin{aligned} i_{\epsilon_s,0}^* i_{\epsilon_s}^! \bar{\mathcal{A}}_{\mathbf{d}} &= i_0^! \bar{\mathcal{A}}_{\mathbf{d}} = (\mathbb{k}_{\{0\}} \otimes L) \oplus (\mathbb{k}_{\{0\}}[-1] \otimes S_5) \oplus (\mathbb{k}_{\{0\}}[-1] \otimes S_1), \\ i_{\epsilon_1,0}^* i_{\epsilon_1}^! \bar{\mathcal{A}}_{\mathbf{d}} &= (\mathbb{k}_{\{0\}} \otimes L) \oplus (\mathbb{k}_{\{0\}}[-1] \otimes S_5) \oplus (\mathbb{k}_{\{0\}}[1] \otimes S_1), \\ i_{\epsilon_2,0}^* i_{\epsilon_2}^! \bar{\mathcal{A}}_{\mathbf{d}} &= (\mathbb{k}_{\{0\}} \otimes L) \oplus (\mathbb{k}_{\{0\}}[1] \otimes S_5) \oplus (\mathbb{k}_{\{0\}}[-1] \otimes S_1), \\ i_{\epsilon_c,0}^* i_{\epsilon_c}^! \bar{\mathcal{A}}_{\mathbf{d}} &= i_0^* \bar{\mathcal{A}}_{\mathbf{d}} = (\mathbb{k}_{\{0\}} \otimes L) \oplus (\mathbb{k}_{\{0\}}[1] \otimes S_5) \oplus (\mathbb{k}_{\{0\}}[1] \otimes S_1). \end{aligned}$$

6.5.3. We consider the case of Section 3.7.3, where $\mathbf{d} \in \mathbb{N}^{\oplus I}$ is given by $d_{1,0} = d_{1,2} = 2$ and $d_{1,2k} = 0$ if $k \notin \{0, 1\}$. The affine graded quiver variety $\mathfrak{M}_0^{\bullet}(\mathbf{d})$ coincides with the 4-dimensional linear space $E := E(\mathbf{d}) = \text{Hom}_{\mathbb{C}}(\mathbb{C}^2, \mathbb{C}^2) \simeq \text{Mat}_2(\mathbb{C})$ and the stratification is given by $E = (E \setminus X) \sqcup (X \setminus \{0\}) \sqcup \{0\}$, where $X := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}_2(\mathbb{C}) \mid ad = bc \right\}$. We have

$$\bar{\mathcal{A}}_{\mathbf{d}} \simeq (\mathbb{k}_{\{0\}} \otimes L) \oplus (\text{IC}(X, \mathbb{k}) \otimes K) \oplus (\mathbb{k}_E[4] \otimes \mathbf{1}),$$

where $L = L(Y_{1,0}^2 Y_{1,2}^2)$, $K = L(Y_{1,0} Y_{1,2})$, and $\mathbf{1} = L(1)$. Note that $\text{IC}(X, \mathbb{k})$ fits into an exact triangle (cf. [1, Exercise 3.10.6]): $\mathbb{k}_X[3] \rightarrow \text{IC}(X, \mathbb{k}) \rightarrow \mathbb{k}_{\{0\}}[1] \xrightarrow{+1}$. The set $J^{\mathbf{d}}$ consists of 6 elements $\epsilon_s = (3, 3, 1, 1)$, $\epsilon_1 = (3, 1, 3, 1)$, $\epsilon_2 = (3, 1, 1, 3)$, $\epsilon_3 = (1, 3, 3, 1)$, $\epsilon_4 = (1, 3, 1, 3)$, and $\epsilon_c = (1, 1, 3, 3)$. We have the following commutative diagram of inclusions:

$$\begin{array}{ccc} & E(\epsilon_c) = E & \\ & \uparrow & \\ & E(\epsilon_4) = \{c = 0\} & \\ \swarrow & & \nwarrow \\ E(\epsilon_2) = \{a = c = 0\} & & E(\epsilon_3) = \{c = d = 0\} \\ \swarrow & & \searrow \\ & E(\epsilon_1) = \{a = c = d = 0\} & \\ & \uparrow & \\ & E(\epsilon_s) = \{0\} & \end{array}$$

By the fact that $E(\epsilon_k) \subset X$ for $k \in \{1, 2, 3\}$ and Remark 6.5, we can compute as:

$$\begin{aligned} i_{\epsilon_s,0}^* i_{\epsilon_s}^! \bar{\mathcal{A}}_{\mathbf{d}} &= i_0^! \bar{\mathcal{A}}_{\mathbf{d}} = (\mathbb{k}_{\{0\}} \otimes L) \oplus (\mathbb{k}_{\{0\}}[-1] \oplus \mathbb{k}_{\{0\}}[-3]) \otimes K \oplus (\mathbb{k}_{\{0\}}[-4] \otimes \mathbf{1}), \\ i_{\epsilon_1,0}^* i_{\epsilon_1}^! \bar{\mathcal{A}}_{\mathbf{d}} &= (\mathbb{k}_{\{0\}} \otimes L) \oplus (\mathbb{k}_{\{0\}}[-1] \oplus \mathbb{k}_{\{0\}}[-1]) \otimes K \oplus (\mathbb{k}_{\{0\}}[-2] \otimes \mathbf{1}), \\ i_{\epsilon_2,0}^* i_{\epsilon_2}^! \bar{\mathcal{A}}_{\mathbf{d}} &= (\mathbb{k}_{\{0\}} \otimes L) \oplus (\mathbb{k}_{\{0\}}[-1] \oplus \mathbb{k}_{\{0\}}[1]) \otimes K \oplus (\mathbb{k}_{\{0\}}[0] \otimes \mathbf{1}), \\ i_{\epsilon_3,0}^* i_{\epsilon_3}^! \bar{\mathcal{A}}_{\mathbf{d}} &= (\mathbb{k}_{\{0\}} \otimes L) \oplus (\mathbb{k}_{\{0\}}[-1] \oplus \mathbb{k}_{\{0\}}[1]) \otimes K \oplus (\mathbb{k}_{\{0\}}[0] \otimes \mathbf{1}), \\ i_{\epsilon_4,0}^* i_{\epsilon_4}^! \bar{\mathcal{A}}_{\mathbf{d}} &= (\mathbb{k}_{\{0\}} \otimes L) \oplus (\mathbb{k}_{\{0\}}[1] \oplus \mathbb{k}_{\{0\}}[1]) \otimes K \oplus (\mathbb{k}_{\{0\}}[2] \otimes \mathbf{1}), \\ i_{\epsilon_c,0}^* i_{\epsilon_c}^! \bar{\mathcal{A}}_{\mathbf{d}} &= i_0^* \bar{\mathcal{A}}_{\mathbf{d}} = (\mathbb{k}_{\{0\}} \otimes L) \oplus (\mathbb{k}_{\{0\}}[1] \oplus \mathbb{k}_{\{0\}}[3]) \otimes K \oplus (\mathbb{k}_{\{0\}}[4] \otimes \mathbf{1}). \end{aligned}$$

7. PROOF OF THEOREM 4.14

In this section, we give a proof of Theorem 4.14 using the geometric construction of the canonical bases due to Lusztig [49] and the symmetric quiver Hecke algebras due to Varagnolo-Vasserot [59]. We retain the notation from Section 4 above. Throughout this section, we assume that $\mathbf{i} = (i_1, \dots, i_\ell) \in I^\ell$ is a reduced word for an element $w \in W$ adapted to a fixed quiver Q of type \mathfrak{g} (recall Definition 4.13). We set $J := \{k \in \mathbb{Z} \mid 1 \leq k \leq \ell\}$ as before.

7.1. Lusztig's construction of canonical bases. First, we review the geometric construction of the canonical basis \mathbf{B} of $U_t^+(\mathfrak{g})_{\mathbb{Z}[t^{\pm 1}]}$ due to Lusztig [49]. For an I -graded \mathbb{C} -vector space $V = \bigoplus_{i \in I} V_i$, we set $\underline{\dim} V := \sum_{i \in I} (\dim_{\mathbb{C}} V_i) \alpha_i \in \mathbb{Q}^+$. For each $\beta \in \mathbb{Q}^+$, we fix an I -graded vector space $V^\beta = \bigoplus_{i \in I} V_i^\beta$ satisfying $\underline{\dim} V^\beta = \beta$. For $\beta, \beta' \in \mathbb{Q}^+$, we set

$$L(\beta, \beta') := \bigoplus_{i \in I} \mathrm{Hom}_{\mathbb{C}}(V_i^\beta, V_i^{\beta'}), \quad E(\beta, \beta') := \bigoplus_{h \in Q_1} \mathrm{Hom}_{\mathbb{C}}(V_{s(h)}^\beta, V_{t(h)}^{\beta'}).$$

The space $X(\beta) := E(\beta, \beta)$ consists of representations of the quiver Q of dimension vector β , on which we have the natural conjugation action of the group $G(\beta) := \prod_{i \in I} GL(V_i^\beta)$. Note that the Lie algebra of $G(\beta)$ is identical to $L(\beta, \beta)$. It is convenient to introduce the bilinear form $\langle -, - \rangle_Q: \mathbb{Q}^+ \times \mathbb{Q}^+ \rightarrow \mathbb{Z}$ defined by

$$\langle \beta, \beta' \rangle_Q := \dim_{\mathbb{C}} L(\beta, \beta') - \dim_{\mathbb{C}} E(\beta, \beta').$$

For any $\beta, \beta' \in \mathbb{Q}$, we have

$$(7.1) \quad (\beta, \beta') = \langle \beta, \beta' \rangle_Q + \langle \beta', \beta \rangle_Q.$$

For any finite sequence $\beta_1, \dots, \beta_n \in \mathbb{Q}^+$, we write $X(\beta_1, \dots, \beta_n) := X(\beta_1) \times \dots \times X(\beta_n)$ and $G(\beta_1, \dots, \beta_n) := G(\beta_1) \times \dots \times G(\beta_n)$. Assume $\beta = \sum_{k=1}^n \beta_k$. By [49, Section 9], we have an adjoint pair of functors

$$D_{G(\beta_1, \dots, \beta_n)}^b(X(\beta_1, \dots, \beta_n), \mathbb{k}) \begin{array}{c} \xleftarrow{\mathrm{Ind}_{\beta_1, \dots, \beta_n}} \\ \xrightarrow{\mathrm{Res}_{\beta_1, \dots, \beta_n}} \end{array} D_{G(\beta)}^b(X(\beta), \mathbb{k}),$$

with $\mathrm{Ind}_{\beta_1, \dots, \beta_n}$ being left adjoint to $\mathrm{Res}_{\beta_1, \dots, \beta_n}$. We shall recall their construction. Choose an identification $V^\beta = V^{\beta_1} \oplus \dots \oplus V^{\beta_n}$ of I -graded vector spaces. (The resulting functors do not depend on this choice up to isomorphism.) We define an I -graded flag $V^\beta = F_0 \supset F_1 \supset \dots \supset F_n = \{0\}$ given by $F_k = \bigoplus_{i > k} V^{\beta_i}$. Let $F(\beta_1, \dots, \beta_n)$ be the subvariety of $X(\beta)$ consisting of representations x satisfying $x(F_k) \subset F_k$ for $1 \leq k \leq n$. Consider the following diagram

$$(7.2) \quad X(\beta_1, \dots, \beta_n) \xleftarrow{\kappa} F(\beta_1, \dots, \beta_n) \xrightarrow{\iota} X(\beta),$$

where ι is the inclusion, and κ is given by $x \mapsto (x|_{F_{k-1}} \bmod F_k)_{1 \leq k \leq n}$. On the other hand, let $P \subset G(\beta)$ denote the stabilizer of the fixed flag F_\bullet , and U its unipotent radical so that $P/U \simeq G(\beta_1, \dots, \beta_n)$. We consider the following diagram

$$X(\beta_1, \dots, \beta_n) \xleftarrow{p_1} G(\beta) \times^U F(\beta_1, \dots, \beta_n) \xrightarrow{p_2} \tilde{F}(\beta_1, \dots, \beta_n) \xrightarrow{p_3} X(\beta),$$

where $\tilde{F}(\beta_1, \dots, \beta_n) = G(\beta) \times^P F(\beta_1, \dots, \beta_n)$, $p_1(g, x) = \kappa(x)$, $p_2(g, x) = (g, x)$, $p_3(g, x) = g \cdot \iota(x)$ for $g \in G(\beta)$ and $x \in F(\beta_1, \dots, \beta_n)$. Note that p_1 is smooth, p_2 is a $G(\beta_1, \dots, \beta_n)$ -torsor, p_3 is proper. Then, we define

$$\mathrm{Ind}_{\beta_1, \dots, \beta_n} \mathcal{F} := p_{3*} \mathcal{F}'[c + 2 \dim U] \quad \text{and} \quad \mathrm{Res}_{\beta_1, \dots, \beta_n} \mathcal{G} := \kappa_* \iota^! \mathcal{G}[c],$$

where \mathcal{F}' is a unique $G(\beta)$ -equivariant complex on $G(\beta) \times^U F(\beta_1, \dots, \beta_n)$ satisfying $p_2^* \mathcal{F}' \simeq p_1^* \mathcal{F}$, and $c := \sum_{1 \leq j < k \leq n} \langle \beta_j, \beta_k \rangle_Q$. When $n = 2$, we write

$$\mathcal{F}_1 \star \mathcal{F}_2 := \text{Ind}_{\beta_1, \beta_2}(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$$

for $\mathcal{F}_k \in D_{G(\beta_k)}^b(X(\beta_k), \mathbb{k})$, $k \in \{1, 2\}$. Then, we have the (strong) associativity

$$\text{Ind}_{\beta_1, \dots, \beta_n}(\mathcal{F} \boxtimes \mathcal{G}) \simeq \text{Ind}_{\beta_1, \dots, \beta_k}(\mathcal{F}) \star \text{Ind}_{\beta_{k+1}, \dots, \beta_n}(\mathcal{G}).$$

Let $\mathcal{C}_\beta := \mathbb{k}_{X(\beta)}[\dim X(\beta)]$ be the constant perverse sheaf on $X(\beta)$. Recall the notation I^β from Section 4.3. We set

$$(7.3) \quad \mathcal{L}_\beta := \bigoplus_{\nu \in I^\beta} \mathcal{L}_\nu, \quad \mathcal{L}_\nu := \mathcal{C}_{\alpha_{\nu_1}} \star \mathcal{C}_{\alpha_{\nu_2}} \star \dots \star \mathcal{C}_{\alpha_{\nu_{|\beta|}}} = (p_3)_* \mathbb{k}_{\tilde{F}_\nu}[\dim \tilde{F}_\nu]$$

where $\tilde{F}_\nu := \tilde{F}(\alpha_{\nu_1}, \dots, \alpha_{\nu_{|\beta|}})$. By the decomposition theorem, the complex \mathcal{L}_β is a finite direct sum of shifts of simple perverse sheaves on $X(\beta)$. Let \mathcal{Q}_β be the smallest additive, strictly full subcategory of $D_{G(\beta)}^b(X(\beta), \mathbb{k})$ that contains \mathcal{L}_β and is closed under taking shifts and direct summands. One can show that the category $\mathcal{Q} := \bigoplus_{\beta \in Q^+} \mathcal{Q}_\beta$ is stable under the functors $\text{Ind}_{\beta_1, \dots, \beta_n}$ and $\text{Res}_{\beta_1, \dots, \beta_n}$, and hence the operation \star defines the structure of $\mathbb{Z}[t^{\pm 1}]$ -algebra on the Grothendieck group $K(\mathcal{Q}) = \bigoplus_{\beta \in Q^+} K(\mathcal{Q}_\beta)$, where the action of $t^{\pm 1}$ corresponds to the cohomological degree shift $[\mp 1]$.

Theorem 7.1 ([49]). *There is a unique isomorphism of $\mathbb{Z}[t^{\pm 1}]$ -algebras*

$$\chi: U_t^+(\mathfrak{g})_{\mathbb{Z}[t^{\pm 1}]} \simeq K(\mathcal{Q}) \quad \text{given by } \chi(e_i) = [\mathcal{C}_{\alpha_i}] \text{ for any } i \in I,$$

through which the homomorphism r corresponds to $[\bigoplus_{\beta, \beta' \in Q^+} \text{Res}_{\beta, \beta'}]$, and the involution ι corresponds to the Verdier duality $[\bigoplus_{\beta \in Q^+} \mathbb{D}_{X(\beta)}]$.

Remark 7.2. The functor $\text{Res}_{\beta, \beta'}$ is isomorphic to $\mathbb{D}_{X(\beta, \beta')} \circ \text{Res}_{\mathbf{T}, \mathbf{W}}^{\mathbf{V}} \circ \mathbb{D}_{X(\beta + \beta')}$ in Lusztig's notation [49, 9.2.10] with $\mathbf{T} = V^\beta$, $\mathbf{W} = V^{\beta'}$, $\mathbf{V} = V^{\beta + \beta'}$.

By construction, the algebra $K(\mathcal{Q})$ has a basis \mathcal{P} over $\mathbb{Z}[t^{\pm 1}]$ consisting of the classes of simple perverse sheaves. The canonical basis \mathbf{B} of $U_t^+(\mathfrak{g})_{\mathbb{Z}[t^{\pm 1}]}$ is defined by $\mathbf{B} := \chi^{-1}(\mathcal{P})$. Recall the dual canonical basis \mathbf{B}^* and its subset $\mathbf{B}^*(w) = \{B_{\mathbf{i}}^*(\mathbf{d}) \mid \mathbf{d} \in \mathbb{N}^{\oplus J}\}$ from Theorem 4.1. For each $\mathbf{d} \in \mathbb{N}^{\oplus J}$, let $B_{\mathbf{i}}(\mathbf{d})$ denote the element of \mathbf{B} dual to $B_{\mathbf{i}}^*(\mathbf{d})$. We write $\text{IC}(\mathbf{d})$ for a unique simple perverse sheaf in \mathcal{P} satisfying $\chi(B_{\mathbf{i}}(\mathbf{d})) = [\text{IC}(\mathbf{d})]$.

7.2. IC-sheaves corresponding to real positive roots. Let $\text{Rep}(Q)$ be the category of representations of the quiver Q over \mathbb{C} . We abbreviate $\text{Hom}_{\text{Rep}(Q)}(x, y)$ as $\text{Hom}_Q(x, y)$.

For each real positive root $\alpha \in R^+$, there exists an indecomposable representation $x(\alpha)$ of the quiver Q , uniquely up to isomorphism, by Kac's theorem. In what follows, we fix such a representation $x(\alpha)$ for each $\alpha \in R^+$. We often regard $x(\alpha)$ as a geometric point of the affine space $X(\alpha)$. The orbit $O(\alpha) := G(\alpha) \cdot x(\alpha)$ is dense in $X(\alpha)$. Since $\text{Stab}_{G(\alpha)} x(\alpha) = \text{End}_Q(x(\alpha))^\times \simeq \mathbb{C}^\times$ is connected, every simple $G(\alpha)$ -equivariant perverse sheaf whose support contains $O(\alpha)$ is isomorphic to the constant one \mathcal{C}_α . For simplicity, we will use the abbreviation

$$\begin{aligned} h(\alpha, \beta) &:= \dim_{\mathbb{C}} \text{Hom}_Q(x(\alpha), x(\beta)), \\ e(\alpha, \beta) &:= \dim_{\mathbb{C}} \text{Ext}_Q^1(x(\alpha), x(\beta)). \end{aligned}$$

With these notations, we have

$$(7.4) \quad \langle \alpha, \beta \rangle_Q = h(\alpha, \beta) - e(\alpha, \beta)$$

for any $\alpha, \beta \in \mathbb{R}^+$.

Recall that we defined the positive root $\alpha_{\mathbf{i},j} := s_{i_1} s_{i_2} \cdots s_{i_{j-1}}(\alpha_{i_j})$ for each $j \in J$. The next lemma is standard.

Lemma 7.3. *The followings hold.*

(1) *For $j, k \in J$, we have*

$$\begin{aligned} h(\alpha_{\mathbf{i},j}, \alpha_{\mathbf{i},k}) &= 0 & \text{if } j < k, \\ e(\alpha_{\mathbf{i},j}, \alpha_{\mathbf{i},k}) &= 0 & \text{if } j \geq k. \end{aligned}$$

(2) *The additive full subcategory $\text{add}\{x(\alpha_{\mathbf{i},j})\}_{j \in J}$ of $\text{Rep}(Q)$ consisting of representations isomorphic to finite direct sums of the indecomposable representations $\{x(\alpha_{\mathbf{i},j})\}_{j \in J}$ is closed under extensions.*

Proof. For a source $i \in I$ of a quiver Q , we have Bernstein–Gelfand–Ponomarev’s reflection functors $\Sigma_i: \text{Rep}(s_i Q) \rightarrow \text{Rep}(Q)$ and $\Sigma_i^*: \text{Rep}(Q) \rightarrow \text{Rep}(s_i Q)$. If x is an indecomposable representation in $\text{Rep}(s_i Q)$ (resp. $\text{Rep}(Q)$) of dimension vector $\alpha \in \mathbb{Q}^+$, its reflection $\Sigma_i x$ (resp. $\Sigma_i^* x$) is indecomposable of dimension vector $s_i \alpha$ if $\alpha \neq \alpha_i$, and zero otherwise (cf. [11, Theorem 4.3.9]). In particular, we have

$$x(\alpha_{\mathbf{i},j}) \simeq \Sigma_{i_1} \Sigma_{i_2} \cdots \Sigma_{i_{j-1}} x(\alpha_{i_j})$$

for each $j \in J$. Therefore, the assertion (1) follows from the adjunction isomorphism $\text{Hom}_Q(x, \Sigma_i y) \simeq \text{Hom}_{s_i Q}(\Sigma_i^* x, y)$ and its Auslander-Reiten dual $\text{Ext}_Q^1(\Sigma_i y, x) \simeq \text{Ext}_{s_i Q}^1(y, \Sigma_i^* x)$ (cf. [11, Exercise 4.3.6]). Moreover, we see that the category $\text{add}\{x(\alpha_{\mathbf{i},j})\}_{j \in J}$ coincides with the kernel of the right exact functor $\Sigma_{i_\ell}^* \cdots \Sigma_{i_2}^* \Sigma_{i_1}^*: \text{Rep}(Q) \rightarrow \text{Rep}(s_{i_\ell} \cdots s_{i_2} s_{i_1} Q)$, which implies the assertion (2). \square

The following result is due to Lusztig.

Proposition 7.4 ([50]). *For each $j \in J$, we have $\text{IC}(\delta_j) = \mathcal{C}_{\alpha_{\mathbf{i},j}}$.*

Proof. This follows from [50, 9.4], which shows that the correspondence among \mathcal{P} induced from the reflection functor Σ_i coincides through χ with the one among \mathbf{B} induced from the braid symmetry T_i . See [42, Section 3] and [61, Theorem 3.5] for some more details. \square

For each $\beta \in \mathbb{Q}^+$, we set

$$\text{KP}_{\mathbf{i}}(\beta) := \{\mathbf{d} = (d_j)_{j \in J} \in \mathbb{N}^{\oplus J} \mid \sum_{j \in J} d_j \alpha_{\mathbf{i},j} = \beta\}.$$

The perverse sheaf $\text{IC}(\mathbf{d})$ lives in $X(\beta)$ (i.e., belongs to \mathcal{Q}_β) if and only if $\mathbf{d} \in \text{KP}_{\mathbf{i}}(\beta)$.

7.3. Geometric interpretation of some structure constants. Recall the the mixed product $\tilde{E}_{\mathbf{i}}^*(\epsilon)$ in the quantum unipotent coordinate ring defined in (4.9). The purpose of this and next subsections is to describe $\tilde{E}_{\mathbf{i}}^*(\epsilon)$ in terms of the intersection cohomology. Main results are Propositions 7.5 and 7.9. They are analogous to Theorem 6.11 above for the quantum Grothendieck rings.

Let $\beta \in \mathbb{Q}^+$ and fix $\mathbf{d} = (d_j)_{j \in J} \in \text{KP}_{\mathbf{i}}(\beta)$. We choose an identification

$$V^\beta = (V^{\alpha_{\mathbf{i},1}})^{\oplus d_1} \oplus \cdots \oplus (V^{\alpha_{\mathbf{i},\ell}})^{\oplus d_\ell} = \bigoplus_{j \in J} V^{\alpha_{\mathbf{i},j}} \otimes D_j,$$

where D_j is a \mathbb{C} -vector space of dimension d_j (“space of multiplicity”). Let $G_{\mathbf{d}} := \prod_{j \in J} \text{GL}(D_j)$. We have an injective homomorphism $G_{\mathbf{d}} \hookrightarrow G(\beta)$ given by $(g_j)_{j \in J} \mapsto (\text{id}_{V^{\alpha_{\mathbf{i},j}}} \otimes g_j)_{j \in J}$, through which we regard $G_{\mathbf{d}}$ as a subgroup of $G(\beta)$. Thus, the group $G_{\mathbf{d}}$ acts on V^β .

Let $\epsilon_c = (j_1, \dots, j_d)$ denote the unique costandard sequence in $J^{\mathbf{d}}$. We fix a basis $\{v_1, \dots, v_d\}$ of the vector space $\bigoplus_{j \in J} D_j$ such that $v_k \in D_{j_k}$. This yields a maximal torus $T_{\mathbf{d}}$ of $G_{\mathbf{d}}$ consisting of diagonal matrices with respect to the basis. The fixed locus $X(\beta)^{T_{\mathbf{d}}}$ is identical to the space

$$X(\mathbf{d}) := X(\alpha_{\mathbf{i},1})^{d_1} \times X(\alpha_{\mathbf{i},2})^{d_2} \times \cdots \times X(\alpha_{\mathbf{i},\ell})^{d_\ell}.$$

The quiver representation

$$x(\mathbf{d}) := x(\alpha_{\mathbf{i},1})^{\oplus d_1} \oplus x(\alpha_{\mathbf{i},2})^{\oplus d_2} \oplus \cdots \oplus x(\alpha_{\mathbf{i},\ell})^{\oplus d_\ell}$$

is regarded as a geometric point of $X(\mathbf{d}) = X(\beta)^{T_{\mathbf{d}}} \subset X(\beta)$. Let

$$i_{x(\mathbf{d})}: \{x(\mathbf{d})\} \hookrightarrow X(\mathbf{d})$$

denote the inclusion.

Let $d := \sum_{j \in J} d_j$. The symmetric group \mathfrak{S}_d acts on the set $J^{\mathbf{d}}$ by place permutations. For each sequence $\epsilon \in J^{\mathbf{d}}$, let $\sigma_\epsilon \in \mathfrak{S}_d$ denote the element of the smallest length such that $\epsilon = (j_{\sigma_\epsilon(1)}, \dots, j_{\sigma_\epsilon(d)})$. Then, we define a cocharacter $\tau_\epsilon \in X_*(T_{\mathbf{d}})$ by $\tau_\epsilon(s) \cdot v_{\sigma_\epsilon(k)} = s^k v_{\sigma_\epsilon(k)}$ for $1 \leq k \leq d$. In the notation of (5.5), we have

$$\begin{aligned} X(\beta)_{\tau_\epsilon}^0 &= \bigoplus_{1 \leq k \leq d} \bigoplus_{h \in Q_1} \text{Hom}_{\mathbb{C}}(V_{s(h)}^{\alpha_{\mathbf{i},\epsilon_k}} \otimes \mathbb{C}v_{\sigma_\epsilon(k)}, V_{t(h)}^{\alpha_{\mathbf{i},\epsilon_k}} \otimes \mathbb{C}v_{\sigma_\epsilon(k)}) = X(\mathbf{d}), \\ X(\beta)_{\tau_\epsilon}^+ &= \bigoplus_{1 \leq k < l \leq d} \bigoplus_{h \in Q_1} \text{Hom}_{\mathbb{C}}(V_{s(h)}^{\alpha_{\mathbf{i},\epsilon_k}} \otimes \mathbb{C}v_{\sigma_\epsilon(k)}, V_{t(h)}^{\alpha_{\mathbf{i},\epsilon_l}} \otimes \mathbb{C}v_{\sigma_\epsilon(l)}). \end{aligned}$$

In particular, we have an isomorphism

$$F(\epsilon) := X(\beta)_{\tau_\epsilon}^0 \oplus X(\beta)_{\tau_\epsilon}^+ \simeq F(\alpha_{\mathbf{i},\epsilon_1}, \alpha_{\mathbf{i},\epsilon_2}, \dots, \alpha_{\mathbf{i},\epsilon_d}).$$

Let

$$X(\mathbf{d}) \xleftarrow{\kappa_\epsilon} F(\epsilon) \xrightarrow{\iota_\epsilon} X(\beta)$$

be the diagram defined as in (7.2).

Proposition 7.5. *For any $\epsilon = (\epsilon_1, \dots, \epsilon_d) \in J^{\mathbf{d}}$, we have*

$$\tilde{E}_{\mathbf{i}}^*(\epsilon) = t^{c(\epsilon)} \sum_{\mathbf{d}' \in \text{KP}_{\mathbf{i}}(\beta)} \left(\sum_{n \in \mathbb{Z}} t^n \dim_{\mathbb{k}} H^n(i_{x(\mathbf{d})}^! \kappa_{\epsilon*} \iota_{\epsilon}^! \text{IC}(\mathbf{d}')) \right) \tilde{B}_{\mathbf{i}}^*(\mathbf{d}')$$

in the quantum unipotent coordinate ring $A_t[N(w)]_{\mathbb{Z}[t^{\pm 1/2}]}$, where

$$c(\epsilon) = -\dim X(\mathbf{d}) - \sum_{1 \leq k < l \leq d} (\text{h}(\alpha_{\mathbf{i},\epsilon_k}, \alpha_{\mathbf{i},\epsilon_l}) - \text{h}(\alpha_{\mathbf{i},\epsilon_l}, \alpha_{\mathbf{i},\epsilon_k})).$$

Proof. By definition, we have

$$\tilde{E}_{\mathbf{i}}^*(\epsilon) = t^{c_1} E_{\mathbf{i},\epsilon_1}^* \cdots E_{\mathbf{i},\epsilon_d}^* = t^{c_2} \sum_{\mathbf{d}' \in \text{KP}_{\mathbf{i}}(\beta)} \langle E_{\mathbf{i},\epsilon_1}^* \cdots E_{\mathbf{i},\epsilon_d}^*, B_{\mathbf{i}}(\mathbf{d}') \rangle \tilde{B}_{\mathbf{i}}^*(\mathbf{d}'),$$

where

$$\begin{aligned} c_1 &= \sum_{1 \leq k < l \leq d} \gamma_{\mathbf{i}}(\delta_{\epsilon_k}, \delta_{\epsilon_l}) - \frac{1}{4} \sum_{1 \leq k \leq d} (\alpha_{\mathbf{i},\epsilon_k}, \alpha_{\mathbf{i},\epsilon_k}), \\ c_2 &= c_1 + \frac{1}{4}(\beta, \beta) = \sum_{1 \leq k < l \leq d} \left(\gamma_{\mathbf{i}}(\delta_{\epsilon_k}, \delta_{\epsilon_l}) + \frac{1}{2}(\alpha_{\mathbf{i},\epsilon_k}, \alpha_{\mathbf{i},\epsilon_l}) \right). \end{aligned}$$

By Theorem 7.1, for each $\mathbf{d}' \in \mathrm{KP}_i(\beta)$, we have

$$\langle E_{i,\epsilon_1}^* \cdots E_{i,\epsilon_d}^*, B_i(\mathbf{d}') \rangle = \langle E_{i,\epsilon_1}^* \otimes \cdots \otimes E_{i,\epsilon_d}^*, \chi^{-1}[\mathrm{Res}_{\alpha_{i,\epsilon_1}, \dots, \alpha_{i,\epsilon_d}} \mathrm{IC}(\mathbf{d}')] \rangle.$$

This is equal to the graded multiplicity of the constant perverse sheaf

$$\mathcal{C}_{\alpha_{i,\epsilon_1}} \boxtimes \cdots \boxtimes \mathcal{C}_{\alpha_{i,\epsilon_d}} \simeq \underline{\mathbb{k}}_{X(\mathbf{d})}[\dim X(\mathbf{d})]$$

in $\mathrm{Res}_{\alpha_{i,\epsilon_1}, \dots, \alpha_{i,\epsilon_d}} \mathrm{IC}(\mathbf{d}')$ by Proposition 7.4. Since we have

$$i_{x(\mathbf{d})}^! (\underline{\mathbb{k}}_{X(\mathbf{d})}[\dim X(\mathbf{d})]) = \underline{\mathbb{k}}_{\{x(\mathbf{d})\}}[-\dim X(\mathbf{d})]$$

and $\underline{\mathbb{k}}_{X(\mathbf{d})}[\dim X(\mathbf{d})]$ is the unique simple $G_{\mathbf{d}}$ -equivariant perverse sheaf on $X(\mathbf{d})$ with a non-trivial (co)stalk at $x(\mathbf{d})$, the graded multiplicity in question can be computed as the Poincaré polynomial of

$$i_{x(\mathbf{d})}^! \mathrm{Res}_{\alpha_{i,\epsilon_1}, \dots, \alpha_{i,\epsilon_d}} \mathrm{IC}(\mathbf{d}')[\dim X(\mathbf{d})] = i_{x(\mathbf{d})}^! \kappa_{\epsilon^*} \iota_{\epsilon}^! \mathrm{IC}(\mathbf{d}') [c_3],$$

where

$$c_3 = \dim X(\mathbf{d}) + \sum_{1 \leq k < l \leq d} \langle \alpha_{i,\epsilon_k}, \alpha_{i,\epsilon_l} \rangle_Q.$$

Therefore, we get

$$\tilde{E}_i^*(\epsilon) = t^{c_2 - c_3} \sum_{\mathbf{d}' \in \mathrm{KP}_i(\beta)} \left(\sum_{n \in \mathbb{Z}} t^n \dim_{\mathbb{k}} \mathrm{H}^n(i_{x(\mathbf{d})}^! \kappa_{\epsilon^*} \iota_{\epsilon}^! \mathrm{IC}(\mathbf{d}')) \right) \tilde{B}_i^*(\mathbf{d}').$$

It remains to observe

$$\begin{aligned} c_2 - c_3 &= -\dim X(\mathbf{d}) + \sum_{1 \leq k < l \leq d} \left(\gamma_i(\delta_{\epsilon_k}, \delta_{\epsilon_l}) - \frac{1}{2}(\langle \alpha_{i,\epsilon_k}, \alpha_{i,\epsilon_l} \rangle_Q - \langle \alpha_{i,\epsilon_l}, \alpha_{i,\epsilon_k} \rangle_Q) \right) \\ &= -\dim X(\mathbf{d}) + \sum_{1 \leq k < l \leq d} (-\langle \alpha_{i,\epsilon_k}, \alpha_{i,\epsilon_l} \rangle_Q + \langle \alpha_{i,\epsilon_l}, \alpha_{i,\epsilon_k} \rangle_Q - e(\alpha_{i,\epsilon_k}, \alpha_{i,\epsilon_l}) + e(\alpha_{i,\epsilon_l}, \alpha_{i,\epsilon_k})) \\ &= -\dim X(\mathbf{d}) - \sum_{1 \leq k < l \leq d} (\mathrm{h}(\alpha_{i,\epsilon_k}, \alpha_{i,\epsilon_l}) - \mathrm{h}(\alpha_{i,\epsilon_l}, \alpha_{i,\epsilon_k})), \end{aligned}$$

where the first equality follows from (7.1), the second one follows from Lemma 7.6 below, and the last one follows from (7.4). \square

Lemma 7.6. *For any $j, k \in J$, we have*

$$\gamma_i(\delta_j, \delta_k) = -\frac{1}{2}(\langle \alpha_{i,j}, \alpha_{i,k} \rangle_Q - \langle \alpha_{i,k}, \alpha_{i,j} \rangle_Q) - e(\alpha_{i,j}, \alpha_{i,k}) + e(\alpha_{i,k}, \alpha_{i,j}).$$

Proof. Since both sides of the desired equality are skew-symmetric, we may assume that $j < k$. Then, we have $\gamma_i(\delta_j, \delta_k) = (\alpha_{i,j}, \alpha_{i,k})/2$ by definition. Now, the result follows from the formulas (7.1), (7.4), and Lemma 7.3. \square

7.4. Lusztig's transversal slice. In this subsection, we restrict the above geometric setting to a certain transversal slice $S(\mathbf{d})$ in $X(\beta)$ considered by Lusztig [48, Section 10]. This is an important step to apply the facts from Section 5.

To define the transversal slice $S(\mathbf{d})$, first we recall the following general fact about quiver representations. Let $x \in X(\beta), x' \in X(\beta')$ be two representations of Q . We have an exact sequence of \mathbb{C} -vector spaces

$$0 \rightarrow \mathrm{Hom}_Q(x, x') \rightarrow L(\beta, \beta') \rightarrow E(\beta, \beta') \rightarrow \mathrm{Ext}_Q^1(x, x') \rightarrow 0,$$

where the middle map is given by

$$L(\beta, \beta') \ni \varphi = (\varphi_i)_{i \in I} \mapsto (\varphi_{t(h)} x_h - x'_h \varphi_{s(h)})_{h \in Q_1} \in E(\beta, \beta').$$

Note that the equality (7.4) follows from this.

Now, we retain the notation from the previous subsection and consider the special case when $x = x' = x(\mathbf{d})$ and $\beta = \beta' = \sum_{j \in J} d_j \alpha_{i,j}$ to get the exact sequence

$$(7.5) \quad 0 \rightarrow \mathrm{Hom}_Q(x(\mathbf{d}), x(\mathbf{d})) \rightarrow L(\beta, \beta) \xrightarrow{\xi} X(\beta) \rightarrow \mathrm{Ext}_Q^1(x(\mathbf{d}), x(\mathbf{d})) \rightarrow 0.$$

Note that the middle map ξ is $\mathrm{Stab}_{G(\beta)} x(\mathbf{d})$ -equivariant, and hence, as $G_{\mathbf{d}} \subset \mathrm{Stab}_{G(\beta)} x(\mathbf{d})$, it is $G_{\mathbf{d}}$ -equivariant. Since $G_{\mathbf{d}}$ is a reductive group, we can find a $G_{\mathbf{d}}$ -stable linear subspace $E(\mathbf{d})$ of $X(\beta)$ such that

$$X(\beta) = \mathrm{Im} \xi \oplus E(\mathbf{d})$$

as $G_{\mathbf{d}}$ -representation. By (7.5), we have $E(\mathbf{d}) \simeq \mathrm{Ext}_Q^1(x(\mathbf{d}), x(\mathbf{d}))$ as \mathbb{C} -vector spaces. Let

$$S(\mathbf{d}) := x(\mathbf{d}) + E(\mathbf{d})$$

be the affine subspace of $X(\beta)$. Note that each geometric point of $S(\mathbf{d})$ is a quiver representation obtained as an extension of indecomposable ones $\{x(\alpha_{i,j})\}_{j \in J}$. Lemma 7.3 (2) implies that we have a finite stratification

$$(7.6) \quad S(\mathbf{d}) = \bigsqcup_{\mathbf{d}' \in \mathrm{KP}_1(\beta)} O(\mathbf{d}') \cap S(\mathbf{d}).$$

The variety $S(\mathbf{d})$ is a transversal slice through $x(\mathbf{d})$, meaning that it intersects transversally with each orbit $O(\mathbf{d}')$, $\mathbf{d}' \in \mathrm{KP}_1(\beta)$.

For each sequence $\epsilon \in J^{\mathbf{d}}$, we define

$$\begin{aligned} X(\epsilon) &:= x(\mathbf{d}) + X(\beta)_{\tau_\epsilon}^+ = \kappa_\epsilon^{-1}(x(\mathbf{d})), \\ S(\epsilon) &:= x(\mathbf{d}) + E(\mathbf{d})_{\tau_\epsilon}^+ = S(\mathbf{d}) \cap F(\epsilon). \end{aligned}$$

We have a commutative diagram:

$$(7.7) \quad \begin{array}{ccccc} X(\mathbf{d}) & \xleftarrow{\kappa_\epsilon} & F(\epsilon) & \xrightarrow{\iota_\epsilon} & X(\beta) \\ \uparrow i_{x(\mathbf{d})} & & \uparrow i_1 & & \uparrow i_{S(\mathbf{d})} \\ \{x(\mathbf{d})\} & \xleftarrow{p} & X(\epsilon) & & \\ \parallel & & \uparrow i_2 & & \\ \{x(\mathbf{d})\} & \xrightarrow{i_{\epsilon, x(\mathbf{d})}} & S(\epsilon) & \xrightarrow{i_\epsilon} & S(\mathbf{d}). \end{array}$$

Here the arrow p is the obvious map, and the arrows $i_1, i_2, i_3, i_{S(\mathbf{d})}, i_{\epsilon, x(\mathbf{d})}, i_\epsilon$ are the inclusions. Note that the upper left square and the right square are both cartesian. All the varieties in the diagram (7.7) are stable under the action of the maximal torus $T_{\mathbf{d}} \subset G_{\mathbf{d}}$ and all the morphisms in the diagram (7.7) are $T_{\mathbf{d}}$ -equivariant.

Lemma 7.7. *We have a natural isomorphism*

$$(7.8) \quad i_{x(\mathbf{d})}^! \kappa_{\epsilon^*} \iota_\epsilon^! \simeq i_{\epsilon, x(\mathbf{d})}^* i_\epsilon^! i_{S(\mathbf{d})}^! [\dim X(\beta) - \dim S(\mathbf{d}) + c(\epsilon)]$$

of functors from $D_{G(\beta)}^b(X(\beta), \mathbb{k})$ to $D_{T_{\mathbf{d}}}^b(\{x(\mathbf{d})\}, \mathbb{k})$.

Proof. By the base change and Proposition 5.4, we have

$$(7.9) \quad i_{x(\mathbf{d})}^! \kappa_{\epsilon} \iota_{\epsilon}^! \simeq p_* i_1^! \iota_{\epsilon}^! \simeq i_2^* i_1^! \iota_{\epsilon}^! \simeq i_{\epsilon, x(\mathbf{d})}^* i_3^* i_1^! \iota_{\epsilon}^!$$

in the notation from the digram (7.7). Let U_{ϵ} be the unipotent subgroup of $G(\beta)$ whose Lie algebra is $L(\beta, \beta)_{\tau_{\epsilon}}^+$. The varieties $F(\epsilon)$ and $X(\epsilon)$ is stable under the action of U_{ϵ} , and hence they are $(U_{\epsilon} \rtimes T_{\mathbf{d}})$ -varieties. In particular, for any $\mathcal{F} \in D_{G(\beta)}^b(X(\beta), \mathbb{k})$, the $!$ -restriction $i_1^! \iota_{\epsilon}^! \mathcal{F}$ can be seen as an object of $D_{U_{\epsilon} \rtimes T_{\mathbf{d}}}^b(X(\epsilon), \mathbb{k})$. We shall show a natural isomorphism

$$(7.10) \quad i_3^* \simeq i_3^! [2(\dim X(\epsilon) - \dim S(\epsilon))]$$

as functors from $D_{U_{\epsilon} \rtimes T_{\mathbf{d}}}^b(X(\epsilon), \mathbb{k})$ to $D_{T_{\mathbf{d}}}^b(S(\epsilon), \mathbb{k})$. Consider the factorization $i_3 = \pi_3 \circ s_3$:

$$\begin{array}{ccc} S(\epsilon) & \xrightarrow{i_3} & X(\epsilon), \\ & \searrow s_3 & \nearrow \pi_3 \\ & U_{\epsilon} \times S(\epsilon) & \end{array}$$

where s_3 and π_3 are $G_{\mathbf{d}}$ -equivariant morphisms defined by $s_3(x) := (1, x)$ and $\pi_3(g, x) := g \cdot x$. The morphism π_3 is a locally trivial fibration. Indeed, its differential at the point $(1, x(\mathbf{d}))$ is naturally identified with the linear map

$$L(\beta, \beta)_{\tau_{\epsilon}}^+ \oplus E(\mathbf{d})_{\tau_{\epsilon}}^+ \rightarrow X(\beta)_{\tau_{\epsilon}}^+ \quad \text{given by } (u, v) \mapsto \xi(u) + v$$

in the notation of (7.5). This is surjective thanks to the exactness of the sequence obtained from (7.5) by taking $(-)^+_{\tau_{\epsilon}}$ -parts. Since the action of \mathbb{C}^{\times} given by τ_{ϵ} contracts the variety $U_{\epsilon} \times S(\epsilon)$ (resp. $X(\epsilon)$) to the single point $(1, x(\mathbf{d}))$ (resp. $x(\mathbf{d})$), it follows that the morphism π_3 is surjective and its differential is surjective at any points. Thus, π_3 is a locally trivial fibration with smooth fibers, and hence we have

$$\pi_3^* \simeq \pi_3^! [2(\dim X(\epsilon) - \dim(U_{\epsilon} \times S(\epsilon)))]$$

as functors from $D_{U_{\epsilon} \rtimes T_{\mathbf{d}}}^b(X(\epsilon), \mathbb{k})$ to $D_{U_{\epsilon} \rtimes T_{\mathbf{d}}}^b(U_{\epsilon} \times S(\epsilon), \mathbb{k})$. On the other hand, we have the induction equivalence

$$s_3^* \simeq s_3^! [2(\dim(U_{\epsilon} \times S(\epsilon)) - \dim S(\epsilon))]: D_{U_{\epsilon} \rtimes T_{\mathbf{d}}}^b(U_{\epsilon} \times S(\epsilon), \mathbb{k}) \xrightarrow{\sim} D_{T_{\mathbf{d}}}^b(S(\epsilon), \mathbb{k}).$$

Combining the above isomorphisms with the natural isomorphisms $i_3^* \simeq s_3^* \pi_3^*$ and $i_3^! \simeq s_3^! \pi_3^!$, we arrive at the isomorphism (7.10).

Now, the isomorphisms (7.9) and (7.10) yield an isomorphism

$$i_{x(\mathbf{d})}^! \kappa_{\epsilon} \iota_{\epsilon}^! \simeq i_{\epsilon, x(\mathbf{d})}^* i_3^! i_1^! \iota_{\epsilon}^! [2(\dim X(\epsilon) - \dim S(\epsilon))] \simeq i_{\epsilon, x(\mathbf{d})}^* i_{\epsilon}^! i_{S(\mathbf{d})}^! [2(\dim X(\epsilon) - \dim S(\epsilon))].$$

It remains to check that the number $2(\dim X(\epsilon) - \dim S(\epsilon))$ coincides with $\dim X(\beta) - \dim S(\mathbf{d}) + e(\epsilon)$. This is done by noting the equalities

$$\begin{aligned} \dim X(\epsilon) &= \dim X(\beta)_{\tau_{\epsilon}}^+, & \dim X(\beta) &= \dim X(\beta)_{\tau_{\epsilon}}^+ + \dim X(\mathbf{d}) + \dim X(\beta)_{\tau_{\epsilon}}^-, \\ \dim S(\epsilon) &= \dim E(\mathbf{d})_{\tau_{\epsilon}}^+, & \dim S(\mathbf{d}) &= \dim E(\mathbf{d}) = \dim E(\mathbf{d})_{\tau_{\epsilon}}^+ + \dim E(\mathbf{d})_{\tau_{\epsilon}}^-, \end{aligned}$$

and computing as follows:

$$\begin{aligned} & 2(\dim X(\epsilon) - \dim S(\epsilon)) - (\dim X(\beta) - \dim S(\mathbf{d})) \\ &= -\dim X(\mathbf{d}) + (\dim X(\beta)_{\tau_{\epsilon}}^+ - \dim E(\mathbf{d})_{\tau_{\epsilon}}^+) - (\dim X(\beta)_{\tau_{\epsilon}}^- - \dim E(\mathbf{d})_{\tau_{\epsilon}}^-) \\ &= -\dim X(\mathbf{d}) + (\dim L(\beta, \beta)_{\tau_{\epsilon}}^+ - \dim H(\mathbf{d})_{\tau_{\epsilon}}^+) - (\dim L(\beta, \beta)_{\tau_{\epsilon}}^- - \dim H(\mathbf{d})_{\tau_{\epsilon}}^-) \\ &= -\dim X(\mathbf{d}) - \dim H(\mathbf{d})_{\tau_{\epsilon}}^+ + \dim H(\mathbf{d})_{\tau_{\epsilon}}^- = c(\epsilon), \end{aligned}$$

where we put $H(\mathbf{d}) := \text{Hom}_Q(x(\mathbf{d}), x(\mathbf{d}))$. Here, the second equality follows from the exactness of the sequence (7.5), and the third one is due to an obvious equality $\dim L(\beta, \beta)_{\tau_\epsilon}^+ = \dim L(\beta, \beta)_{\tau_\epsilon}^-$. Thus, we obtain the desired isomorphism (7.8). \square

Proposition 7.8. *For any $\mathbf{d} \in \mathbb{N}^{\oplus J}$, we have*

$$\text{IC}(\mathbf{d}) = \text{IC}(\overline{O(\mathbf{d})}, \mathbb{k}).$$

Proof. First we show $\text{IC}(\overline{O(\mathbf{d})}, \mathbb{k}) \in \mathcal{P}$. Let $\epsilon_c = (j_1, \dots, j_d)$ be the unique costandard sequence in $J^{\mathbf{d}}$ as before. Then $\epsilon_s = \epsilon_c^{\text{op}} = (j_d, \dots, j_1)$ is the unique standard sequence in $J^{\mathbf{d}}$. The image of the proper map

$$p_3: \tilde{F}(\epsilon_s) := \tilde{F}(\alpha_{i,j_d}, \dots, \alpha_{i,j_1}) \rightarrow X(\beta),$$

which appeared in the definition of the induction functor $\text{Ind}_{\alpha_{i,j_d}, \dots, \alpha_{i,j_1}}$, contains a dense subset consisting of quiver representations $x \in X(\beta)$ which respects an I -graded flag $V^\beta = F^d \supset F^{d-1} \supset \dots \supset F^1 \supset F^0 = 0$ and satisfies $x|_{F^k/F^{k-1}} \simeq x(\alpha_{i,j_k})$ for $1 \leq k \leq d$. Since $e(\alpha_{i,j_l}, \alpha_{i,j_k}) = 0$ for $1 \leq k \leq l \leq d$ by Lemma 7.3 (1), such a representation x is always isomorphic to $x(\mathbf{d})$, and hence $p_3(\tilde{F}(\epsilon_s)) = \overline{O(\mathbf{d})}$. Thus, the support of the object

$$\mathcal{C}_{\alpha_{i,j_d}} \star \dots \star \mathcal{C}_{\alpha_{i,j_1}} = (p_3)_* \mathbb{k}_{\tilde{F}(\epsilon_s)}[\dim \tilde{F}(\epsilon_s)]$$

coincides with the orbit closure $\overline{O(\mathbf{d})}$. Since the complex $\text{IC}(\overline{O(\mathbf{d})}, \mathbb{k})$ is the unique simple $G(\beta)$ -equivariant perverse sheaf on $X(\beta)$ whose support coincides with $\overline{O(\mathbf{d})}$, some of its shifts must contribute to $\mathcal{C}_{\alpha_{i,j_d}} \star \dots \star \mathcal{C}_{\alpha_{i,j_1}}$ as direct summands. By Theorem 7.1 and Proposition 7.4, the object $\mathcal{C}_{\alpha_{i,j_d}} \star \dots \star \mathcal{C}_{\alpha_{i,j_1}}$ belongs to the category \mathcal{Q} . Therefore, $\text{IC}(\overline{O(\mathbf{d})}, \mathbb{k})$ belongs to \mathcal{P} .

Now, in order to verify $\text{IC}(\mathbf{d}) = \text{IC}(\overline{O(\mathbf{d})}, \mathbb{k})$, it suffice to show the equality

$$(7.11) \quad \langle E_{\mathbf{i}}^*(\mathbf{d}), \chi^{-1}[\text{IC}(\overline{O(\mathbf{d})}, \mathbb{k})] \rangle = 1$$

by the characterization of $B_{\mathbf{i}}^*(\mathbf{d})$ in Theorem 4.1. By the same computation as in the proof of Proposition 7.5, the LHS of (7.11) is equal to

$$\begin{aligned} & t^{c(\epsilon_s)} \sum_{n \in \mathbb{Z}} t^n \dim_{\mathbb{k}} \text{H}^n(i_{x(\mathbf{d})}^! \kappa_{\epsilon_s}^* i_{\epsilon_s}^! \text{IC}(\overline{O(\mathbf{d})}, \mathbb{k})) \\ &= \sum_{n \in \mathbb{Z}} t^n \dim_{\mathbb{k}} \text{H}^n(i_{\epsilon_s}^! i_{S(\mathbf{d})}^! \text{IC}(\overline{O(\mathbf{d})}, \mathbb{k})[\dim X(\beta) - \dim S(\mathbf{d})]) \\ &= \sum_{n \in \mathbb{Z}} t^n \dim_{\mathbb{k}} \text{H}^n((i_{S(\mathbf{d})} \circ i_{\epsilon_s})^! \text{IC}(\overline{O(\mathbf{d})}, \mathbb{k})[\dim O(\mathbf{d})]), \end{aligned}$$

where the first equality is due to Lemma 7.7. Note that $S(\epsilon_s) = \{x(\mathbf{d})\}$ and $i_{S(\mathbf{d})} \circ i_{\epsilon_s}$ is the inclusion $\{x(\mathbf{d})\} \hookrightarrow X(\beta)$. In particular, we have

$$(i_{S(\mathbf{d})} \circ i_{\epsilon_s})^! \text{IC}(\overline{O(\mathbf{d})}, \mathbb{k})[\dim O(\mathbf{d})] \simeq \mathbb{k}_{\{x(\mathbf{d})\}}$$

and hence the desired equality (7.11) follows. \square

Proposition 7.9. *For any $\mathbf{d} \in \text{KP}_{\mathbf{i}}(\beta)$ and $\epsilon \in J^{\mathbf{d}}$, we have the equality*

$$(7.12) \quad \tilde{E}_{\mathbf{i}}^*(\epsilon) = \sum_{\mathbf{d}' \in \text{KP}_{\mathbf{i}}(\beta)} \left(\sum_{n \in \mathbb{Z}} t^n \dim_{\mathbb{k}} \text{H}^n(i_{\epsilon, x(\mathbf{d}')}^* i_{\epsilon}^! \text{IC}(\overline{O(\mathbf{d}')} \cap S(\mathbf{d}'), \mathbb{k})) \right) \tilde{B}_{\mathbf{i}}^*(\mathbf{d}')$$

in the quantum unipotent coordinate ring $A_t[N(w)]_{\mathbb{Z}[t^{\pm 1/2}]}$.

Proof. Since $S(\mathbf{d})$ is a transversal slice, we have

$$i_{S(\mathbf{d})}^! \mathrm{IC}(\overline{O(\mathbf{d}')} , \mathbb{k})[\dim X(\beta) - \dim S(\mathbf{d})] \simeq \mathrm{IC}(\overline{O(\mathbf{d}')} \cap S(\mathbf{d}), \mathbb{k})$$

for any $\mathbf{d}' \in \mathrm{KP}_1(\beta)$ (cf. [25, Theorem 5.4.1]). Therefore, the assertion follows from Proposition 7.5 together with Lemma 7.7 and Proposition 7.8. \square

7.5. Geometric realization of symmetric quiver Hecke algebras. Now, we briefly review the geometric interpretation of the symmetric quiver Hecke algebras to Varagnolo-Vasserot [59]. Let $\beta \in \mathbb{Q}^+$. Recall the complex \mathcal{L}_β defined in (7.3).

Theorem 7.10 ([59]). *There is an isomorphism of graded \mathbb{k} -algebras*

$$(7.13) \quad H_\beta \simeq \mathrm{Hom}_{G_\beta}^\bullet(\mathcal{L}_\beta, \mathcal{L}_\beta).$$

For each $b \in \mathbb{B}$, we fix a representative $\mathrm{IC}_b \in \mathcal{Q}$ of the class $\chi(b)$. Proposition 7.8 implies $\mathrm{IC}_b \simeq \mathrm{IC}(\overline{O(\mathbf{d})}, \mathbb{k})$ if $b = B_1(\mathbf{d})$ for some $\mathbf{d} \in \mathbb{N}^{\oplus J}$. By the decomposition theorem, we have

$$(7.14) \quad \mathcal{L}_\beta \simeq \bigoplus_{b \in \mathbb{B}_\beta} \mathrm{IC}_b \otimes_{\mathbb{k}} L_b^\bullet$$

for some finite-dimensional self-dual graded vector space L_b^\bullet . Through the isomorphism (7.13) in Theorem 7.10, we can regard L_b as a graded simple H_β -module. The set $\{L_b^\bullet \mid b \in \mathbb{B}_\beta\}$ gives a complete system of representatives of the self-dual simple isomorphism classes of the category $\mathcal{M}_{\mathfrak{f}, \beta}^\bullet$. Under the isomorphism (4.3) in Theorem 4.3, the class $[L_b^\bullet]$ corresponds to the dual element $b^* \in \mathbb{B}^*$. Taking the total perverse cohomology, we define

$$\bar{\mathcal{L}}_\beta := \bigoplus_{k \in \mathbb{Z}} {}^p \mathcal{H}^k(\mathcal{L}_\beta) = \bigoplus_{b \in \mathbb{B}_\beta} \mathrm{IC}_b \otimes_{\mathbb{k}} L_b,$$

where L_b denotes the ungraded finite-dimensional \mathbb{C} -vector space obtained from L_b^\bullet by forgetting the grading. Since $\bar{\mathcal{L}}_\beta$ is a semisimple perverse sheaf, its Yoneda algebra

$$\mathrm{Hom}_{G_\beta}^\bullet(\bar{\mathcal{L}}_\beta, \bar{\mathcal{L}}_\beta)$$

is non-negatively graded, whose degree zero part is isomorphic to the semisimple algebra $\bigoplus_{b \in \mathbb{B}_\beta} \mathrm{End}_{\mathbb{k}}(L_b)$. Let $\mathrm{Hom}_{G_\beta}^\bullet(\bar{\mathcal{L}}_\beta, \bar{\mathcal{L}}_\beta)^\wedge$ denote its completion along the grading.

Corollary 7.11. *There is an isomorphism of \mathbb{k} -algebras*

$$\widehat{H}_\beta \simeq \mathrm{Hom}_{G_\beta}^\bullet(\bar{\mathcal{L}}_\beta, \bar{\mathcal{L}}_\beta)^\wedge.$$

The set $\{L_b \mid b \in \mathbb{B}\}$ gives a complete system of representative of the simple isomorphism classes of the category $\mathcal{M}_{\mathfrak{f}}^{\mathrm{nilp}} = \widehat{H}\text{-mod}$. Through the isomorphism in Corollary 4.4, the class $[L_b]$ corresponds to the specialized element $b^*|_{t=1}$.

When $b = B_1(\mathbf{d})$ for some $\mathbf{d} \in \mathbb{N}^{\oplus J}$, we write $L_1(\mathbf{d})$ for L_b . Note that this notation is compatible with the previous one in Section 4.4.

7.6. Geometric realization of mixed convolution products. Let $\beta \in \mathbb{Q}^+$ and $\mathbf{d} \in \mathrm{KP}_1(\beta)$. In this subsection, we establish a geometric realization of the mixed products $M_1(\epsilon)$ and their deformations $\tilde{M}_1(\epsilon)$ for any $\epsilon \in J^{\mathbf{d}}$.

Let M^\bullet be a graded H_β -module and z an indeterminate of degree 2. Endow the graded \mathbb{k} -vector space $M^\bullet[z] := M^\bullet \otimes \mathbb{k}[z]$ with an H_β -module structure by the same formulas as

(4.6) with $a(z) = z$. The resulting graded H_β -module $M^\bullet[z]$ is called the *affinization* of M^\bullet . Note that, for any $j \in J$, we have an isomorphism

$$\tilde{L}_{\mathbf{i},j} = (L_{\mathbf{i},j})_{jz} \simeq L_{\mathbf{i},j}^\bullet[z] \otimes_{\mathbb{k}[z]} \mathbb{O}$$

of $\widehat{H}_{\alpha_{\mathbf{i},j}}$ -modules, where $\mathbb{k}[z] \rightarrow \mathbb{O} = \mathbb{k}[[z]]$ is given by $z \mapsto jz$. A proof of the following lemma is given later in Section 7.9.

Lemma 7.12. *For each $j \in J$, we have an isomorphism of graded $H_{\alpha_{\mathbf{i},j}}$ -modules*

$$L_{\mathbf{i},j}^\bullet[z] \simeq \mathrm{Hom}_{G(\alpha_{\mathbf{i},j})}^\bullet((i_{O(\alpha_{\mathbf{i},j})})! \underline{\mathbb{k}}_{O(\alpha_{\mathbf{i},j})}, \mathcal{L}_{\alpha_{\mathbf{i},j}}) \langle \dim X(\alpha_{\mathbf{i},j}) \rangle,$$

where $i_{O(\alpha_{\mathbf{i},j})}: O(\alpha_{\mathbf{i},j}) \hookrightarrow X(\alpha_{\mathbf{i},j})$ denotes the inclusion.

We retain the notation from the previous subsections. Let us consider a cocharacter $\rho^\vee \in X_*(T_{\mathbf{d}})$ given by

$$\rho^\vee(s)|_{D_j} = s^j \cdot \mathrm{id}_{D_j}$$

for any $j \in J$. In what follows, we regard a $T_{\mathbf{d}}$ -variety as a \mathbb{C}^\times -variety through $\rho^\vee: \mathbb{C}^\times \rightarrow T_{\mathbf{d}}$.

Proposition 7.13. *For each $\epsilon \in J^{\mathbf{d}}$, we have an isomorphism of \widehat{H}_β -modules*

$$\tilde{M}_{\mathbf{i}}(\epsilon) \simeq \widehat{H}_{\mathbb{C}^\times}^\bullet(i_{x(\mathbf{d})}^! \kappa_{\epsilon*} l_\epsilon^! \bar{\mathcal{L}}_\beta),$$

which specializes to

$$M_{\mathbf{i}}(\epsilon) \simeq H^\bullet(i_{x(\mathbf{d})}^! \kappa_{\epsilon*} l_\epsilon^! \bar{\mathcal{L}}_\beta).$$

Proof. From the definition, we have

$$\tilde{M}_{\mathbf{i}}(\epsilon) \simeq (L_{\mathbf{i},\epsilon_1}^\bullet[z_1] \star \cdots \star L_{\mathbf{i},\epsilon_d}^\bullet[z_d]) \otimes_{\mathbb{C}[z_1, \dots, z_d]} \mathbb{O}.$$

Here the tensor product $- \otimes_{\mathbb{k}[z_1, \dots, z_d]} \mathbb{O}$ is taken with respect to the \mathbb{k} -algebra homomorphism $\mathbb{k}[z_1, \dots, z_d] \rightarrow \mathbb{O} = \mathbb{k}[[z]]$ given by $z_k \mapsto \epsilon_k z$ for $1 \leq k \leq d$, which is identified with the homomorphism $H_{T_{\mathbf{d}}}^\bullet(\mathrm{pt}, \mathbb{k}) \rightarrow \widehat{H}_{\mathbb{C}^\times}^\bullet(\mathrm{pt}, \mathbb{k})$ induced from the cocharacter $\rho^\vee: \mathbb{C}^\times \rightarrow T_{\mathbf{d}}$. Unpacking the definition, we have

$$L_{\mathbf{i},\epsilon_1}^\bullet[z_1] \star \cdots \star L_{\mathbf{i},\epsilon_d}^\bullet[z_d] = H_\beta e(\epsilon) \otimes_{H_\epsilon} (L_{\mathbf{i},\epsilon_1}^\bullet[z_1] \otimes \cdots \otimes L_{\mathbf{i},\epsilon_d}^\bullet[z_d])$$

where we abbreviate $e(\epsilon) := e(\alpha_{\mathbf{i},\epsilon_1}, \dots, \alpha_{\mathbf{i},\epsilon_d})$ and $H_\epsilon := H_{\alpha_{\mathbf{i},\epsilon_1}, \dots, \alpha_{\mathbf{i},\epsilon_d}}$. Thanks to Theorem 7.10, Lemma 7.12 and [1, Proposition 6.7.5], we have the graded isomorphisms

$$H_\beta e(\epsilon) \simeq \mathrm{Hom}_{G(\beta)}^\bullet(\mathcal{L}_{\alpha_{\mathbf{i},\epsilon_1}} \star \cdots \star \mathcal{L}_{\alpha_{\mathbf{i},\epsilon_d}}, \mathcal{L}_\beta) \simeq \mathrm{Hom}_{G^{\mathbf{d}}}^\bullet(\mathcal{L}^{\mathbf{d}}, \kappa_{\epsilon*} l_\epsilon^! \mathcal{L}_\beta) \langle c \rangle,$$

$$H_\epsilon \simeq \mathrm{Hom}_{G^{\mathbf{d}}}^\bullet(\mathcal{L}^{\mathbf{d}}, \mathcal{L}^{\mathbf{d}}),$$

$$L_{\mathbf{i},\epsilon_1}^\bullet[z_1] \otimes \cdots \otimes L_{\mathbf{i},\epsilon_d}^\bullet[z_d] \simeq \mathrm{Hom}_{G^{\mathbf{d}}}^\bullet((i_{O^{\mathbf{d}}})! \underline{\mathbb{k}}_{O^{\mathbf{d}}}, \mathcal{L}^{\mathbf{d}}) \langle \dim X(\mathbf{d}) \rangle,$$

where $G^{\mathbf{d}} := G(\alpha_{\mathbf{i},1})^{d_1} \times \cdots \times G(\alpha_{\mathbf{i},\ell})^{d_\ell}$, $O^{\mathbf{d}} := O(\alpha_{\mathbf{i},1})^{d_1} \times \cdots \times O(\alpha_{\mathbf{i},\ell})^{d_\ell} \subset X(\mathbf{d})$, $\mathcal{L}^{\mathbf{d}} := \mathcal{L}_{\alpha_{\mathbf{i},1}}^{\boxtimes d_1} \boxtimes \cdots \boxtimes \mathcal{L}_{\alpha_{\mathbf{i},\ell}}^{\boxtimes d_\ell} \in D_{G^{\mathbf{d}}}^b(X(\mathbf{d}), \mathbb{k})$ and $c := \sum_{1 \leq k < l \leq d} \langle \alpha_{\mathbf{i},\epsilon_k}, \alpha_{\mathbf{i},\epsilon_l} \rangle_Q$. (Be aware that the group $G^{\mathbf{d}}$ is different from the group $G_{\mathbf{d}}$ from the previous sections. In fact, we have $G_{\mathbf{d}} \cap G^{\mathbf{d}} = T_{\mathbf{d}}$.) Note that both $\mathcal{L}^{\mathbf{d}}$ and $\kappa_{\epsilon*} l_\epsilon^! \mathcal{L}_\beta$ belong to the category $\mathcal{Q}^{\mathbf{d}} := \mathcal{Q}_{\alpha_{\mathbf{i},1}}^{\boxtimes d_1} \boxtimes \cdots \boxtimes \mathcal{Q}_{\alpha_{\mathbf{i},\ell}}^{\boxtimes d_\ell} \subset D_{G^{\mathbf{d}}}^b(X(\mathbf{d}), \mathbb{k})$, and every indecomposable object of $\mathcal{Q}^{\mathbf{d}}$ appears up to shift as a direct summand of the object

$\mathcal{L}^{\mathbf{d}}$ by definition. Therefore, we have

$$\begin{aligned}
& L_{\mathbf{i}, \epsilon_1}^\bullet [z_1] \star \cdots \star L_{\mathbf{i}, \epsilon_d}^\bullet [z_d] \\
& \simeq \mathrm{Hom}_{G^{\mathbf{d}}}^\bullet(\mathcal{L}^{\mathbf{d}}, \kappa_{\epsilon^\star} \iota_\epsilon^! \mathcal{L}_\beta) \langle c \rangle \otimes_{\mathrm{Hom}_{G^{\mathbf{d}}}^\bullet(\mathcal{L}^{\mathbf{d}}, \mathcal{L}^{\mathbf{d}})} \mathrm{Hom}_{G^{\mathbf{d}}}^\bullet((i_{O^{\mathbf{d}}})_! \mathbb{k}_{O^{\mathbf{d}}}, \mathcal{L}^{\mathbf{d}}) \langle \dim X(\mathbf{d}) \rangle \\
& \simeq \mathrm{Hom}_{G^{\mathbf{d}}}^\bullet((i_{O^{\mathbf{d}}})_! \mathbb{k}_{O^{\mathbf{d}}}, \kappa_{\epsilon^\star} \iota_\epsilon^! \mathcal{L}_\beta) \langle c + \dim X(\mathbf{d}) \rangle \\
& \simeq \mathrm{H}_{G^{\mathbf{d}}}^\bullet(i_{O^{\mathbf{d}}}^! \kappa_{\epsilon^\star} \iota_\epsilon^! \mathcal{L}_\beta) \langle c + \dim X(\mathbf{d}) \rangle \\
& \simeq \mathrm{H}_{T_{\mathbf{d}}}^\bullet(i_{x(\mathbf{d})}^! \kappa_{\epsilon^\star} \iota_\epsilon^! \mathcal{L}_\beta) \langle c + \dim X(\mathbf{d}) \rangle
\end{aligned}$$

as graded H_β -modules, where the last equality is due to the induction equivalence with $\mathrm{Stab}_{G^{\mathbf{d}}} x(\mathbf{d}) = T_{\mathbf{d}}$. As a consequence, we obtain

$$\tilde{M}_{\mathbf{i}}(\epsilon) \simeq \mathrm{H}_{T_{\mathbf{d}}}^\bullet(i_{x(\mathbf{d})}^! \kappa_{\epsilon^\star} \iota_\epsilon^! \mathcal{L}_\beta) \otimes_{\mathrm{H}_{T_{\mathbf{d}}}^\bullet(\mathrm{pt}, \mathbb{k})} \widehat{\mathrm{H}}_{\mathbb{C}^\times}^\bullet(\mathrm{pt}, \mathbb{k}) \simeq \widehat{\mathrm{H}}_{\mathbb{C}^\times}^\bullet(i_{x(\mathbf{d})}^! \kappa_{\epsilon^\star} \iota_\epsilon^! \bar{\mathcal{L}}_\beta)$$

as \widehat{H}_β -modules. Here the last isomorphism follows from [1, Lemma 6.7.4]. The same lemma in [1] also yields the specialized isomorphism

$$M_{\mathbf{i}}(\epsilon) \simeq \mathrm{H}_{\mathbb{C}^\times}^\bullet(i_{x(\mathbf{d})}^! \kappa_{\epsilon^\star} \iota_\epsilon^! \bar{\mathcal{L}}_\beta) \otimes_{\mathrm{H}_{\mathbb{C}^\times}^\bullet(\mathrm{pt}, \mathbb{k})} \mathbb{k} \simeq \mathrm{H}^\bullet(i_{x(\mathbf{d})}^! \kappa_{\epsilon^\star} \iota_\epsilon^! \bar{\mathcal{L}}_\beta),$$

which completes the proof. \square

Now, restricting to the transversal slice $S(\mathbf{d}) \subset X(\beta)$ considered in Section 7.4, we define

$$\begin{aligned}
(7.15) \quad \bar{\mathcal{A}}_{\mathbf{d}} & := i_{S(\mathbf{d})}^! \bar{\mathcal{L}}_\beta [\dim X(\beta) - \dim S(\mathbf{d})] \\
& \simeq \bigoplus_{\mathbf{d}' \in \mathrm{KP}_1(\beta)} \mathrm{IC}(\overline{O(\mathbf{d}')} \cap S(\mathbf{d}), \mathbb{k}) \otimes L_{\mathbf{i}}(\mathbf{d}'),
\end{aligned}$$

which is a $T_{\mathbf{d}}$ -equivariant semisimple perverse sheaf. Here, the isomorphism is due to (7.6), Proposition 7.8 and [25, Theorem 5.4.1]. The functor $i_{S(\mathbf{d})}^!$ induces a \mathbb{k} -algebra homomorphism

$$(7.16) \quad \widehat{H}_\beta \simeq \mathrm{Hom}_{G(\beta)}^\bullet(\bar{\mathcal{L}}_\beta, \bar{\mathcal{L}}_\beta)^\wedge \rightarrow \mathrm{Hom}_{T_{\mathbf{d}}}^\bullet(\bar{\mathcal{A}}_{\mathbf{d}}, \bar{\mathcal{A}}_{\mathbf{d}})^\wedge.$$

Proposition 7.14. *For each $\epsilon \in J^{\mathbf{d}}$, we have an isomorphism of \widehat{H}_β -modules*

$$\tilde{M}_{\mathbf{i}}(\epsilon) \simeq \widehat{\mathrm{H}}_{\mathbb{C}^\times}^\bullet(i_{\epsilon, x(\mathbf{d})}^* i_\epsilon^! \bar{\mathcal{A}}_{\mathbf{d}}),$$

where the \widehat{H}_β -module structure on the RHS is given through the homomorphism (7.16). Specializing at $z = 0$, we obtain

$$M_{\mathbf{i}}(\epsilon) \simeq \mathrm{H}^\bullet(i_{\epsilon, x(\mathbf{d})}^* i_\epsilon^! \bar{\mathcal{A}}_{\mathbf{d}}).$$

Proof. The assertion follows from Proposition 7.13 together with Lemma 7.7. \square

7.7. Geometric interpretation of R -matrices. In this subsection, we establish a geometric interpretation of the renormalized R -matrices between the deformed mixed tensor products, analogous to Theorem 6.7 for the quantum loop algebras. First, we need a lemma.

Lemma 7.15. *For any $j, k \in J$, we have*

$$\alpha(j, k) = e(\alpha_{\mathbf{i}, j}, \alpha_{\mathbf{i}, k}) + e(\alpha_{\mathbf{i}, k}, \alpha_{\mathbf{i}, j}).$$

Proof. We may assume $j > k$ without loss of generality. Then, we have $e(\alpha_{\mathbf{i}, j}, \alpha_{\mathbf{i}, k}) = 0$ by Lemma 7.3 (1). We have to show $\alpha(j, k) = e := e(\alpha_{\mathbf{i}, k}, \alpha_{\mathbf{i}, j})$. Let $\mathbf{d} = \boldsymbol{\delta}_j + \boldsymbol{\delta}_k \in \mathbb{N}^{\oplus J}$, $\epsilon = \epsilon_s(\mathbf{d}) = (j, k)$ and $\epsilon' = \epsilon_c(\mathbf{d}) = (k, j)$. We abbreviate $E = E(\mathbf{d}) \simeq \mathbb{C}^e$, $x = x(\mathbf{d})$, and $S = S(\mathbf{d}) = x + E$. We have $S(\epsilon) = \{x\}$ and $S(\epsilon') = S$. The action of $s \in \mathbb{C}^\times$ on E (through

ρ^\vee) is simply the multiplication by s^{j-k} . By Proposition 7.14 and Proposition 5.4, we have isomorphisms

$$\begin{aligned}\tilde{L}_{i,j} \star_{\mathbb{O}} \tilde{L}_{i,k} &\simeq \widehat{H}_{\mathbb{C}^\times}^\bullet(i^! \bar{\mathcal{A}}_d) \simeq \text{Hom}_{\mathbb{C}^\times}^\bullet(\mathbb{k}_{\{x\}}, i^! \bar{\mathcal{A}}_d)^\wedge \simeq \text{Hom}_{\mathbb{C}^\times}^\bullet(i_* i^* \mathbb{k}_S, \bar{\mathcal{A}}_d)^\wedge, \\ \tilde{L}_{i,k} \star_{\mathbb{O}} \tilde{L}_{i,j} &\simeq \widehat{H}_{\mathbb{C}^\times}^\bullet(i^* \bar{\mathcal{A}}_d) \simeq \text{Hom}_{\mathbb{C}^\times}^\bullet(\mathbb{k}_{\{x\}}, p_* \bar{\mathcal{A}}_d)^\wedge \simeq \text{Hom}_{\mathbb{C}^\times}^\bullet(\mathbb{k}_S, \bar{\mathcal{A}}_d)^\wedge,\end{aligned}$$

where $i: \{x\} \hookrightarrow S$ and $p: S \rightarrow \{x\}$ are the trivial maps. The adjunction morphisms

$$\mathbb{k}_S \rightarrow i_* i^* \mathbb{k}_S, \quad \text{and} \quad i_* i^* \mathbb{k}_S \simeq i_! i^! \mathbb{k}_S[2e] \rightarrow \mathbb{k}_S[2e]$$

respectively give rise to the homomorphisms

$$\begin{aligned}\mathcal{R}_{j,k} &: \text{Hom}_{\mathbb{C}^\times}^\bullet(i_* i^* \mathbb{k}_S, \bar{\mathcal{A}}_d) \rightarrow \text{Hom}_{\mathbb{C}^\times}^\bullet(\mathbb{k}_S, \bar{\mathcal{A}}_d), \\ \mathcal{R}_{k,j} &: \text{Hom}_{\mathbb{C}^\times}^\bullet(\mathbb{k}_S, \bar{\mathcal{A}}_d) \rightarrow \text{Hom}_{\mathbb{C}^\times}^\bullet(i_* i^* \mathbb{k}_S, \bar{\mathcal{A}}_d)(-2e)\end{aligned}$$

of graded $\text{Hom}_{T_d}^\bullet(\bar{\mathcal{A}}_d, \bar{\mathcal{A}}_d)$ -modules. Let $\bar{\mathcal{R}}_{j,k}$ and $\bar{\mathcal{R}}_{k,j}$ denote their specializations at $z = 0$ respectively, which are obtained simply by forgetting the \mathbb{C}^\times -equivariance. Since S has the finite stratification (7.6), there is a unique $\mathbf{d}' \in \text{KP}_i(\beta)$ with $\beta = \alpha_{i,j} + \alpha_{i,k}$ such that $S = \overline{O(\mathbf{d}')} \cap S$. Then, the decomposition (7.15) tells us that the perverse sheaf $\bar{\mathcal{A}}_d$ contains both $\mathbb{k}_{\{x\}} = i_* i^* \mathbb{k}_S$ and $\mathbb{k}_S[e]$ as summands. Thus, it follows that the specializations $\bar{\mathcal{R}}_{j,k}$ and $\bar{\mathcal{R}}_{k,j}$ are both non-zero, and hence the completions of $\mathcal{R}_{j,k}$ and $\mathcal{R}_{k,j}$ are identical to the renormalized R -matrices $R_{j,k}$ and $R_{k,j}$ respectively (up to multiples in \mathbb{O}^\times). In particular, we have $\mathcal{R}_{k,j} \circ \mathcal{R}_{j,k} = a z^{\alpha(j,k)} \text{id}$ for some $a \in \mathbb{k}^\times$. On the other hand, $\mathcal{R}_{k,j} \circ \mathcal{R}_{j,k}$ is a graded homomorphism of degree $2e$ by construction. Therefore, we get $\alpha(j,k) = e$ as desired. \square

Recall the preorder \lesssim of J^d from Section 2.4. Lemmas 7.3 (1) and 7.15 implies the following.

Corollary 7.16. *For $\epsilon, \epsilon' \in J^d$, we have $S(\epsilon) \subset S(\epsilon')$ if and only if $\epsilon \lesssim \epsilon'$. For the standard (resp. costandard) sequence ϵ_s (resp. ϵ_c), we have $S(\epsilon_s) = \{x(\mathbf{d})\}$ (resp. $S(\epsilon_c) = S(\mathbf{d})$).*

For $\epsilon, \epsilon' \in J^d$ satisfying $\epsilon \lesssim \epsilon'$, let $i_{\epsilon',\epsilon}: S(\epsilon) \hookrightarrow S(\epsilon')$ denote the inclusion. Note that we have $i_{\epsilon,x(\mathbf{d})} = i_{\epsilon,\epsilon_s}$ and $i_\epsilon = i_{\epsilon_c,\epsilon}$ for any $\epsilon \in J^d$. We have the following diagram of inclusions

$$\begin{array}{ccc} & S(\epsilon_c) = S(\mathbf{d}) & \\ & \nearrow i_\epsilon & \nwarrow i_{\epsilon'} \\ S(\epsilon) & \xrightarrow{i_{\epsilon',\epsilon}} & S(\epsilon'). \\ & \nwarrow i_{\epsilon,x(\mathbf{d})} & \nearrow i_{\epsilon',x(\mathbf{d})} \\ & S(\epsilon_s) = \{x(\mathbf{d})\} & \end{array}$$

Then the canonical morphism of functors $i_{\epsilon',\epsilon}^! \rightarrow i_{\epsilon',\epsilon}^*$ induces a morphism

$$i_{\epsilon,x(\mathbf{d})}^! i_{\epsilon'}^! \bar{\mathcal{A}}_d = i_{\epsilon,x(\mathbf{d})}^* i_{\epsilon',\epsilon}^! i_{\epsilon'}^! \bar{\mathcal{A}}_d \rightarrow i_{\epsilon,x(\mathbf{d})}^* i_{\epsilon',\epsilon}^* i_{\epsilon'}^! \bar{\mathcal{A}}_d = i_{\epsilon',x(\mathbf{d})}^* i_{\epsilon'}^! \bar{\mathcal{A}}_d.$$

Taking the cohomology, we obtain a homomorphism of graded H_β -modules:

$$\mathcal{R}_{\epsilon',\epsilon}: H_{\mathbb{C}^\times}^\bullet(i_{\epsilon,0}^* i_{\epsilon'}^! \bar{\mathcal{A}}_d) \rightarrow H_{\mathbb{C}^\times}^\bullet(i_{\epsilon',0}^* i_{\epsilon'}^! \bar{\mathcal{A}}_d).$$

A proof of the following proposition can be the same as Proposition 6.7.

Proposition 7.17. *Let $\epsilon, \epsilon' \in J^d$ satisfying $\epsilon \lesssim \epsilon'$. The completion $\widehat{\mathcal{R}}_{\epsilon',\epsilon}$ of the homomorphism $\mathcal{R}_{\epsilon',\epsilon}$ is identical to the intertwiner $R_{\epsilon',\epsilon}$ up to multiples in \mathbb{O}^\times .*

7.8. Proof of Theorem 4.14. Now, we are ready to prove our main theorem. One can verify the following analog of Theorem 6.9 by the same argument using Propositions 7.14 and 7.17 instead of Propositions 6.4 and 6.7 respectively.

Theorem 7.18. *Assume that \mathbf{i} is a reduced word for w adapted to the quiver Q . For any $\mathbf{d} \in \text{KP}_{\mathbf{i}}(\beta)$ and $\epsilon \in J^{\mathbf{d}}$, we have the following equality in $K(\mathcal{C}_w)_t$:*

$$(7.17) \quad [M_{\mathbf{i}}(\epsilon)]_t = \sum_{\mathbf{d}' \in \text{KP}_{\mathbf{i}}(\beta)} \left(\sum_{n \in \mathbb{Z}} t^n \dim_{\mathbb{k}} \mathbf{H}^n(i_{\epsilon, x(\mathbf{d})}^* i_{\epsilon}^! \text{IC}(\overline{O(\mathbf{d}')} \cap S(\mathbf{d}), \mathbb{k})) \right) [L_{\mathbf{i}}(\mathbf{d}')].$$

Comparing (7.12) with (7.17), we obtain the desired equality (4.10) when our reduced word \mathbf{i} is adapted to Q . Thus, we have proved Theorem 4.14.

As a byproduct of the proof, we also obtain the following analog of Corollary 6.10.

Corollary 7.19. *When \mathbf{i} is adapted to a quiver of type \mathfrak{g} , the filtration layer $\text{Gr}_n^F M_{\mathbf{i}}(\epsilon) = F_n M_{\mathbf{i}}(\epsilon) / F_{n+1} M_{\mathbf{i}}(\epsilon)$ is a semisimple \widehat{H} -module for any $\epsilon \in J^{\mathbf{d}}$ and $n \in \mathbb{Z}$.*

7.9. Proof of Lemma 7.12. In this subsection, we give a proof of Lemma 7.12 above. First we recall the construction of the isomorphism (7.13). Let $\beta \in \mathbb{Q}^+$ and recall the $G(\beta)$ -variety \tilde{F}_{ν} , which we identify with the variety of pairs (x, F^{\bullet}) of $x \in X(\beta)$ and I -graded flag $F^{\bullet} = (V^{\beta} = F^0 \supset F^1 \supset \dots \supset F^{|\beta|} = 0)$ such that $\dim F^{n-1}/F^n = \alpha_{\nu_n}$ and $x(F^n) \subset F^n$ for any $1 \leq n \leq |\beta|$. Then the proper morphism $p_3: \tilde{F}_{\nu} \rightarrow X(\beta)$ is simply the projection $(x, F^{\bullet}) \mapsto x$. We consider the convolution algebra of the $G(\beta)$ -equivariant Borel-Moore homologies:

$$Z_{\beta} := \bigoplus_{\nu, \nu' \in I^{\beta}} \mathbf{H}_{\bullet}^{G(\beta)}(\tilde{F}_{\nu} \times_{X(\beta)} \tilde{F}_{\nu'}, \mathbb{k}).$$

With this notation, the isomorphism (7.13) in Theorem 7.10 is constructed as the composition of two isomorphisms of \mathbb{k} -algebras:

$$H_{\beta} \simeq Z_{\beta} \simeq \text{Hom}_{G(\beta)}^{\bullet}(\mathcal{L}_{\beta}, \mathcal{L}_{\beta}).$$

Through the first isomorphism $H_{\beta} \simeq Z_{\beta}$, the idempotent $e(\nu)$ goes to the fundamental class of the diagonal $[\Delta(\tilde{F}_{\nu})] = \Delta_*[\tilde{F}_{\nu}]$, where Δ is the diagonal embedding, and the element $\tau_k e(\nu)$ goes to the fundamental class of a certain subvariety of $\tilde{F}_{\sigma_k \nu} \times \tilde{F}_{\nu}$. The element $x_n e(\nu)$ goes to $\Delta_* c_1^{G(\beta)}(\mathcal{O}_{\nu}(n))$, where $c_1^{G(\beta)}(\mathcal{O}_{\nu}(n))$ denotes the first equivariant Chern class of the $G(\beta)$ -equivariant line bundle $\mathcal{O}_{\nu}(n)$ on \tilde{F}_{ν} whose fiber at (x, F^{\bullet}) is F^{n-1}/F^n . The second isomorphism $Z_{\beta} \simeq \text{Hom}_{G(\beta)}^{\bullet}(\mathcal{L}_{\beta}, \mathcal{L}_{\beta})$ is an equivariant version of the isomorphism in [10, Section 8.6].

In what follows, we fix $j \in J$ and put $\beta = \alpha_{\mathbf{i}, j}$ for the sake of brevity. Let M^{\bullet} denote the RHS of the desired isomorphism. We have

$$\begin{aligned} M^{\bullet} &= \text{Hom}_{G(\beta)}^{\bullet}((i_{O(\beta)})_! \mathbb{k}_{O(\beta)}, \mathcal{L}_{\beta}) \langle \dim X(\beta) \rangle \\ &\simeq \mathbf{H}_{G(\beta)}^{\bullet}(i_{O(\beta)}^* \mathcal{L}_{\beta}[-\dim X(\beta)]) \\ &\simeq \mathbf{H}_{\mathbb{C}^{\times}}^{\bullet}(i_{x(\beta)}^* \mathcal{L}_{\beta}[-\dim X(\beta)]), \end{aligned}$$

where the last isomorphism comes from the induction equivalence together with $\text{Stab}_{G(\beta)} x(\beta) = \mathbb{C}^{\times} \text{id}_{V^{\beta}} \simeq \mathbb{C}^{\times}$. By the definition of \mathcal{L}_{β} and the decomposition theorem, we have

$$(7.18) \quad i_{x(\beta)}^* \mathcal{L}_{\beta}[-\dim X(\beta)] \simeq \bigoplus_{\nu \in I^{\beta}} \mathbf{H}^{\bullet}(\tilde{F}_{\nu}(x(\beta)), \mathbb{k}) \langle -d(\nu, \beta) \rangle \otimes \mathbb{k}_{\{x(\beta)\}},$$

where $\tilde{F}_\nu(x(\beta)) := p_3^{-1}(x(\beta)) \subset \tilde{F}_\nu$ denotes the variety of I -graded flags stable under $x(\beta)$, and $d(\nu, \beta) := -\dim \tilde{F}_\nu + \dim X(\beta)$. Note that $i_{x(\beta)}^* \mathrm{IC}(\mathbf{d})$ is isomorphic to $\mathbb{k}_{\{x(\beta)\}}[\dim X(\beta)]$ if $\mathbf{d} = \delta_j$, and zero otherwise. Thus, the decomposition (7.14) implies an isomorphism

$$i_{x(\beta)}^* \mathcal{L}_\beta[-\dim X(\beta)] \simeq L_{\mathbf{i},k}^\bullet \otimes \mathbb{k}_{\{x(\beta)\}}.$$

Comparing this with (7.18), we get an isomorphism

$$(7.19) \quad L_{\mathbf{i},j}^\bullet \simeq \bigoplus_{\nu \in I^\beta} \mathbf{H}^\bullet(\tilde{F}_\nu(x(\beta)), \mathbb{k})\langle -d(\nu, \beta) \rangle$$

of graded vector spaces. By construction, the H_β -action on $L_{\mathbf{i},j}^\bullet$ corresponds to the convolution action of Z_β on the RHS of (7.19). On the other hand, from (7.18), we get

$$(7.20) \quad M^\bullet \simeq \bigoplus_{\nu \in I^\beta} \mathbf{H}^\bullet(\tilde{F}_\nu(x(\beta)), \mathbb{k})\langle -d(\nu, \beta) \rangle \otimes \mathbf{H}_{\mathbb{C}^\times}^\bullet(\mathrm{pt}, \mathbb{k}).$$

Through the isomorphisms (7.13) and (7.20), the H_β -action on M^\bullet is translated into the convolution action of Z_β on the RHS of (7.20). In particular, the action of the element $x_n e(\nu)$ on M^\bullet corresponds to the multiplication of the equivariant Chern class $c_1^{\mathbb{C}^\times}(\mathcal{O}_\nu(n)|_{\tilde{F}_\nu(x(\beta))})$ on the RHS of (7.20). Let us make the identification $\mathbf{H}_{\mathbb{C}^\times}^\bullet = \mathbb{k}[z]$ with z representing the fundamental weight of \mathbb{C}^\times . Since the \mathbb{C}^\times -action on the fibers of $\mathcal{O}_\nu(n)$ is of weight 1, we have

$$c_1^{\mathbb{C}^\times}(\mathcal{O}_\nu(n)|_{\tilde{F}_\nu(x(\beta))}) = c_1(\mathcal{O}_\nu(n)|_{\tilde{F}_\nu(x(\beta))}) + z.$$

This matches with the formula (4.6) (with $a(z) = z$) defining the action of x_n on the affinization $L_{\mathbf{i},j}^\bullet[z]$. Thus, the isomorphisms (7.19) and (7.20) yield an isomorphism $M^\bullet \simeq L_{\mathbf{i},j}^\bullet[z]$ of graded H_β -modules, which completes the proof.

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