Quantum Kac-Moody algebras and categorifications

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May 13, 2013

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- To name a few : Macdonald identities, Conformal field theory, Exactly solvable models...

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- In this talk : q is not a root of unity.

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- Example of a quantum Serre relation in $\mathcal{U}_q(\hat{sl_2})$:

$$0 = E_0 E_1^3 - (q^2+1+q^{-2}) E_1 E_0 E_1^2 + (q^2+1+q^{-2}) E_1^2 E_0 E_1 - E_1^3 E_0.$$

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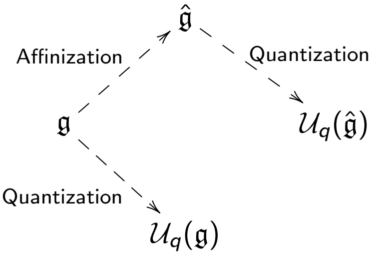
- 

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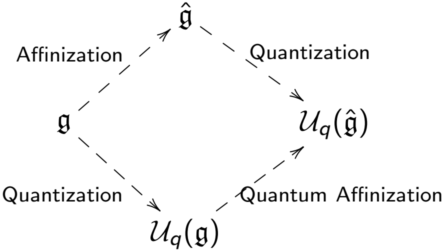
- 

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Example associated to $\mathcal{U}_q(\hat{sl}_2)$:

$$\mathcal{R}(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{q(u-1)}{u-q^2} & \frac{1-q^2}{u-q^2} & 0 \\ 0 & \frac{u(1-q^2)}{u-q^2} & \frac{q(u-1)}{u-q^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

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- Today : an application via categorification.

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- There are more global understanding of the category $\mathcal{C}$.

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Theorem (Frenkel-Reshetikhin 98)

The Grothendieck ring $K_0(\mathcal{C})$ is commutative.

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- Example : $G = SL_3(\mathbb{C}) \supset N = \{\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} | x, y, z \in \mathbb{C}\}$.
- $\mathbb{C}[N] = \mathbb{C}[x, y, z]$.
- Lusztig, 90 : definition of a canonical basis $\mathcal{B}$ of $\mathbb{C}[N]$ (by using intersection cohomology of certain quiver representation spaces).
- Example for $N \subset SL_3(\mathbb{C})$:
  $\mathcal{B} = \{x^a z^b (xy - z)^c | a, b, c \in \mathbb{N}\} \cup \{y^a z^b (xy - z)^c | a, b, c \in \mathbb{N}\}$.
- Important question : properties of the constant structures of $(\mathbb{C}[N], \mathcal{B})$.

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- It is a geometric realization of $K_0(\mathcal{C}')$ with its natural basis.

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- Application of quantum Kac-Moody algebras to a *classical* problem (only finite-dimensional Lie groups appear in the Corollary !).