Representations of shifted quantum groups (or quantized Coulomb branches) and Baxter polynomials

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We explain the application of polynomiality of Q-operators to representations of truncated shifted quantum affine algebras (quantized K-theoretical Coulomb branches). The Q-operators are transfer-matrices associated to prefundamental representations of the Borel subalgebra of a quantum affine algebra, via the standard R-matrix construction. In a joint work with E. Frenkel, we have proved that, up to a scalar multiple, they act polynomially on simple finite-dimensional representations of a quantum affine algebra. This establishes the existence of Baxter polynomial in a general setting. In the framework of the study of K-theoretical Coulomb branches, Finkelberg-Tsymbaliuk introduced remarkable new algebras, the shifted quantum affine algebras and their truncations. We propose a conjectural parametrization of simple modules of a non simply-laced truncation in terms of the Langlands dual quantum affine Lie algebra (this has various motivations, including the symplectic duality relating Coulomb branches and quiver varieties). We prove that a simple finite-dimensional representation of a shifted quantum affine algebra descends to a truncation as predicted by this conjecture. This is derived from the existence of Baxter polynomial.

Shifted quantum affine algebras and their truncations arose [FT] in the study of quantized K-theoretic Coulomb branches of 3d $N = 4$ SUSY quiver gauge theories in the sense of Braverman-Finkelberg-Nakajima which are at the center of current important developments. A presentation of (truncated) shifted quantum affine algebras by generators and relations was given by Finkelberg-Tsymbaliuk.

Let $\mathfrak{g}$ be a simple complex finite-dimensional Lie algebra, and $\hat{\mathfrak{g}}$ the corresponding untwisted affine Kac-Moody algebra, central extension of the loop algebra $\mathcal{L}\mathfrak{g} = \mathfrak{g}\otimes\mathbb{C}[t^{\pm 1}]$. Drinfeld and Jimbo associated to each complex number $q \in \mathbb{C}^*$ a Hopf algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ called a quantum affine algebra. Shifted quantum affine algebras $\mathcal{U}_q^{\mu_+,\mu_-}(\hat{\mathfrak{g}})$ can be seen as variations of $\mathcal{U}_q(\hat{\mathfrak{g}})$, but depending on two coweights μ_+, μ_- of the underlying simple Lie algebra $\mathfrak{g}$. These coweights corresponding to shifts of formal power series in the Cartan-Drinfeld elements (that is quantum analogs of the $t^r h \in \mathcal{L}\mathfrak{g}$, with $r \in \mathbb{Z}$ and $h \in \mathfrak{h}$ in the Cartan subalgebra of $\mathfrak{g}$). In particular $\mathcal{U}_q^{0,0}(\hat{\mathfrak{g}})$ is a central extension of the ordinary quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$. Up to isomorphism, $\mathcal{U}_q^{\mu_+,\mu_-}(\hat{\mathfrak{g}})$ only depends on $\mu = \mu_+ + \mu_-$ and will be denoted simply by $\mathcal{U}_q^{\mu}(\hat{\mathfrak{g}})$.

The truncations are quotients of $\mathcal{U}_q^{\mu}(\hat{\mathfrak{g}})$ and depend on additional parameters, including a dominant coweight λ and additional "flavour" parameters (which are complex numbers). In this context, these parameters λ and μ can be interpreted as parameters for generalized slices of the affine Grassmannian $\overline{\mathcal{W}}_\mu^\lambda$ (usual slices when μ is dominant).

In [H], we develop the representation theory of shifted quantum affine algebras. We establish several analogies with the representation theory of ordinary quantum affine algebras, but our approach is also based on several techniques. In particular, we relate these representations to representations of the Borel subalgebra $\mathcal{U}_q(\hat{\mathfrak{b}})$ of the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$. Consider a category $\mathcal{O}_\mu$ of representations of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ which is an analog of the ordinary category $\mathcal{O}$. We obtain induction/restriction functors to the category $\mathcal{O}$ of $\mathcal{U}_q(\hat{\mathfrak{b}})$-modules.

For general untwisted types, the category $\mathcal{O}$ of representations of the quantum affine Borel algebra $\mathcal{U}_q(\hat{\mathfrak{b}})$ was introduced and studied in [HJ]. Some representations in this category extend to a representation of the whole quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$, but many do not, including the prefundamental representations constructed in [HJ] and whose transfer-matrices have remarkable properties for the corresponding quantum integrable systems [FH].

Let us now discuss truncated shifted quantum affine algebras, quotients of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$. For simply-laced types, simple representations of truncated shifted Yangians have been parametrized in terms of Nakajima monomial crystals [KTWWY]. See the Introduction of [H] for comments on related results in [NW].

We will use Baxter polynomiality in quantum integrable systems for an approach to general types. Let us recall that to each representation V of $\mathcal{U}_q(\hat{\mathfrak{b}})$ in the category $\mathcal{O}$, is attached a transfer-matrix $t_V(z)$ which is a formal power series in a formal parameter z with coefficients in $\mathcal{U}_q(\hat{\mathfrak{g}})$. Given another simple finite-dimensional representation W of $\mathcal{U}_q(\hat{\mathfrak{g}})$, we get a family of commuting operators on $W[[z]]$. This is a quantum integrable model generalizing the XXZ-model. It is established in [FH], the spectrum of this system, that is the eigenvalues of the transfer-matrices, can be described in terms of certain polynomials, generalizing Baxter's polynomials associated to the XXZ-model. These Baxter's polynomials are obtained from the eigenvalues of transfer-matrices associated to prefundamental representations of $\mathcal{U}_q(\hat{\mathfrak{b}})$. Moreover, this Baxter polynomiality implies the polynomiality of certain series of Cartan-Drinfeld elements acting on finite-dimensional representations [FH]. We relate this result to the structures of representations of truncated shifted quantum affine algebras. In particular, we give in [H] a uniform proof of the finiteness of the number of simple isomorphism classes for truncations.

In non-simply-laced types, we propose a parametrization of these simple representations. We use a limit obtained from interpolating (q, t)-characters. The latter were defined by Frenkel and the author as an incarnation of Frenkel-Reshetikhin deformed $\mathcal{W}$-algebras interpolating between q-characters of a non simply-laced quantum affine algebra and its Langlands dual. They lead to the definition of an interpolation between the Grothendieck ring $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ of finite-dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ (at $t = 1$) and the Grothendieck ring $\mathrm{Rep}(\mathcal{U}_t({}^L\hat{\mathfrak{g}}))$ of finite-dimensional representations of the Langlands dual algebra quantum affine algebra $\mathcal{U}_t({}^L\hat{\mathfrak{g}})$ (at $q = \epsilon$ a certain root of 1) :

$$\begin{array}{ccc} & \mathfrak{K}_{q,t} & \\ {\scriptstyle t=1} \swarrow & & \searrow {\scriptstyle q=\epsilon} \\ \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) & & \mathrm{Rep}(\mathcal{U}_t({}^L\hat{\mathfrak{g}})) \end{array}$$

Here $\mathfrak{K}_{q,t}$ is the ring of interpolating (q,t)-characters.

To describe our parametrization, we found it is relevant to use a different specialization of interpolating (q,t)-characters that we call Langlands dual q-characters (with $t=1$ for variables but $q=\epsilon$ for coefficients).

The interpolating (q,t)-characters are closely related to the deformed $\mathcal{W}$-algebras which appeared in the context of the geometric Langlands program. Note also that the parametrization in [KTWWY] for simply-laced types can be understood in the context of symplectic duality (more precisely from the equivariant version of the Hikita conjecture for the symplectic duality formed by an affine Grassmannian slice and a quiver variety). Hence the statement of our conjecture is also motivated by the symplectic duality and the Langlands duality.

Conjecture [H] *The simple modules of a truncation are explicitly parametrized by monomials in the Langlands dual q-character of a finite-dimensional representation of the Langlands dual quantum affine algebra.*

Based on results by Nakajima, Nakajima-Weekes [NW] gave a bijection between more general simple representations of a non simply-laced quantized Coulomb branch and those for simply-laced types and so a parametrization of simple representations in category $\mathcal{O}$ of truncated non simply-laced shifted Yangians (and quantum affine algebras). After using the comparison between simply-laced and twisted q-characters by the author, one can consider a possible relation between the two parametrizations. In small examples discussed in a correspondence between Nakajima, this different method seems to give the same parametrization as our result.

A main evidence for the Conjecture is the following, obtained as a consequence of the Baxter polynomiality.

Theorem [H] *A finite-dimensional simple representation in $\mathcal{O}_\mu$ descends to a certain explicit truncation as predicted by the Conjecture.*

References

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