LEVEL 0 MONOMIAL CRYSTALS

DAVID HERNANDEZ AND HIRAKU NAKAJIMA

Dedicated to Professor George Lusztig on his 60th birthday

Introduction

In this paper we study the monomial crystal $\mathcal{M}$ defined by the second author [32]. We show that each component of $\mathcal{M}$ can be embedded into a crystal $\mathcal{B}(\lambda)$ of an extremal weight module $V(\lambda)$ introduced by Kashiwara [18] (Theorem 2.2). This result was originally conjectured by Kashiwara, when the second author discussed the result of [32] with him. We prove this result by showing that the monomial crystal is equivalent to the combinatorial crystal appeared in Kashiwara's embedding theorem [17]. (See Proposition 2.6.) We then study the case of extremal weight modules of level 0. We realize the crystal $\mathcal{B}(\varpi_\ell)$ of a level 0 fundamental representation via the monomial crystal (Theorem 3.2). And we determine all monomials appearing in the corresponding component of the monomial crystal for all fundamental representations except for some fundamental representations for $E_6^{(2)}$, $E_7^{(1)}$, $E_8^{(1)}$. Thus we obtain explicit descriptions of the crystals in these examples. For classical types, we give them in terms of tableaux. For exceptional types, we list up all monomials. Most of them have been calculated already in the literature ([14, 38, 24, 12, 36, 27, 37, 3]), but we have a few new examples in exceptional types. And our method works for arbitrary fundamental representations in principle, though we certainly need to use a computer with *huge* memory for the triple node of $E_8^{(1)}$.

One of motivations of this work comes from the study of q–characters of finite dimensional modules of the quantum affine algebra, introduced by Knight [25], Frenkel-Reshetikhin [7], and have been intensively studied for example in [6, 26, 28, 29, 31, 32, 8, 9, 10, 5] and the references therein. In the combinatorial algorithm to compute q–characters for arbitrary irreducible representations [29, 31], the first step was to compute (t–analogs of) q–characters for level 0 fundamental representations. Therefore it would be nice if we could give their explicit forms. They can be calculated by a computer, but we hope to see a structure by examining their possible relations to the crystal bases.

In simply-laced type examples given in this paper, we construct *explicit* bijections between monomials in q–characters, counted with multiplicities and the crystal bases. (The existences of *abstract* bijections are trivial as both have the same cardinality as dimensions of modules.) In fact, the computation of the crystal base has been done with help of explicit knowledge of q–characters. This is opposite to our motivation, and we need a further study to achieve it.

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1. Background

In this section we give backgrounds on quantized enveloping algebras, extremal weight modules.

1.1. **Cartan matrix.** Let $C = (C_{i,j})_{1\leq i,j\leq n}$ be a *generalized Cartan matrix*, i.e., $C_{i,j} \in \mathbb{Z}$, $C_{i,i} = 2$, $C_{i,j} \leq 0$ for $i \neq j$ and $C_{i,j} = 0$ if and only if $C_{j,i} = 0$. We set $I = \{1, \dots, n\}$ and $l = \mathrm{rank}(C)$. In the following we suppose that C is symmetrizable, that is to say that there is a matrix $D = \mathrm{diag}(r_1, \dots, r_n)$ ($r_i \in \mathbb{N}^*$) such that $B = DC$ is symmetric.

We consider a *realization* $(\mathfrak{h}, \Pi, \Pi^\vee)$ of C (see [13]): $\mathfrak{h}$ is a $2n - l$ dimensional $\mathbb{Q}$-vector space, $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$ (set of the simple roots) and $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$ (set of simple coroots) are set so that $\alpha_j(\alpha_i^\vee) = C_{i,j}$ for $1 \leq i, j \leq n$. Let $\Lambda_1, \dots, \Lambda_n \in \mathfrak{h}^*$ (resp. the $\Lambda_1^\vee, \dots, \Lambda_n^\vee \in \mathfrak{h}$) be the fundamental weights (resp. coweights) : $\Lambda_i(\alpha_j^\vee) = \alpha_i(\Lambda_j^\vee) = \delta_{i,j}$.

Let $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(\alpha_i^\vee) \in \mathbb{Z} \text{ for all } i \in I\}$ be the weight lattice and $P^+ = \{\lambda \in P \mid \lambda(\alpha_i^\vee) \geq 0 \text{ for all } i \in I\}$ the semigroup of dominant weights. Let $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i \subset P$ (the root lattice) and $Q^+ = \sum_{i\in I} \mathbb{N}\alpha_i \subset Q$. For $\lambda, \mu \in \mathfrak{h}^*$, write $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$.

1.2. **Quantized enveloping algebras.** In the following we suppose that $q \in \mathbb{C}^*$ is not a root of unity.

Let $q_i = q^{r_i}$. For $l \in \mathbb{Z}, r \geq 0, m \geq m' \geq 0$ we introduce the following monomials in $\mathbb{Z}[q^\pm]$:

$$[l]_q = \frac{q^l - q^{-l}}{q - q^{-1}} \in \mathbb{Z}[q^\pm],\ [r]_q! = [r]_q[r-1]_q \dots [1]_q,\ \begin{bmatrix} m \\ m' \end{bmatrix}_q = \frac{[m]_q!}{[m-m']_q![m']_q!}.$$

Definition 1.1. The *quantized enveloping algebra* $\mathcal{U}_q(\mathfrak{g})$ is the $\mathbb{C}$-algebra with generators k_h ($h \in \mathfrak{h}$), $x_i^\pm$ ($i \in I$) and relations

$$k_h k_{h'} = k_{h+h'},\ k_0 = 1,\ k_h x_j^\pm k_{-h} = q^{\pm\alpha_j(h)} x_j^\pm,$$

$$[x_i^+, x_j^-] = \delta_{i,j} \frac{k_{r_i\alpha_i^\vee} - k_{-r_i\alpha_i^\vee}}{q_i - q_i^{-1}},$$

$$\sum_{r=0}^{1-C_{i,j}} (-1)^r \begin{bmatrix} 1 - C_{i,j} \\ r \end{bmatrix}_{q_i} (x_i^\pm)^{1-C_{i,j}-r} x_j^\pm (x_i^\pm)^r = 0 \text{ (for } i \neq j).$$

This algebra was introduced independently by Drinfeld and Jimbo.

We use the notation $k_i^\pm = k_{\pm r_i\alpha_i^\vee}$ and for $l \geq 0$ we set $(x_i^\pm)^{(l)} = (x_i^\pm)^l/[l]_{q_i}!$.

For $J \subset I$ we denote by $\mathfrak{g}_J$ the Kac-Moody algebra of Cartan matrix $(C_{i,j})_{i,j\in J}$.

Let $\mathcal{U}_q(\mathfrak{h})$ the commutative subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by the k_h ($h \in \mathfrak{h}$).

For V a $\mathcal{U}_q(\mathfrak{h})$-module and $\omega \in P$ we denote by V_ω the weight space of weight ω defined by

$$V_\omega = \{v \in V \mid k_h v = q^{\omega(h)} v \text{ for all } h \in \mathfrak{h}\}.$$

In particular for $v \in V_\omega$ we have $k_i v = q_i^{\omega(\alpha_i^\vee)} v$ and for $i \in I$ we have $x_i^\pm V_\omega \subset V_{\omega\pm\alpha_i}$.

We say that V is $\mathcal{U}_q(\mathfrak{h})$-*diagonalizable* if $V = \bigoplus_{\omega\in P} V_\omega$.

1.3. **Extremal weight modules.** In this section we recall the definition of extremal weight modules given by Kashiwara [18, 19].

Definition 1.2. A $\mathcal{U}_q(\mathfrak{g})$-module V is said to be *integrable* if V is $\mathcal{U}_q(\mathfrak{h})$-diagonalizable, the weight subspace $V_\omega \subset V$ is finite dimensional for all $\omega \in P$, and for $\mu \in P$, $i \in I$ there is $R \geq 0$ such that $V_{\mu \pm r\alpha_i} = \{0\}$ for $r \geq R$.

Definition 1.3. For V an integrable $\mathcal{U}_q(\mathfrak{g})$-module and $\lambda \in P$, a vector $v \in V_\lambda$ is called *extremal of weight* λ if there are vectors $\{v_w\}_{w\in W}$ such that $v_{\mathrm{Id}} = v$ and

$$x_i^{\pm} v_w = 0 \text{ if } \pm w(\lambda)(\alpha_i^\vee) \geq 0 \text{ and } (x_i^{\mp})^{\pm(w(\lambda)(\alpha_i^\vee))} v_w = v_{s_i(w)}.$$

In the same way one can define the notion of extremal elements in a crystal. Note that if v is extremal of weight λ, then for $w \in W$, v_w is extremal of weight $w(\lambda)$.

Definition 1.4. For $\lambda \in P$, the *extremal weight module* $V(\lambda)$ *of extremal weight* λ is the $\mathcal{U}_q(\mathfrak{g})$-module generated by a vector v_λ with the defining relations that v_λ is extremal of weight λ.

Example. If λ is dominant, $V(\lambda)$ is the simple highest weight module of highest weight λ.

Theorem 1.5 ([18])**.** *For $\lambda \in P$, the module $V(\lambda)$ is integrable and has a crystal basis $\mathcal{B}(\lambda)$.*

Note that $u_\lambda \in \mathcal{B}(\lambda)$ (which represents v_λ) is extremal of weight λ in the crystal $\mathcal{B}(\lambda)$.

2. Monomial crystal

In this section we recall the definition of the monomial crystal and show that each connected component can be embedded in the crystal of an extremal weight module (Theorem 2.2).

In this section we suppose that C is without odd cycles, i.e., there is a function $s\colon I \to \{0,1\}$ ($i \mapsto s_i$) such that $C_{i,j} \leq -1$ implies $s_i + s_j = 1$. This situation includes all Cartan matrices of finite type and all Cartan matrices of affine type except $A_{2l}^{(1)}$ ($l \geq 1$).

2.1. **Construction.** Consider formal variables $Y_{i,l}^{\pm}$, e^λ ($i \in I, l \in \mathbb{Z}, \lambda \in P$) and let A be the set of monomials of the form $m = e^{\omega(m)}\prod_{i\in I, l\in\mathbb{Z}} Y_{i,l}^{u_{i,l}(m)}$ where $u_{i,l}(m) \in \mathbb{Z}$, $\omega(m) \in P$ such that

$$\sum_{l\in\mathbb{Z}} u_{i,l}(m) = \omega(m)(\alpha_i^\vee). \tag{2.1}$$

For $m \in A$ and $i \in I$ we set $u_i(m) = \sum_{l\in\mathbb{Z}} u_{i,l}(m)$.

For example, $Y_{i,l}^{\pm} e^{\pm\Lambda_i} \in A$ and $A_{i,l} = e^{\alpha_i} Y_{i,l-1} Y_{i,l+1} \prod_{j\neq i} Y_{j,l}^{C_{j,i}} \in A$.

We call l the *grade* of the variable $Y_{i,l}$.

Remark 2.1. (1) If we fix a monomial m and consider only monomials m' which are products of m with various $A_{i,l}^{\pm}$'s (as we shall do in this paper), $\omega(m')$ is uniquely determined by $\omega(m)$ and $u_{i,l}(m')$. Indeed let z be a formal variable and consider the modified quantized Cartan matrix $C(z) = (C_{i,j}(z))_{i,j}$ defined by $C_{i,i}(z) = [2]_z$, and for $i \neq j$, $C_{i,j}(z) = C_{i,j}$. For $P(z) \in \mathbb{Z}[z^{\pm}]$, let $P(z) = \sum_{l\in\mathbb{Z}} P_l z^l$.

$C(z)$ is invertible because $(\det(C(q)))_n = 1 \neq 0$. Let $\tilde{C}(z) = (\tilde{C}_{i,j}(z))_{i,j}$ be its inverse. If $m'm^{-1} = e^{\omega(m')-\omega(m)} \prod_{i\in I, l\in\mathbb{Z}} A_{i,l}^{v_{i,l}}$ (with $v_{i,l} \in \mathbb{Z}$) we have $v_{i,l} = \sum_{j\in I, l'\in\mathbb{Z}} u_{j,l'}(m'm^{-1})(z^l\tilde{C}_{i,j}(z))_{l'}$. So we can safely omit $e^{\omega(m')}$.

(2) The group A appears, in an equivalent form, in [31] for q–characters at roots of unity, and also in [9] to study the q–characters of integrable representations of general quantum affinizations. The additional term e^λ (denoted by k_λ there) appears by looking at a part of a "universal $\mathcal{R}$-matrix".

A monomial m is said to be J-*dominant* if for all $j \in J, l \in \mathbb{Z}$ we have $u_{j,l}(m) \geq 0$. An I-dominant monomials is said to be *dominant*. Let B_J is the set of J-dominant monomials, B is the set of dominant monomials.

Consider the subgroup $\mathcal{M} \subset A$ defined by

$$\mathcal{M} = \{m \in A \mid u_{i,l}(m) = 0 \text{ if } l \equiv s_i + 1 \bmod [2]\}.$$

(For the shortness of notations, we have replaced the condition $l \equiv s_i \bmod [2]$ of [32] by $l \equiv s_i + 1 \bmod [2]$).

Let us define $\mathrm{wt}\colon A \to P$ and $\varepsilon_i, \varphi_i, p_i, q_i\colon A \to \mathbb{Z}\cup\{\infty\}\cup\{-\infty\}$ for $i \in I$ by ($m \in A$)

$$\mathrm{wt}(m) = \omega(m),$$
$$\varphi_{i,L}(m) = \sum_{l\leq L} u_{i,l}(m), \quad \varphi_i(m) = \max\{\varphi_{i,L}(m) \mid L \in \mathbb{Z}\} \geq 0,$$
$$\varepsilon_{i,L}(m) = -\sum_{l\geq L} u_{i,l}(m), \quad \varepsilon_i(m) = \max\{\varepsilon_{i,L}(m) \mid L \in \mathbb{Z}\} \geq 0,$$
$$p_i(m) = \max\{L \in \mathbb{Z} \mid \varepsilon_{i,L}(m) = \varepsilon_i(m)\} = \max\{L \in \mathbb{Z} \mid \sum_{l<L} u_{i,l}(m) = \varphi_i(m)\},$$
$$q_i(m) = \min\{L \in \mathbb{Z} \mid \varphi_{i,L}(m) = \varphi_i(m)\} = \min\{L \in \mathbb{Z} \mid -\sum_{l>L} u_{i,l}(m) = \varepsilon_i(m)\}.$$

Then we define $\tilde{e}_i, \tilde{f}_i\colon A \to A\cup\{0\}$ for $i \in I$ by

$$\tilde{e}_i(m) = \begin{cases} 0 & \text{if } \varepsilon_i(m) = 0, \\ mA_{i,p_i(m)-1} & \text{if } \varepsilon_i(m) > 0, \end{cases}$$
$$\tilde{f}_i(m) = \begin{cases} 0 & \text{if } \varphi_i(m) = 0, \\ mA^{-1}_{i,q_i(m)+1} & \text{if } \varphi_i(m) > 0. \end{cases}$$

By [32, 21] $(\mathcal{M}, \mathrm{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i)$ is a crystal (called the monomial crystal).

2.2. **Connected components of $\mathcal{M}$ and monomial realization of highest weight crystals.** For $m \in \mathcal{M}$ we denote by $\mathcal{M}(m)$ the subcrystal of $\mathcal{M}$ generated by m.

By [32, 21] the crystal $\mathcal{M}(m)$ is isomorphic to the crystal $\mathcal{B}(\mathrm{wt}(m))$ of the highest weight module of highest weight $\mathrm{wt}(m)$, if m is dominant.

The aim of sections 2.3 and 3 is to "generalize" this result for general $m \in \mathcal{M}$.

2.3. **Embedding of $\mathcal{M}(m)$ into $\mathcal{B}(\lambda)$.** In this section we prove the following:

Theorem 2.2. *For $m \in \mathcal{M}$, the crystal $\mathcal{M}(m)$ is isomorphic to a connected component of the crystal $\mathcal{B}(\lambda)$ of an extremal weight module for some $\lambda \in P$.*

Note that it is proved in [4, Theorem 4.15] that for quantum affine algebras, all the connected components of $\mathcal{B}(\lambda)$ are isomorphic to each other modulo shift of weight by δ.

The proof is a slight modification of Kashiwara's proof of the above mentioned result.

Definition 2.3. A *shift* on I is the data $(\leq, \varphi)$ of a total ordering $\leq$ on I and of a map $\varphi\colon I \to \mathbb{Z}$ such that

(1) $\varphi(i) \geq \varphi(j)$ for $i \leq j$,
(2) if $C_{i,j} \leq -1$ and $i \leq j$, then $\varphi(i) = \varphi(j) + 1$,
(3) for $i \in I$, $s_i \equiv \varphi(i) \bmod [2]$.

For $\varphi\colon I \to \mathbb{Z}$, one says that a total ordering $\leq$ on I is *adapted* to φ if $(\leq, \varphi)$ is a shift.

Lemma 2.4. *Let $\varphi\colon I \to \mathbb{Z}$ such that $\varphi(i) - \varphi(j) \in \{\pm 1\}$ if $C_{i,j} \leq -1$ and $s_i \equiv \varphi(i) \bmod [2]$ for $i \in I$. Then there is at least one total ordering on I adapted to φ.*

Proof. For each $r \in \mathbb{Z}$ choose a total ordering on $\{j \in I \mid \varphi(j) = r\}$, and for each $(i,j) \in I^2$ such that $\varphi(i) < \varphi(j)$, put $i > j$. □

Note that in general there is at least one shift. Put $\varphi(i) = s_i$, and Lemma 2.4 gives a shift $(\varphi, \leq)$.

In the following we fix a shift $(\leq, \varphi)$ in I. We put a numbering $I = \{i_1, \dots, i_n\}$ so that $i_1 < i_2 < \cdots < i_n$.

For $i \in I$, let $\mathcal{B}_i$ be the crystal $\mathcal{B}_i = \{b_i(l) \mid l \in \mathbb{Z}\}$ with $\mathrm{wt}(b_i(l)) = l\alpha_i$ and $(j \neq i)$

$$\varepsilon_i(b_i(l)) = -l,\ \varphi_i(b_i(l)) = l,\ \tilde{e}_i(b_i(l)) = b_i(l+1),\ \tilde{f}_i(b_i(l)) = b_i(l-1),$$
$$\varepsilon_j(b_i(l)) = \varphi_j(b_i(l)) = -\infty,\ \tilde{e}_j(b_i(l)) = \tilde{f}_j(b_i(l)) = 0.$$

Let $\mathcal{B}(\infty)$ be the crystal of $\mathcal{U}_q^-(\mathfrak{g})$ and let $T_\lambda = \{t_\lambda\}$ $(\lambda \in P)$ be the crystal defined by $\mathrm{wt}(t_\lambda) = \lambda$, $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty$ and $\tilde{e}_i(t_\lambda) = \tilde{f}_i(t_\lambda) = 0$.

Let $\mathcal{C}$ be the crystal consisting of a single element c with $\mathrm{wt}(c) = 0$, $\varepsilon_i(c) = \varphi_i(c) = 0$, $\tilde{e}_i(c) = \tilde{f}_i(c) = 0$.

For $m \in A$ we define the crystal $K_m = \mathcal{C} \otimes \cdots \otimes K_2 \otimes K_1 \otimes K_0 \otimes T_\alpha \otimes K_{-1} \otimes K_{-2} \otimes \cdots \otimes \mathcal{C}$ where for $l \in \mathbb{Z}$, $K_l = \mathcal{B}_{i_1} \otimes \mathcal{B}_{i_2} \otimes \cdots \otimes \mathcal{B}_{i_n} \otimes T_{\lambda(l)}$ and $\lambda(l) = \sum_{i \in I} \lambda_i(l)\Lambda_i = \sum_{i \in I} u_{i,2l+\varphi(i)}(m)\Lambda_i$ and $\alpha = \mathrm{wt}(m) - \sum_{i \in I, l \in \mathbb{Z}} u_{i,l}(m)\Lambda_i$.

We also denote $\langle \lambda(l), \alpha_i^\vee \rangle$ by $\lambda_i(l)$.

Definition 2.5. Let us define $\Phi_m^\varphi\colon \mathcal{M}(m) \to K_m$ as follows: for $m' \in \mathcal{M}(m)$ with

$$m' = e^{\mathrm{wt}(m')} \prod_{i \in I, k \in \mathbb{Z}} Y_{i,2k+\varphi(i)}^{\lambda_i(k)} \prod_{i \in I, k \in \mathbb{Z}} A_{i,2k+\varphi(i)+1}^{z_i(k)}$$

we define $\Phi_m^\varphi(m') = b$ by

$$b = c \otimes \cdots \otimes b_2 \otimes b_1 \otimes b_0 \otimes t_\alpha \otimes b_{-1} \otimes b_{-2} \otimes \cdots \otimes c,$$
$$\text{where } b_l = b_{i_1}(z_{i_1}(l)) \otimes \cdots \otimes b_{i_n}(z_{i_n}(l)) \otimes t_{\lambda(l)}.$$

The map Φ_m^φ is well-defined as the $z_i(k)$ depend only of m' (see Remark 2.1).

Proposition 2.6. *Φ_m^φ is a strict embedding of the crystal.*

When m is dominant, this result appeared in [30, 8.5] in an equivalent form. More precisely, we parametrize $\operatorname{Irr}\widetilde{\mathfrak{Z}}^\diamond$ there by monomials as explained in [32, §3]. Then the above is exactly [30, 8.5].

Although the proof is exactly the same, we reproduce it here in our current notation for the sake of the reader.

Proof. The injectivity is obvious. Let $m' \in \mathcal{M}(m)$ and $b = \Phi_m^\varphi(m')$. First we have

$$\operatorname{wt}(b) = \alpha + \sum_{i\in I, l\in\mathbb{Z}} u_{i,l}(m)\Lambda_i + \sum_{i\in I, l\in\mathbb{Z}} z_i(l)\alpha_i = \operatorname{wt}(m) + \operatorname{wt}(m'm^{-1}) = \operatorname{wt}(m').$$

Let us prove the following formulas ($i \in I, L \in \mathbb{Z}$):

$$\varepsilon_i(b_{L-1}) - \sum_{l\geq L} \operatorname{wt}(b_l)(\alpha_i^\vee) = -\sum_{l\geq 2L+\varphi(i)} u_{i,l}(m'), \tag{2.2}$$

$$\varphi_i(b_L) + \sum_{l<L} \operatorname{wt}(b_l)(\alpha_i^\vee) = \sum_{l\leq 2L+\varphi(i)} u_{i,l}(m'). \tag{2.3}$$

The equation (2.2) can be checked as

$$\begin{aligned}
&-\sum_{l\geq L}\{z_i(l) + z_i(l-1)\} - \sum_{l\geq L, j>i} C_{i,j}z_j(l) - \sum_{l\geq L-1, j<i} C_{i,j}z_j(l) - \sum_{l\geq L}\lambda_i(l)\\
&= -z_i(L-1) - \sum_{j<i} C_{i,j}z_j(L-1) + \sum_{l\geq L}(-\sum_{j\in I} C_{i,j}z_j(l) - \lambda_i(l))\\
&= \varepsilon_i(b_{L-1}) - \sum_{l\geq L}\operatorname{wt}(b_l)(\alpha_i^\vee).
\end{aligned}$$

The equation (2.3) can be checked exactly in the same way.

The equation (2.2) implies

$$\varepsilon_i(b) = \max_{L\in\mathbb{Z}}\{\varepsilon_i(b_{L-1}) - \sum_{l\geq L}\operatorname{wt}(b_l)(\alpha_i^\vee)\} = \max_{L\in\mathbb{Z}}\{-\sum_{l\geq 2L+\varphi(i)} u_{i,l}(m')\} = \varepsilon_i(m').$$

Similarly the equation (2.3) implies $\varphi_i(b) = \varphi_i(m')$.

Let us prove the compatibility with the operators $\tilde{e}_i, \tilde{f}_i$.

If $\varepsilon_i(m') = \varepsilon_i(b) = 0$, then both $\tilde{e}_i(m')$ and $\tilde{e}_i(b)$ are 0. Suppose otherwise. Then $\tilde{e}_i(b)$ is given by replacing $z_i(L_i)$ by $z_i(L_i)+1$ where $L_i = \max\{L \in \mathbb{Z} \mid \varepsilon_i(b_L) - \sum_{l>L}\operatorname{wt}(b_l)(\alpha_i^\vee) = \varepsilon_i(b)\}$. Therefore $\tilde{e}_i(b) = \Phi_m^\varphi(m'A_{i,2L_i+\varphi(i)+1})$. But it follows from the equation (2.2) that $2L_i + \varphi(i) + 2 = p_i(m')$, and so $\tilde{e}_i(b) = \Phi_m^\varphi(m'A_{i,p_i(m')-1}) = \Phi_m^\varphi(\tilde{e}_i(m'))$. Similarly $\tilde{f}_i$ is compatible. □

Let $\mathcal{B} = \mathcal{B}_{i_1}\otimes\mathcal{B}_{i_2}\otimes\cdots\otimes\mathcal{B}_{i_n}$, and let $\mathcal{P}$ (resp. $\mathcal{P}^-$) be the subcrystal of $\mathcal{C}\otimes\cdots\otimes\mathcal{B}\otimes\mathcal{B}$ (resp. of $\mathcal{B}\otimes\mathcal{B}\otimes\cdots\otimes\mathcal{C}$) of elements of the form $c\otimes\cdots\otimes b(0)\otimes b(0)\otimes b_l\otimes b_{l-1}\otimes\cdots\otimes b_1$ (resp. $b_1\otimes\cdots\otimes b_{l-1}\otimes b_l\otimes b(0)\otimes b(0)\otimes\cdots\otimes c$) where $b_{l'} \in B$ $(1 \leq l' \leq l)$ and $b(0) = b_{i_1}(0)\otimes\cdots\otimes b_{i_n}(0)$.

Proof of Theorem 2.2. By the crystal isomorphism $T_\lambda \otimes \mathcal{B}_i \simeq \mathcal{B}_i \otimes T_{s_i(\lambda)}$ given by $t_\lambda \otimes b_i(l) \mapsto b_i(l + \lambda(\alpha_i^\vee)) \otimes t_{s_i(\lambda)}$, our crystal K_m is isomorphic to $\mathcal{P}\otimes T_{\lambda'}\otimes\mathcal{P}^-$ for some $\lambda' \in P$.

It is known that $\mathcal{P}$ is isomorphic to $\bigsqcup_{\tilde{e}_i(b)=0}\mathcal{B}(\infty)\otimes T_{\operatorname{wt}(b)}$. (See [20, 7.2.4] for example.) Similarly $\mathcal{P}^-$ is $\bigsqcup_{\tilde{f}_i(b)=0} T_{\operatorname{wt}(b)}\otimes\mathcal{B}(-\infty)$. Therefore $\mathcal{P}\otimes T_{\lambda'}\otimes\mathcal{P}^-$ is a disjoint union of various $\mathcal{B}(\infty)\otimes T_\lambda\otimes B(-\infty)$. The crystal of the modified enveloping algebra $\tilde{\mathcal{U}}_q(\mathfrak{g})$ is equal to $\bigsqcup_{\lambda\in P}\mathcal{B}(\infty)\otimes T_\lambda\otimes\mathcal{B}(-\infty)$ and its connected components can be embedded into some $\mathcal{B}(\lambda)$ ([18, Corollary 9.3.4]). Therefore our assertion follows. □

3. Monomial realization of the level 0 extremal fundamental weight crystals

In this section we study in more details extremal weight crystals (Proposition 3.1) for quantum affine algebras. We prove that the crystal of a level 0 fundamental extremal weight module can be realized in the monomial crystal (Theorem 3.2).

We omit $e^{\omega(m')}$ hereafter by Remark 2.1(1).

3.1. **Extremal monomials.** When m is dominant, the component $\mathcal{M}(m)$ is isomorphic to $\mathcal{B}(\lambda)$ where λ is the weight of m. But the situation is different in general, as not all $m\in\mathcal{M}$ are extremal, even if the monomial is dominant or antidominant for each $i\in I$. For example in the case $D_4^{(1)}$, $m=Y_{2,0}Y_{0,3}^{-2}$ is not extremal. Indeed suppose that m is extremal. Then we have

$$m_{s_2}=\tilde{f}_2(m)=Y_{2,2}^{-1}Y_{0,1}Y_{0,3}^{-2}Y_{1,1}Y_{4,1}Y_{3,1}.$$

But $(\mathrm{wt}(m_{s_2}))(\alpha_0^\vee)=-1\leq 0$ and $\tilde{f}_0(m_{s_2})=Y_{0,3}^{-3}Y_{1,1}Y_{4,1}Y_{3,1}\neq 0$, and so m_{s_2} is not extremal, we have a contradiction.

However we have the following consequence of Theorem 2.2.

Proposition 3.1. *Let $(\varphi,\leq)$ be a shift. Then for $(l_1,\dots,l_n)\in\mathbb{Z}^n$, the monomial $m=\prod_{i\in I}Y_{i,\varphi(i)}^{l_i}\in\mathcal{M}$ is extremal and $\mathcal{M}(m)$ is isomorphic to the connected component of $\mathcal{B}(\mathrm{wt}(m))$ generated by $u_{\mathrm{wt}(m)}$.*

Proof. Consider the morphism Φ_m^φ. It follows from Theorem 2.2 that it gives an embedding $\mathcal{M}(m)\subset\mathcal{B}(\lambda)$ where $\lambda\in P$. But for this particular m we have $\Phi_m^\varphi(m)=c\otimes\cdots\otimes b(0)\otimes b(0)\otimes t_{\mathrm{wt}(m)}\otimes b(0)\otimes b(0)\otimes\cdots\otimes c$ in Proposition 2.6. So m is sent to $u_{\mathrm{wt}(m)}\in\mathcal{B}(\mathrm{wt}(m))$ which is extremal. □

3.2. **Monomial realization of the level 0 extremal fundamental weight crystals.** We suppose that C is of affine type. Let us number the set of simple roots as $I=\{0,1,\dots,n\}$. We choose the extra vertices 0 so that $a_0=a_0^\vee=1$ (except $A_{2n}^{(2)}$, $a_0=2$, $a_0^\vee=1$), and the index number of the vertices are the notations of [13] (for untwisted cases $X^{(1)}$ we use the enumeration of finite type of [13] for the sub-Dynkin diagram of type X). This choice is unique up to an automorphism of the Dynkin diagram. We set $I_0=I\setminus\{0\}$.

We also consider a new type $A_{2n}^{(2)\dagger}$, which is the same as $A_{2n}^{(2)}$, but we take the opposite numbering convention from [13], i.e., the vertex i in $A_{2n}^{(2)\dagger}$ is the the vertex $n-i$ in $A_{2n}^{(2)}$. In particular, the extra vertex 0 is the vertex n in $A_{2n}^{(2)}$, and we have $a_0=1$, $a_0^\vee=2$. We need to distinguish these as we consider the restriction of representations to $\mathcal{U}_q(\mathfrak{g}_{I_0})$. Note also that this convention was taken in [4].

Let $Q^\vee=\sum_{i\in I}\mathbb{Z}\alpha_i^\vee$. There is a unique $c\in\sum_{i\in I}\mathbb{N}\alpha_i^\vee$ such that $\{h\in Q^\vee\mid\alpha_i(h)=0\text{ for all }i\in I\}=\mathbb{Z}c$. We write $c=\sum_{i\in I}a_i^\vee\alpha_i^\vee$. In the same way one can define $\delta=\sum_{i\in I}a_i\alpha_i\in Q$. The a_i are given in [13], the $a_i^\vee$ are the a_i of the transposed Cartan matrix.

We have $\{\omega\in P\mid\omega(\alpha_i^\vee)=0\text{ for all }i\in I\}=\mathbb{Q}\delta\cap P$. Put $P_{\mathrm{cl}}=P/(\mathbb{Q}\delta\cap P)$.

Let $P^0 = \{\lambda \in P \mid \lambda(c) = 0\}$ be the set of level 0 weights.

Let $\mathcal{U}_q(\mathfrak{g})'$ be the subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by $x_i^\pm$ and k_h ($h \in \sum \mathbb{Q}\alpha_i^\vee$). This has P_{cl} as a weight lattice. We have the corresponding definition of the crystal. When we want to distinguish crystals of $\mathcal{U}_q(\mathfrak{g})$ and $\mathcal{U}_q(\mathfrak{g})'$, we call the former a P-crystal, and the latter a P_{cl}-crystal.

For $i \in I_0$, let us define a *level* 0 *fundamental weight* ϖ_i by $\Lambda_i - a_i^\vee \Lambda_0 \in P^0$ when $\mathfrak{g} \neq A_{2n}^{(2)\dagger}$ and

$$\varpi_i = \Lambda_i - \Lambda_0 \quad (i \neq n), \qquad \varpi_n = 2\Lambda_n - \Lambda_0$$

when $\mathfrak{g} = A_{2n}^{(2)\dagger}$. The corresponding extremal weight module $V(\varpi_i)$ are called a *level* 0 *fundamental extremal weight module*. Those representations and their crystal have been intensively studied, see [1, 2, 4, 19, 22, 34, 35].

We identify these with (usual) fundamental weights of the finite dimensional Lie algebra $\mathfrak{g}_{I_0}$ when $(\mathfrak{g}, i) \neq (A_{2n}^{(2)\dagger}, n)$. For $(\mathfrak{g}, i) = (A_{2n}^{(2)\dagger}, n)$, we identify ϖ_n with the *twice* of the n^{th} fundamental weight. We denote by $V_{I_0}(\varpi_i)$ the corresponding irreducible $\mathcal{U}_q(\mathfrak{g}_{I_0})$-module, and by $\mathcal{B}_{I_0}(\varpi_i)$ its crystal base, for either case.

As $\mathcal{B}(\varpi_i)$ is connected (see [19]), it follows from Proposition 3.1 that

Theorem 3.2. *Let $(\leq, \varphi)$ be a shift on I. For $i \in I_0$, let M be the monomial given by $Y_{i,\varphi(i)} Y_{0,\varphi(0)}^{-a_i^\vee}$ for $\mathfrak{g} \neq A_{2n}^{(2)\dagger}$, $M = Y_{i,\varphi(i)} Y_{0,\varphi(0)}^{-1}$ for $\mathfrak{g} = A_{2n}^{(2)\dagger}$, $i \neq n$, and $M = Y_{n,\varphi(n)}^2 Y_{0,\varphi(0)}^{-1}$ for $\mathfrak{g} = A_{2n}^{(2)\dagger}$, $i = n$. Then M is extremal in $\mathcal{M}$ and $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_i)$.*

This result establishes a monomial realization of the level 0 extremal fundamental weight crystals $\mathcal{B}(\varpi_i)$. We will give some examples in Sect. 5.

Not all monomials of weight ϖ_i give a crystal isomorphic to $\mathcal{B}(\varpi_i)$ (see the example in Sect. 3.1). However there are some other monomials which generate the same crystal as we will see in the next subsection.

3.3. **Other monomial realizations.** For $i \in I$, let $\theta_i \geq 0$ be the distance between i and 0, that is to say the minimum $p \geq 0$ such that there exists a sequence $\{0 = j_0, j_1, \ldots, j_p = i\}$ of distinct elements of I satisfying $C_{j_l, j_{l+1}} \leq -1$.

Suppose $\mathfrak{g} \neq A_{2n}^{(2)\dagger}$ for brevity.

Corollary 3.3. *Let $i \in I_0$ and $l, l' \in \mathbb{Z}$ such that $l - l' \in \{-\theta_i, -\theta_i + 2, \ldots, \theta_i\}$ and $l' \equiv s_0$ mod [2]. We have $\mathcal{M}(Y_{i,l} Y_{0,l'}^{-a_i^\vee}) \simeq \mathcal{B}(\varpi_i)$.*

Proof. It follows from Theorem 3.2 that it suffices to show that there is a shift $(\leq, \varphi)$ such that $\varphi(i) = l$ and $\varphi(0) = l'$. Suppose that $l - l' \leq 0$ (the proof is the same for $l - l' \geq 0$) and let $a = (\theta_i + l - l')/2$. Define $\varphi \colon I \to \mathbb{Z}$ by $\varphi(j) = l' + \theta_j$ if $\theta_j \leq a$ and $\varphi(j) = l' + 2a - \theta_j$ if $\theta_j \geq a$. We can conclude with Lemma 2.4. □

For example in all cases we have the following:

(1) if $\theta_i \in 2\mathbb{Z}$ then $\mathcal{M}(Y_{i,0} Y_{0,0}^{-a_i^\vee}) \simeq \mathcal{B}(\varpi_i)$,
(2) if $\theta_i \in 2\mathbb{Z} + 1$ then $\mathcal{M}(Y_{i,0} Y_{0,1}^{-a_i^\vee}) \simeq \mathcal{B}(\varpi_i)$.

Proposition 3.4. *Suppose that C is of type $D_n^{(1)}$ ($n \geq 4$) and let $i \in \{2, \ldots, n-2\}$. Then $M = Y_{i,0} Y_{0,i-1}^{-1} Y_{0,i+1}^{-1}$ is extremal and $\mathcal{B}(M) \simeq \mathcal{B}(\varpi_i)$.*

Proof. First suppose that $i \leq n-3$. Consider

$$m = (\tilde{f}_2 \cdots \tilde{f}_{i-1}\tilde{f}_i)(M) = Y_{0,i+1}^{-1} Y_{1,i-1} Y_{2,i}^{-1} Y_{i+1,1}.$$

Let us define $\varphi \colon I \to \mathbb{Z}$ by $\varphi(0) = i+1, \varphi(2) = i, \varphi(1) = \varphi(3) = i-1, \varphi(4) = i-2, \ldots, \varphi(n-2) = i-n+4, \varphi(n) = \varphi(n-1) = i-n+5$. Lemma 2.4 gives a shift $(\varphi, \leq)$. So it follows from Proposition 3.1 that m is extremal, and so $M = m_{s_i s_{i-1} \ldots s_2}$ is extremal.

If $i = n-2$, in the same way we consider

$$m = (\tilde{f}_2 \cdots \tilde{f}_{i-1}\tilde{f}_i)(M) = Y_{0,n-1}^{-1} Y_{1,n-3} Y_{2,n-2}^{-1} Y_{n-1,1} Y_{n,1}.$$

□

In the following we will see various examples of realizations of the level 0 extremal fundamental weight crystals.

4. Finite dimensional crystals – start

Kashiwara has shown that there are a $\mathcal{U}_q(\mathfrak{g})'$-automorphism z_ℓ of the level 0 fundamental extremal weight module $V(\varpi_\ell)$ preserving the global crystal base, and the induced P_{cl}-crystal automorphism, denoted also by z_ℓ, on the crystal $\mathcal{B}(\varpi_\ell)$ [19]. The weight of z_ℓ in the P-crystal is $d_\ell\delta$ where $d_\ell = \max(1, a_\ell^\vee/a_\ell)$ except $d_\ell = 1$ for $(\mathfrak{g}, \ell) = (A_{2n}^{(2)}, n)$. The quotient $\mathcal{B}(\varpi_\ell)/z_\ell$ is the crystal of the finite dimensional irreducible $\mathcal{U}_q(\mathfrak{g})'$-module $W(\varpi_\ell) = V(\varpi_\ell)/(z_\ell - 1)V(\varpi_\ell)$. We denote it by $\mathcal{B}(W(\varpi_\ell))$. We call $W(\varpi_\ell)$ the *level* 0 *fundamental representation*.

After Theorem 3.2 it is natural to ask the followings.

(1) Give an explicit description of monomials appearing in $\mathcal{M}(M)$.
(2) Give an explicit description of the automorphism z_ℓ.

Note that the automorphism z_ℓ is defined as a composite of operators $\tilde{e}_i$, $\tilde{f}_i$'s. But we require more explicit description.

We do not answer these questions in general, but we give examples in the next sections. These are motivated by known descriptions of level 0 crystals in terms of tableaux [14, 24, 37] in part, but closer to those of q–characters [32].

Before giving examples, we define P_{cl}-crystal automorphisms on the monomial crystal $\mathcal{M}$. For $p \in \mathbb{Z}, \alpha \in \mathbb{Q}\delta \cap P$ let $\tau_{2p,\alpha}$ denote the map $\tau_{2p,\alpha} \colon \mathcal{M} \to \mathcal{M}$ defined by $\tau_{2p,\alpha}(e^\lambda \prod Y_{i,n}^{u_{i,n}}) = e^{\lambda+\alpha} \prod Y_{i,n+2p}^{u_{i,n}}$. This clearly preserves the compatibility condition (2.1) and is a P_{cl}-crystal automorphism. In the following, we omit α from the notation and denote simply by τ_{2p}.

Suppose that $\mathcal{M}(M)$ is a monomial crystal isomorphic to $\mathcal{B}(\varpi_\ell)$ such that M is an extremal vector with $\tilde{e}_i M = 0$ for all $i \in I_0$. If we have a monomial $m \in \mathcal{M}(M)$ with $\mathrm{wt}(m) = \mathrm{wt}(M) + N d_\ell \delta$ for $N \in \mathbb{Z}$, then we have $m = z_\ell^N(M)$. This follows from [19, §5.2]. In particular, if $\mathcal{M}(M)$ is isomorphic to $\mathcal{B}(\varpi_\ell)$ and preserved under τ_{2p}, then τ_{2p} is equal to a power of z_ℓ.

In the following examples, we answer the above questions (1),(2) in the following manner:

(1) First show that $\mathcal{M}(M)$ is invariant under τ_{2p} for some p. Then $\mathcal{M}(M)/\tau_{2p} \simeq \mathcal{B}(\varpi_\ell)/z_\ell^N$ for some N.
(2) We determine all monomials in $\mathcal{M}(M)/\tau_{2p}$ and give z_ℓ explicitly in these monomials.

We thus obtain explicit descriptions of crystals of some finite dimensional representations of $\mathcal{U}_q(\mathfrak{g})'$: we treat all fundamental representations except some fundamental representations for $E_6^{(2)}$, $E_7^{(1)}$, $E_8^{(1)}$. However it is natural to hope this procedure works for any fundamental representations with appropriate choices of the initial monomials m.

Note that the uniqueness of the crystal base for $W(\varpi_\ell)$ is not known so far. But all the examples where we compare the crystal base with those existing in the literature, we can always prove that the crystal base is isomorphic.

4.1. Let us illustrate our description in type $A_{2r+1}^{(1)}$ with $n = 2r+1$ ($r \geq 0$).

Mimicking the definition in [32, 15], we define

$$\boxed{k}_p = Y_{k-1,p+k}^{-1} Y_{k,p+k-1} \qquad \text{for } 1 \leq k \leq n+1,\ p \in \mathbb{Z},$$

where $Y_{n+1,p}$ is understood as $Y_{0,p}$.

4.1.1. Let us consider the first level 0 fundamental extremal weight module $V(\varpi_1)$. Let $M = Y_{1,p}Y_{0,p+1}^{-1}$. We have $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_1)$ by Corollary 3.3.

Then the crystal graph of $\mathcal{M}(M)$ is given in Figure 1. Here $0[n+1]$ means

$$\begin{tikzcd} \boxed{1}_p \arrow[r, "1"] & \boxed{2}_p \arrow[r, "2"] & \cdots\cdots \arrow[r, "n-1"] & \boxed{n}_p \arrow[r, "n"] & \boxed{n+1}_p \arrow[llll, bend left, "{0[n+1]}"] \end{tikzcd}$$

FIGURE 1. (Type $A_n^{(1)}$) the crystal $\mathcal{B}(\varpi_1)$ of the vector representation

$\tilde{f}_0 \boxed{n+1}_p = \boxed{1}_{p+n+1}$, i.e., the suffix is shifted by $n+1$. In particular $\mathcal{M}(M)$ is preserved under τ_{n+1}, which has weight $-\delta$. Therefore we have $z_1 = \tau_{-n-1}$ and $\mathcal{M}(M)/\tau_{n+1} \simeq \mathcal{B}(W(\varpi_1))$.

4.1.2. Next consider $\mathcal{B}(\varpi_\ell)$ for $\ell \leq r+1$. (The description for the remaining case $\ell > r+1$ can be obtained from these cases by applying a diagram automorphism.) Let $M_0 = Y_{\ell,0}Y_{0,\ell}^{-1}$. It follows from Corollary 3.3 that $\mathcal{M}(M_0) \simeq \mathcal{B}(\varpi_\ell)$. We set

$$\begin{aligned} M_j &= Y_{\ell,2j} Y_{0,n-\ell+1+2j}^{-1} Y_{j,\ell+j}^{-1} Y_{j,n-\ell+1+j} \\ &= \left(\boxed{1}_{n-\ell+2j} \boxed{2}_{n-\ell+2j-2} \cdots \boxed{j}_{n-\ell+2}\right) \times \left(\boxed{j+1}_{\ell-1} \boxed{j+2}_{\ell-3} \cdots \boxed{\ell}_{1-\ell+2j}\right) \\ &= \prod_{p=1}^{j} \boxed{p}_{n-\ell-2p+2j+2} \times \prod_{p=j+1}^{\ell} \boxed{p}_{\ell+1-2p+2j} \end{aligned}$$

with $0 \leq j \leq \ell$. Note that $M_\ell = Y_{\ell,n+1}Y_{0,n+1+\ell}^{-1} = \tau_{n+1}(M_0)$. Note also that $M_1 = \tau_2(M_0)$ for $\ell = r+1$.

For an increasing sequence $T = (1 \leq i_1 < i_2 < \cdots < i_\ell \leq n+1)$ of integers (i.e., a Young tableaux of shape (ℓ)) we assign

$$m_{T;j} = \prod_{p=1}^{j} \boxed{i_p}_{n-\ell-2p+2j+2} \times \prod_{p=j+1}^{\ell} \boxed{i_p}_{\ell+1-2p+2j} \qquad \text{for } 0 \leq j \leq \ell - 1.$$

Then one can directly check that

(1) $\mathcal{M}_{I_0}(M_j)$ consists of $m_{T;j}$ for various sequences T (cf. [32, 4.6]),
(2) $\tilde{f}_0(m_{T;j})$ with $T = (2, 3, \ldots, \ell, n+1)$ is equal to M_{j+1},
(3) the automorphism σ defined by $m_{T;j} \mapsto m_{T;j+1}$ (j, $j+1$ are understood modulo ℓ) is a P_{cl}-crystal automorphism.

Here, for $J \subset I$ and $m \in \mathcal{M}$ we denote by $\mathcal{M}_J(m)$ the set of monomials obtained by applying $\tilde{e}_j$, $\tilde{f}_j$ with $j \in J$ to m. It is a crystal for the Lie subalgebra $\mathfrak{g}_J$ associated with J.

>From (2) all M_j (and hence $m_{T;j}$ by (1)) are in $\mathcal{M}(M_0)$ by induction. Computing the weights, we find that $M_j = (z_\ell)^{-j}(M_0)$ as explained above. In particular, $\tau_{n+1} = (z_\ell)^{-\ell}$. In the case $\ell = r+1$, we have $\tau_2 = z_\ell^{-1}$. Therefore $\mathcal{M}(M_0)/\tau_2 \simeq \mathcal{B}(W(\varpi_{r+1}))$. By the same reason mentioned above, σ is equal to z_ℓ. Therefore $\mathcal{M}(M_0)/\sigma \simeq \mathcal{B}(W(\varpi_\ell))$.

Let us describe Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ in terms of tableaux. This can be done by transfering the definition of those operators on monomials to tableaux. For $i \neq 0$ we have $\tilde{e}_i m_{T;j} = m_{T';j}$ or 0. Here T' is obtained from T by replacing i by $i-1$. If it is not possible (say, when we have both $i-1$ and i in T), then it is zero. Similarly $\tilde{f}_i = m_{T'';j}$ or 0, where T'' is given by replacing i by $i+1$. We can also describe the action of $\tilde{e}_0$, $\tilde{f}_0$:

$$\tilde{e}_0(m_{T;j}) = \begin{cases} 0 & \text{if } i_1 \neq 1 \text{ or } i_\ell = n+1, \\ m_{(i_2,\cdots,i_\ell,n+1);j-1} & \text{if } i_1 = 1 \text{ and } i_\ell \neq n+1, \end{cases}$$

$$\tilde{f}_0(m_{T;j}) = \begin{cases} 0 & \text{if } i_1 = 1 \text{ or } i_\ell \neq n+1, \\ m_{(1,i_1,\cdots,i_{\ell-1});j+1} & \text{if } i_1 \neq 1 \text{ and } i_\ell = n+1. \end{cases}$$

Here we extend the definition of $m_{T;j}$ from $0 \leq j \leq \ell-1$ to all $j \in \mathbb{Z}$ so that $m_{T;j+\ell} = \tau_{n+1} m_{T;j}$.

As a corollary we get a description of $\mathcal{B}(W(\varpi_\ell))$ in terms of tableaux. This coincides with one in [14]. We also get an isomorphism of I_0-crystals $\mathcal{B}(W(\varpi_\ell)) \simeq \mathcal{B}_{I_0}(\varpi_\ell)$. This is a well-known result.

Comparing the above descriptions with the tableaux sum expressions of q–characters in [32], we see that there is a bijection between $\mathcal{M}(M_0)/\sigma \simeq \mathcal{B}(W(\varpi_\ell))$ and monomials appearing the q–characters of $W(\varpi_\ell)$. In fact, the bijection is simply given by putting $Y_{0,*} = 1$ in $m_{T;0}$.

5. Finite dimensional crystals – classical types

In this section we treat all classical types.

5.1. Type $D_n^{(1)}$. Let $\mathbf{B} = \{1, \ldots, n, \overline{n}, \ldots, \overline{1}\}$. We give the ordering $\prec$ on the set $\mathbf{B}$ by

$$1 \prec 2 \prec \cdots \prec n-1 \prec \begin{matrix} n \\ \\ \overline{n} \end{matrix} \prec \overline{n-1} \prec \cdots \prec \overline{2} \prec \overline{1}.$$

Remark that there is no order between n and $\overline{n}$.

For $p \in \mathbb{Z}$, mimicking the definition in [32, 15], we define

$$\boxed{1}_p = Y_{0,p+2}^{-1} Y_{1,p}, \quad \boxed{2}_p = Y_{0,p+2}^{-1} Y_{1,p+2}^{-1} Y_{2,p+1},$$
$$\boxed{i}_p = Y_{i-1,p+i}^{-1} Y_{i,p+i-1} \qquad (3 \le i \le n-2),$$
$$\boxed{n{-}1}_p = Y_{n-2,p+n-1}^{-1} Y_{n-1,p+n-2} Y_{n,p+n-2}, \qquad \boxed{\overline{n{-}1}}_p = Y_{n-2,p+n-1} Y_{n-1,p+n}^{-1} Y_{n,p+n}^{-1},$$
$$\boxed{n}_p = Y_{n-1,p+n}^{-1} Y_{n,p+n-2}, \qquad \boxed{\overline{n}}_p = Y_{n-1,p+n-2} Y_{n,p+n}^{-1},$$
$$\boxed{\overline{i}}_p = Y_{i-1,p+2n-2-i} Y_{i,p+2n-1-i}^{-1} \qquad (3 \le i \le n-2),$$
$$\boxed{\overline{2}}_p = Y_{0,p+2n-4} Y_{1,p+2n-4} Y_{2,p+2n-3}^{-1}, \qquad \boxed{\overline{1}}_p = Y_{0,p+2n-4} Y_{1,p+2n-2}^{-1}.$$

We define the *i-grade* $\mathrm{gr}_i(\boxed{*}_p)$ as the grade of the variable $Y_{i,*}$ appearing in $\boxed{*}_p$. If $Y_{i,*}$ does not appear, it is not defined. As variables appear at most once, it is well-defined. When the suffix is clear from the context, we may omit it and simply write $\mathrm{gr}_i(\boxed{*})$.

5.1.1. First consider the case $\ell = 1$. We take $M = Y_{1,p} Y_{0,p+2}^{-1}$. It follows from Corollary 3.3 that $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_1)$. The crystal graph of $\mathcal{M}(M)$ is given in Figure 2.

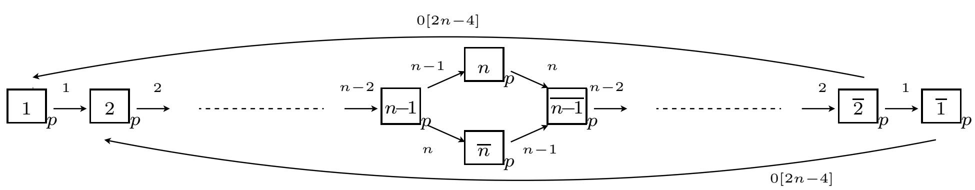

FIGURE 2. (Type $D_n^{(1)}$) the crystal $\mathcal{B}(\varpi_1)$ of the vector representation

We have

$$\tilde{f}_0(\boxed{\overline{2}}_p) = Y_{1,p+2n-4} Y_{0,p+2n-2}^{-1} = \boxed{1}_{p+2n-4},$$
$$\tilde{f}_0(\boxed{\overline{1}}_p) = Y_{0,p+2n-2}^{-1} Y_{1,p+2n-2}^{-1} Y_{2,p+2n-3} = \boxed{2}_{p+2n-4}.$$

Therefore $\mathcal{M}(M)$ is preserved under τ_{2n-4}. Computing weights as above, we find that $z_1 = \tau_{4-2n}$ and so we have $\mathcal{M}(M)/\tau_{2n-4} \simeq \mathcal{B}(W(\varpi_1))$. We also get an isomorphism of I_0-crystals $\mathcal{B}(W(\varpi_1)) \simeq \mathcal{B}_{I_0}(\varpi_1)$.

5.1.2. *Preliminary results for crystals of finite type D.* As is illustrated in examples in type $A_n^{(1)}$, we first need to describe the I_0-crystal structure on the monomials. This will be given in this subsection. All the results on the I_0-crystal are independent of the information on $Y_{0,*}$, so we set $Y_{0,*}$ as 1 in this subsection. Note also that results can be modified in an obvious manner so that the suffixes of $\boxed{}_*$ can be shifted simultaneously. We will use the results in these modified forms in later subsections.

Theorem 5.1. *Let $1 \le \ell \le n-2$ and*

$$M = \boxed{1}_{\ell-1} \boxed{2}_{\ell-3} \cdots \boxed{\ell}_{1-\ell}.$$

Then $\mathcal{M}_{I_0}(M)$ *is isomorphic to* $\mathcal{B}_{I_0}(\varpi_\ell)$ *and is equal to the set of monomials*

$$m_T = \boxed{i_1}_{\ell-1}\boxed{i_2}_{\ell-3}\cdots\boxed{i_\ell}_{1-\ell},$$

indexed by the set $D_{\ell,0,0}$ *of tableaux* $T=(i_1,\dots,i_\ell)$ *satisfying the conditions*

(1) $i_a \in \mathbf{B}, i_1 \not\geq i_2 \not\geq \cdots \not\geq i_\ell$,

(2) *there is no pair* a, b *such that* $1 \le a < b \le \ell$ *and* $i_a = k$, $i_b = \overline{k}$ *and* $b-a = n-1-k$.

Moreover the map $T \mapsto m_T$ *defines a bijection between* $D_{\ell,0,0}$ *and* $\mathcal{M}_{I_0}(M)$.

This result follows from [32, 3.4, 5.5]. It was also proved by Kang-Kim-Shin [15] in the present form. We briefly recall their argument for a later purpose. They checked the following statements:

(a) The set of monomials m_T with T satisfying (1), but not necessarily (2), is preserved by $\tilde{e}_i$, $\tilde{f}_i$.
(b) If a monomial m_T satisfies $\tilde{e}_i m_T = 0$ for all $i = 1,\dots,n$, then m_T must be equal to M.
(c) For a tableau T satisfying (1), there exists a tableau T' satisfying (1),(2) and $m_T = m_{T'}$.
(d) The tableau T satisfying (1),(2) is uniquely determined from the monomial m_T.

The statement (d) is not explicitly stated in [15], but follows from [15, Prop. 3.2] or the argument below.

Let us give an example for the procedure (c). Suppose $n=7$ and $T=(2,3,4,\overline{3},\overline{2})$. Using the relation $\boxed{k}_p\boxed{\overline{k}}_{p-2(n-1-k)} = \boxed{k{+}1}_p\boxed{\overline{k{+}1}}_{p-2(n-1-k)}$ several times, we get

$$\boxed{2}_4\boxed{3}_2\boxed{4}_0\boxed{\overline{3}}_{-2}\boxed{\overline{2}}_{-4} = \boxed{4}_4\boxed{5}_2\boxed{6}_0\boxed{\overline{6}}_{-2}\boxed{\overline{5}}_{-4}.$$

Thus $T'=(4,5,6,\overline{6},\overline{5})$. In general, we replace the pair $\boxed{k}_p\boxed{\overline{k}}_{p-2(n-1-k)}$ by $\boxed{k{+}1}_p\boxed{\overline{k{+}1}}_{p-2(n-1-k)}$ repeatedly from $k=1$ to $n-2$.

As we saw in examples in type $A_n^{(1)}$, we need to study a tableau whose suffixes may *jump*. For $1 \le \ell \le n-2$, $0 \le r \le n-\ell-1$, $0 \le h \le \ell$ let

$$\begin{aligned} &M_{\ell,h,r} = Y_{h,\ell-h}Y_{h,\ell-h-2r}^{-1}Y_{\ell,-2r} \\ &= \left(\boxed{1}_{\ell-1}\boxed{2}_{\ell-3}\cdots\boxed{h}_{\ell-2h+1}\right)\times\left(\boxed{h{+}1}_{\ell-2h-2r-1}\boxed{h{+}2}_{\ell-2h-2r-3}\cdots\boxed{\ell}_{1-\ell-2r}\right) \\ &= \prod_{p=1}^{h}\boxed{p}_{\ell-2p+1}\times\prod_{p=h+1}^{\ell}\boxed{p}_{\ell+1-2p-2r} \end{aligned}$$

and consider a monomial

$$m_T = \left(\boxed{i_1}_{\ell-1}\boxed{i_2}_{\ell-3}\cdots\boxed{i_h}_{\ell-2h+1}\right)\times\left(\boxed{i_{h+1}}_{\ell-2h-2r-1}\boxed{i_{h+2}}_{\ell-2h-2r-3}\cdots\boxed{i_\ell}_{1-\ell-2r}\right)$$

appearing in $\mathcal{M}_{I_0}(M_{\ell,h,r})$. When $h=0$ or ℓ, these are obtained from $\mathcal{M}_{I_0}(M)$ in Theorem 5.1 by the simultaneous shift of grades.

We should consider T as a tableau of shape $(h, \ell-h)$ (one column with h boxes and one column with $\ell-h$ boxes), where the second column is shifted below by $h+r$ boxes. But we simply denote it by $T=((i_1,\dots,i_h),(i_{h+1},\dots,i_\ell))$ or by $T=(i_1,\dots,i_\ell)$ for the sake of spaces.

>From the proof of Theorem 5.1 in [15], we have

$$\begin{aligned} &i_1 \not\succeq i_2 \not\succeq \cdots \not\succeq i_h, \\ &i_{h+1} \not\succeq i_{h+2} \not\succeq \cdots \not\succeq i_\ell. \end{aligned} \tag{D.1}$$

Let us study the order between i_h and i_{h+1}. The following example shows that $i_h \not\succeq i_{h+1}$ may not be satisfied in general: Let $n = 7$, $\ell = 3$, $r = n - \ell - 2 = 2$. Consider the starting monomial $M = \boxed{1}_2\boxed{2}_{-4}\boxed{3}_{-6}$. It gives in the crystal $\mathcal{M}_{I_0}(M)$ the monomial $m = \boxed{3}_2\boxed{4}_{-4}\boxed{\overline{4}}_{-6} = Y_{2,5}^{-1}Y_{3,4}Y_{3,0}^{-1}Y_{4,-1}Y_{3,2}Y_{4,3}^{-1}$. If we apply $\tilde{f}_3$, we get the monomial $m' = Y_{3,6}^{-1}Y_{4,5}Y_{3,0}^{-1}Y_{4,-1}Y_{3,2}Y_{4,3}^{-1}$ and this monomial can only be written in the form $m' = \boxed{4}_2\boxed{4}_{-4}\boxed{\overline{4}}_{-6}$.

The condition (2) in Theorem 5.1 also needs to be modified for the pair a, b with $a \le h$, $h+1 \le b$ as the suffix jump. A naive guess is to replace $n-1-k$ by $n-r-k-1$, but this change does not work as indicated by the following example: Consider the case $n = 6$, $\ell = 4$, $r = 1$ and the starting monomial $m = \boxed{1}_3\boxed{2}_{-1}\boxed{3}_{-3}\boxed{4}_{-5}$. Then consider the monomial $m' = \boxed{1}_3\boxed{2}_{-1}\boxed{3}_{-3}\boxed{\overline{1}}_{-5} = Y_{1,3}Y_{1,5}^{-1}Y_{1,1}^{-1}Y_{3,-1}$. For $b = 4$ and $a = 1$, we have $b-a = n-r-k-1 = 3$. Thus this monomial violates the condition (2) of Theorem 5.1. But if we replace the pair $(i_1, i_4) = (1, \overline{1})$ to $(2, \overline{2})$ as before, we get $\boxed{2}_3\boxed{2}_{-1}\boxed{3}_{-3}\boxed{\overline{2}}_{-5}$, which does not satisfies the condition (1) of Theorem 5.1. In the original situation we can further replace the pair $(i_2, i_4) = (2, \overline{2})$ to $(3, \overline{3})$, and then further $(i_3, i_4) = (3, \overline{3})$ to $(4, \overline{4})$ to achieve the condition (1). But we cannot make this replacement as $\boxed{2}_{-1}\boxed{\overline{2}}_{-5} \neq \boxed{3}_{-1}\boxed{\overline{3}}_{-5}$.

We modify the condition (2) as follows.

(D.2) There is no pair a, b such that $1 \le a < b \le h$ and $i_a = k$, $i_b = \overline{k}$ and $b - a = n - 1 - k$.

(D.3) There is no pair a, b such that $h + 1 \le a < b \le \ell$ and $i_a = k$, $i_b = \overline{k}$ and $b - a = n - 1 - k$.

(D.4) There is no pair a, b such that $a \le h$, $h + 1 \le b$, $i_a = k$, $i_b = \overline{k}$ and $b - a = n - \max(r, 1) - k$.

The conditions (D.2,3) can be achieved without changing the corresponding monomial by the procedure explained above. For (D.4) (when $r \ge 1$), we replace a pair $(i_a, i_b) = (k, \overline{k})$ with $b - a = n - \max(r, 1) - k$ by $(k - 1, \overline{k-1})$. If there are several such pairs or this procedure yields a new such pair, we replace them repeatedly starting from $k = n - 1$, then $k = n - 2$,..., and finally to $k = 2$. (Note that this is converse to the order of the procedure for (D.2,3).) As $r \le n - \ell - 1$, the condition (D.4) always holds for $k = 1$.

Our approach to determine all monomials appearing in $\mathcal{M}_{I_0}(M_{\ell,h,r})$ is to relate them to monomials in $\mathcal{M}_{I_0}(M_{\ell,h,r-1})$. Since we understand the case $r = 0$, we know a general case inductively.

In order to accomplish this approach, we first remark that the crystal structure on the monomials can be transfered to that on the tableaux satisfying (D.1 $\sim$ 4).

Lemma 5.2. *There exists a unique crystal structure on the set of tableaux T satisfying* (D.1 $\sim$ 4) *such that $T \mapsto m_T$ is a strict morphism, i.e., it preserves ε_k, φ_k,* wt *and commutes with $\tilde{f}_k$, $\tilde{e}_k$.*

Proof. We transfer ε_k, φ_k, wt on monomials to those on tableaux via $T \mapsto m_T$.

Let us define $\tilde{f}_k$ on tableaux. ($\tilde{e}_k$ can be defined in the same way.) In general, $\tilde{f}_k m_T \neq 0$ can be written as $m_{T'}$ for a tableau T' which is obtained by replacing an entry i_a in T by a new one according to the rule described in Figure 2. To define $\tilde{f}_k$ on tableaux, we need to specify the entry i_a to be replaced. There might be ambiguity when we have a pair $(i_a, i_b) = (k, \overline{k+1})$ with $\mathrm{gr}_k(\boxed{i_a}) = \mathrm{gr}_k(\boxed{i_b})$. This happens when $b-a = n-1-k$ for $a, b \le h$ or $h+1 \le a, b$ and $b-a = n-1-k-r$ for $a \le h$, $h+1 \le b$. In the first case (or the second case with $r=0$) we replace k by $k+1$. In the second case with $r \neq 0$ we replace $\overline{k+1}$ by $\overline{k}$. Note that we are forced to take these choices by (D.2 $\sim$ 4). Now the assertion is clear. $\square$

Let us prove the statement (d) after Theorem 5.1 as we promised. From (a),(c) we have a surjective map $T \mapsto m_T$. Since it commutes with $\tilde{e}_i$ and $\tilde{f}_i$, the injectivity follows if we check that $\tilde{e}_i T = 0$ for all i implies $T = (1, \dots, \ell)$. But the proof of the statement (b) in [15], in fact, gives this statement.

Let us next define a map $\sigma_{\ell,h,r}$ from tableaux satisfying (D.1 $\sim$ 4) to those where we increase r by 1, i.e., each $\boxed{i_c}_{\ell-2r-2c+1}$ is replaced by $\boxed{i_c}_{\ell-2r-2c-1}$ for $c \ge h+1$. Almost all the cases, $\sigma_{\ell,h,r}(T)$ is just T. But the condition (D.4) is violated if there is a pair $(i_a, i_b) = (k, \overline{k})$ such that $a \le h$, $h+1 \le b$ and $b-a = n-r-k-1$. We replace it by $(k+1, \overline{k+1})$. If there are several such pairs or this procedure yields a new such pair, we replace them repeatedly starting from $k=1$ to $n-r-1$. We define $T' = \sigma_{\ell,h,r}(T)$ as the final result. As we have

$$\boxed{k}_{\ell-2a+1}\boxed{\overline{k}}_{\ell-2r-2b+1} = \boxed{k+1}_{\ell-2a+1}\boxed{\overline{k+1}}_{\ell-2r-2b+1},$$

the procedure keeps the corresponding monomial unchanged if we do not change r for the map $T \mapsto m_T$.

Let us check that $\sigma_{\ell,h,r}$ intertwines $\tilde{f}_k$. By definition, $\sigma_{\ell,h,r}\tilde{f}_k T$ is possibly different from $\tilde{f}_k \sigma_{\ell,h,r} T$ if there is a pair (i_a, i_b) with $a \le h$, $h+1 \le b$ such that the order of k-grades $p = \mathrm{gr}_k(\boxed{i_a})$, $q = \mathrm{gr}_k(\boxed{i_b})$ are changed by $\sigma_{\ell,h,r}$. If both i_a and i_b contribute to $Y_{k,*}$ in positive or negative powers, the rule for $\tilde{f}_k T$ is changed accordingly. (See the proof of Lemma 5.2 how $\tilde{f}_k T$ is defined.) Thus it is enough to study the case when one contributes in positive, and the other in negative. For $k = n-1$, n such a change cannot occur. As grades can only be shifted by 2, for $k \le n-2$ we have a possible change only when $p+2 = q$ for $(i_a, i_b) = (k, \overline{k})$, and $p = q$ for $(i_a, i_b) = (k+1, \overline{k+1})$. These are equivalent to

$$\begin{cases} b-a = n-r-k-1 & \text{if } (i_a, i_b) = (k, \overline{k}), \\ b-a = n-r-k-2 & \text{if } (i_a, i_b) = (k+1, \overline{k+1}). \end{cases}$$

Therefore if there is no pair $(i_a, i_b) = (k, \overline{k})$ with $a \le h$, $h+1 \le b$ and $b-a = n-r-k-1$ for any k, then $\tilde{f}_k m_T$ is unchanged when we increase r by 1. But we have defined $\sigma_{\ell,h,r}$ exactly so that this condition is achieved. Thus we have

$$\sigma_{\ell,h,r}\tilde{f}_k T = \tilde{f}_k \sigma_{\ell,h,r} T \qquad \text{for all } k \in I_0.$$

This equality holds even if $\tilde{f}_k T = 0$.

Similarly we define $\sigma'_{\ell,h,r}(T)$ as follows. When $r = 1$, we simply set it T. Assume $r > 1$ hereafter. Suppose that there is a pair $(i_a, i_b) = (k, \overline{k})$ such that $a \le h$, $h+1 \le b$ and $b-a = n-r-k+1$. We replace it by $(k-1, \overline{k-1})$. If there are several such pairs or this procedure yields a new such pair, we replace them repeatedly starting from $k = n - r$ to 3. We define $\sigma'_{\ell,h,r}(T)$ as the final result. As $r \le n - \ell - 1$, we have $\ell + k \ge b - a + k + 1 = n - r + 2 \ge \ell + 3$. Therefore $k \le 2$ cannot happen, so $k - 1 \in \mathbf{B}$.

These maps are somewhat similar to one defined in [15, Prop. 3.2].

Now we introduce new conditions:

(D.5) Suppose that $i_{h+1} = k \in \{1, \dots, n-1\}$ and $i_h \succeq i_{h+1}$. Then $i_h = k'$ is also in $\{1, \dots, n-1\}$, and the successive part $(\overline{k'}, \overline{k'-1}, \dots, \overline{k})$ appears as $(i_{b'}, i_{b'+1}, \dots, i_b)$ with $n - r - k < b - h \le n - k - 1$.

(D.6) Suppose that $i_{h+1} = \overline{k} \in \{\overline{1}, \dots, \overline{n-1}\}$ and $i_h \succeq i_{h+1}$. Then $i_h = \overline{k'}$ is also in $\{\overline{1}, \dots, \overline{n-1}\}$, and the successive part $(k', k'+1, \dots, k)$ appears as $(i_{a'}, i_{a'+1}, \dots, i_a)$ with $n - r - k \le h - a < n - k - 1$.

(D.7) If $i_{h+1} = n$ or $\overline{n}$, then $i_h \not\succeq i_{h+1}$.

Note that (D.1) implies that the successive part in (D.5) occurs in $b' > h + 1$. This together with the second inequality (and $b + k = b' + k'$) implies $k' < n - 2$. Thus $i_h = n - 1, n - 2$ cannot happen in (D.5). Similarly $i_{h+1} = \overline{n-1}, \overline{n-2}$, cannot happen in (D.6).

Definition 5.3. Let $D_{\ell,h,r}$ be the set of tableaux T satisfying (D.1 $\sim$ 7).

Remark 5.4. When $r = 0$, the conditions (D.1 $\sim$ 7) are equivalent to (1),(2) in Theorem 5.1.

Proposition 5.5. *$\sigma_{\ell,h,r}$ defines a crystal isomorphism from $D_{\ell,h,r}$ to $D_{\ell,h,r+1}$. Its inverse is given by $\sigma'_{\ell,h,r+1}$.*

As a corollary we have

Theorem 5.6. *The map $T \mapsto m_T$ induces a crystal isomorphism between $D_{\ell,h,r}$ and $\mathcal{M}_{I_0}(M_{\ell,h,r})$.*

Proof. We first prove that the image of $D_{\ell,h,r}$ is contained in $\mathcal{M}_{I_0}(M_{\ell,h,r})$ by the induction on r. This is true for $r = 0$ by Theorem 5.1. Suppose it is true for r. First note that $\sigma_{\ell,h,r}$ maps $T = (1, \dots, \ell)$ to $(1, \dots, \ell)$. Take $T \in D_{\ell,h,r+1}$. By the induction hypothesis $m_{\sigma'_{\ell,h,r+1}(T)}$ can be written as

$$m_{\sigma'_{\ell,h,r+1}(T)} = \tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_N} M_{\ell,h,r}$$

for $N \ge 0$, $i_p \in I_0$. We then have

$$m_T = \tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_N} M_{\ell,h,r+1}.$$

This shows $m_T \in \mathcal{M}_{I_0}(M_{\ell,h,r+1})$.

As the crystal graph of $\mathcal{M}_{I_0}(M_{\ell,h,r})$ is connected by its definition, the map is surjective.

By the induction on r, it follows that the only tableau T with $\tilde{e}_i T = 0$ for all $i \in I_0$ is the highest one $T = (1, \dots, \ell)$. This shows that the strict crystal morphism $T \mapsto m_T$ is injective. □

Proof of Proposition 5.5. It is enough to show that $\sigma_{\ell,h,r}$ is a set theoretical bijection, as we already observed that it is a strict crystal morphism.

When $r = 0$, there is no pair $(i_a, i_b) = (k, \overline{k})$ to replace by (D.2 $\sim$ 4). Thus $\sigma_{\ell,h,0}$ is just an identity. Also $\sigma'_{\ell,h,1}$ is an identity by definition. On the other hand, the conditions (D.1 $\sim$ 7) are the same for $r = 0$ and 1. Therefore the assertion is true for $r = 0$. We assume $r > 0$ hereafter.

Suppose T satisfies (D.1 $\sim$ 7). We show that $\sigma_{\ell,h,r}(T)$ also satisfies (D.1 $\sim$ 7). The condition (D.1) is clearly satisfied. The condition (D.4) with r replaced by $r+1$ is satisfied by the definition of $\sigma_{\ell,h,r}$.

We study the cases $i_h \succeq i_{h+1}$ and $i_h \not\succeq i_{h+1}$ separately.

Case (1): $i_h \succeq i_{h+1}$.

We assume $i_{h+1} = k \in \{1,\dots,n-1\}$. By (D.5) $i_h = k' \in \{1,\dots,n-1\}$ and there exists a successive part $(\overline{k'},\dots,\overline{k}) = (i_{b'},\dots,i_b)$ with $h+1 \le b'$, $n-r-k < b-h \le n-1-k$. The condition (D.2) automatically holds as $i_h \in \{1,\dots,n-1\}$.

Suppose that i_h is replaced during the procedure. Then in the middle of the procedure, we find an entry i'_B with $i'_B = \overline{k}$, $B \ge h+1$, $B-h = n-r-k-1$. As i'_B is obtained by replacing i_B, we have $i'_B \preceq i_B$. Therefore $B \ge b$. But this contradicts with (D.5) as

$$n-r-k-1 = B-h \ge b-h > n-r-k.$$

Therefore i_h remains unchanged during the procedure. Therefore the procedure is performed for pairs $(K,\overline{K})$ with $K<k$, so all $(i_h,i_{h+1},\dots,i_b)$ are also unchanged. Thus (D.5) remains true. Suppose (D.3) is violated, i.e., there exists $(i_A,i_B) = (K,\overline{K})$ with $B>A\ge h+1$, $B-A = n-1-K$. As $K \ge k$, such a pair can appear only in $(i_{h+1},\dots,i_b)$. But this part is unchanged, so (D.3) for r implies that this cannot happen. Thus (D.3) is also satisfied.

We can similarly check the assertion when $i_{h+1} \in \{\overline{1},\dots,\overline{n-1}\}$.

Case (2): $i_h \not\succeq i_{h+1}$.

Suppose that we apply the above procedure to a tableau $T = ((i_1,\cdots,i_h),(i_{h+1},\cdots,i_\ell))$ to get a new tableau $T' = ((j_1,\cdots,j_h),(j_{h+1},\cdots,j_\ell))$. We separate the cases according to the order among j_h and j_{h+1}.

Subcase (2.1): $j_h \succeq j_{h+1}$ and $i_{h+1} \in \{1,\dots,n\}$.

As i_{h+1} is unchanged, $j_h \succeq j_{h+1}$ can happen only when i_h is replaced during the procedure. Suppose that i_h is replaced from k' to m with $m \ge k'+1$. Then the procedure yields a successive part $(j_b,\dots,j_{b''}) = (\overline{m},\dots,\overline{k'+1})$ with $b-h = n-r-m$. We have

$$m = j_h \succeq j_{h+1} = i_{h+1} \not\preceq i_h = k'$$

Thus $\overline{j_{h+1}}$ can appear only in the successive part, so (D.5) is satisfied with r replaced by $r+1$.

The condition (D.2) is automatic. Suppose that (D.3) is violated, i.e., there exists $(j_A,j_B) = (K,\overline{K})$ with $B>A>h$, $B-A = n-1-K$. We have $K \ge j_{h+1} = i_{h+1} \not\preceq i_h = k'$. Therefore j_B can occur only in $B \le b''$. If j_B appears outside of the successive part, then $j_B = i_B$ and we have a contradiction with (D.3) for the original tableau. If j_B appears in the successive part, we have

$$n-1-K = B-A < B-h = n-r-K.$$

As $r \ge 1$, we have a contradiction.

Similarly we can check the assertion $j_h \succeq j_{h+1}$ and $i_h \in \{\overline{1},\dots,\overline{n}\}$. When $i_h \in \{1,\dots,n\}$, $i_{h+1} \in \{\overline{1},\dots,\overline{n}\}$, the inequality $j_h \succeq j_{h+1}$ cannot happen. Thus we checked the assertion when $j_h \succeq j_{h+1}$.

Subcase (2.2): $j_h \not\succeq j_{h+1}$.

The conditions (D.5 $\sim$ 7) are satisfied by the assumption. Let $(j_a,j_b) = (k+1,\overline{k+1})$ with $b-a = n-r-k-1$ be the pair obtained by the last replacement in the procedure. We suppose that (D.2) is violated, i.e., we have a pair $A<B\le h$ such that $j_A = K$ and $j_B = \overline{K}$ and $B-A = n-1-K$. As i_c for $a<c<b$ is

unchanged by the above procedure, the condition (D.2) for T implies that j_A can appear only in $A \leq a$. Then $n-1-K = B-A = (a-A)+(B-b)+n-r-k-1$, so $K+a-A = b-B+r+k > k+1$. This inequality contradicts with (D.1) as

$$k+1 = j_a \geq j_A + (a-A) > k+1.$$

Thus (D.2) is satisfied. In the same way (D.3) is satisfied.

Next we show that $\sigma'_{\ell,h,r}(T)$ also satisfies (D.1 $\sim$ 7). We may suppose $r \geq 2$. The condition (D.1) is clearly satisfied. The condition (D.4) with r replaced by $r-1$ is satisfied by the definition of $\sigma'_{\ell,h,r}$.

Suppose that we apply the above procedure to a tableau $T = ((i_1, \cdots, i_h), (i_{h+1}, \cdots, i_\ell))$ to get a new tableau $T' = ((j_1, \cdots, j_h), (j_{h+1}, \cdots, j_\ell))$. Let $(i_a, i_b) = (k, \overline{k})$ with $a \leq h$, $h+1 \leq b$, $b-a = n-r-k+1$ be the first pair replaced in the procedure. Suppose that (D.2) is violated, i.e., we have a pair $A < B \leq h$ such that $j_A = K$ and $j_B = \overline{K}$ and $B-A = n-1-K$. As i_c for $a < c < b$ is unchanged by the above procedure, we have $A \leq a$. If $i_A = j_A$, i.e., i_A is not unchanged, we have a contradiction with (D.2) for T. Therefore $i_A \geq j_A + 1$. We have $n-1-K = B-A = (a-A)+(B-b)+n-r-k+1$, so $K+a-A = b-B+r+k-2 > k-1$. This inequality contradicts with (D.1) as

$$k = i_a \geq i_A + (a-A) \geq j_A + 1 + (a-A) > k.$$

So (D.2) is satisfied by T'. In the same way (D.3) is satisfied by T'.

In order to check the remaining conditions, we treat the cases separately according the ordering among i_h, i_{h+1}.

Case (a): $i_h \not\succeq i_{h+1}$.

This inequality is preserved during the procedure. Therefore we have $j_h \not\succeq j_{h+1}$, so (D.5 $\sim$ 7) are preserved.

Case (b): $i_h \succeq i_{h+1}$ and $i_{h+1} = k \in \{1, \dots, n-1\}$.

Take the successive part $(\overline{k'}, \overline{k'-1}, \dots, \overline{k}) = (i_{b'}, i_{b'+1}, \dots, i_b)$ with $i_h = k'$ as in (D.5). Suppose that an entry in the successive part is replaced during the procedure, i.e., we replace a pair $(i_A, i_B) = (K, \overline{K})$ with $A \leq h$, $b' \leq B \leq b$ with $B-A = n-r-K+1$. The inequality in (D.5) implies

$$n-r-K+1 \leq b-h+k-K = B-h \leq B-A.$$

So this can happen only when two inequalities are equalities, i.e., $n-r-k+1 = b-h$ and $A = h$. And in such case, we really replace the pair by the definition of $\sigma'_{\ell,h,r}$.

Subcase (b.1): i_h is unchanged.

As we observed above, the successive part remains unchanged. By (D.5) we have $n-r-k < b-h \leq n-k-1$. And the case $b-h = n-r-k+1$ is excluded as we have just observed. Therefore the left hand side of the inequality can be improved to $n-r-k+1$. This shows that (D.5) with r replaced by $r-1$ is satisfied.

Subcase (b.2): i_h is changed.

Suppose that i_h is changed, say from k' to $j_h = m$ with $m \leq k'-1$. Then $i_{b'} = \overline{k'}$ is replaced by $\overline{k-1}$, $i_{b'+1}$ is replaced by $\overline{k-2}$, and so on. This procedure continues at least until we replace i_b by $\overline{k-1}$. Thus $m < k$. This is equivalent to $j_h < j_{h+1}$. Thus we have (D.2 $\sim$ 4).

If $i_h \not\succeq i_{h+1}$, we get $j_h \not\succeq j_{h+1}$ as the procedure preserves this inequality. Thus If $i_h \succeq i_{h+1}$, we have a successive part $(k', k'+1, \dots, k) = (i_{a'}, i_{a'+1}, \dots, i_a)$ with $i_{h+1} = \overline{k}$. Therefore the procedure continues at least until i_{h+1} is replaced by $k'-1$, i.e., $m < k'$. Therefore $j_h \not\succeq j_{h+1}$.

Case (c): $i_h \succeq i_{h+1}$ and $i_{h+1} \in \{\overline{1}, \dots, \overline{n}\}$.

This case can be proved in the same way as in case (b).

Finally it is clear that $\sigma_{\ell,h,r}$ and $\sigma'_{\ell,h,r+1}$ are mutually inverse. All replaced pairs $(k+1, \overline{k+1})$ are returned back to $(k, \overline{k})$. And we do not have extra replacements by (D.4). $\square$

When $r = 0$, $\mathcal{M}_{I_0}(M_{\ell,h,0})$ is independent of h. Therefore we get a crystal isomorphism between any pair $\mathcal{M}_{I_0}(M_{\ell,h,r})$ and $\mathcal{M}_{I_0}(M_{\ell,h',r'})$ as a composite of various $\sigma_{\ell,h'',r''}$ and $\sigma'_{\ell,h'',r''}$.

For a later purpose we explicitly write down the crystal isomorphism

$$\tau_{\ell,h,r} \colon D_{\ell,h,r} \cong \mathcal{M}_{I_0}(M_{\ell,h,r}) \to D_{\ell,h+1,r} \cong \mathcal{M}_{I_0}(M_{\ell,h+1,r}).$$

This is the composite $\sigma_{\ell,h+1,r-1}\sigma_{\ell,h+1,r-2} \cdots \sigma_{\ell,h+1,0}\sigma'_{\ell,h,1}\sigma'_{\ell,h,2} \cdots \sigma'_{\ell,h,r}$. All replaced pairs $(k-1, \overline{k-1})$ are returned back to $(k, \overline{k})$ except for those $i_{h+1} = \overline{k-1}$. Also we may have extra replacements for $i_h = k-1$.

Let $T = ((i_1, \dots, i_h), (i_{h+1}, \dots, i_\ell))$. We describe $\tau_{\ell,h,r}(T)$ in the following three cases separately.

- (D.a) $i_{h+1} = k \in \{1, \dots, n-1\}$ and there is an entry $i_b = \overline{k}$ with $n-r-k < b-h \le n-1-k$.
- (D.b) $i_{h+1} = \overline{k} \in \{\overline{1}, \dots, \overline{n-1}\}$ and there is an entry $i_a = k$ with $n-r-k \le h-a < n-1-k$.
- (D.c) Neither (D.a) nor (D.b) is not satisfied.

In the case (D.c) we simply have

$$\tau_{\ell,h,r}(T) = ((i_1, \cdots, i_{h+1}), (i_{h+2}, \cdots, i_\ell)).$$

Next suppose we are in the case (D.a). As was explained in the paragraph just after (D.7), the inequalities imply $b > h+1$ and $k < n-2$. Starting from i_b, we go back i_{b-1}, i_{b-2}, ... while entries are successive. Let $i_{b''}$ be the ending entry, so $(i_{b''}, i_{b''+1}, \dots, i_b)$ are successive as $(\overline{k''}, \overline{k''+1}, \dots, \overline{k})$ and $i_{b''-1} \neq \overline{k''-1}$. Also by the same reasoning as above, we have $k'' < n-2$. We then have

$$\begin{aligned}\tau_{\ell,h,r}(T) = ((i_1, \cdots, i_h, k''{+}1),& \\ &(i_{h+2}, \cdots, i_{b''-1}, \overline{k''{+}1}, \overline{k''}, \cdots, \overline{k{+}1}, i_{b+1}, \cdots, i_\ell)).\end{aligned}$$

Similarly in the case (D.b), we take $i_{a''}$ so that $(i_{a''}, i_{a''+1}, \dots, i_a) = (k'', k''+1, \dots, k)$ and $i_{a''-1} \neq k''-1$. We have $k < n-2$. We then have

$$\begin{aligned}\tau_{\ell,h,r}(T) = ((i_1, \cdots, i_{a''-1}, k''{-}1, \cdots, k{-}1, i_{a+1}, \cdots, i_h,& \\ &\overline{k''{-}1}), (i_{h+2}, \cdots, i_\ell)).\end{aligned}$$

5.1.3. Now we study $\mathcal{B}(\varpi_\ell)$ for $2 \le \ell \le n-2$. Let $M_{0,0} = Y_{\ell,0}Y^{-1}_{0,\ell-1}Y^{-1}_{0,\ell+1} = \boxed{1}_{\ell-1}\boxed{2}_{\ell-3} \cdots \boxed{\ell}_{1-\ell}$. Then $\mathcal{M}(M_{0,0}) \simeq \mathcal{B}(\varpi_\ell)$ by Proposition 3.4.

For $0 \le j \le \ell$, we set

$$\begin{aligned} M_{j,0} &= \left(\boxed{1}_{2n-\ell+2j-5}\boxed{2}_{2n-\ell+2j-7}\cdots\boxed{j}_{2n-\ell-3}\right)\times\left(\boxed{j+1}_{\ell-1}\boxed{j+2}_{\ell-3}\cdots\boxed{\ell}_{1-\ell+2j}\right) \\ &= \prod_{a=1}^{j}\boxed{a}_{2n-\ell-2a+2j-3}\times\prod_{a=j+1}^{\ell}\boxed{a}_{\ell-2a+2j+1} \\ &= \begin{cases} Y_{\ell,0}Y_{0,\ell-1}^{-1}Y_{0,\ell+1}^{-1} & \text{if } j=0, \\ Y_{\ell,2}Y_{0,\ell+1}^{-1}Y_{0,2n-\ell-1}^{-1}Y_{1,\ell+1}^{-1}Y_{1,2n-\ell-3} & \text{if } j=1, \\ Y_{\ell,2j}Y_{0,2n-\ell+2j-5}^{-1}Y_{0,2n-\ell+2j-3}^{-1}Y_{j,\ell+j}^{-1}Y_{j,2n-\ell+j-4} & \text{otherwise.} \end{cases} \end{aligned}$$

Note that $M_{\ell,0} = \tau_{2n-4}(M_{0,0})$.

For a tableau $T = ((i_1,\dots,i_j),(i_{j+1},\dots,i_\ell))$ we define $m_{T;j,0}$ by replacing the a^{th}-entry by i_a.

Claim. We have $M_{j,0} \in \mathcal{M}(M_{0,0})$ for $0 \le j \le \ell$.

In fact, by Theorem 5.6 we have $m_{T;j,0}$ with $T = (3,\dots,\ell+1,\overline{2})$ is contained in $\mathcal{M}_{I_0}(M_{j,0})$ as $M_{j,0} = m_{(1,\cdots,\ell);j,0}$. Then we get $\tilde{f}_0 m_{T;j,0} = m_{T';j+1,0}$ with $T' = (1,3,4,\dots,\ell+1)$. Again by Theorem 5.6 this is contained in $\mathcal{M}_{I_0}(M_{j+1,0})$ as $M_{j+1,0} = m_{(1,\cdots,\ell);j+1,0}$. By induction we obtain the claim.

We have $\mathrm{wt}(M_{j,0}) = \varpi_\ell - j\delta$. Thus $M_{1,0} = z_\ell^{-1}(M_{0,0})$. As $M_{\ell,0} = \tau_{2n-4}(M_{0,0})$, $\mathcal{B}(M_{0,0})$ is preserved under τ_{2n-4} and we have $(z_\ell)^{-\ell} = \tau_{2n-4}$. As in type $A_n^{(1)}$, it is enough to study $\mathcal{M}(M_{0,0})/\tau_{2n-4}$. We extend the definition of $M_{j,0}$ from $0 \le j \le \ell$ to all $j \in \mathbb{Z}$ so that $M_{j+\ell,0} = \tau_{2n-4}M_{j,0}$. The same applies to other various other monomials introduced below though we do not mention it hereafter.

If we apply $\tilde{e}_0$ to $M_{j,0}$, we get the monomial given by replacing $\boxed{2}_*$ by $\boxed{\overline{1}}_{4-2n+*}$, that is

$$\begin{aligned} \tilde{e}_0(M_{j,0}) = \prod_{a=3}^{j}\boxed{a}_{2n-\ell-2a+2j-3}\times\prod_{a=\max(j+1,3)}^{\ell}\boxed{a}_{\ell-2a+2j+1} \\ \times\begin{cases} \boxed{1}_{\ell-1}\boxed{\overline{1}}_{\ell-2n+1} & \text{if } j=0, \\ \boxed{1}_{2n-\ell-3}\boxed{\overline{1}}_{\ell-2n+3} & \text{if } j=1, \\ \boxed{1}_{2n-\ell+2j-5}\boxed{\overline{1}}_{-\ell+2j-3} & \text{if } 2\le j\le\ell. \end{cases} \end{aligned}$$

Let $N_{j,1}$ denote $\boxed{1}_*\boxed{\overline{1}}_*$ in the right hand side. We have

$$N_{j,1} = \begin{cases} Y_{0,\ell-3}Y_{0,\ell+1}^{-1} & \text{if } j=0, \\ Y_{0,\ell-1}Y_{0,2n-\ell-1}^{-1}Y_{1,\ell+1}^{-1}Y_{1,2n-\ell-3} & \text{if } j=1, \\ Y_{0,2n-\ell+2j-7}Y_{0,2n-\ell+2j-3}^{-1} & \text{if } 2\le j\le\ell. \end{cases}$$

We define $M_{j,1}$ by replacing $\boxed{a}_*$ by $\boxed{a-2}_*$ for $a \ge 3$ and $\boxed{2}_*$ by $\boxed{\overline{1}}_{4-2n+*}$ in $M_{j,0}$, that is

$$M_{j,1} = \prod_{a=1}^{j-2}\boxed{a}_{2n-\ell-2a+2j-7}\times\prod_{a=\max(j-1,1)}^{\ell-2}\boxed{a}_{\ell-2a+2j-3}\times N_{j,1}.$$

We have

$$\begin{aligned}\mathrm{wt}(M_{j,1}) &= \varpi_\ell - j\delta + \alpha_0 + \sum_{a=1}^{\ell-2}(\alpha_a + \alpha_{a+1}) \\ &= \varpi_\ell - (j-1)\delta - \alpha_{\ell-1} - 2\alpha_\ell - 2\alpha_{\ell+1} - \cdots - 2\alpha_{n-2} - \alpha_{n-1} - \alpha_n \\ &= \varpi_{\ell-2} - (j-1)\delta.\end{aligned}$$

We recursively define $M_{j,k}$ by replacing $\boxed{a}_*$ by $\boxed{a\text{–}2}_*$ for $a \geq 3$ and $\boxed{2}_*$ by $\boxed{\bar{1}}_{4-2n+*}$ in $M_{j,k-1}$ until all boxes are either $\boxed{1}_*$ or $\boxed{\bar{1}}_*$. We have $k = 0, \ldots, \lfloor \ell/2 \rfloor$ where $\lfloor \ell/2 \rfloor$ is the largest integer which does not exceed $\ell/2$ (the integer part of $\ell/2$). We define $N_{j,k}$ in the same way. We have $\mathrm{wt}(M_{j,k}) = \varpi_{\ell-2k} - (j-k)\delta$.

Let us give $M_{j,k}$, $N_{j,k}$ explicitly.

(1) $k < \lfloor j/2 \rfloor$:

$$\begin{aligned}N_{j,k} &= \prod_{a=1}^{k}\left(\boxed{1}_{2n-\ell-4a+2j-1}\boxed{\bar{1}}_{-\ell-4a+2j+1}\right) \\ &= Y_{0,2n-\ell-4k+2j-3}Y^{-1}_{0,2n-\ell+2j-3}, \\ M_{j,k} &= N_{j,k} \times \prod_{a=1}^{j-2k}\boxed{a}_{2n-\ell-4k-2a+2j-3} \times \prod_{a=j-2k+1}^{\ell-2k}\boxed{a}_{\ell-2(a-j+2k)+1}, \\ &= Y_{\ell-2k,2j-2k}Y^{-1}_{0,2n-\ell-4k+2j-5}Y^{-1}_{0,2n-\ell+2j-3} \\ &\qquad \times Y^{-1}_{j-2k,j+\ell-2k}Y_{j-2k,2n-\ell+j-2k-4}\end{aligned}$$

(2) j is odd and $k = (j-1)/2$:

$$\begin{aligned}N_{j,(j-1)/2} &= Y_{0,2n-\ell-5}Y^{-1}_{0,2n-\ell+2j-3}, \\ M_{j,(j-1)/2} &= N_{j,(j-1)/2} \times \boxed{1}_{2n-\ell-3} \times \prod_{a=2}^{\ell-j+1}\boxed{a}_{\ell-2a+3} \\ &= Y^{-1}_{0,\ell+1}Y^{-1}_{0,2n-\ell+2j-3}Y^{-1}_{1,\ell+1}Y_{1,2n-\ell-3}Y_{\ell-j+1,j+1},\end{aligned}$$

(3) j is even and $k \geq j/2$:

$$\begin{aligned}N_{j,k} &= N_{j,j/2} \times \prod_{a=1}^{k-j/2}\left(\boxed{1}_{\ell-4a+3}\boxed{\bar{1}}_{\ell-4a-2n+5}\right) \\ &= Y_{0,\ell-4k+2j+1}Y^{-1}_{0,\ell+1}Y_{0,2n-\ell-3}Y^{-1}_{0,2n-\ell+2j-3}, \\ M_{j,k} &= N_{j,k} \times \prod_{a=1}^{\ell-2k}\boxed{a}_{\ell-2a-4k+2j+1} \\ &= \begin{cases} Y_{0,-\ell+2j+1}Y^{-1}_{0,\ell+1}Y_{0,2n-\ell-3}Y^{-1}_{0,2n-\ell+2j-3} & \text{if } k = \ell/2, \\ Y^{-1}_{0,\ell+1}Y_{0,2n-\ell-3}Y^{-1}_{0,2n-\ell+2j-3}Y_{1,-\ell+2j+1} & \text{if } k = (\ell-1)/2, \\ Y^{-1}_{0,\ell-4k+2j-1}Y^{-1}_{0,\ell+1}Y_{0,2n-\ell-3}Y^{-1}_{0,2n-\ell+2j-3}Y_{\ell-2k,2j-2k} & \text{otherwise,} \end{cases}\end{aligned}$$

(4) j is odd and $k \geq (j+1)/2$:

$$\begin{aligned} N_{j,k} &= N_{j,(j-1)/2} \times \boxed{1}_{2n-\ell-3} \boxed{\bar{1}}_{\ell-2n+3} \\ &\quad \times \prod_{a=1}^{k-(j+1)/2} \left(\boxed{1}_{\ell-4a+1} \boxed{\bar{1}}_{\ell-4a-2n+3} \right) \\ &= Y_{0,\ell-4k+2j+1} Y_{0,2n-\ell+2j-3}^{-1} Y_{1,\ell+1}^{-1} Y_{1,2n-\ell-3}, \\ M_{j,k} &= N_{j,k} \times \prod_{a=1}^{\ell-2k} \boxed{a}_{\ell-2a-4k+2j+1} \\ &= \begin{cases} Y_{0,-\ell+2j+1} Y_{0,2n-\ell+2j-3}^{-1} Y_{1,\ell+1}^{-1} Y_{1,2n-\ell-3} & \text{if } k = \ell/2, \\ Y_{0,2n-\ell+2j-3}^{-1} Y_{1,-\ell+2j+1} Y_{1,\ell+1}^{-1} Y_{1,2n-\ell-3} & \text{if } k = (\ell-1)/2, \\ Y_{0,\ell-4k+2j-1}^{-1} Y_{0,2n-\ell+2j-3}^{-1} Y_{1,\ell+1}^{-1} Y_{1,2n-\ell-3} Y_{\ell-2k,2j-2k} & \text{otherwise.} \end{cases} \end{aligned}$$

All $M_{j,k}$ satisfy $\tilde{e}_i M_{j,k} = 0$ for all $i \in I_0$. The monomials appearing in $\mathcal{M}_{I_0}(M_{j,k}) \cong \mathcal{B}_{I_0}(\varpi_{\ell-2k})$ can be described as in the previous subsection. Indeed for $k \neq \ell/2$ let us define a monomial $m_{T;j,k}$ associated with a tableau $T = ((i_1, \dots, i_{j-2k}), (i_{j-2k+1}, \cdots, i_{\ell-2k}) \in D_{\ell-2k,j-2k,n-\ell-2}$ by

(1) $k < \lfloor j/2 \rfloor$:

$$m_{T;j,k} = N_{j,k} \prod_{a=1}^{j-2k} \boxed{i_a}_{2n-\ell-4k-2a+2j-3} \times \prod_{a=j-2k+1}^{\ell-2k} \boxed{i_a}_{\ell-2(a-j+2k)+1},$$

(2) j is odd and $k = (j-1)/2$:

$$m_{T;j,(j-1)/2} = N_{j,k} \times \boxed{i_1}_{2n-\ell-3} \prod_{a=2}^{\ell-j+1} \boxed{i_a}_{\ell-2a+3},$$

(3) j is even and $k \geq j/2$:

$$m_{T;j,k} = N_{j,k} \times \prod_{a=1}^{\ell-2k} \boxed{i_a}_{\ell-2a-4k+2j+1},$$

(4) j is odd and $k \geq (j+1)/2$:

$$m_{T;j,k} = N_{j,k} \times \prod_{a=1}^{\ell-2k} \boxed{i_a}_{\ell-2a-4k+2j+1}.$$

(For the case (4) the $Y_{1,\ell+1}^{-1} Y_{1,2n-\ell-3}$ does not change anything because all other $Y_{1,r}^{\pm}$ satisfy $r < \ell+1$.) For $k = \ell/2$ we set $D_{0,j-\ell,n-\ell-2} = \{\emptyset\}$ and define $m_{\emptyset;j,k}$ by the same formula as in (3),(4) where the last product is understood as 1. If $k > j/2$, we set $D_{\ell-2k,j-2k,n-\ell-2} = D_{\ell-2k,0,0}$, i.e., the set of tableaux whose suffixes do not jump.

As $M_{j,0} \in \mathcal{M}(M_{0,0})$, it becomes clear by induction on k that all $M_{j,k}$ are in $\mathcal{M}(M_{0,0})$, and so by using Theorem 5.6 all $m_{T;j,k}$ are in $\mathcal{M}(M_{0,0})$ (the argument is similar to one for the type $A_n^{(1)}$).

As $\mathrm{wt}(M_{j,0}) = \varpi_\ell - j\delta$, we have $z_\ell^{-1}(M_{j,0}) = M_{j+1,0}$ by the reason explained in the beginning of this section. Then from the definition of $M_{j,k}$ we have $z_\ell^{-1}(M_{j,k}) = M_{j+1,k}$. Let us consider $\tau_{\ell-2k,j-2k,n-\ell-2}$, where we understand it as the identity map when $k > j/2$. It maps $M_{j,k}$ to $M_{j+1,k}$ and respects the I_0-crystal structure. Since such a map is unique, we have $z_\ell^{-1} = \tau_{\ell-2k,j-2k,n-\ell-2} \colon \mathcal{M}_{I_0}(M_{j,k}) \to \mathcal{M}_{I_0}(M_{j+1,k})$.

We can describe Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ in terms of tableaux as in type $A_n^{(1)}$. For $i \neq 0$, it is basically explained in the proof of Lemma 5.2. So let us consider the case $\tilde{e}_0$, $\tilde{f}_0$. We get that $\tilde{e}_0(m_{T;j,k})$ is equal to

$$\begin{cases} m_{(i_3,\cdots,i_{\ell-2k});j,k+1} & \text{if } i_2 = 2 \text{ and } i_{\ell-2k-1} \not\succeq \overline{2}, \\ m_{(i_1,\cdots,i_{\ell-2k},\overline{2},\overline{1});j-2,k-1} & \text{if } i_2 \not\preceq 2,\ i_{\ell-2k} \not\succeq \overline{2} \text{ and } k > 0, \\ m_{(i_2,\cdots,i_\ell,\overline{3-i_1});j-1,0} & \text{if } i_1 \preceq 2,\ i_2 \not\preceq 2,\ i_\ell \not\succeq \overline{2} \text{ and } k = 0, \\ 0 & \text{otherwise}, \end{cases}$$

and that $\tilde{f}_0(m_{T;j,k})$ is equal to

$$\begin{cases} m_{(1,2,i_1,\cdots,i_{\ell-2k});j,k-1} & \text{if } i_1 \not\preceq 2,\ i_{\ell-2k-1} \not\succeq \overline{2} \text{ and } k > 0, \\ m_{(i_1,\cdots,i_{\ell-2k-2});j+2,k+1} & \text{if } i_{\ell-2k-1} = \overline{2} \text{ and } i_2 \not\preceq 2, \\ m_{(3-\overline{i_\ell},i_1,\cdots,i_{\ell-1});j+1,0} & \text{if } i_1 \not\preceq 2,\ i_{\ell-1} \not\succeq \overline{2},\ i_\ell \succeq \overline{2} \text{ and } k = 0, \\ 0 & \text{otherwise}, \end{cases}$$

where we denote $\overline{\overline{1}} = 1$ and $\overline{\overline{2}} = 2$, and we extend the definition of $m_{T;j,k}$ from $0 \leq j \leq \ell - 1$ to all $j \in \mathbb{Z}$ so that $m_{T;j+\ell} = \tau_{2n-4} m_{T;j}$. We understood that the condition '$i_2 = 2$' is not satisfied when $k = \lfloor \ell/2 \rfloor$ (and hence there is no entry i_2). Similarly '$i_2 \not\preceq 2$' is satisfied when $k = \lfloor \ell/2 \rfloor$. The same rules apply to other conditions. And we will use the same conventions for other classical types.

As we have checked the stability for operators $\tilde{e}_0$, $\tilde{f}_0$, all the monomials appearing in $\mathcal{M}(M_{0,0})$ are the τ_{2n-4}-images of the $m_{T;j,k}$.

In particular, we can describe the I_0-crystal structure of $\mathcal{B}(W(\varpi_\ell))$ as

$$\begin{aligned} \mathcal{B}(W(\varpi_\ell)) &\simeq \mathcal{M}(M_{0,0})/z_\ell = \mathcal{M}_{I_0}(M_{0,0}) \sqcup \mathcal{M}_{I_0}(M_{0,1}) \sqcup \cdots \sqcup \mathcal{M}_{I_0}(M_{0,\lfloor \ell/2 \rfloor}) \\ &\simeq \mathcal{B}_{I_0}(\varpi_\ell) \sqcup \mathcal{B}_{I_0}(\varpi_{\ell-2}) \sqcup \cdots \sqcup \begin{cases} \mathcal{B}_{I_0}(\varpi_1) & \text{if } \ell \text{ is odd}, \\ \mathcal{B}_{I_0}(0) & \text{if } \ell \text{ is even}. \end{cases} \end{aligned}$$

In fact, this last result is well-known.

As an application of the description of what we just obtained, we construct an explicit bijection between two sets of monomials, one is $\mathcal{M}(M_{0,0})/z_\ell$, the other is those appearing in the q–characters of $W(\varpi_\ell)$ counted with multiplicities. Recall the conditions (1),(2) in Theorem 5.1. In [32] we proved that the q–character of $W(\varpi_\ell)$ is given by the sum of monomials corresponding to $T = (i_1, \ldots, i_\ell)$ satisfying (1) alone. We then defined $l(T)$ as the number of pairs as in (2). Now we define the bijection

$$\begin{aligned} &\{T = (i_1, \ldots, i_\ell) \mid T \text{ satisfies } (1), l(T) = d\} \\ &\qquad\qquad \longleftrightarrow \{T' = (i_1, \ldots, i_{\ell-2d}) \mid T' \text{ satisfies } (1),(2)\} \end{aligned}$$

by letting T' be the tableaux obtained by removing all the pairs violating (2) in T.

This bijection cannot be expressed in terms of monomials in a simple way unlike type A case.

As another application, we get a description of the crystal $\mathcal{B}(W(\varpi_\ell))$ in terms of tableaux. Namely we identify it with $\{m_{T;0,k} \mid 0 \leq k \leq \lfloor \ell/2 \rfloor\}$. Then we express $\tilde{e}_0 m_{T;0,k}$, $\tilde{f}_0 m_{T;0,k}$ as $m_{T';0,k'}$, $m_{T'';0,k''}$ by the above formula composed with the crystal automorphism $\tau_{\ell,h,r}$ for suitable h, r. This description is similar to one in [24, 37], probably the same if we use the isomorphism between our $D_{\ell,0,0}$ and Kashiwara-Nakashima's tableaux [23] in [15]. Note that the uniqueness of the crystal base of $W(\varpi_\ell)$ was proved in [24].

5.1.4. *Spin representations.* Finally we consider the case $\ell = n-1$ or n. Following [32, 15] we define the half size numbered box as

$$\boxed{i}_p = \begin{cases} Y_{1,p-1} & \text{if } i = 1, \\ Y_{1,p+1}^{-1} Y_{2,p} Y_{0,p+1}^{-1} & \text{if } i = 2, \\ Y_{i-1,p+i-1}^{-1} Y_{i,p+i-2} & \text{if } 3 \le i \le n-2, \\ Y_{n-2,p+n-2}^{-1} & \text{if } i = n-1, \\ Y_{n,p+n-1} & \text{if } i = n, \end{cases}$$

$$\boxed{\overline{i}}_p = \begin{cases} Y_{0,p+2n-1} & \text{if } i = 1, \\ 1 & \text{if } 2 \le i \le n-2, \\ Y_{n-1,p+n+1}^{-1} Y_{n,p+n+1}^{-1} & \text{if } i = n-1, \\ Y_{n-1,p+n-1} & \text{if } i = n. \end{cases}$$

Let $M = Y_{\ell,0} Y_{0,n-2}^{-1} = \prod_{a=1}^{n-1} \boxed{a}_{n+1-2a} \times \boxed{n}_{1-n}$ ($\ell = n$) or $\prod_{a=1}^{n-1} \boxed{a}_{n+1-2a} \times \boxed{\overline{n}}_{1-n}$ ($\ell = n-1$). We have $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_\ell)$ by Corollary 3.3.

Let $\mathcal{B}_{\mathrm{sp}}^{+}$ (resp. $\mathcal{B}_{\mathrm{sp}}^{-}$) be the set of tableaux $T = (i_1, \dots, i_n)$ satisfying

(1) $i_a \in \mathbf{B}$, $i_1 \prec i_2 \prec \cdots \prec i_n$,
(2) i and $\overline{i}$ do not appear simultaneously,
(3) if $i_a = n$, $n-a$ is even (resp. odd),
(4) if $i_a = \overline{n}$, $n-a$ is odd (resp. even).

We define m_T by

$$m_T = \prod_{a=1}^{n} \boxed{i_a}_{n+1-2a}.$$

Then $\mathcal{B}_{I_0}(M)$ is $\{m_T \mid T \in \mathcal{B}_{\mathrm{sp}}^{\pm}\}$, where $\pm$ is $-$ if $\ell = n-1$ and $+$ if $\ell = n$. Let $T = (3, 4, 5, \dots, n-1, n, \overline{2}, \overline{1})$ for $\ell = n$ or $T = (3, 4, 5, \dots, n-1, \overline{n}, \overline{2}, \overline{1})$ for $\ell = n-1$. Then $m_T = Y_{2,n+1}^{-1} Y_{\ell,4} Y_{0,n}$. Applying $\tilde{f}_0$ to m_T, we get $Y_{\ell,4} Y_{0,n+2}^{-1} = \tau_4(M)$. As this has weight $\mathrm{wt}(M) - \delta$, it follows that $\tau_4(M) = z_\ell^{-1}(M)$ as before. As a consequence, we have $z_\ell = \tau_{-4}$ and $\mathcal{B}(W(\varpi_\ell)) \simeq \mathcal{M}(M)/\tau_4$.

We describe the action of $\tilde{e}_0$, $\tilde{f}_0$. We have

$$\tilde{e}_0(m_T) = \begin{cases} \tau_{-4}(m_{(i_3, \cdots, i_n, \overline{2}, \overline{1})}) & \text{if } i_2 = 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{f}_0(m_T) = \begin{cases} \tau_4(m_{(1,2,i_1,\cdots,i_{n-2})}) & \text{if } i_{n-1} = \overline{2}, \\ 0 & \text{otherwise.} \end{cases}$$

So the above are all the monomials in $\mathcal{M}(M)/\tau_4$. So we recover a well-known result $\mathcal{B}(W(\varpi_\ell)) \simeq \mathcal{B}_{I_0}(\varpi_\ell)$. The map $Y_{0,*} \mapsto 1$ gives a bijection between $\{m_T \mid T \in \mathcal{B}_{\mathrm{sp}}^{\pm}\}$ and the monomials appearing in q–characters of $W(\varpi_\ell)$, where all the multiplicities are 1 in these cases.

5.2. Type $B_n^{(1)}$. We can describe monomial crystals of other classical types by a similar method. We just state the result without proofs.

Let $\mathbf{B} = \{1, \ldots, n, 0, \overline{n}, \ldots, \overline{1}\}$. We give the ordering $\prec$ on the set $\mathbf{B}$ by

$$1 \prec 2 \prec \cdots \prec n \prec 0 \prec \overline{n} \prec \cdots \prec \overline{2} \prec \overline{1}.$$

For $p \in \mathbb{Z}$, we define

$$\begin{aligned}
&\boxed{1}_p = Y_{0,p+2}^{-1} Y_{1,p}, \quad \boxed{2}_p = Y_{0,p+2}^{-1} Y_{1,p+2}^{-1} Y_{2,p+1}, \\
&\boxed{i}_p = Y_{i-1,p+i}^{-1} Y_{i,p+i-1} \qquad (3 \le i \le n-1), \\
&\boxed{n}_p = Y_{n-1,p+n}^{-1} Y_{n,p+n-1}^{2}, \\
&\boxed{0}_p = Y_{n,p+n+1}^{-1} Y_{n,p+n-1}, \\
&\boxed{\overline{n}}_p = Y_{n-1,p+n} Y_{n,p+n+1}^{-2}, \\
&\boxed{\overline{i}}_p = Y_{i-1,p+2n-i} Y_{i,p+2n+1-i}^{-1} \qquad (3 \le i \le n-1), \\
&\boxed{\overline{2}}_p = Y_{0,p+2n-2} Y_{1,p+2n-2} Y_{2,p+2n-1}^{-1}, \qquad \boxed{\overline{1}}_p = Y_{0,p+2n-2} Y_{1,p+2n}^{-1}.
\end{aligned}$$

5.2.1. First consider the case $\ell = 1$. Let $M = Y_{1,0} Y_{0,2}^{-1}$. It follows from Corollary 3.3 that $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_\ell)$. The crystal graph of $\mathcal{M}(M)$ is given in Figure 3. We find $\tau_{2n-2} = z_\ell^{-1}$ and $\mathcal{M}(M)/\tau_{2n-2} = \mathcal{M}_{I_0}(M)$.

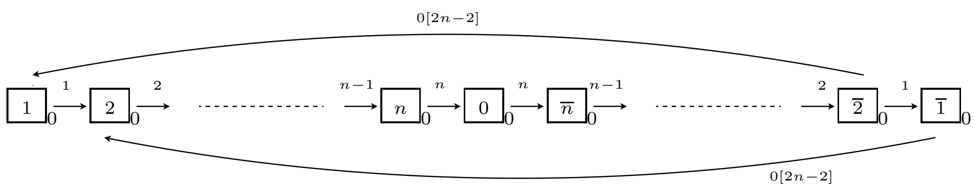

FIGURE 3. (Type $B_n^{(1)}$) the crystal $\mathcal{B}(\varpi_1)$

5.2.2. *Preliminary results for crystals of finite type B.* Let $1 \le \ell \le n-1$, $0 \le r \le n-\ell$ and $0 \le h \le \ell$. Consider the monomial

$$\begin{aligned}
M_{\ell,h,r} &= Y_{h,\ell-h} Y_{h,\ell-h-2r}^{-1} Y_{\ell,-2r} \\
&= \left(\boxed{1}_{\ell-1} \boxed{2}_{\ell-3} \cdots \boxed{h}_{\ell-2h+1} \right) \times \left(\boxed{h{+}1}_{\ell-2h-2r-1} \boxed{h{+}2}_{\ell-2h-2r-3} \cdots \boxed{\ell}_{1-\ell-2r} \right) \\
&= \prod_{p=1}^{h} \boxed{p}_{\ell-2p+1} \times \prod_{p=h+1}^{\ell} \boxed{p}_{\ell+1-2p-2r}.
\end{aligned}$$

For $T = ((i_1, \cdots, i_h), (i_{h+1}, \cdots, i_\ell))$ such that $i_p \in \mathbf{B}$, we define the monomial

$$m_T = \boxed{i_1}_{\ell-1} \boxed{i_2}_{\ell-3} \cdots \boxed{i_h}_{\ell-2h+1} \boxed{i_{h+1}}_{\ell-1-2h-2r} \boxed{i_{h+2}}_{\ell-2h-3-2r} \cdots \boxed{i_\ell}_{-\ell+1-2r}.$$

Let $B_{\ell,h,r}$ be the set of tableaux T satisfying the following conditions

(B.1) $i_a \in \mathbf{B}$, $i_1 \prec i_2 \prec \cdots \prec i_h$ but 0 can be repeated, and $i_{h+1} \prec i_{h+2} \prec \cdots \prec i_\ell$ but 0 can be repeated.

(B.2) There is no pair a, b such that $1 \le a < b \le h$ and $i_a = k$, $i_b = \overline{k}$ and $b - a = n - k$.

(B.3) There is no pair a, b such that $h+1 \leq a < b \leq \ell$ and $i_a = k$, $i_b = \overline{k}$ and $b - a = n - k$.
(B.4) There is no pair a, b such that $a \leq h$, $h+1 \leq b$, $i_a = k$, $i_b = \overline{k}$ and $b - a = n + 1 - \max(r, 1) - k$.
(B.5) Suppose that $i_{h+1} = k \in \{1, \dots, n\}$ and $i_h \succeq i_{h+1}$. Then $i_h = k'$ is also in $\{1, \dots, n\}$, and the successive part $(\overline{k'}, \overline{k'-1}, \dots, \overline{k})$ appears as $(i_{b'}, i_{b'+1}, \dots, i_b)$ with $n - r - k + 1 < b - h \leq n - k$.
(B.6) Suppose that $i_{h+1} = \overline{k} \in \{\overline{1}, \dots, \overline{n}\}$ and $i_h \succeq i_{h+1}$. Then $i_h = \overline{k'}$ is also in $\{\overline{1}, \dots, \overline{n}\}$, and the successive part $(k', k'+1, \dots, k)$ appears as $(i_{a'}, i_{a'+1}, \dots, i_a)$ with $n - r - k + 1 \leq h - a < n - k$.
(B.7) If $i_{h+1} = 0$, then $i_h \preceq 0$.

Note that the conditions above are the same as the ones in [15] when $r = 0$.

For $T = ((i_1, \dots, i_h), (i_{h+1}, \dots, i_\ell)) \in B_{\ell,h,r}$ we define the tableau $\tau_{\ell,h,r}(T)$ in the following three cases separately.

(B.a) $i_{h+1} = k \in \{1, \dots, n\}$ and there is an entry $i_b = \overline{k}$ with $n - r - k + 1 < b - h \leq n - k$.
(B.b) $i_{h+1} = \overline{k} \in \{\overline{1}, \dots, \overline{n}\}$ and there is an entry $i_a = k$ with $n - r - k + 1 \leq h - a < n - k$.
(B.c) Neither (B.a) nor (B.b) is not satisfied.

In the case (B.a), let b'' such that $(i_{b''}, i_{b''+1}, \dots, i_b)$ are successive as $(\overline{k''}, \overline{k''+1}, \dots, \overline{k})$ and $i_{b''-1} \neq \overline{k''-1}$. We have $k'' < n-1$. We set

$$\tau_{\ell,h,r}(T) = ((i_1, \cdots, i_h, k''+1), \\ (i_{h+2}, \cdots, i_{b''-1}, \overline{k''+1}, \overline{k''}, \cdots, \overline{k+1}, i_{b+1}, \cdots, i_\ell)).$$

Similarly in the case (B.b), we take $i_{a''}$ so that $(i_{a''}, i_{a''+1}, \dots, i_a) = (k'', k''+1, \dots, k)$ and $i_{a''-1} \neq k''-1$. We have $k < n-1$. We then set

$$\tau_{\ell,h,r}(T) = ((i_1, \cdots, i_{a''-1}, k''-1, \cdots, k-1, i_{a+1}, \cdots, i_h, \\ \overline{k''-1}), (i_{h+2}, \cdots, i_\ell)).$$

In the case (B.c) we set

$$\tau_{\ell,h,r}(T) = ((i_1, \cdots, i_{h+1}), (i_{h+2}, \cdots, i_\ell)).$$

Theorem 5.7. (1) *The map $T \mapsto m_T$ induces a crystal isomorphism between $B_{\ell,h,r}$ and $\mathcal{M}_{I_0}(M_{\ell,h,r})$.*

(2) *$\tau_{\ell,h,r}$ induces a crystal isomorphism $\mathcal{M}_{I_0}(M_{\ell,h,r})$ to $\mathcal{M}_{I_0}(M_{\ell,h+1,r})$.*

5.2.3. Now we study $\mathcal{B}(\varpi_\ell)$ for $2 \leq \ell \leq n-1$. Let $M_{0,0} = Y_{\ell,0} Y_{0,\ell-1}^{-1} Y_{0,\ell+1}^{-1} = \boxed{1}_{\ell-1} \boxed{2}_{\ell-3} \cdots \boxed{\ell}_{1-\ell}$. For $\ell \neq n-1$ we have $\tilde{f}_2 \tilde{f}_3 \cdots \tilde{f}_\ell M_{0,0} = Y_{0,\ell+1}^{-1} Y_{1,\ell-1} Y_{2,\ell}^{-1} Y_{\ell+1,1}$ and for $\ell = n-1$ we have $\tilde{f}_2 \tilde{f}_3 \cdots \tilde{f}_\ell M_{0,0} = Y_{0,\ell+1}^{-1} Y_{1,\ell-1} Y_{2,\ell}^{-1} Y_{n,1}^2$. By a method similar to the proof of Proposition 3.4 we have $\mathcal{M}(M_{0,0}) \simeq \mathcal{B}(\varpi_\ell)$.

For $0 \leq j < \ell$, $0 \leq k < \ell/2$, let us define the monomial $m_{T;j,k}$ associated with $T = ((i_1, \dots, i_{j-2k}), (i_{j-2k+1}, \cdots, i_{\ell-2k})) \in B_{\ell-2k, j-2k, n-\ell-1}$ by

(1) $k < \lfloor j/2 \rfloor$:

$$m_{T;j,k} = Y_{0,2n-\ell-4k+2j-1} Y^{-1}_{0,2n-\ell+2j-1} \prod_{a=1}^{j-2k} \boxed{i_a}_{2n-\ell-4k-2a+2j-1} \times \prod_{a=j-2k+1}^{\ell-2k} \boxed{i_a}_{\ell-2(a-j+2k)+1},$$

(2) j is odd and $k = (j-1)/2$:

$$m_{T;j,(j-1)/2} = Y_{0,2n-\ell-3} Y^{-1}_{0,2n-\ell+2j-1} \boxed{i_1}_{2n-\ell-1} \prod_{a=2}^{\ell-j+1} \boxed{i_a}_{\ell-2a+3},$$

(3) j is even and $k \geq j/2$:

$$m_{T;j,k} = Y_{0,\ell-4k+2j+1} Y^{-1}_{0,\ell+1} Y_{0,2n-\ell-1} Y^{-1}_{0,2n-\ell+2j-1} \prod_{a=1}^{\ell-2k} \boxed{i_a}_{\ell-2a-4k+2j+1},$$

(4) j is odd and $k \geq (j+1)/2$:

$$m_{T;j,k} = Y_{0,\ell-4k+2j+1} Y^{-1}_{0,2n-\ell+2j-1} Y^{-1}_{1,\ell+1} Y_{1,2n-\ell-1} \prod_{a=1}^{\ell-2k} \boxed{i_a}_{\ell-2a-4k+2j+1}.$$

For $\ell = k/2$ we set $B_{0,j-\ell,n-\ell-1} = \{\emptyset\}$ and define $m_{\emptyset;j,k}$ by the same formula as in (3),(4) where the last product is understood as 1. We extend the definition of $m_{T;j,k}$ for all $j \in \mathbb{Z}$ so that $m_{T;j+\ell,k} = \tau_{2n-2} m_{T;j,k}$.

We describe the action of $\tilde{e}_0$, $\tilde{f}_0$. We get that $\tilde{e}_0(m_{T;j,k})$ is equal to

$$\begin{cases} m_{(i_3,\cdots,i_{\ell-2k});j,k+1} & \text{if } i_2 = 2 \text{ and } i_{\ell-2k-1} \not\succeq \overline{2}, \\ m_{(i_1,\cdots,i_{\ell-2k},\overline{2},\overline{1});j-2,k-1} & \text{if } i_2 \not\preceq 2,\ i_{\ell-2k} \not\succeq \overline{2} \text{ and } k > 0, \\ m_{(i_2,\cdots,i_\ell,\overline{3-i_1});j-1,0} & \text{if } i_1 \preceq 2,\ i_2 \not\preceq 2,\ i_\ell \not\succeq \overline{2} \text{ and } k = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and that $\tilde{f}_0(m_{T;j,k})$ is equal to

$$\begin{cases} m_{(1,2,i_1,\cdots,i_{\ell-2k});j,k-1} & \text{if } i_1 \not\preceq 2,\ i_{\ell-2k-1} \not\succeq \overline{2} \text{ and } k > 0, \\ m_{(i_1,\cdots,i_{\ell-2k-2});j+2,k+1} & \text{if } i_{\ell-2k-1} = \overline{2} \text{ and } i_2 \not\preceq 2, \\ m_{(3-\overline{i_\ell},i_1,\cdots,i_{\ell-1});j+1,0} & \text{if } i_1 \not\preceq 2,\ i_{\ell-1} \not\succeq \overline{2},\ i_\ell \succeq \overline{2} \text{ and } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

So all monomials of $\mathcal{M}(M_{0,0})$ are connected to either $M_{j,k}$ ($0 \leq j < \ell$, $0 \leq k \leq \lfloor \ell/2 \rfloor$) or their τ_{2n-2} images in the I_0-crystal, thus

$$\mathcal{M}(M_{0,0})/\tau_{2n-2} = \bigsqcup_{0 \leq j < \ell, 0 \leq k \leq \lfloor \ell/2 \rfloor} \mathcal{M}_{I_0}(M_{j,k}).$$

Moreover for $0 \leq j \leq \ell - 1$, $0 \leq k < \lfloor \ell/2 \rfloor$ we have

$$(z_\ell)^{-1}(m_{T;j,k}) = m_{\tau_{\ell-2k,j-2k,n-\ell-1}(T);j+1,k}.$$

We have $\tau_{2n-2} = (z_\ell)^{-\ell}$, and all monomials in $\mathcal{M}(M_{0,0})/\tau_{2n-2}$ are written as $m_{T;j,k}$. The crystal automorphism z_ℓ is given by $\tau^{-1}_{\ell-2k,j-2k,n-\ell-1}$.

So we get

$$\mathcal{B}(W(\varpi_\ell)) \simeq \mathcal{B}_{I_0}(\varpi_\ell) \sqcup \mathcal{B}_{I_0}(\varpi_{\ell-2}) \sqcup \cdots \sqcup \begin{cases} \mathcal{B}_{I_0}(\varpi_1) & \text{if } \ell \text{ is odd,} \\ \mathcal{B}_{I_0}(0) & \text{if } \ell \text{ is even.} \end{cases}$$

Our crystal structure described here is probably the same as one in [24] if we use the isomorphism between our $B_{\ell,0,0}$ and Kashiwara-Nakashima's tableaux [23] in [15]. Note that the uniqueness of the crystal base of $W(\varpi_\ell)$ was proved in [24].

5.2.4. Finally we consider the case $\ell = n$. Let $M = Y_{n,0}Y_{0,n-1}^{-1}$. It follows from Corollary 3.3 that $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_\ell)$.

Let

$$\boxed{i}_p = \begin{cases} Y_{1,p-1} & \text{if } i = 1, \\ Y_{1,p+1}^{-1} Y_{2,p} Y_{0,p+1}^{-1} & \text{if } i = 2, \\ Y_{i-1,p+i-1}^{-1} Y_{i,p+i-2} & \text{if } 3 \le i \le n-1, \\ Y_{n-1,p+n-1}^{-1} & \text{if } i = n, \\ Y_{n,p+n} & \text{if } i = 0, \end{cases}$$

$$\boxed{\overline{i}}_p = \begin{cases} Y_{0,p+2n+1} & \text{if } i = 1, \\ 1 & \text{if } 2 \le i \le n-1, \\ Y_{n,p+n+2}^{-2} & \text{if } i = n. \end{cases}$$

Then the monomials appearing in $\mathcal{M}_{I_0}(M)$ are $m_T = \prod_{a=1}^{n+1} \boxed{i_a}_{n+2-2a}$ associated with a tableau $T = (i_1, \dots, i_{n+1})$ satisfying the conditions

(1) $i_a \in \mathbf{B}$, $i_1 \prec i_2 \prec \cdots \prec i_{n+1}$,
(2) i and $\overline{i}$ do not appear simultaneously.

We have $z_\ell = \tau_{-4}$. We describe the action of $\tilde{e}_0$, $\tilde{f}_0$: we have

$$\tilde{e}_0(m_T) = \begin{cases} \tau_{-4}(m_{(i_3,\cdots,i_{n+1},\overline{2},\overline{1})}) & \text{if } i_2 = 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{f}_0(m_T) = \begin{cases} \tau_4(m_{(1,2,i_1,\cdots,i_{n-1})}) & \text{if } i_n = \overline{2}, \\ 0 & \text{otherwise.} \end{cases}$$

So all monomials in $\mathcal{M}(M)/\tau_4$ are written as m_T. As an application, we recover a known result $\mathcal{B}(W(\varpi_\ell)) = \mathcal{B}_{I_0}(\varpi_\ell)$.

By the condition (2) there is always an entry $i_a = 0$. If we remove this entry, we get the tableaux description in [23].

5.3. Type $C_n^{(1)}$. Let $\mathbf{B} = \{1, \dots, n, \overline{n}, \dots, \overline{1}\}$. We give the ordering $\prec$ on the set $\mathbf{B}$ by

$$1 \prec 2 \prec \cdots \prec n \prec \overline{n} \prec \cdots \prec \overline{2} \prec \overline{1}.$$

For $p \in \mathbb{Z}$, we define

$$\boxed{i}_p = Y_{i-1,p+i}^{-1} Y_{i,p+i-1} \qquad (1 \le i \le n),$$

$$\boxed{\overline{i}}_p = Y_{i-1,p+2n-i} Y_{i,p+2n+1-i}^{-1} \qquad (1 \le i \le n).$$

5.3.1. First consider the case $\ell = 1$. Let $M = Y_{0,1}^{-1} Y_{1,0}$. It follows from Corollary 3.3 that $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_\ell)$. The crystal graph of $\mathcal{M}(M)$ is given in Figure 4. We find $\tau_{2n} = z_\ell^{-1}$ and $\mathcal{M}(M)/\tau_{2n} = \mathcal{M}_{I_0}(M)$.

$$\begin{tikzcd} \boxed{1}_0 \arrow[r, "1"] & \boxed{2}_0 \arrow[r, "2"] & \cdots \arrow[r, "n-1"] & \boxed{n}_0 \arrow[r, "n"] & \boxed{\overline{n}}_0 \arrow[r, "n-1"] & \cdots \arrow[r, "2"] & \boxed{\overline{2}}_0 \arrow[r, "1"] & \boxed{\overline{1}}_0 \arrow[lllllll, bend left, "0{[2n]}"] \end{tikzcd}$$

FIGURE 4. (Type $C_n^{(1)}$) the crystal $\mathcal{B}(\varpi_1)$

5.3.2. *Preliminary results for crystals of finite type C.* Let $1 \leq \ell \leq n$, $0 \leq r \leq n-\ell$ and $0 \leq h \leq \ell$. Consider the monomial

$$\begin{aligned} &M_{\ell,h,r} = Y_{h,\ell-h} Y_{h,\ell-h-2r}^{-1} Y_{\ell,-2r} \\ &= \left(\boxed{1}_{\ell-1} \boxed{2}_{\ell-3} \cdots \boxed{h}_{\ell-2h+1}\right) \times \left(\boxed{h{+}1}_{\ell-2h-2r-1} \boxed{h{+}2}_{\ell-2h-2r-3} \cdots \boxed{\ell}_{1-\ell-2r}\right) \\ &= \prod_{p=1}^{h} \boxed{p}_{\ell-2p+1} \times \prod_{p=h+1}^{\ell} \boxed{p}_{\ell+1-2p-2r}. \end{aligned}$$

For $T = ((i_1, \cdots, i_h), (i_{h+1}, \cdots, i_\ell))$ such that $i_p \in \mathbf{B}$, we define the monomial

$$m_T = \boxed{i_1}_{\ell-1} \boxed{i_2}_{\ell-3} \cdots \boxed{i_h}_{\ell-2h+1} \boxed{i_{h+1}}_{\ell-1-2h-2r} \boxed{i_{h+2}}_{\ell-2h-3-2r} \cdots \boxed{i_\ell}_{-\ell+1-2r}.$$

Let $C_{\ell,h,r}$ be the set of tableaux T satisfying the following conditions

(C.1) $i_a \in \mathbf{B}$, $i_1 \prec i_2 \prec \cdots \prec i_h$, and $i_{h+1} \prec i_{h+2} \prec \cdots \prec i_\ell$.

(C.2) There is no pair a, b such that $1 \leq a < b \leq h$ and $i_a = k$, $i_b = \overline{k}$ and $b - a = n - k$.

(C.3) There is no pair a, b such that $h+1 \leq a < b \leq \ell$ and $i_a = k$, $i_b = \overline{k}$ and $b - a = n - k$.

(C.4) There is no pair a, b such that $a \leq h$, $h+1 \leq b$, $i_a = k$, $i_b = \overline{k}$ and $b - a = n + 1 - \max(r, 1) - k$.

(C.5) Suppose that $i_{h+1} = k \in \{1, \ldots, n\}$ and $i_h \succeq i_{h+1}$. Then $i_h = k'$ is also in $\{1, \ldots, n\}$, and the successive part $(\overline{k'}, \overline{k'-1}, \ldots, \overline{k})$ appears as $(i_{b'}, i_{b'+1}, \ldots, i_b)$ with $n - r - k + 1 < b - h \leq n - k$.

(C.6) Suppose that $i_{h+1} = \overline{k} \in \{\overline{1}, \ldots, \overline{n}\}$ and $i_h \succeq i_{h+1}$. Then $i_h = \overline{k'}$ is also in $\{\overline{1}, \ldots, \overline{n}\}$, and the successive part $(k', k'+1, \ldots, k)$ appears as $(i_{a'}, i_{a'+1}, \ldots, i_a)$ with $n - r - k + 1 \leq h - a < n - k$.

Note that the conditions above are the same as the ones in [15] when $r = 0$. Note also that we only have $r = 0$ when $\ell = n$.

For $T = ((i_1, \ldots, i_h), (i_{h+1}, \ldots, i_\ell)) \in C_{\ell,h,r}$ we define the tableau $\tau_{\ell,h,r}(T)$ in the following three cases separately.

(C.a) $i_{h+1} = k \in \{1, \ldots, n\}$ and there is an entry $i_b = \overline{k}$ with $n - r - k + 1 < b - h \leq n - k$.

(C.b) $i_{h+1} = \overline{k} \in \{\overline{1}, \ldots, \overline{n}\}$ and there is an entry $i_a = k$ with $n - r - k + 1 \leq h - a < n - k$.

(C.c) Neither (C.a) nor (C.b) is not satisfied.

In the case (C.a), let b'' such that $(i_{b''}, i_{b''+1}, \dots, i_b)$ are successive as $(\overline{k''}, \overline{k''+1}, \dots, \overline{k})$ and $i_{b''-1} \neq \overline{k''-1}$. We have $k'' < n-1$. We set

$$\tau_{\ell,h,r}(T) = ((i_1, \cdots, i_h, k''{+}1), \\ (i_{h+2}, \cdots, i_{b''-1}, \overline{k''{+}1}, \overline{k''}, \cdots, \overline{k{+}1}, i_{b+1}, \cdots, i_\ell)).$$

Similarly in the case (C.b), we take $i_{a''}$ so that $(i_{a''}, i_{a''+1}, \dots, i_a) = (k'', k''+1, \dots, k)$ and $i_{a''-1} \neq k''-1$. We have $k < n-1$. We then set

$$\tau_{\ell,h,r}(T) = ((i_1, \cdots, i_{a''-1}, k''{-}1, \cdots, k{-}1, i_{a+1}, \cdots, i_h, \\ \overline{k''{-}1}), (i_{h+2}, \cdots, i_\ell)).$$

In the case (C.c) we set

$$\tau_{\ell,h,r}(T) = ((i_1, \cdots, i_{h+1}), (i_{h+2}, \cdots, i_\ell)).$$

Theorem 5.8. (1) *The map $T \mapsto m_T$ induces a crystal isomorphism between $C_{\ell,h,r}$ and $\mathcal{M}_{I_0}(M_{\ell,h,r})$.*

(2) *$\tau_{\ell,h,r}$ induces a crystal isomorphism $\mathcal{M}_{I_0}(M_{\ell,h,r})$ to $\mathcal{M}_{I_0}(M_{\ell,h+1,r})$.*

5.3.3. Now we study $\mathcal{B}(\varpi_\ell)$ for $1 \le \ell \le n$. Let $M_0 = Y_{\ell,0} Y_{0,\ell}^{-1} = \boxed{1}_{\ell-1} \boxed{2}_{\ell-3} \cdots \boxed{\ell}_{1-\ell}$. It follows from Corollary 3.3 that $\mathcal{M}(M_0) \simeq \mathcal{B}(\varpi_\ell)$.

For $0 \le j < \ell$, let us define the monomial $m_{T;j}$ associated with $T = ((i_1, \dots, i_{\ell-j}), (i_{\ell-j+1}, \cdots, i_\ell)) \in C_{\ell,\ell-j,n-\ell}$ by

$$m_{T;j} = \prod_{a=1}^{\ell-j} \boxed{i_a}_{-2j+\ell+1-2a} \times \prod_{a=\ell-j+1}^{\ell} \boxed{i_a}_{3\ell+1-2n-2j-2a}.$$

We extend the definition of $m_{T;j}$ for all $j \in \mathbb{Z}$ so that $m_{T;j+\ell} = \tau_{2n} m_{T;j}$.

We describe the action of $\tilde{e}_0$, $\tilde{f}_0$ by computation on monomials. We get that $\tilde{e}_0(m_{T;j})$ is equal to

$$\begin{cases} m_{(i_2,\cdots,i_\ell,\overline{1});j+1} & \text{if } i_1 = 1 \text{ and } i_\ell \neq \overline{1}, \\ 0 & \text{otherwise,} \end{cases}$$

and that $\tilde{f}_0(m_{T;j})$ is equal to

$$\begin{cases} m_{(1,i_1,\cdots,i_{\ell-1});j-1} & \text{if } i_1 \neq 1 \text{ and } i_\ell = \overline{1}, \\ 0 & \text{otherwise.} \end{cases}$$

We have $\tau_{2n} = z_\ell^{-\ell}$ and all monomials in $\mathcal{M}(M_0)/\tau_{2n}$ are written as $m_{T;j}$. The case $\ell = n$ is exceptional. We have $\tau_2 = z_n^{-1}$, so $\mathcal{M}(M_0)/\tau_2 \simeq \mathcal{B}(W(\varpi_n))$. The P_{cl}-crystal automorphism z_ℓ is given by $\tau_{\ell,\ell-j-1,n-\ell}^{-1}$.

As an application, we have

$$\mathcal{B}(W(\varpi_\ell)) \simeq \mathcal{B}_{I_0}(\varpi_\ell).$$

A conjectural description of the crystal of $\mathcal{B}(W(\varpi_\ell))$ was proposed in [36]. As their description is given by relating the crystal to an $A^{(1)}_{2n+1}$-crystal, it is not clear, at least to authors, whether their conjecture is true or not.

5.4. Type $A_{2n}^{(2)}$ $(n \geq 1)$. Let $\mathbf{B} = \{1, \dots, n, \overline{n}, \dots, \overline{1}\}$. We give the ordering $\prec$ on the set $\mathbf{B}$ by

$$1 \prec 2 \prec \cdots \prec n \prec \overline{n} \prec \cdots \prec \overline{2} \prec \overline{1}.$$

For $p \in \mathbb{Z}$, we define

$$\boxed{1}_p = Y_{1,p} Y_{0,p+1}^{-2}, \quad \boxed{\overline{1}}_p = Y_{0,p+2n-1}^{2} Y_{1,p+2n}^{-1},$$
$$\boxed{i}_p = Y_{i,p+i-1} Y_{i-1,p+i}^{-1} \qquad (2 \leq i \leq n),$$
$$\boxed{\overline{i}}_p = Y_{i-1,p+2n-i} Y_{i,p+2n-i+1}^{-1} \qquad (2 \leq i \leq n).$$

5.4.1. First consider the case $\ell = 1$. Let $M = Y_{1,0} Y_{0,1}^{-2}$. It follows from Corollary 3.3 that $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_\ell)$. Let $M' = \tilde{e}_0(M) = Y_{0,-1} Y_{0,1}^{-1}$. The crystal graph of $\mathcal{M}(M)$ is given in Figure 5. We find $\tau_{2n} = z_\ell^{-1}$ and $\mathcal{M}(M)/\tau_{2n} = \mathcal{M}_{I_0}(M) \sqcup \mathcal{M}_{I_0}(M')$.

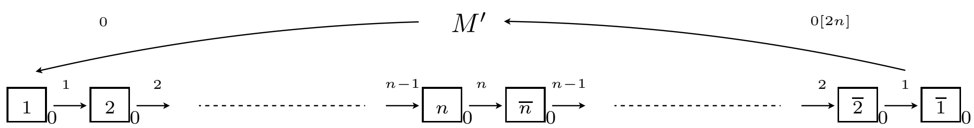

FIGURE 5. (Type $A_{2n}^{(2)}$) the crystal $\mathcal{B}(\varpi_1)$

5.4.2. Now we study $\mathcal{B}(\varpi_\ell)$ for $1 \leq \ell \leq n$. Let $M_{0,0} = Y_{\ell,0} Y_{0,\ell}^{-2} = \boxed{1}_{\ell-1} \boxed{2}_{\ell-3} \cdots \boxed{\ell}_{1-\ell}$. It follows from Corollary 3.3 that $\mathcal{M}(M_{0,0}) \simeq \mathcal{B}(\varpi_\ell)$.

For $0 \leq j < \ell$, $0 \leq k < \ell$, let us define the monomial $m_{T;j,k}$ associated with $T = ((i_1, \dots, i_{\ell-j-k}), (i_{j-2k+1}, \cdots, i_{\ell-k})) \in C_{\ell-k,\ell-j-k,n-\ell}$ by

(1) $0 \leq k \leq \ell - j - 1$:

$$m_{T;j,k} = (Y_{0,\ell-2j}^{-1} Y_{0,\ell-2j-2k}) \prod_{a=1}^{\ell-j-k} \boxed{i_a}_{-2j+\ell+1-2a-2k} \times \prod_{a=\ell-j-k+1}^{\ell-k} \boxed{i_a}_{3\ell+1-2n-2j-2a-2k},$$

(2) $\ell - j \leq k \leq \ell - 1$:

$$m_{T;j,k} = (Y_{0,\ell-2j}^{-1} Y_{0,-\ell} Y_{0,\ell-2n}^{-1} Y_{0,-2n+3\ell-2j-2k}) \prod_{a=1}^{\ell-k} \boxed{i_a}_{3\ell+1-2n-2j-2a-2k}.$$

For $k = \ell$ we set $C_{0,-j,n-\ell} = \{\emptyset\}$ and define $m_{\emptyset;j,k}$ by the same formula as in (1),(2) where the last product is understood as 1. We extend the definition of $m_{T;j,k}$ for all $j \in \mathbb{Z}$ so that $m_{T;j+\ell,k} = \tau_{2n} m_{T;j,k}$.

We describe the action of $\tilde{e}_0$, $\tilde{f}_0$. We get that $\tilde{e}_0(m_{T;j,k})$ is equal to

$$\begin{cases} m_{(i_2,\cdots,i_{\ell-k});j,k+1} & \text{if } i_1 = 1 \text{ and } i_{\ell-k} \neq \overline{1}, \\ m_{(i_1,\cdots,i_{\ell-k},\overline{1});j+1,k-1} & \text{if } i_1 \neq 1,\ i_{\ell-k} \neq \overline{1} \text{ and } k > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and that $\tilde{f}_0(m_{T;j,k})$ is equal to

$$\begin{cases} m_{(i_1,\cdots,i_{\ell-k-1});j-1,k+1} & \text{if } i_1 \neq 1 \text{ and } i_{\ell-k} = \overline{1}, \\ m_{(1,i_1,\cdots,i_{\ell-k});j,k-1} & \text{if } i_1 \neq 1,\ i_{\ell-k} \neq \overline{1} \text{ and } k > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We have $\tau_{2n} = (z_\ell)^{-\ell}$ and all the monomials in $\mathcal{M}(M_{0,0})/\tau_{2n}$ are written as $m_{T;j,k}$. The case $\ell = n$ is exceptional. We have $\tau_2 = z_n^{-1}$, so $\mathcal{M}(M_0)/\tau_2 \simeq \mathcal{B}(W(\varpi_n))$. For $\ell \neq n$, the crystal automorphism z_ℓ is given by $\tau^{-1}_{\ell-k,\ell-j-k-1,n-\ell}$. As an application, we have

$$\mathcal{B}(W(\varpi_\ell)) \simeq \mathcal{B}_{I_0}(\varpi_\ell) \sqcup \mathcal{B}_{I_0}(\varpi_{\ell-1}) \sqcup \cdots \sqcup \mathcal{B}_{I_0}(\varpi_1) \sqcup \mathcal{B}_{I_0}(0).$$

A conjectural description of the crystal of $\mathcal{B}(W(\varpi_\ell))$ was proposed in [36]. As their description is given by relating the crystal to an $A^{(1)}_{2n+1}$-crystal, it is not clear, at least to authors, whether their conjecture is true or not.

5.5. Type $A^{(2)\dagger}_{2n}$ $(n \geq 1)$. Let $\mathbf{B} = \{1, \dots, n, 0, \overline{n}, \dots, \overline{1}\}$. We give the ordering $\prec$ on the set $\mathbf{B}$ by

$$1 \prec 2 \prec \cdots \prec n \prec 0 \prec \overline{n} \prec \cdots \prec \overline{2} \prec \overline{1}.$$

For $p \in \mathbb{Z}$, we define

$$\begin{aligned} \boxed{i}_p &= Y^{-1}_{i-1,p+i} Y_{i,p+i-1} \qquad (1 \leq i \leq n-1), \\ \boxed{n}_p &= Y^{-1}_{n-1,p+n} Y^2_{n,p+n-1}, \\ \boxed{0}_p &= Y^{-1}_{n,p+n+1} Y_{n,p+n-1}, \\ \boxed{\overline{n}}_p &= Y_{n-1,p+n} Y^{-2}_{n,p+n+1}, \\ \boxed{\overline{i}}_p &= Y_{i-1,p+2n-i} Y^{-1}_{i,p+2n+1-i} \qquad (1 \leq i \leq n-1). \end{aligned}$$

5.5.1. First consider the case $\ell = 1$. Let $M = Y^{-1}_{0,1} Y_{1,0}$. It follows from Corollary 3.3 that $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_\ell)$. The crystal graph of $\mathcal{M}(M)$ is given in Figure 6. We find $\tau_{2n} = z_\ell^{-1}$ and $\mathcal{M}(M)/\tau_{2n} = \mathcal{M}_{I_0}(M)$.

$$\begin{tikzcd} \boxed{1}_0 \arrow[r, "1"] & \boxed{2}_0 \arrow[r, "2"] & \cdots \arrow[r, "n-1"] & \boxed{n}_0 \arrow[r, "n"] & \boxed{0}_0 \arrow[r, "n"] & \boxed{\overline{n}}_0 \arrow[r, "n-1"] & \cdots \arrow[r, "2"] & \boxed{\overline{2}}_0 \arrow[r, "1"] & \boxed{\overline{1}}_0 \arrow[llllllll, bend left, "{0[2n]}"] \end{tikzcd}$$

FIGURE 6. (Type $A^{(2)\dagger}_{2n}$) the crystal $\mathcal{B}(\varpi_1)$

5.5.2. Now we study $\mathcal{B}(\varpi_\ell)$ for $1 \le \ell \le n-1$. Let $M_0 = Y_{\ell,0}Y_{0,\ell}^{-1} = \boxed{1}_{\ell-1}\boxed{2}_{\ell-3}\cdots\boxed{\ell}_{1-\ell}$. It follows from Corollary 3.3 that $\mathcal{M}(M_0) \simeq \mathcal{B}(\varpi_\ell)$.

For $0 \le j < \ell$, let us define the monomial $m_{T;j}$ associated with $T = ((i_1, \dots, i_{\ell-j}), (i_{\ell-j+1}, \cdots, i_\ell)) \in B_{\ell,\ell-j,n-\ell}$ by

$$m_{T;j} = \prod_{a=1}^{\ell-j} \boxed{i_a}_{-2j+\ell+1-2a} \times \prod_{a=\ell-j+1}^{\ell} \boxed{i_a}_{3\ell+1-2n-2j-2a}.$$

We extend the definition of $m_{T;j}$ for all $j \in \mathbb{Z}$ so that $m_{T;j+\ell} = \tau_{2n} m_{T;j}$. We describe the action of $\tilde{e}_0$, $\tilde{f}_0$ by computation on monomials. We get that $\tilde{e}_0(m_{T;j})$ is equal to

$$\begin{cases} m_{(i_2,\cdots,i_\ell,\overline{1});j+1} & \text{if } i_1 = 1 \text{ and } i_\ell \neq \overline{1}, \\ 0 & \text{otherwise,} \end{cases}$$

and that $\tilde{f}_0(m_{T;j})$ is equal to

$$\begin{cases} m_{(1,i_1,\cdots,i_{\ell-1});j-1} & \text{if } i_1 \neq 1 \text{ and } i_\ell = \overline{1}, \\ 0 & \text{otherwise.} \end{cases}$$

We have $\tau_{2n} = z_\ell^{-\ell}$ and all monomials in $\mathcal{M}(M_0)/\tau_{2n}$ are written as $m_{T;j}$. The P_{cl}-crystal automorphism z_ℓ is given by $\tau_{\ell,\ell-j-1,n-\ell}^{-1}$. As an application, we have

$$\mathcal{B}(W(\varpi_\ell)) \simeq \mathcal{B}_{I_0}(\varpi_\ell).$$

A conjectural description of the crystal of $\mathcal{B}(W(\varpi_\ell))$ was proposed in [36]. As their description is given by relating the crystal to an $A_{2n+1}^{(1)}$-crystal, it is not clear, at least to authors, whether their conjecture is true or not.

5.5.3. Finally we consider the case $\ell = n$. Let $M = Y_{n,0}^2 Y_{0,\ell}^{-1} = \boxed{1}_{n-1}\boxed{2}_{n-3}\cdots\boxed{n}_{1-n}$. It follows from Corollary 3.3 that $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_n)$. Let us define the monomial $m_T = \prod_{a=1}^n \boxed{i_a}_{n+1-2a}$ associated with $T = (i_1, \cdots, i_n)$ satisfying (1) $i_a \in \mathbf{B}$ and $i_1 \prec i_2 \prec \cdots \prec i_n$ but 0 can be repeated, and (2) there is no pair a, b such that $i_a = k$, $i_b = \overline{k}$ and $b - a = n - k$. The above exhausts all monomials in $\mathcal{M}_{I_0}(M)$ (see [15, Proposition 2.10]). We describe the action of $\tilde{e}_0$, $\tilde{f}_0$ on these monomials : we have

$$\tilde{e}_0(m_T) = \begin{cases} \tau_{-2}(m_{(i_2,\cdots,i_n,\overline{1})}) & \text{if } i_1 = 1 \text{ and } i_n \neq \overline{1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{f}_0(m_T) = \begin{cases} \tau_2(m_{(1,i_1,\cdots,i_{n-1})}) & \text{if } i_n = \overline{1} \text{ and } i_1 \neq 1, \\ 0 & \text{otherwise.} \end{cases}$$

So the above exhausts all the monomials in $\mathcal{M}(M)/\tau_2$. We have $\tau_2 = z_n^{-1}$, so $\mathcal{M}(M)/\tau_2 \simeq \mathcal{B}(W(\varpi_n))$. As an application, we have $\mathcal{B}(W(\varpi_n)) \simeq \mathcal{B}_{I_0}(\varpi_n)$. Note that ϖ_n is identified with the *twice* of the n^{th} fundamental weight of $\mathfrak{g}_{I_0}$.

5.6. Type $A_{2n-1}^{(2)}$ ($n \ge 3$). Let $\mathbf{B} = \{1, \dots, n, \overline{n}, \dots, \overline{1}\}$. We give the ordering $\prec$ on the set $\mathbf{B}$ by

$$1 \prec 2 \prec \cdots \prec n \prec \overline{n} \prec \cdots \prec \overline{2} \prec \overline{1}.$$

For $p \in \mathbb{Z}$, we define

$$\boxed{1}_p = Y_{0,p+2}^{-1} Y_{1,p}, \quad \boxed{2}_p = Y_{0,p+2}^{-1} Y_{1,p+2}^{-1} Y_{2,p+1},$$
$$\boxed{i}_p = Y_{i-1,p+i}^{-1} Y_{i,p+i-1} \qquad (3 \le i \le n),$$
$$\boxed{\overline{i}}_p = Y_{i-1,p+2n-i} Y_{i,p+2n+1-i}^{-1} \qquad (3 \le i \le n),$$
$$\boxed{\overline{2}}_p = Y_{0,p+2n-2} Y_{1,p+2n-2} Y_{2,p+2n-1}^{-1}, \qquad \boxed{\overline{1}}_p = Y_{0,p+2n-2} Y_{1,p+2n}^{-1}.$$

5.6.1. First consider the case $\ell = 1$. Let $M = Y_{1,0} Y_{0,2}^{-1}$. It follows from Corollary 3.3 that $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_1)$. The crystal graph of $\mathcal{M}(M)$ is given in Figure 7. We find $\tau_{2n-2} = z_\ell^{-1}$ and $\mathcal{M}(M)/\tau_{2n-2} = \mathcal{M}_{I_0}(M)$.

$$\begin{tikzcd}
\boxed{1}_0 \arrow[r, "1"] & \boxed{2}_0 \arrow[r, "2"] & \cdots \arrow[r, "n-1"] & \boxed{n}_0 \arrow[r, "n"] & \boxed{\overline{n}}_0 \arrow[r, "n-1"] & \cdots \arrow[r, "2"] & \boxed{\overline{2}}_0 \arrow[r, "1"] \arrow[llllll, bend right, "0[2n-2]"'] & \boxed{\overline{1}}_0 \arrow[llllll, bend left, "0[2n-2]"]
\end{tikzcd}$$

FIGURE 7. (Type $A_{2n-1}^{(2)}$) the crystal $\mathcal{B}(\varpi_1)$

5.6.2. Now we study $\mathcal{B}(\varpi_\ell)$ for $2 \le \ell \le n-1$. Let $M_{0,0} = Y_{\ell,0} Y_{0,\ell-1}^{-1} Y_{0,\ell+1}^{-1} = \boxed{1}_{\ell-1} \boxed{2}_{\ell-3} \cdots \boxed{\ell}_{1-\ell}$. As $\tilde{f}_2 \tilde{f}_3 \cdots \tilde{f}_\ell(M_{0,0}) = Y_{0,\ell+1}^{-1} Y_{0,\ell-1} Y_{1,\ell}^{-1} Y_{\ell+1,1}$, we see as in Proposition 3.4 that $\mathcal{M}(M_{0,0}) \simeq \mathcal{B}(\varpi_\ell)$.

For $0 \le j < \ell$, $0 \le k < \ell/2$, let us define the monomial $m_{T;j,k}$ associated with $T = ((i_1, \ldots, i_{j-2k}), (i_{j-2k+1}, \cdots, i_{\ell-2k})) \in C_{\ell-2k, j-2k, n-\ell-1}$ by

(1) $k < \lfloor j/2 \rfloor$:

$$m_{T;j,k} = Y_{0,2n-\ell-4k+2j-1} Y_{0,2n-\ell+2j-1}^{-1} \prod_{a=1}^{j-2k} \boxed{i_a}_{2n-\ell-4k-2a+2j-1} \times \prod_{a=j-2k+1}^{\ell-2k} \boxed{i_a}_{\ell-2(a-j+2k)+1},$$

(2) j is odd and $k = (j-1)/2$:

$$m_{T;j,(j-1)/2} = Y_{0,2n-\ell-3} Y_{0,2n-\ell+2j-1}^{-1} \boxed{i_1}_{2n-\ell-1} \prod_{a=2}^{\ell-j+1} \boxed{i_a}_{\ell-2a+3},$$

(3) j is even and $k \ge j/2$:

$$m_{T;j,k} = Y_{0,\ell-4k+2j+1} Y_{0,\ell+1}^{-1} Y_{0,2n-\ell-1} Y_{0,2n-\ell+2j-1}^{-1} \prod_{a=1}^{\ell-2k} \boxed{i_a}_{\ell-2a-4k+2j+1},$$

(4) j is odd and $k \ge (j+1)/2$:

$$m_{T;j,k} = Y_{0,\ell-4k+2j+1} Y_{0,2n-\ell+2j-1}^{-1} Y_{1,\ell+1}^{-1} Y_{1,2n-\ell-1} \prod_{a=1}^{\ell-2k} \boxed{i_a}_{\ell-2a-4k+2j+1}.$$

For $k = \ell/2$ we set $C_{0,j-\ell,n-\ell-1} = \{\emptyset\}$ and define $m_{\emptyset;j,k}$ by the same formula as in (3),(4) where the last product is understood as 1. We extend the definition of $m_{T;j,k}$ for all $j \in \mathbb{Z}$ so that $m_{T;j+\ell,k} = \tau_{2n-2} m_{T;j,k}$.

We describe the action of $\tilde{e}_0$, $\tilde{f}_0$. We get that $\tilde{e}_0(m_{T;j,k})$ is equal to

$$\begin{cases} m_{(i_3,\cdots,i_{\ell-2k});j,k+1} & \text{if } i_2 = 2 \text{ and } i_{\ell-2k-1} \not\succeq \overline{2}, \\ m_{(i_1,\cdots,i_{\ell-2k},\overline{2},\overline{1});j-2,k-1} & \text{if } i_2 \not\preceq 2,\ i_{\ell-2k} \not\succeq \overline{2} \text{ and } k > 0, \\ m_{(i_2,\cdots,i_\ell,\overline{3-i_1});j-1,0} & \text{if } i_1 \preceq 2,\ i_2 \not\preceq 2,\ i_\ell \not\succeq \overline{2} \text{ and } k = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and that $\tilde{f}_0(m_{T;j,k})$ is equal to

$$\begin{cases} m_{(1,2,i_1,\cdots,i_{\ell-2k});j,k-1} & \text{if } i_1 \not\preceq 2,\ i_{\ell-2k-1} \not\succeq \overline{2} \text{ and } k > 0, \\ m_{(i_1,\cdots,i_{\ell-2k-2});j+2,k+1} & \text{if } i_{\ell-2k-1} = \overline{2} \text{ and } i_2 \not\preceq 2, \\ m_{(3-\overline{i_\ell},i_1,\cdots,i_{\ell-1});j+1,0} & \text{if } i_1 \not\preceq 2,\ i_{\ell-1} \not\succeq \overline{2},\ i_\ell \succeq \overline{2} \text{ and } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We have $\tau_{2n-2} = (z_\ell)^{-\ell}$, and all monomials in $\mathcal{M}(M_{0,0})/\tau_{2n-2}$ are written as $m_{T;j,k}$. The crystal automorphism z_ℓ is given by $\tau^{-1}_{\ell-2k,j-2k,n-\ell-1}$. As an application, we have

$$\mathcal{B}(W(\varpi_\ell)) \simeq \mathcal{B}_{I_0}(\varpi_\ell) \sqcup \mathcal{B}_{I_0}(\varpi_{\ell-2}) \sqcup \cdots \sqcup \begin{cases} \mathcal{B}_{I_0}(\varpi_1) & \text{if } \ell \text{ is odd,} \\ \mathcal{B}_{I_0}(0) & \text{if } \ell \text{ is even.} \end{cases}$$

A crystal base on $W(\varpi_\ell)$ was constructed in [12]. A key fact used there is that $W(\varpi_\ell)$ remains irreducible when it is restricted to $\mathcal{U}_q(\overline{\mathfrak{g}})$ for the finite dimensional Lie algebra $\overline{\mathfrak{g}}$ obtained by removing the vertex n. They showed that the crystal base for the restriction is preserved also by $\tilde{e}_n$, $\tilde{f}_n$. By the uniqueness of the crystal base for an irreducible $\mathcal{U}_q(\overline{\mathfrak{g}})$-module we conclude that their crystal base is isomorphic to the $\mathcal{B}(W(\varpi_\ell))$. However their description of the Kashiwara operators was given in terms of $\overline{\mathfrak{g}}$, it is not obvious to compare our description to theirs.

5.6.3. Finally we consider the case $\ell = n$. Let $M_0 = Y_{n,0} Y_{0,n-1}^{-2} = \prod_{a=1}^{n} \boxed{a}_{\,n+1-2a}$. It follows from Corollary 3.3 that $\mathcal{M}(M_0) \simeq \mathcal{B}(\varpi_\ell)$.

For $0 \le k \le \lfloor n/2 \rfloor$, let us define the monomial $m_{T;k}$ associated with $T = (i_1, \ldots, i_{n-2k}) \in C_{n-2k,0,0}$ by

$$m_{T;k} = Y_{0,n-1}^{-1} Y_{0,n+1-4k} \times \prod_{a=1}^{n-2k} \boxed{i_a}_{\,n+1-4k-2a},$$

where the case $n = 2k$ is understood as before.

We describe the action of $\tilde{e}_0$, $\tilde{f}_0$. We get that $\tilde{e}_0(m_{T;k})$ is equal to

$$\begin{cases} m_{(i_3,\cdots,i_{n-2k});k+1} & \text{if } i_2 = 2 \text{ and } i_{n-2k-1} \not\succeq \overline{2}, \\ \tau_{-4}(m_{(i_1,\cdots,i_{n-2k},\overline{2},\overline{1});k-1}) & \text{if } i_2 \not\preceq 2,\ i_{n-2k} \not\succeq \overline{2} \text{ and } k > 0, \\ \tau_{-2}(m_{(i_2,\cdots,i_n,\overline{3-i_1});0}) & \text{if } i_1 \preceq 2,\ i_2 \not\preceq 2,\ i_n \not\succeq \overline{2} \text{ and } k = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and that $\tilde{f}_0(m_{T;k})$ is equal to

$$\begin{cases} m_{(1,2,i_1,\cdots,i_{n-2k});k-1} & \text{if } i_1 \not\preceq 2,\ i_{n-2k-1} \not\succeq \overline{2} \text{ and } k>0, \\ \tau_4(m_{(i_1,\cdots,i_{n-2k-2});k+1}) & \text{if } i_{n-2k-1} = \overline{2} \text{ and } i_2 \not\preceq 2, \\ \tau_2(m_{(3-\overline{i_\ell},i_1,\cdots,i_{\ell-1});0}) & \text{if } i_1 \not\preceq 2,\ i_{n-1} \not\succeq \overline{2},\ i_n \succeq \overline{2} \text{ and } k=0, \\ 0 & \text{otherwise.} \end{cases}$$

We find that $z_\ell = \tau_{-2}$ and the monomials appearing in $\mathcal{M}(M_0)/\tau_2$ are written as $m_{T;k}$. As an application, we have

$$\mathcal{B}(W(\varpi_\ell)) \simeq \mathcal{B}_{I_0}(\varpi_\ell) \sqcup \mathcal{B}_{I_0}(\varpi_{\ell-2}) \sqcup \cdots \sqcup \begin{cases} \mathcal{B}_{I_0}(\varpi_1) & \text{if } \ell \text{ is odd,} \\ \mathcal{B}_{I_0}(0) & \text{if } \ell \text{ is even.} \end{cases}$$

5.7. Type $D_{n+1}^{(2)}$ ($n \geq 2$). Let $\mathbf{B} = \{1, \dots, n, 0, \overline{n}, \dots, \overline{1}\}$. We give the ordering $\prec$ on the set $\mathbf{B}$ by

$$1 \prec 2 \prec \cdots \prec n \prec 0 \prec \overline{n} \prec \cdots \prec \overline{2} \prec \overline{1}.$$

For $p \in \mathbb{Z}$, we define

$$\begin{aligned} \boxed{1}_p &= Y_{1,p} Y_{0,p+1}^{-2}, \\ \boxed{i}_p &= Y_{i,p+i-1} Y_{i-1,p+i}^{-1} \qquad (2 \leq i \leq n-1), \\ \boxed{n}_p &= Y_{n-1,p+n}^{-1} Y_{n,p+n-1}^{2}, \\ \boxed{0}_p &= Y_{n,p+n-1} Y_{n,p+n+1}^{-1}, \\ \boxed{\overline{n}}_p &= Y_{n-1,p+n} Y_{n,p+n+1}^{-2}, \\ \boxed{\overline{i}}_p &= Y_{i-1,p+2n-i} Y_{i,p+2n-i+1}^{-1} \qquad (2 \leq i \leq n-1), \\ \boxed{\overline{1}}_p &= Y_{0,p+2n-1}^{2} Y_{1,p+2n}^{-1}. \end{aligned}$$

5.7.1. First consider the case $\ell = 1$. Let $M = Y_{1,0} Y_{0,1}^{-2}$. It follows from Corollary 3.3 that $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_\ell)$. Let $M' = \tilde{e}_0(M) = Y_{0,-1} Y_{0,1}^{-1}$. The crystal graph of $\mathcal{M}(M)$ is given in Figure 8. We find $\tau_{2n} = z_\ell^{-1}$ and $\mathcal{M}(M)/\tau_{2n} = \mathcal{M}_{I_0}(M) \sqcup \mathcal{M}_{I_0}(M')$.

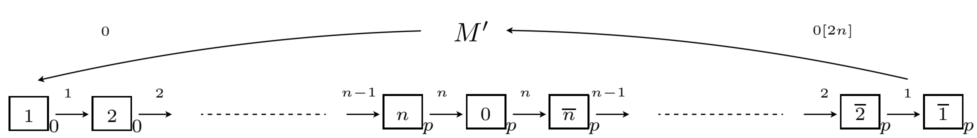

FIGURE 8. (Type $D_{n+1}^{(2)}$) the crystal $\mathcal{B}(\varpi_1)$

5.7.2. Now we study $\mathcal{B}(\varpi_\ell)$ for $1 \le \ell \le n-1$. Let $M_{0,0} = Y_{\ell,0}Y_{0,\ell}^{-2} = \boxed{1}_{\ell-1}\boxed{2}_{\ell-3}\cdots\boxed{\ell}_{1-\ell}$. It follows from Corollary 3.3 that $\mathcal{M}(M_{0,0}) \simeq \mathcal{B}(\varpi_\ell)$.

For $0 \le j < \ell$, $0 \le k < \ell$, let us define the monomial $m_{T;j,k}$ associated with $T = ((i_1, \dots, i_{\ell-j-k}), (i_{\ell-j-k+1}, \cdots, i_{\ell-k})) \in B_{\ell-k,\ell-j-k,n-\ell}$ by

(1) $0 \le k \le \ell - j - 1$:

$$m_{T;j,k} = (Y_{0,\ell-2j}^{-1}Y_{0,\ell-2j-2k}) \prod_{a=1}^{\ell-j-k} \boxed{i_a}_{-2j+\ell+1-2a-2k}$$
$$\times \prod_{a=\ell-j-k+1}^{\ell-k} \boxed{i_a}_{3\ell+1-2n-2j-2a-2k},$$

(2) $\ell - j \le k \le \ell - 1$:

$$m_{T;j,k} = (Y_{0,\ell-2j}^{-1}Y_{0,-\ell}Y_{0,\ell-2n}^{-1}Y_{0,-2n+3\ell-2j-2k}) \prod_{a=1}^{\ell-k} \boxed{i_a}_{3\ell+1-2n-2j-2a-2k}.$$

For $k = \ell$ we set $B_{0,-j,n-\ell} = \{\emptyset\}$ and define $m_{\emptyset;j,k}$ by the same formula as in (1),(2) where the last product is understood as 1. We extend the definition of $m_{T;j,k}$ for all $j \in \mathbb{Z}$ so that $m_{T;j+\ell,k} = \tau_{2n}m_{T;j,k}$.

We describe the action of $\tilde{e}_0$, $\tilde{f}_0$. We get that $\tilde{e}_0(m_{T;j,k})$ is equal to

$$\begin{cases} m_{(i_2,\cdots,i_{\ell-k});j,k+1} & \text{if } i_1 = 1 \text{ and } i_{\ell-k} \neq \overline{1}, \\ m_{(i_1,\cdots,i_{\ell-k},\overline{1});j+1,k-1} & \text{if } i_1 \neq 1,\ i_{\ell-k} \neq \overline{1} \text{ and } k > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and that $\tilde{f}_0(m_{T;j,k})$ is equal to

$$\begin{cases} m_{(i_1,\cdots,i_{\ell-k-1});j-1,k+1} & \text{if } i_1 \neq 1 \text{ and } i_{\ell-k} = \overline{1}, \\ m_{(1,i_1,\cdots,i_{\ell-k});j,k-1} & \text{if } i_1 \neq 1,\ i_{\ell-k} \neq \overline{1} \text{ and } k > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We have $\tau_{2n} = (z_\ell)^{-\ell}$ and all monomials in $\mathcal{M}(M_{0,0})/\tau_{2n}$ are written as $m_{T;j,k}$. The crystal automorphism z_ℓ is given by $\tau_{\ell-k,\ell-j-k-1,n-\ell}^{-1}$. As an application, we have

$$\mathcal{B}(W(\varpi_\ell)) \simeq \mathcal{B}_{I_0}(\varpi_\ell) \sqcup \mathcal{B}_{I_0}(\varpi_{\ell-1}) \sqcup \cdots \sqcup \mathcal{B}_{I_0}(\varpi_1) \sqcup \mathcal{B}_{I_0}(0).$$

A conjectural description of the crystal of $\mathcal{B}(W(\varpi_\ell))$ was proposed in [36]. As their description is given by relating the crystal to an $A_{2n+1}^{(1)}$-crystal, it is not clear, at least to authors, whether their conjecture is true or not.

5.7.3. Finally we consider the case $\ell = n$. Let $M = Y_{n,0}Y_{0,n}^{-1}$. It follows from Corollary 3.3 that $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_\ell)$.

Let

$$\boxed{i}_p = \begin{cases} Y_{i-1,p+i-1}^{-1}Y_{i,p+i-2} & \text{if } 1 \le i \le n-1, \\ Y_{n-1,p+n-1}^{-1} & \text{if } i = n, \\ Y_{n,p+n} & \text{if } i = 0, \end{cases}$$

$$\boxed{\overline{i}}_p = \begin{cases} Y_{0,p+2n} & \text{if } i = 1, \\ 1 & \text{if } 2 \le i \le n-1, \\ Y_{n,p+n+2}^{-2} & \text{if } i = n. \end{cases}$$

Then the monomials appearing in $\mathcal{M}_{I_0}(M)$ are $m_T = \prod_{a=1}^{n+1} \boxed{i_a}_{n+2-2a}$ associated with a tableau $T = (i_1, \dots, i_{n+1})$ satisfying the conditions

(1) $i_a \in \mathbf{B},\ i_1 \prec i_2 \prec \cdots \prec i_{n+1}$,
(2) i and $\overline{i}$ do not appear simultaneously.

We describe the action of $\tilde{e}_0$, $\tilde{f}_0$ on these monomials. We have

$$\tilde{e}_0(m_T) = \begin{cases} \tau_{-2}(m_{(i_2,\cdots,i_{n+1},\overline{1})}) & \text{if } i_1 = 1, \\ 0 & \text{otherwise}, \end{cases}$$

$$\tilde{f}_0(m_T) = \begin{cases} \tau_2(m_{(1,i_1,\cdots,i_n)}) & \text{if } i_{n+1} = \overline{1}, \\ 0 & \text{otherwise}. \end{cases}$$

We have $\tau_2 = (z_\ell)^{-1}$ and the above monomials are those appearing in $\mathcal{M}(M)/\tau_2$. As an application, we have $\mathcal{B}(W(\varpi_n)) \simeq \mathcal{B}_{I_0}(\varpi_n)$.

6. Finite dimensional crystals – exceptional types

In this section we treat all exceptional cases (except some nodes of type $E_7^{(1)}$, $E_8^{(1)}$, and for one node of type $E_6^{(2)}$ where we do not get the decomposition in I_0-crystals at this moment). We enumerate the nodes of the Dynkin diagram as explained in section 3.2.

6.1. Type $E_n^{(1)}$. Recall $V_{I_0}(\lambda)$ denotes the irreducible $\mathcal{U}_q(\mathfrak{g}_{I_0})$-module with the highest weight λ. To save the space, we write i_p instead of $Y_{i,p}$ in some places.

6.1.1. Let ℓ be a nonzero vertex with $a_\ell = 1$, i.e., $\ell = 1$ or 5 for $E_6^{(1)}$ and $\ell = 6$ for $E_7^{(1)}$. In these cases it is known that the corresponding level 0 fundamental representation $W(\varpi_\ell)$ is restricted to the irreducible $\mathcal{U}_q(\mathfrak{g}_{I_0})$-module $V_{I_0}(\varpi_\ell)$. Let us consider $\mathcal{M}(M)$ for $M = Y_{\ell,0}Y_{0,\theta_\ell}^{-1}$ where θ_ℓ is the distance of 0 and ℓ. By Corollary 3.3 we have $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_\ell)$. Moreover an explicit calculation shows that $Y_{\ell,p}Y_{0,\theta_\ell+p}^{-1} = \tau_p(M)$ appears in $\mathcal{M}(M)$ where $p = 6$ for $E_6^{(1)}$ and $p = 8$ for $E_7^{(1)}$. By the weight calculation we have $z_\ell = \tau_{-p}$. Hence $\mathcal{M}(M)/\tau_p \simeq \mathcal{B}(W(\varpi_\ell))$. We can check that all monomials are connected to some $\tau_p^N(M)$ in the I_0-crystal. This recovers the above mentioned result that $W(\varpi_\ell)$ is restricted to $V_{I_0}(\varpi_\ell)$.

Let us explain the $E_6^{(1)}$ case for an illustration. Let $M = Y_{5,0}Y_{0,4}^{-1}$. Then a calculation shows that the following 27 monomials appear in $\mathcal{B}_{I_0}(M)$:

$5_0 0_4^{-1}$, $4_1 5_2^{-1} 0_4^{-1}$, $3_2 4_3^{-1} 0_4^{-1}$, $6_3 2_3 3_4^{-1} 0_4^{-1}$, $6_5^{-1} 2_3$, $1_4 6_3 2_5^{-1} 0_4^{-1}$, $1_4 6_5^{-1} 2_5^{-1} 3_4$, $1_6^{-1} 6_3 0_4^{-1}$, $1_6^{-1} 6_5^{-1} 3_4$, $1_4 3_6^{-1} 4_5$, $1_6^{-1} 2_5 3_6^{-1} 4_5$, $1_4 4_7^{-1} 5_6$, $2_7^{-1} 4_5$, $1_6^{-1} 2_5 4_7^{-1} 5_6$, $1_4 5_8^{-1}$, $2_7^{-1} 3_6 4_7^{-1} 5_6$, $1_6^{-1} 2_5 5_8^{-1}$, $6_7 3_8^{-1} 5_6$, $2_7^{-1} 3_6 5_8^{-1}$, $6_9^{-1} 5_6 0_8$, $6_7 3_8^{-1} 4_7 5_8^{-1}$, $6_9^{-1} 4_7 5_8^{-1} 0_8$, $6_7 4_9^{-1}$, $6_9^{-1} 3_8 4_9^{-1} 0_8$, $2_9 3_{10}^{-1} 0_8$, $1_{10} 2_{11}^{-1} 0_8$, $1_{12}^{-1} 0_8$.

Applying $\tilde{f}_0$ to $6_9^{-1} 5_6 0_8$, we get $5_6 0_{10}^{-1} = \tau_6(M)$. It is also clear that all monomials are connected to either M or its τ_6-images in the I_0-crystal.

Remark 6.1. (1) For a level 0 fundamental representation $W(\varpi_\ell)$ the corresponding quiver varieties are moduli spaces of vector bundles of rank a_ℓ on ALE spaces. In particular, they are moduli spaces of line bundles for the cases studied here. Then each component is a single point, and it is a geometric reason why $W(\varpi_\ell)$ is restricted to the irreducible representation of $\mathcal{U}_q(\mathfrak{g}_{I_0})$.

(2) This crystal has been studied in [27].

6.1.2. Let ℓ be the vertex adjacent to the vertex 0, i.e., $\ell = 6$ for $E_6^{(1)}$, 1 for $E_7^{(1)}$ and $E_8^{(1)}$. We have $a_\ell = 2$. It is known that $W(\varpi_\ell)$ is restricted to the direct sum of the adjoint representation $V_{I_0}(\varpi_\ell)$ and the trivial representation $V_{I_0}(0)$ of $\mathcal{U}_q(\mathfrak{g}_{I_0})$. We can check this, for example, by using the algorithm for the t–analog of q–characters [29]. All the coefficients of monomials are 1 except one, whose coefficient is $1+t^2$. The exceptional monomial is $Y_{3,5}Y_{3,7}^{-1}$ for $E_6^{(1)}$, $Y_{3,8}Y_{3,10}^{-1}$ for $E_7^{(1)}$ and $Y_{5,14}Y_{5,16}^{-1}$ for $E_8^{(1)}$, if the l–highest weight monomial is $Y_{\ell,0}$.

Let $M = Y_{\ell,0}Y_{0,1}^{-1}Y_{0,p}^{-1}$ where $p = 5$ for $E_6^{(1)}$, 7 for $E_7^{(1)}$ and 11 for $E_8^{(1)}$. We have

$$\begin{aligned} E_6: &\quad \tilde{f}_3\tilde{f}_6 M = Y_{4,2}Y_{2,2}Y_{3,3}^{-1}Y_{0,5}^{-1}, \\ E_7: &\quad \tilde{f}_3\tilde{f}_2\tilde{f}_1 M = Y_{7,3}Y_{4,3}Y_{3,4}^{-1}Y_{0,7}^{-1}, \\ E_8: &\quad \tilde{f}_5\tilde{f}_4\tilde{f}_3\tilde{f}_2\tilde{f}_1 M = Y_{6,5}Y_{8,5}Y_{5,6}^{-1}Y_{0,11}^{-1}. \end{aligned}$$

By the same argument as in the proof of Proposition 3.4, we see that M is extremal. Therefore $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_\ell)$. A direct calculation shows that the monomial corresponding to the lowest weight vector in the adjoint representation is $m = Y_{\ell,h^\vee}^{-1}Y_{0,h^\vee-p}Y_{0,h^\vee-1}$ where $h^\vee$ is the dual Coxeter number, i.e., $h^\vee = 12$ for $E_6^{(1)}$, 18 for $E_7^{(1)}$ and 30 for $E_8^{(1)}$. Applying $\tilde{f}_0$ to m, we get $Y_{0,h^\vee-p}Y_{0,h^\vee+1}^{-1}$, which corresponds to the trivial representation.

We have $\tilde{f}_0\tilde{e}_\ell m = Y_{0,h^\vee-p+2}^{-1}Y_{\ell,h^\vee-p+1}Y_{\ell,h^\vee-2}Y_{i,h^\vee-1}^{-1}$, where i is the vertex adjacent to ℓ different from 0. A direct calculation shows that

$$Y_{0,(h^\vee-p+3)/2}^{-1}Y_{\ell,(h^\vee-p+1)/2}Y_{\ell,(h^\vee+p-5)/2}Y_{i,(h^\vee+p-3)/2}^{-1} = \tau_{-(h^\vee-p+1)/2}(\tilde{f}_0\tilde{e}_\ell m)$$

is in $\mathcal{M}_{I_0}(M)$. (Note that $(h^\vee-p+1)/2$ is 4 for $E_6^{(1)}$, 6 for $E_7^{(1)}$ and 10 for $E_8^{(1)}$.) Therefore $\tau_{(h^\vee-p+1)/2}(M)$ is contained in $\mathcal{M}(M)$. The weight of $\tau_{(h^\vee-p+1)/2}(M)$ is equal to $\mathrm{wt}(M)-\delta$. Therefore this is $z_\ell^{-1}(M)$ and we have $\tau_{(h^\vee-p+1)/2} = z_\ell^{-1}$ and $\mathcal{M}(M)/\tau_{(h^\vee-p+1)/2} \simeq \mathcal{B}(W(\varpi_\ell))$.

We can also check that $\mathcal{M}(M)/\tau_{(h^\vee-p+1)/2} \simeq \mathcal{M}_{I_0}(M) \sqcup \{Y_{0,h^\vee-p}Y_{0,h^\vee+1}^{-1}\}$. Therefore we recover that $W(\varpi_\ell)$ is restricted to $V_{I_0}(\varpi_\ell)\oplus V_{I_0}(0)$.

Let us give $E_6^{(1)}$ case for an illustration. The following monomials appear in $\mathcal{B}_{I_0}(6_0 0_1^{-1} 0_5^{-1})$:

$6_0 0_1^{-1} 0_5^{-1}$, $6_2^{-1} 3_1 0_5^{-1}$, $2_2 3_3^{-1} 4_2 0_5^{-1}$, $1_3 2_4^{-1} 4_2 0_5^{-1}$, $2_2 4_4^{-1} 5_3 0_5^{-1}$, $1_5^{-1} 4_2 0_5^{-1}$, $1_3 2_4^{-1} 3_3 4_4^{-1} 5_3 0_5^{-1}$, $2_2 5_5^{-1} 0_5^{-1}$, $1_5^{-1} 3_3 4_4^{-1} 5_3 0_5^{-1}$, $1_3 6_4 3_5^{-1} 5_3 0_5^{-1}$, $1_3 2_4^{-1} 3_3 5_5^{-1} 0_5^{-1}$, $1_5^{-1} 6_4 2_4 3_5^{-1} 5_3 0_5^{-1}$, $1_5^{-1} 3_3 5_5^{-1} 0_5^{-1}$, $1_3 6_6^{-1} 5_3$, $1_3 6_4 3_5^{-1} 4_4 5_5^{-1} 0_5^{-1}$, $1_5^{-1} 6_6^{-1} 2_4 5_3$, $6_4 2_6^{-1} 5_3 0_5^{-1}$, $1_5^{-1} 6_4 2_4 3_5^{-1} 4_4 5_5^{-1} 0_5^{-1}$, $1_3 6_6^{-1} 4_4 5_5^{-1}$, $1_3 6_4 4_6^{-1} 0_5^{-1}$, $6_6^{-1} 2_6^{-1} 3_5 5_3$, $1_5^{-1} 6_6^{-1} 2_4 4_4 5_5^{-1}$, $6_4 2_6^{-1} 4_4 5_5^{-1} 0_5^{-1}$, $1_5^{-1} 6_4 2_4 4_6^{-1} 0_5^{-1}$, $1_3 6_6^{-1} 3_5 4_6^{-1}$, $3_7^{-1} 4_6 5_3$, $6_6^{-1} 2_6^{-1} 3_5 4_4 5_5^{-1}$, $1_5^{-1} 6_6^{-1} 2_4 3_5 4_6^{-1}$, $6_4 2_6^{-1} 3_5 4_6^{-1} 0_5^{-1}$, $1_3 2_6 3_7^{-1}$, $4_8^{-1} 5_3 5_7$, $3_7^{-1} 4_4 4_6 5_5^{-1}$, $6_6^{-1} 2_6^{-1} 3_5^2 4_6^{-1}$, $1_5^{-1} 2_4 2_6 3_7^{-1}$, $6_4 6_6 3_7^{-1} 0_5^{-1}$, $1_3 1_7 2_8^{-1}$, $5_3 5_9^{-1}$, $4_4 4_8^{-1} 5_5^{-1} 5_7$, $3_5 3_7^{-1}$, $1_5^{-1} 1_7 2_4 2_8^{-1}$, $6_4 6_8^{-1} 0_5^{-1}$, $1_3 1_9^{-1}$, $4_4 5_5^{-1} 5_9^{-1}$, $3_5 4_6^{-1} 4_8^{-1} 5_7$, $6_6 2_6 3_7^{-2} 4_6$, $1_7 2_6^{-1} 2_8^{-1} 3_5$, $6_6^{-1} 6_8^{-1} 3_5 0_7$, $1_5^{-1} 1_9^{-1} 2_4$, $3_5 4_6^{-1} 5_9^{-1}$, $6_6 2_6 3_7^{-1} 4_8^{-1} 5_7$, $6_8^{-1} 2_6 3_7^{-1} 4_6 0_7$, $1_7 6_6 2_8^{-1} 3_7^{-1} 4_6$, $1_9^{-1} 2_6^{-1} 3_5$, $6_6 2_6 3_7^{-1} 5_9^{-1}$, $6_8^{-1} 2_6 4_8^{-1} 5_7 0_7$, $1_7 6_8^{-1} 2_8^{-1} 4_6 0_7$, $1_7 6_6 2_8^{-1} 4_8^{-1} 5_7$, $1_9^{-1} 6_6 3_7^{-1} 4_6$, $6_8^{-1} 2_6 5_9^{-1} 0_7$, $1_7 6_8^{-1} 2_8^{-1} 3_7 4_8^{-1} 5_7 0_7$, $1_9^{-1} 6_8^{-1} 4_6 0_7$, $1_7 6_6 2_8^{-1} 5_9^{-1}$, $1_9^{-1} 6_6 4_8^{-1} 5_7$, $1_7 6_8^{-1} 2_8^{-1} 3_7 5_9^{-1} 0_7$, $1_9^{-1} 6_8^{-1} 3_7 4_8^{-1} 5_7 0_7$, $1_7 3_9^{-1} 5_7 0_7$, $1_9^{-1} 6_6 5_9^{-1}$, $1_9^{-1} 6_8^{-1} 3_7 5_9^{-1} 0_7$, $1_9^{-1} 2_8 3_9^{-1} 5_7 0_7$, $1_7 3_9^{-1} 4_8 5_9^{-1} 0_7$, $1_9^{-1} 2_8 3_9^{-1} 4_8 5_9^{-1} 0_7$, $2_{10}^{-1} 5_7 0_7$, $1_7 4_{10}^{-1} 0_7$, $2_{10}^{-1} 4_8 5_9^{-1} 0_7$, $1_9^{-1} 2_8 4_{10}^{-1} 0_7$, $2_{10}^{-1} 3_9 4_{10}^{-1} 0_7$, $6_{10} 3_{11}^{-1} 0_7$, $6_{12}^{-1} 0_7 0_{11}$.

There is $6_4 6_6 3_7^{-1} 0_5^{-1}$ as claimed. We can also check that all monomials are connected to either M, $0_7 0_{13}^{-1}$ or their τ_4-images in the I_0-crystal.

Remark 6.2. (1) In this example, the corresponding quiver varieties are either a single point or an ALE space of type E_n. The graded quiver varieties, which are fixed point sets of a $\mathbb{C}^*$-action, are single points or a complex projective line. The latter gives the monomial with coefficient $1+t^2$.

(2) The crystal structure here is isomorphic to one studied recently in [3]. As the crystal graph is connected, we conclude that the crystal base constructed in [3] are isomorphic to $\mathcal{B}(W(\varpi_\ell))$.

6.1.3. Let $\mathfrak{g} = E_6^{(1)}$ and $\ell = 2$. The t–analog of q–character of $W(\varpi_2)$ has 351 monomials among which the following 27 monomials have coefficients $1+t^2$ and others have 1:

$3_3 3_5^{-1} 5_3$, $3_3 3_5^{-1} 4_4 5_5^{-1}$, $3_3 4_6^{-1}$, $6_4 2_4 3_5^{-1} 4_4 4_6^{-1}$, $6_6^{-1} 2_4 4_4 4_6^{-1}$, $1_5 6_4 2_6^{-1} 4_4 4_6^{-1}$, $1_5 6_6^{-1} 2_6^{-1} 3_5 4_4 4_6^{-1}$, $1_7^{-1} 6_4 4_4 4_6^{-1}$, $1_7^{-1} 6_6^{-1} 3_5 4_4 4_6^{-1}$, $1_5 3_7^{-1} 4_4$, $1_7^{-1} 2_6 3_7^{-1} 4_4$, $1_5 3_5 3_7^{-1} 4_6^{-1} 5_5$, $2_8^{-1} 4_4$, $1_7^{-1} 2_6 3_5 3_7^{-1} 4_6^{-1} 5_5$, $1_5 3_5 3_7^{-1} 5_7^{-1}$, $2_8^{-1} 3_5 4_6^{-1} 5_5$, $1_7^{-1} 2_6 3_5 3_7^{-1} 5_7^{-1}$, $2_8^{-1} 3_5 5_7^{-1}$, $6_6 2_6 2_8^{-1} 3_7^{-1} 5_5$, $6_6 2_6 2_8^{-1} 3_7^{-1} 4_6 5_7^{-1}$, $6_8^{-1} 2_6 2_8^{-1} 5_5$, $6_8^{-1} 2_6 2_8^{-1} 4_6 5_7^{-1}$, $6_6 2_6 2_8^{-1} 4_8^{-1}$, $6_8^{-1} 2_6 2_8^{-1} 3_7 4_8^{-1}$, $2_6 3_9^{-1}$, $1_7 2_8^{-1} 3_7 3_9^{-1}$, $1_9^{-1} 3_7 3_9^{-1}$.

>From this (or by other methods) we can see that $W(\varpi_2)$ is restricted to $V_{I_0}(\varpi_2) \oplus V_{I_0}(\varpi_5)$.

Let us consider the monomial crystal $\mathcal{M}(M)$ with $M = 2_0 0_3^{-1} 0_5^{-1}$. >From $\tilde{f}_6 \tilde{f}_3 \tilde{f}_2 M = 1_1 4_2 6_4^{-1} 0_5^{-1}$, we see that M is extremal by the argument in the proof of Proposition 3.4. Therefore $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_2)$.

There is a monomial

$$m = 1_5 6_6^{-1} 6_8^{-1} 4_4 0_7 = \tilde{f}_6 \tilde{f}_6 \tilde{f}_3 \tilde{f}_2 \tilde{f}_3 \tilde{f}_2 \tilde{f}_1 \tilde{f}_4 \tilde{f}_5 \tilde{f}_3 \tilde{f}_4 \tilde{f}_6 \tilde{f}_3 \tilde{f}_2 M$$

in $\mathcal{M}_{I_0}(M)$. We have $\tilde{e}_2 \tilde{e}_3 \tilde{e}_6 \tilde{f}_0 m = 1_3^{-1} 1_5 2_2 0_5^{-1} 0_9^{-1}$. By the weight calculation, we find that this is $z_\ell^{-1}(M)$. Let us denote this by M_1.

In $\mathcal{M}_{I_0}(M_1)$ we can find a monomial

$$m' = 1_7 6_{10}^{-2} 4_8 0_9 = \tilde{f}_6 \tilde{f}_6 \tilde{f}_3 \tilde{f}_2 \tilde{f}_3 \tilde{f}_2 \tilde{f}_1 \tilde{f}_4 \tilde{f}_5 \tilde{f}_3 \tilde{f}_4 \tilde{f}_6 \tilde{f}_3 \tilde{f}_2 M_1.$$

We have $\tilde{e}_2 \tilde{e}_3 \tilde{e}_6 \tilde{f}_0 m' = 2_6 0_9^{-1} 0_{11}^{-1} = \tau_6(M)$. This is equal to $z_\ell^{-2}(M)$.

We have

$$5_3 0_{11}^{-1} = \tilde{e}_5 \tilde{e}_4 \tilde{e}_3 \tilde{e}_6 \tilde{e}_0 \cdot \tau_6(M)$$

in $\mathcal{M}(M)$. Write this $M_{0;1}$. Then $\mathcal{M}_{I_0}(M_{0;1})$ consists of the following 27 monomials:

$5_3 0_{11}^{-1}$, $4_4 5_5^{-1} 0_{11}^{-1}$, $3_5 4_6^{-1} 0_{11}^{-1}$, $6_6 2_6 3_7^{-1} 0_{11}^{-1}$, $6_8^{-1} 2_6 0_7 0_{11}^{-1}$, $1_7 6_6 2_8^{-1} 0_{11}^{-1}$, $1_7 6_8^{-1} 2_8^{-1} 3_7 0_7 0_{11}^{-1}$, $1_9^{-1} 6_6 0_{11}^{-1}$, $1_9^{-1} 6_8^{-1} 3_7 0_7 0_{11}^{-1}$, $1_7 3_9^{-1} 4_8 0_7 0_{11}^{-1}$, $1_9^{-1} 2_8 3_9^{-1} 4_8 0_7 0_{11}^{-1}$, $1_7 4_{10}^{-1} 5_9 0_7 0_{11}^{-1}$, $2_{10}^{-1} 4_8 0_7 0_{11}^{-1}$, $1_9^{-1} 2_8 4_{10}^{-1} 5_9 0_7 0_{11}^{-1}$, $1_7 5_{11}^{-1} 0_7 0_{11}^{-1}$, $2_{10}^{-1} 3_9 4_{10}^{-1} 5_9 0_7 0_{11}^{-1}$, $1_9^{-1} 2_8 5_{11}^{-1} 0_7 0_{11}^{-1}$, $2_{10}^{-1} 3_9 5_{11}^{-1} 0_7 0_{11}^{-1}$, $6_{10} 3_{11}^{-1} 5_9 0_7 0_{11}^{-1}$, $6_{10} 3_{11}^{-1} 4_{10} 5_{11}^{-1} 0_7 0_{11}^{-1}$, $6_{12}^{-1} 5_9 0_7 0_{11}^{-1}$, $6_{12}^{-1} 4_{10} 5_{11}^{-1} 0_7$, $6_{10} 4_{12}^{-1} 0_7 0_{11}^{-1}$, $6_{12}^{-1} 3_{11} 4_{12}^{-1} 0_7$, $2_{12} 3_{13}^{-1} 0_7$, $1_{13} 2_{14}^{-1} 0_7$, $1_{15}^{-1} 0_7$.

We have $\tilde{e}_5 \tilde{e}_4 \tilde{e}_3 \tilde{e}_6 \tilde{e}_0 \cdot M_1 = 1_3^{-1} 1_5 5_{-1} 0_9^{-1}$. Set this $M_{1;1}$. Then $\mathcal{M}_{I_0}(M_{1;1})$ consists of

$1_3^{-1} 1_5 5_{-1} 0_9^{-1}$, $1_3^{-1} 1_5 4_0 5_1^{-1} 0_9^{-1}$, $1_3^{-1} 1_5 3_1 4_2^{-1} 0_9^{-1}$, $1_3^{-1} 1_5 2_2 3_3^{-1} 6_2 0_9^{-1}$, $1_3^{-1} 1_5 2_2 6_4^{-1} 0_9^{-1}$, $1_5 2_4^{-1} 6_2 0_9^{-1}$, $1_5 2_4^{-1} 3_3 6_4^{-1} 0_3 0_9^{-1}$, $1_7^{-1} 2_4^{-1} 2_6 6_2 0_9^{-1}$, $1_7^{-1} 2_4^{-1} 2_6 3_3 6_4^{-1} 0_3 0_9^{-1}$, $1_5 3_5^{-1} 4_4 0_3 0_9^{-1}$, $1_7^{-1} 2_6 3_5^{-1} 4_4 0_3 0_9^{-1}$, $1_5 4_6^{-1} 5_5 0_3 0_9^{-1}$, $2_8^{-1} 3_5^{-1} 3_7 4_4 0_3 0_9^{-1}$, $1_7^{-1} 2_6 4_6^{-1} 5_5 0_3 0_9^{-1}$, $1_5 5_7^{-1} 0_3 0_9^{-1}$, $2_8^{-1} 3_7 4_6^{-1} 5_5 0_3 0_9^{-1}$, $1_7^{-1} 2_6 5_7^{-1} 0_3 0_9^{-1}$, $2_8^{-1} 3_7 5_7^{-1} 0_3 0_9^{-1}$, $3_9^{-1} 4_6^{-1} 4_8 5_5 6_8 0_3 0_9^{-1}$, $3_9^{-1} 4_8 5_7^{-1} 6_8 0_3 0_9^{-1}$, $4_6^{-1} 4_8 5_5 6_{10}^{-1} 0_3$, $4_8 5_7^{-1} 6_{10}^{-1} 0_3$, $4_{10}^{-1} 5_7^{-1} 5_9 6_8 0_3 0_9^{-1}$, $3_9 4_{10}^{-1} 5_7^{-1} 5_9 6_{10}^{-1} 0_3$, $2_{10} 3_{11}^{-1} 5_7^{-1} 5_9 0_3$, $1_{11} 2_{12}^{-1} 5_7^{-1} 5_9 0_3$, $1_{13}^{-1} 5_7^{-1} 5_9 0_3$.

These have different weights, so there is only one way to make a bijection to the above polynomials with coefficients $1 + t^2$ preserving weights. It is the bijection given in order.

Also it should be possible to make the bijection between $\mathcal{M}_{I_0}(M)$ and $\mathcal{M}_{I_0}(M_1)$ *explicit*, though we do not do here, as both are 351 monomials.

Thus we have

$$\mathcal{M}(M)/\tau_6 \simeq \mathcal{M}_{I_0}(M) \sqcup \mathcal{M}_{I_0}(M_1) \sqcup \mathcal{M}_{I_0}(M_{0;1}) \sqcup \mathcal{M}_{I_0}(M_{1;1}),$$

and we have a crystal isomorphism τ interchanging $\mathcal{M}_{I_0}(M) \leftrightarrow \mathcal{M}_{I_0}(M_1)$ and $\mathcal{M}_{I_0}(M_{0;1}) \leftrightarrow \mathcal{M}_{I_0}(M_{1;1})$. These follow from the known results, but should be possible to check directly from the above computation.

6.1.4. Let $\mathfrak{g} = E_6^{(1)}$ and $\ell = 3$. It is known that $W(\varpi_3)$ restricts to $V_{I_0}(\varpi_3) \oplus V_{I_0}(\varpi_6)^{\oplus 2} \oplus V_{I_0}(\varpi_1 + \varpi_5) \oplus V_{I_0}(0)$ as a $\mathcal{U}_q(\mathfrak{g}_{I_0})$-module.

Let $M = 3_0 0_2^{-1} 0_4^{-1} 0_6^{-1}$. We have

$$m = \tilde{f}_6 \tilde{f}_3^2 \tilde{f}_6 \tilde{f}_4 \tilde{f}_2 \tilde{f}_3 M = 1_2 2_3 3_4^{-1} 4_3 5_2 6_5^{-1} 0_6^{-1}.$$

By the same argument as in the proof of Proposition 3.4, we see that M is extremal. Therefore $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_3)$.

We have

$$m' = \tilde{e}_3 \tilde{f}_6^2 \tilde{f}_3^3 \tilde{f}_4^2 \tilde{f}_2^2 \tilde{f}_5 \tilde{f}_1 m = 1_4 3_4 5_4 6_5^{-1} 6_7^{-2} 0_6$$

in $\mathcal{M}_{I_0}(M)$. Then

$$\tilde{e}_3 \tilde{e}_4 \tilde{e}_2 \tilde{e}_3 \tilde{e}_6 \tilde{e}_6 \tilde{f}_0 m' = 3_2 0_4^{-1} 0_6^{-1} 0_8^{-1} = \tau_2(M).$$

By the weight calculation, this is $z_\ell^{-1}(M)$, so we have $z_\ell = \tau_{-2}$ and $\mathcal{M}(M)/\tau_2 \simeq \mathcal{B}(W(\varpi_\ell))$.

Let

$$M_1 = \tilde{e}_6 \tilde{e}_0 \tau_2(M) = 6_1 0_6^{-1} 0_8^{-1}.$$

Then $\mathcal{M}_{I_0}(M_1)$ is the crystal of the adjoint representation of $\mathfrak{g}_{I_0}$. By 6.1.2 the lowest weight vector is $6_{13}^{-1} 0_8 0_{12} \times 0_2 0_8^{-1} = 6_{13}^{-1} 0_2 0_{12}$. Applying $\tau_{-2}\tilde{f}_0$, we get $M_2 = 0_0 0_{12}^{-1}$. Applying $\tilde{f}_0$ again, we get $M_3 = 6_1 0_2^{-1} 0_{12}^{-1}$. Looking at monomials in 6.1.2, we find $1_4 6_7^{-1} 5_4 0_6 0_{12}^{-1}$ in $\mathcal{M}_{I_0}(M_3)$. Applying $\tilde{f}_0$, we get $M_4 = 1_4 5_4 0_8^{-1} 0_{12}^{-1}$. This monomial generates the I_0-crystal of $V_{I_0}(\varpi_1 + \varpi_5)$.

Thus

$$\mathcal{M}(M)/\tau_2 = \mathcal{M}_{I_0}(M) \sqcup \mathcal{M}_{I_0}(M_1) \sqcup \mathcal{M}_{I_0}(M_2) \sqcup \mathcal{M}_{I_0}(M_3) \sqcup \mathcal{M}_{I_0}(M_4).$$

This follows from $\operatorname{Res} W_0(\varpi_3) \simeq V_{I_0}(\varpi_3) \oplus V_{I_0}(\varpi_6)^{\oplus 2} \oplus V_{I_0}(\varpi_1 + \varpi_5) \oplus V_{I_0}(0)$, but it is probably possible to check directly from the above computation.

Remark 6.3. The authors do not find the last two examples in the literature. One can probably check their *perfectness*, though we have not done yet.

6.2. Type $G_2^{(1)}$.

6.2.1. First we consider $\ell = 1$. Let $M = Y_{1,0}Y_{0,1}^{-1}Y_{0,3}^{-1}$. As $\tilde{f}_1(M) = Y_{1,2}^{-1}Y_{2,1}^3Y_{0,3}^{-1}$, we see as in Proposition 3.4 that $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_\ell)$.

As $\tilde{e}_1\tilde{e}_2^3\tilde{e}_1^2\tilde{e}_0\tilde{f}_1 M = \tau_{-2}(M)$, $\mathcal{M}(M)$ is preserved under τ_{-2}. It has weight δ, so $z_\ell = \tau_{-2}$ and hence $\mathcal{M}(m)/\tau_2 \simeq \mathcal{B}(W(\varpi_\ell))$.

Let $M' = \tilde{e}_0(M) = Y_{0,-1}Y_{0,3}^{-1}$. We have $\mathcal{M}_{I_0}(M') = \{M'\}$. The following 14 monomials appear in $\mathcal{M}_{I_0}(M)$:

$1_0 0_1^{-1} 0_3^{-1}$, $2_1^3 1_2^{-1} 0_3^{-1}$, $2_1^2 2_3^{-1} 0_3^{-1}$, $2_1 2_3^{-2} 1_2 0_3^{-1}$, $2_3^{-3} 1_2^2 0_3^{-1}$, $2_1 2_3 1_4^{-1}$, $1_2 1_4^{-1}$, $2_1 2_5^{-1}$, $1_4^{-2} 2_3^3 0_3$, $2_3^{-1} 2_5^{-1} 1_2$, $2_3^2 2_5^{-1} 1_4^{-1} 0_3$, $2_3 2_5^{-2} 0_3$, $2_5^{-3} 0_3 1_4$, $0_3 0_5 1_6^{-1}$.

By direct calculation, we find that these 14 monomials and M' are all monomials of $\mathcal{M}(M)/\tau_2$. As an application, we get

$$\mathcal{B}(W(\varpi_\ell)) \simeq \mathcal{B}_{I_0}(\varpi_\ell) \sqcup \mathcal{B}_{I_0}(0).$$

This crystal was described in [38, 3]. The crystal base is isomorphic to ours by the same reason as in 6.1.2.

6.2.2. Now we consider the case $\ell = 2$. Let $M = Y_{2,0}Y_{0,2}^{-1}$. It follows from Corollary 3.3 that $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_\ell)$. The following 7 monomials appear in $\mathcal{M}_{I_0}(M)$:

$M = 2_0 0_2^{-1}$, $m_2 = 1_1 2_2^{-1} 0_2^{-1}$, $m_3 = 1_3^{-1} 2_2^2$, $m_4 = 2_2 2_4^{-1}$, $m_5 = 2_4^{-2} 1_3$, $m_6 = 1_5^{-1} 2_4 0_4$, $m_7 = 2_6^{-1} 0_4$.

The crystal graph of $\mathcal{M}(M)$ is given in Figure 9. We find $z_\ell = \tau_{-4}$ and $\mathcal{M}(M)/\tau_4 = \mathcal{M}_{I_0}(M)$.

The authors do not find a description of this crystal structure in the literature (probably because it is not perfect), but one can easily obtain it from the description of its I_0-crystal structure in [16].

$$\begin{tikzcd}
M \arrow[r, "2"] & m_2 \arrow[r, "1"] & m_3 \arrow[r, "2"] & m_4 \arrow[r, "2"] & m_5 \arrow[r, "1"] & m_6 \arrow[r, "2"] \arrow[lllll, bend right, "{0[4]}"'] & m_7 \arrow[lllll, bend left, "{0[4]}"]
\end{tikzcd}$$

FIGURE 9. (Type $G_2^{(1)}$) the crystal $\mathcal{B}(\varpi_2)$

6.3. Type $F_4^{(1)}$.

6.3.1. First let $\ell = 1$ and $M = Y_{1,0}Y_{0,1}^{-1}Y_{0,5}^{-1}$. We have $\tilde{f}_2\tilde{f}_1 M = Y_{0,5}^{-1}Y_{2,3}^{-1}Y_{3,2}^2$ and so we see as in Proposition 3.4 that $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_\ell)$. As $\tilde{e}_1\tilde{e}_2\tilde{e}_3^2\tilde{e}_4^2\tilde{e}_2\tilde{e}_3\tilde{e}_3\tilde{e}_2\tilde{e}_1\tilde{e}_1\tilde{e}_0\tilde{f}_1 M = \tau_{-4}(M)$, $\mathcal{M}(M)$ is preserved under τ_4, which has weight δ. Therefore we have $z_\ell = \tau_{-4}$ and $\mathcal{M}(m)/\tau_4 \simeq \mathcal{B}(W(\varpi_\ell))$.

Let $M' = \tilde{e}_0(M) = Y_{0,-1}Y_{0,5}^{-1}$. We have $\mathcal{M}_{I_0}(M') = \{M'\}$.

The following 52 monomials appear in $\mathcal{M}_{I_0}(M)$:

$1_0 0_1^{-1} 0_5^{-1}$, $1_2^{-1} 2_1 0_5^{-1}$, $2_3^{-1} 3_2^2 0_5^{-1}$, $3_2 3_4^{-1} 4_3 0_5^{-1}$, $2_3 3_4^{-2} 4_3^2 0_5^{-1}$, $3_2 4_5^{-1} 0_5^{-1}$, $1_4 2_5^{-1} 4_3^2 0_5^{-1}$, $2_3 3_4^{-1} 4_3 4_5^{-1} 0_5^{-1}$, $1_6^{-1} 4_3^2$, $1_4 2_5^{-1} 4_3 4_5^{-1} 3_4 0_5^{-1}$, $2_3 4_5^{-2} 0_5^{-1}$, $1_6^{-1} 3_4 4_3 4_5^{-1}$, $1_4 2_5^{-1} 4_5^{-2} 3_4^2 0_5^{-1}$, $1_4 3_6^{-1} 4_3 0_5^{-1}$, $1_6^{-1} 2_5 3_6^{-1} 4_3$, $1_6^{-1} 3_4^2 4_5^{-2}$, $1_4 3_4 3_6^{-1} 4_5^{-1} 0_5^{-1}$, $2_7^{-1} 3_6 4_3$, $1_6^{-1} 2_5 3_6^{-1} 3_4 4_5^{-1}$, $1_4 2_5 3_6^{-2} 0_5^{-1}$, $3_8^{-1} 4_7 4_3$, $2_7^{-1} 3_6 3_4 4_5^{-1}$, $1_6^{-1} 2_5^2 3_6^{-2}$, $1_4 1_6 2_7^{-1} 0_5^{-1}$, $4_9^{-1} 4_3$, $3_8^{-1} 3_4 4_5^{-1} 4_7$, $2_5 2_7^{-1}$, $1_4 1_8^{-1} 0_7 0_5^{-1}$, $3_4 4_5^{-1} 4_9^{-1}$, $2_5 3_6^{-1} 3_8^{-1} 4_7$, $1_6 2_7^{-2} 3_6^2$, $1_6^{-1} 1_8^{-1} 2_5 0_7$, $2_5 3_6^{-1} 4_9^{-1}$, $0_7 1_8^{-1} 2_7^{-1} 3_6^2$, $1_6 2_7^{-1} 3_6 3_8^{-1} 4_7$,

$1_62_7^{-1}3_64_9^{-1}$, $1_8^{-1}3_63_8^{-1}4_70_7$, $1_63_8^{-2}4_7^2$, $1_8^{-1}3_64_9^{-1}0_7$, $1_63_8^{-1}4_74_9^{-1}$, $1_8^{-1}2_73_8^{-2}4_7^20_7$, $1_8^{-1}2_73_8^{-1}4_74_9^{-1}0_7$, $1_64_9^{-2}$, $2_9^{-1}4_7^20_7$, $2_9^{-1}3_84_74_9^{-1}0_7$, $1_8^{-1}2_74_9^{-2}0_7$, $3_{10}^{-1}4_70_7$, $2_9^{-1}3_8^24_9^{-2}0_7$, $3_83_{10}^{-1}4_9^{-1}0_7$, $2_93_{10}^{-2}0_7$, $1_{10}2_{11}^{-1}0_7$, $1_{12}^{-1}0_70_{11}$.

These 52 monomials, M' and their τ_4-images are all monomials of $\mathcal{M}(M)$. As an application we have

$$\mathcal{B}(W(\varpi_1)) \simeq \mathcal{B}_{I_0}(\varpi_1) \sqcup \mathcal{B}_{I_0}(0).$$

The crystal base is isomorphic to one in [3] by the same reason as in 6.1.2.

6.3.2. Let us consider $\ell = 2$ and $M = Y_{2,0}Y_{0,2}^{-1}Y_{0,4}^{-1}Y_{0,6}^{-1}$. We have $\tilde{f}_1\tilde{f}_2^2\tilde{f}_3^2\tilde{f}_1\tilde{f}_2M = Y_{4,2}^2Y_{3,3}^2Y_{2,4}^{-1}Y_{1,5}^{-1}Y_{0,6}^{-1}$ and so we see as in Proposition 3.4 that $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_\ell)$. As $\tilde{e}_2\tilde{e}_3^2\tilde{e}_2\tilde{e}_1^2\tilde{e}_2\tilde{e}_4^2\tilde{e}_3^4\tilde{e}_2^3\tilde{e}_1^2\tilde{e}_0\tilde{f}_1\tilde{f}_2^2\tilde{f}_3^2\tilde{f}_1\tilde{f}_2M = \tau_{-2}(M)$, $\mathcal{M}(M)$ is preserved under τ_2, which has weight δ. Therefore $\mathcal{M}(M)/\tau_2 \simeq \mathcal{B}(W(\varpi_\ell))$.

Let $M_2 = \tilde{e}_1\tilde{e}_0M = Y_{0,4}^{-1}Y_{0,6}^{-1}Y_{1,-1}$. The following 52 monomials appear in $\mathcal{M}_{I_0}(M_2)$:

$1_{-1}0_4^{-1}0_6^{-1}$, $1_1^{-1}2_00_00_4^{-1}0_6^{-1}$, $2_2^{-1}3_1^20_00_4^{-1}0_6^{-1}$, $3_13_3^{-1}4_20_00_4^{-1}0_6^{-1}$, $2_23_3^{-2}4_2^20_00_4^{-1}0_6^{-1}$, $3_14_4^{-1}0_00_4^{-1}0_6^{-1}$, $1_32_4^{-1}4_2^20_00_4^{-1}0_6^{-1}$, $2_23_3^{-1}4_24_4^{-1}0_00_4^{-1}0_6^{-1}$, $1_5^{-1}4_2^20_00_6^{-1}$, $1_32_4^{-1}4_24_4^{-1}3_30_5^{-1}0_00_4^{-1}0_6^{-1}$, $2_24_4^{-2}0_00_4^{-1}0_6^{-1}$, $1_5^{-1}3_34_34_4^{-1}0_00_6^{-1}$, $1_32_4^{-1}4_4^{-2}3_3^20_5^{-1}0_00_4^{-1}0_6^{-1}$, $1_33_5^{-1}4_20_5^{-1}0_00_4^{-1}0_6^{-1}$, $1_5^{-1}2_43_5^{-1}4_20_00_6^{-1}$, $1_5^{-1}3_3^24_4^{-2}0_00_6^{-1}$, $1_33_33_5^{-1}4_4^{-1}0_00_4^{-1}0_6^{-1}$, $2_6^{-1}3_54_20_00_6^{-1}$, $1_5^{-1}2_43_5^{-1}3_34_4^{-1}0_00_6^{-1}$, $1_32_43_5^{-2}0_5^{-1}0_00_4^{-1}0_6^{-1}$, $3_7^{-1}4_64_20_00_6^{-1}$, $2_6^{-1}3_53_34_4^{-1}0_00_6^{-1}$, $1_5^{-1}2_4^23_5^{-2}0_00_6^{-1}$, $1_31_52_6^{-1}0_5^{-1}0_00_4^{-1}0_6^{-1}$, $4_8^{-1}4_20_00_6^{-1}$, $3_7^{-1}3_34_4^{-1}4_60_00_6^{-1}$, $2_42_6^{-1}0_00_6^{-1}$, $1_31_7^{-1}0_00_6^{-1}$, $3_34_4^{-1}4_8^{-1}0_00_6^{-1}$, $2_43_5^{-1}3_7^{-1}4_60_00_6^{-1}$, $1_52_6^{-2}3_5^20_00_6^{-1}$, $1_5^{-1}1_7^{-1}2_40_00_6^{-1}$, $2_43_5^{-1}4_8^{-1}0_00_6^{-1}$, $1_7^{-1}2_6^{-1}3_5^20_0$, $1_52_6^{-1}3_53_7^{-1}4_60_00_6^{-1}$, $1_52_6{-}13_54_8^{-1}0_00_6^{-1}$, $1_7^{-1}3_53_7^{-1}4_60_0$, $1_53_7^{-2}4_6^20_00_6^{-1}$, $1_7^{-1}3_54_8^{-1}0_0$, $1_53_7^{-1}4_64_8^{-1}0_00_6^{-1}$, $1_7^{-1}2_63_7^{-2}4_6^20_0$, $1_7^{-1}2_63_7^{-1}4_64_8^{-1}0_0$, $1_54_8^{-2}0_00_6^{-1}$, $2_8^{-1}4_6^20_0$, $2_8^{-1}3_74_74_8^{-1}0_0$, $1_7^{-1}2_64_8^{-2}0_0$, $3_9^{-1}4_60_0$, $2_8^{-1}3_7^24_8^{-2}0_0$, $3_73_9^{-1}4_8^{-1}0_0$, $2_83_9^{-2}0_0$, $1_92_{10}^{-1}0_0$, $1_{11}^{-1}0_00_{10}$.

Let $M_3 = \tilde{e}_1\tilde{e}_2\tilde{e}_3^2\tilde{e}_2\tilde{e}_4^2\tilde{e}_3^2\tilde{e}_2\tilde{e}_1\tilde{e}_0M_2 = Y_{1,-5}Y_{0,6}^{-1}Y_{0,-4}^{-1}$. The following 52 monomials appear in $\mathcal{M}_{I_0}(M_3)$:

$1_{-5}0_6^{-1}0_{-4}^{-1}$, $1_{-3}^{-1}2_{-4}0_6^{-1}$, $2_{-2}^{-1}3^2_{-3}0_6^{-1}$, $3_{-3}3_{-1}^{-1}4_{-2}0_6^{-1}$, $2_{-2}3_{-1}^{-2}4^2_{-2}0_6^{-1}$, $3_{-3}4_0^{-1}0_6^{-1}$, $1_{-1}2_0^{-1}4^2_{-2}0_6^{-1}$, $2_{-2}3_{-1}^{-1}4_{-2}4_0^{-1}0_6^{-1}$, $1_1^{-1}4^2_{-2}0_00_6^{-1}$, $1_{-1}2_0^{-1}4_{-2}4_0^{-1}3_{-1}0_6^{-1}$, $2_{-2}4_0^{-2}0_6^{-1}$, $1_1^{-1}3_{-1}4_{-2}4_0^{-1}0_00_6^{-1}$, $1_{-1}2_0^{-1}4_0^{-2}3^2_{-1}0_6^{-1}$, $1_{-1}3_1^{-1}4_{-2}0_6^{-1}$, $1_1^{-1}2_03_1^{-1}4_{-2}0_00_6^{-1}$, $1_1^{-1}3^2_{-1}4_0^{-2}0_00_6^{-1}$, $1_{-1}3_{-1}3_1^{-1}4_0^{-1}0_6^{-1}$, $2_2^{-1}3_14_{-2}0_00_6^{-1}$, $1_1^{-1}2_03_1^{-1}3_{-1}4_0^{-1}0_00_6^{-1}$, $1_{-1}2_03_1^{-2}0_6^{-1}$, $3_3^{-1}4_24_{-2}0_00_6^{-1}$, $2_2^{-1}3_13_{-1}4_0^{-1}0_00_6^{-1}$, $1_1^{-1}2_0^23_1^{-2}0_00_6^{-1}$, $1_{-1}1_12_2^{-1}0_6^{-1}$, $4_4^{-1}4_{-2}0_00_6^{-1}$, $3_3^{-1}3_{-1}4_0^{-1}4_20_00_6^{-1}$, $2_02_2^{-1}0_00_6^{-1}$, $1_{-1}1_3^{-1}0_20_6^{-1}$, $3_{-1}4_0^{-1}4_4^{-1}0_00_6^{-1}$, $2_03_1^{-1}3_3^{-1}4_20_00_6^{-1}$, $1_12_2^{-2}3_1^20_00_6^{-1}$, $1_1^{-1}1_3^{-1}2_00_20_00_6^{-1}$, $2_03_1^{-1}4_4^{-1}0_00_6^{-1}$, $1_3^{-1}2_2^{-1}3_1^20_00_20_6^{-1}$, $1_12_2^{-1}3_13_2^{-1}4_20_00_6^{-1}$, $1_12_2^{-1}3_14_4^{-1}0_00_6^{-1}$, $1_3^{-1}3_13_3^{-1}4_20_00_20_6^{-1}$, $1_13_3^{-2}4_2^20_00_6^{-1}$, $1_3^{-1}3_14_4^{-1}0_20_00_6^{-1}$, $1_13_3^{-1}4_24_4^{-1}0_00_6^{-1}$, $1_3^{-1}2_23_3^{-2}4_2^20_20_00_6^{-1}$, $1_3^{-1}2_23_3^{-1}4_24_4^{-1}0_20_00_6^{-1}$, $1_14_4^{-2}0_00_6^{-1}$, $2_4^{-1}4_2^20_20_00_6^{-1}$, $2_4^{-1}3_34_24_4^{-1}0_20_00_6^{-1}$, $1_3^{-1}2_24_4^{-2}0_20_00_6^{-1}$, $3_5^{-1}4_20_20_00_6^{-1}$, $2_4^{-1}3_3^24_4^{-2}0_20_00_6^{-1}$, $3_33_5^{-1}4_4^{-1}0_20_00_6^{-1}$, $2_43_5^{-2}0_20_00_6^{-1}$, $1_52_6^{-1}0_20_00_6^{-1}$, $1_7^{-1}0_20_0$.

Let $M_4 = \tilde{e}_0M_3 = Y_{0,-6}Y_{0,6}^{-1}$ and $M_5 = \tilde{e}_4^2\tilde{e}_3^2\tilde{e}_2^2\tilde{e}_1^2\tilde{e}_0\tilde{f}_1\tilde{f}_2M = Y_{4,-2}^2Y_{0,2}^{-1}Y_{0,6}^{-1}$. We have $\mathcal{M}_{I_0}(M_4) = \{M_4\}$. We do not give the list of monomials of $\mathcal{M}_{I_0}(M)$ and $\mathcal{M}_{I_0}(M_5)$ (a total of 1598 monomials).

All monomials of $\mathcal{M}(M)/\tau_2$ are connected to either M, M_2, M_3, M_4, M_5 in the I_0-crystal (it is possible to check from the above computation; or it also follows from $\mathrm{Res}W(\varpi_2) = V_{I_0}(\varpi_2) \oplus V_{I_0}(\varpi_1)^{\oplus 2} \oplus V_{I_0}(0) \oplus V_{I_0}(2\varpi_4)$).

6.3.3. Let us consider $\ell = 3$ and $M = Y_{3,0}Y_{0,3}^{-1}Y_{0,5}^{-1}$. We have $\tilde{f}_1\tilde{f}_2\tilde{f}_3M = Y_{4,1}Y_{3,2}Y_{1,4}^{-1}Y_{0,5}^{-1}$ and so we see as in Proposition 3.4 that $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_\ell)$. Let $M_1 = \tilde{e}_3\tilde{e}_2\tilde{e}_3\tilde{e}_4^2\tilde{e}_1\tilde{e}_2\tilde{e}_3^3\tilde{e}_2^2\tilde{e}_1^2\tilde{e}_0\tilde{f}_1\tilde{f}_2\tilde{f}_3M = (Y_{4,-1}Y_{4,-3}^{-1})Y_{3,-4}Y_{0,3}^{-1}Y_{0,-1}^{-1}$. This has weight wt$M+\delta$ and hence $z_\ell(M)=M_1$. As

$$\tilde{e}_3\tilde{e}_2\tilde{e}_3\tilde{e}_4^2\tilde{e}_1\tilde{e}_2\tilde{e}_3^3\tilde{e}_2^2\tilde{e}_1^2\tilde{e}_0\tilde{f}_1\tilde{f}_2\tilde{f}_3M_1=Y_{3,-6}Y_{0,-1}^{-1}Y_{0,-3}^{-1}=\tau_{-6}(M),$$

$\mathcal{M}(m)$ is preserved under τ_6 and we have $(z_\ell)^{-2}=\tau_6$.

Let us define the monomials $M_3=\tilde{e}_4\tilde{e}_3\tilde{e}_2\tilde{e}_1\tilde{e}_0M=Y_{0,5}^{-1}Y_{4,-3}$ and $M_4=\tilde{e}_4\tilde{e}_3\tilde{e}_2\tilde{e}_1\tilde{e}_0M_1=Y_{4,-1}Y_{4,-3}^{-1}Y_{4,-7}Y_{0,3}^{-1}$. In particular, as z_ℓ is compatible with the operators $\tilde{e}_i$, it follows from $z_\ell(M)=M_1$ that $z_\ell(M_3)=M_4$.

The following 26 monomials appear in $\mathcal{M}_{I_0}(M_3)$:

$4_{-3}0_5^{-1}$, $3_{-2}4_{-1}^{-1}0_5^{-1}$, $2_{-1}3_0^{-1}0_5^{-1}$, $1_02_1^{-1}3_00_5^{-1}$, $1_03_2^{-1}4_10_5^{-1}$, $1_2^{-1}3_00_5^{-1}0_1$, $1_04_3^{-1}0_5^{-1}$, $1_2^{-1}2_13_2^{-1}4_10_5^{-1}0_1$, $1_2^{-1}2_14_3^{-1}0_5^{-1}0_1$, $2_3^{-1}3_24_10_5^{-1}0_1$, $2_3^{-1}3_2^24_3^{-1}0_5^{-1}0_1$, $3_4^{-1}4_14_30_5^{-1}0_1$, $3_23_4^{-1}0_5^{-1}0_1$, $4_14_5^{-1}0_5^{-1}0_1$, $2_33_4^{-2}4_30_5^{-1}0_1$, $3_24_3^{-1}4_5^{-1}0_5^{-1}0_1$, $1_42_5^{-1}4_30_5^{-1}0_1$, $2_33_4^{-1}4_5^{-1}0_5^{-1}0_1$, $1_6^{-1}4_30_1$, $1_42_5^{-1}3_44_5^{-1}0_5^{-1}0_1$, $1_6^{-1}3_44_8^{-1}0_1$, $1_43_6^{-1}0_5^{-1}0_1$, $1_6^{-1}2_53_6^{-1}0_1$, $2_7^{-1}3_60_1$, $3_8^{-1}4_70_1$, $4_9^{-1}0_1$.

The following 26 monomials appear in $\mathcal{M}_{I_0}(M_4)$:

$4_{-1}4_{-3}^{-1}4_{-7}0_3^{-1}$, $3_{-6}4_{-5}^{-1}4_{-1}4_{-3}^{-1}0_3^{-1}$, $2_{-5}3_{-4}^{-1}4_{-1}4_{-3}^{-1}0_3^{-1}$, $1_{-4}2_{-3}^{-1}3_{-4}4_{-1}4_{-3}^{-1}0_3^{-1}$, $1_{-4}3_{-2}^{-1}4_{-1}0_3^{-1}$, $1_{-2}^{-1}3_{-4}4_{-1}4_{-3}^{-1}0_{-3}0_3^{-1}$, $1_{-4}3_{-2}^{-1}3_04_1^{-1}0_3^{-1}$, $1_{-2}^{-1}2_{-3}3_{-2}^{-1}4_{-1}0_{-3}0_3^{-1}$, $1_{-2}^{-1}2_{-3}3_{-2}^{-1}3_04_1^{-1}0_{-3}0_3^{-1}$, $2_{-1}^{-1}3_{-2}4_{-1}0_{-3}0_3^{-1}$, $2_{-1}^{-1}3_{-2}3_04_1^{-1}0_{-3}0_3^{-1}$, $3_0^{-1}4_{-1}^20_{-3}0_3^{-1}$, $2_{-1}^{-1}2_13_{-2}3_2^{-1}0_{-3}0_3^{-1}$, $4_{-1}4_1^{-1}0_{-3}0_3^{-1}$, $2_13_0^{-1}3_2^{-1}4_{-1}0_{-3}0_3^{-1}$, $3_04_1^{-2}0_{-3}0_3^{-1}$, $1_22_3^{-1}3_0^{-1}3_24_{-1}0_{-3}0_3^{-1}$, $2_13_2^{-1}4_1^{-1}0_{-3}0_3^{-1}$, $1_4^{-1}3_0^{-1}3_24_{-1}0_{-3}$, $1_22_3^{-1}3_24_1^{-1}0_{-3}0_3^{-1}$, $1_4^{-1}3_24_1^{-1}0_{-3}$, $1_23_4^{-1}4_1^{-1}4_30_{-3}0_3^{-1}$, $1_4^{-1}2_33_4^{-1}4_1^{-1}4_30_{-3}$, $2_5^{-1}3_44_1^{-1}4_30_{-3}$, $3_6^{-1}4_54_1^{-1}4_30_{-3}$, $4_7^{-1}4_1^{-1}4_30_{-3}$.

The crystal isomorphism z_ℓ is given in order.

It should also be possible to make explicit the bijection between $\mathcal{M}_{I_0}(M)$ and $\mathcal{M}_{I_0}(M_1)$ (but to do not write it in the paper as there are 273 monomials).

All monomials of $\mathcal{M}(M)/\tau_6$ are connected to either M, M_1, M_3, M_4 in the I_0-crystal (it is possible to check from the above computation; this follows also from $\mathrm{Res}W_0(\varpi_3)=V_{I_0}(\varpi_3)\oplus V_{I_0}(\varpi_4)$).

6.3.4. Finally consider $\ell=4$ and $M=Y_{4,0}Y_{0,4}^{-1}$. It follows from Corollary 3.3 that $\mathcal{M}(M)\simeq\mathcal{B}(\varpi_\ell)$. As $\tilde{e}_4\tilde{e}_3\tilde{e}_2\tilde{e}_1\tilde{e}_3\tilde{e}_2\tilde{e}_4\tilde{e}_3^2\tilde{e}_2\tilde{e}_1\tilde{e}_0M=\tau_{-6}(M)$, $\mathcal{M}(M)$ is preserved under τ_6, which is of weight δ. So $z_\ell=\tau_{-6}$ and $\mathcal{M}(M)/\tau_6\simeq\mathcal{B}(W(\varpi_\ell))$.

The following 26 monomials appear in $\mathcal{M}_{I_0}(M)$:

$4_00_4^{-1}$, $3_14_2^{-1}0_4^{-1}$, $2_23_3^{-1}0_4^{-1}$, $1_32_4^{-1}3_30_4^{-1}$, $1_33_5^{-1}4_40_4^{-1}$, $1_5^{-1}3_3$, $1_34_6^{-1}0_4^{-1}$, $1_5^{-1}2_43_5^{-1}4_4$, $1_5^{-1}2_44_6^{-1}$, $2_6^{-1}3_54_4$, $2_6^{-1}3_5^24_6^{-1}$, $3_7^{-1}4_44_6$, $3_53_7^{-1}$, $4_44_8^{-1}$, $2_63_7^{-2}4_6$, $3_54_6^{-1}4_8^{-1}$, $1_72_8^{-1}4_6$, $2_63_7^{-1}4_8^{-1}$, $1_9^{-1}4_60_8$, $1_72_8^{-1}3_74_8^{-1}$, $1_9^{-1}3_74_8^{-1}0_8$, $1_73_9^{-1}$, $1_9^{-1}2_83_9^{-1}0_8$, $2_{10}^{-1}3_90_8$, $3_{11}^{-1}4_{10}0_8$, $4_{12}^{-1}0_8$.

These are the monomials appearing in $\mathcal{M}(M)/\tau_6$. We thus have $\mathcal{B}(W(\varpi_\ell))\simeq\mathcal{B}_{I_0}(\varpi_\ell)$.

Remark 6.4. The authors do not find the last three examples in the literature. One can probably check whether they are *perfect* or not, though we have not done yet.

6.4. Type $E_6^{(2)}$.

6.4.1. First let $\ell=1$ and $M=Y_{1,0}Y_{0,1}^{-1}Y_{0,5}^{-1}$. We have $\tilde{f}_2\tilde{f}_1M=Y_{0,5}^{-1}Y_{2,3}^{-1}Y_{3,2}$ and so we see as in Proposition 3.4 that $\mathcal{M}(M)\simeq\mathcal{B}(\varpi_\ell)$. As $\tilde{e}_1\tilde{e}_2\tilde{e}_3\tilde{e}_4\tilde{e}_2\tilde{e}_3\tilde{e}_2\tilde{e}_1\tilde{e}_1\tilde{e}_0\tilde{f}_1M=\tau_{-4}(M)$, $\mathcal{M}(m)$ is preserved under τ_4, which is of weight δ. Thus we have $z_\ell=\tau_{-4}$ and $\mathcal{M}(m)/\tau_4\simeq\mathcal{B}(W(\varpi_\ell))$.

Let $M'=\tilde{e}_0M=Y_{0,-1}Y_{0,5}^{-1}$. We have $\mathcal{M}_{I_0}(M')=\{M'\}$.

The following 26 monomials appear in $\mathcal{M}_{I_0}(M)$:
$1_0 0_1^{-1} 0_5^{-1}$, $2_1 1_2^{-1} 0_5^{-1}$, $3_2 2_3^{-1} 0_5^{-1}$, $4_3 3_4^{-1} 2_3 0_5^{-1}$, $4_3 2_5^{-1} 1_4 0_5^{-1}$, $4_5^{-1} 2_3 0_5^{-1}$, $4_3 1_6^{-1}$, $4_5^{-1} 3_4 2_5^{-1} 1_4 0_5^{-1}$, $4_5^{-1} 3_4 1_6^{-1}$, $3_6^{-1} 2_5 1_4 0_5^{-1}$, $3_6^{-1} 2_5^2 1_6^{-1}$, $2_7^{-1} 1_4 1_6 0_5^{-1}$, $2_5 2_7^{-1}$, $1_4 1_8^{-1} 0_5^{-1} 0_7$, $3_6 2_7^{-2} 1_6$, $2_5 1_6^{-1} 1_8^{-1} 0_7$, $4_7 3_8^{-1} 1_6$, $3_6 2_7^{-1} 1_8^{-1} 0_7$, $4_9^{-1} 1_6$, $4_7 3_8^{-1} 2_7 1_8^{-1} 0_7$, $4_9^{-1} 2_7 1_8^{-1} 0_7$, $4_7 2_9^{-1} 0_7$, $4_9^{-1} 3_8 2_9^{-1} 0_7$, $3_{10}^{-1} 2_9 0_7$, $2_{11}^{-1} 1_{10} 0_7$, $1_{12}^{-1} 0_7 0_{11}$.

These 26 monomials, M' are all monomials of $\mathcal{M}(M)/\tau_4$. As an application we have

$$\mathcal{B}(W(\varpi_1)) \simeq \mathcal{B}_{I_0}(\varpi_1) \sqcup \mathcal{B}_{I_0}(0).$$

The crystal structure here is isomorphic to one studied recently in [3]. As the crystal graph is connected, we conclude that the crystal base constructed in [3] are isomorphic to $\mathcal{B}(W(\varpi_\ell))$.

6.4.2. Now we consider $\ell = 2$ and $M = Y_{2,0}Y_{0,2}^{-1}Y_{0,4}^{-1}Y_{0,6}^{-1}$. We have $\tilde{f}_1\tilde{f}_2^2\tilde{f}_3\tilde{f}_1\tilde{f}_2M = Y_{4,2}Y_{3,3}Y_{2,4}^{-1}Y_{1,5}^{-1}Y_{0,6}^{-1}$ and so we see as in Proposition 3.4 that $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_\ell)$. As $\tilde{e}_2\tilde{e}_3\tilde{e}_2\tilde{e}_1^2\tilde{e}_2\tilde{e}_4\tilde{e}_3^2\tilde{e}_2^3\tilde{e}_1^2\tilde{e}_0\tilde{f}_1\tilde{f}_2^2\tilde{f}_3\tilde{f}_1\tilde{f}_2M = \tau_{-2}(M)$, $\mathcal{M}(M)$ is preserved under τ_2, which is of weight δ. Therefore we have $z_\ell = \tau_{-2}$ and $\mathcal{M}(M)/\tau_2 \simeq \mathcal{B}(W(\varpi_\ell))$.

Let $M_2 = \tilde{e}_1\tilde{e}_0M = Y_{1,-1}Y_{0,4}^{-1}Y_{0,6}^{-1}$. The following 26 monomials appear in $\mathcal{M}_{I_0}(M_2)$:
$1_{-1} 0_4^{-1} 0_6^{-1}$, $2_0 1_1^{-1} 0_0 0_4^{-1} 0_6^{-1}$, $3_1 2_2^{-1} 0_0 0_4^{-1} 0_6^{-1}$, $4_2 3_3^{-1} 2_2 0_0 0_4^{-1} 0_6^{-1}$, $4_2 2_4^{-1} 1_3 0_0 0_4^{-1} 0_6^{-1}$, $4_4^{-1} 2_2 0_0 0_4^{-1} 0_6^{-1}$, $4_2 1_5^{-1} 0_0 0_6^{-1}$, $4_4^{-1} 3_3 2_4^{-1} 1_3 0_0 0_4^{-1} 0_6^{-1}$, $4_4^{-1} 3_3 1_5^{-1} 0_0 0_6^{-1}$, $3_5^{-1} 2_4 1_3 0_0 0_4^{-1} 0_6^{-1}$, $3_5^{-1} 2_4^2 1_5^{-1} 0_0 0_6^{-1}$, $2_6^{-1} 1_3 1_5 0_0 0_4^{-1} 0_6^{-1}$, $2_4 2_6^{-1} 0_0 0_6^{-1}$, $1_3 1_7^{-1} 0_0 0_4^{-1}$, $3_5 2_6^{-2} 1_5 0_0 0_6^{-1}$, $2_4 1_5^{-1} 1_7^{-1} 0_0$, $4_6 3_7^{-1} 1_5 0_0 0_6^{-1}$, $3_5 2_6^{-1} 1_7^{-1} 0_0$, $4_8^{-1} 1_5 0_0 0_6^{-1}$, $4_6 3_7^{-1} 2_6 1_7^{-1} 0_0$, $4_8^{-1} 2_6 1_7^{-1} 0_0$, $4_6 2_8^{-1} 0_0$, $4_8^{-1} 3_7 2_8^{-1} 0_0$, $3_9^{-1} 2_8 0_0$, $2_{10}^{-1} 1_9 0_0$, $1_{11}^{-1} 0_0 0_{10}$.

Let $M_3 = \tilde{e}_1\tilde{e}_2\tilde{e}_3\tilde{e}_2\tilde{e}_4\tilde{e}_3\tilde{e}_2\tilde{e}_1\tilde{e}_0M_2 = Y_{1,-5}Y_{0,6}^{-1}Y_{0,-4}^{-1}$. The following 26 monomials appear in $\mathcal{M}_{I_0}(M_3)$:
$1_{-5} 0_{-4}^{-1} 0_6^{-1}$, $2_{-4} 1_{-3}^{-1} 0_6^{-1}$, $3_{-3} 2_{-2}^{-1} 0_6^{-1}$, $4_{-2} 3_{-1}^{-1} 2_{-2} 0_6^{-1}$, $4_{-2} 2_0^{-1} 1_{-1} 0_6^{-1}$, $4_0^{-1} 2_{-2} 0_6^{-1}$, $4_{-2} 1_1^{-1} 0_0 0_6^{-1}$, $4_0^{-1} 3_{-1} 2_0^{-1} 1_{-1} 0_6^{-1}$, $4_0^{-1} 3_{-1} 1_1^{-1} 0_0 0_6^{-1}$, $3_1^{-1} 2_0 1_{-1} 0_6^{-1}$, $3_1^{-1} 2_0^2 1_1^{-1} 0_0 0_6^{-1}$, $2_2^{-1} 1_{-1} 1_1 0_6^{-1}$, $2_0 2_2^{-1} 0_0 0_6^{-1}$, $1_{-1} 1_3^{-1} 0_6^{-1} 0_2$, $3_1 2_2^{-2} 1_1 0_0 0_6^{-1}$, $2_0 1_1^{-1} 1_3^{-1} 0_6^{-1} 0_0 0_2$, $4_2 3_3^{-1} 1_1 0_0 0_6^{-1}$, $3_1 2_2^{-1} 1_3^{-1} 0_0 0_6^{-1} 0_2$, $4_4^{-1} 1_1 0_0 0_6^{-1}$, $4_2 3_3^{-1} 2_2 1_3^{-1} 0_0 0_6^{-1} 0_2$, $4_4^{-1} 2_2 1_3^{-1} 0_0 0_6^{-1} 0_2$, $4_2 2_4^{-1} 0_0 0_6^{-1} 0_2$, $4_4^{-1} 3_3 2_4^{-1} 0_0 0_6^{-1} 0_2$, $3_5^{-1} 2_4 0_0 0_6^{-1} 0_2$, $2_6^{-1} 1_5 0_0 0_6^{-1} 0_2$, $1_7^{-1} 0_0 0_2$.

Let $M_4 = \tilde{e}_0M_3 = Y_{0,-6}Y_{0,6}^{-1}$. We have $\mathcal{M}_{I_0}(M_4) = \{M_4\}$.

Let $M_5 = \tilde{e}_4\tilde{e}_3\tilde{e}_2^2\tilde{e}_1^2\tilde{e}_0\tilde{f}_1\tilde{f}_2M = Y_{4,-2}Y_{0,2}^{-1}Y_{0,6}^{-1}$. The following 52 monomials appear in $\mathcal{M}_{I_0}(M_5)$:
$4_{-2} 0_2^{-1} 0_6^{-1}$, $4_0^{-1} 3_{-1} 0_2^{-1} 0_6^{-1}$, $3_1^{-1} 2_0^2 0_2^{-1} 0_6^{-1}$, $2_0 2_2^{-1} 1_1 0_2^{-1} 0_6^{-1}$, $3_1 2_2^{-2} 1_1^2 0_2^{-1} 0_6^{-1}$, $2_0 1_3^{-1} 0_6^{-1}$, $4_2 3_3^{-1} 1_1^2 0_2^{-1} 0_6^{-1}$, $3_1 2_2^{-1} 1_1 1_3^{-1} 0_6^{-1}$, $4_4^{-1} 1_1^2 0_2^{-1} 0_6^{-1}$, $4_2 3_4^{-1} 1_1 1_3^{-1} 2_2 0_6^{-1}$, $3_1 1_3^{-2} 0_2 0_6^{-1}$, $4_4^{-1} 2_2 1_1 1_3^{-1} 0_6^{-1}$, $4_2 3_3^{-1} 1_3^{-2} 2_2^2 0_2 0_6^{-1}$, $4_2 2_4^{-1} 1_1 0_6^{-1}$, $4_4^{-1} 3_3 2_4^{-1} 1_1 0_6^{-1}$, $4_4^{-1} 2_2^2 1_3^{-2} 0_2 0_6^{-1}$, $4_2 2_2 2_4^{-1} 1_3^{-1} 0_2 0_6^{-1}$, $3_5^{-1} 2_4 1_1 0_6^{-1}$, $4_4^{-1} 3_3 2_4^{-1} 2_2 1_3^{-1} 0_2 0_6^{-1}$, $4_2 3_3 2_4^{-2} 0_2 0_6^{-1}$, $2_6^{-1} 1_5 1_1 0_6^{-1}$, $3_5^{-1} 2_4 2_2 1_3^{-1} 0_2 0_6^{-1}$, $4_4^{-1} 3_3^2 2_4^{-2} 0_2 0_6^{-1}$, $4_2 4_4 3_5^{-1} 0_2 0_6^{-1}$, $1_7^{-1} 1_1$, $2_6^{-1} 2_2 1_3^{-1} 1_5 0_2 0_6^{-1}$, $3_3 3_5^{-1} 0_2 0_6^{-1}$, $4_2 4_6^{-1} 0_2 0_6^{-1}$, $2_2 1_3^{-1} 1_7^{-1} 0_2$, $3_3 2_4^{-1} 2_6^{-1} 1_5 0_2 0_6^{-1}$, $4_4 3_5^{-2} 2_4^2 0_2 0_6^{-1}$, $4_4^{-1} 4_6^{-1} 3_3 0_2 0_6^{-1}$, $3_3 2_4^{-1} 1_7^{-1} 0_2$, $4_6^{-1} 3_5^{-1} 2_4^2 0_2 0_6^{-1}$, $4_4 3_5^{-1} 2_4 2_6^{-1} 1_5 0_2 0_6^{-1}$, $4_4 3_5^{-1} 2_4 1_7^{-1} 0_2$, $4_6^{-1} 2_4 3_6^{-1} 1_5 0_2 0_6^{-1}$, $4_4 2_6^{-2} 1_5^2 0_2 0_6^{-1}$, $4_6^{-1} 2_4 1_7^{-1} 0_2$, $4_4 2_6^{-1} 1_5 1_7^{-1} 0_2$, $4_6^{-1} 3_5 2_6^{-2} 1_5^2 0_2 0_6^{-1}$, $4_6^{-1} 3_5 2_6^{-1} 1_5 1_7^{-1} 0_2$, $4_4 1_7^{-2} 0_2 0_6$, $3_7^{-1} 1_5^2 0_2 0_6^{-1}$, $3_7^{-1} 2_6 1_5 1_7^{-1} 0_2$, $4_6^{-1} 3_5 1_7^{-2} 0_2 0_6$, $2_8^{-1} 1_5 0_2$, $3_7^{-1} 2_6^2 1_7^{-2} 0_2 0_6$, $2_6 2_8^{-1} 1_7^{-1} 0_2 0_6$, $3_7 2_8^{-2} 0_2 0_6$, $4_8 3_9^{-1} 0_2 0_6$, $4_{10}^{-1} 0_2 0_6$.

We do not list the 273 monomials of $\mathcal{M}_{I_0}(M)$, but we can check that all monomials of $\mathcal{M}(M)/\tau_2$ are connected to either M, M_2, M_3, M_4, M_5 in the I_0-crystal. As an application we have

$$\mathcal{B}(W(\varpi_2)) \simeq \mathcal{B}_{I_0}(\varpi_2) \sqcup \mathcal{B}_{I_0}(\varpi_1) \sqcup \mathcal{B}_{I_0}(\varpi_1) \sqcup \mathcal{B}_{I_0}(0) \sqcup \mathcal{B}_{I_0}(\varpi_4).$$

6.4.3. We consider $\ell = 3$ and $M = Y_{3,0}Y_{0,3}^{-2}Y_{0,5}^{-2}$. We have $\tilde{f}_1^2\tilde{f}_2^2\tilde{f}_3 M = Y_{4,1}Y_{3,2}Y_{1,4}^{-2}Y_{0,5}^{-2}$ and so we see as in Proposition 3.4 that $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_\ell)$. As $\tilde{e}_3\tilde{e}_2^2\tilde{e}_4\tilde{e}_3^2\tilde{e}_2^2\tilde{e}_1^4\tilde{e}_2^2\tilde{e}_3^2\tilde{e}_2^3\tilde{e}_4^3\tilde{e}_3^3\tilde{e}_2^4\tilde{e}_1^4\tilde{e}_0^4 M = \tau_{-6}(M)$, $\mathcal{M}(M)$ is preserved under τ_6, which is of weight $-4\delta = -2d_\ell\delta$. Therefore we have $(z_\ell)^{-2} = \tau_6$. Let $M_1 = \tilde{e}_3\tilde{e}_2^2\tilde{e}_1^2\tilde{e}_3\tilde{e}_2^2\tilde{e}_4^2\tilde{e}_3^3\tilde{e}_2^4\tilde{e}_1^4\tilde{e}_0^2\tilde{f}_1^2\tilde{f}_2^2\tilde{f}_3 M = Y_{0,3}^{-2}Y_{0,-1}^{-2}Y_{3,-4}$. This has weight $\mathrm{wt}(M) + d_\ell\delta$, hence we have $z_\ell(M) = M_1$.

We do not determine the I_0-crystal components of $\mathcal{M}(M)/\tau_6$ at this moment.

6.4.4. Finally let us consider $\ell = 4$. Let $M = Y_{4,0}Y_{0,4}^{-2}$. It follows from Corollary 3.3 that $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_\ell)$. As $\tilde{e}_4\tilde{e}_3\tilde{e}_2^2\tilde{e}_1^2\tilde{e}_3\tilde{e}_2^2\tilde{e}_4\tilde{e}_3^2\tilde{e}_2^2\tilde{e}_1^2\tilde{e}_0^2 M = \tau_{-6}(M)$, $\mathcal{M}(M)$ is preserved under τ_6, which is of weight $-2\delta = -d_\ell\delta$. Therefore we have $z_\ell = \tau_{-6}$ and $\mathcal{M}(M)/\tau_6 \simeq \mathcal{B}(W(\varpi_\ell))$.

The following 52 monomials appear in $\mathcal{M}_{I_0}(M)$:

$4_0 0_4^{-2}$, $2_2^{-1}3_1 0_4^{-2}$, $3_3^{-1}2_2^2 0_4^{-2}$, $2_2 2_4^{-1}1_3 0_4^{-2}$, $3_3 2_4^{-2}1_3^2 0_4^{-2}$, $2_2 1_5^{-1}0_4^{-1}$, $4_4 3_5^{-1}1_3^2 0_4^{-2}$, $3_3 2_4^{-1}1_3 1_5^{-1}0_4^{-1}$, $4_6^{-1}1_3^2 0_4^{-2}$, $4_4 3_5^{-1}1_3 1_5^{-1}2_4 0_4^{-1}$, $3_3 1_5^{-2}$, $4_6^{-1}2_4 1_3 1_5^{-1}0_4^{-1}$, $4_4 3_5^{-1}1_5^{-2}2_4^2$, $4_4 2_6^{-1}1_3 0_4^{-1}$, $4_6^{-1}3_5 2_6^{-1}1_3 0_4^{-1}$, $4_6^{-1}2_4^2 1_5^{-2}$, $4_4 2_4 2_6^{-1}1_5^{-1}$, $3_7^{-1}2_6 1_3 0_4^{-1}$, $4_6^{-1}3_5 2_6^{-1}2_4 1_5^{-1}$, $4_4 3_5 2_6^{-2}$, $2_8^{-1}1_7 1_3 0_4^{-1}$, $3_7^{-1}2_6 2_4 1_5^{-1}$, $4_6^{-1}3_5^2 2_6^{-2}$, $4_4 4_6 3_7^{-1}$, $1_9^{-1}1_3 0_8 0_4^{-1}$, $2_8^{-1}2_4 1_5^{-1}1_7$, $3_5 3_7^{-1}$, $4_4 4_8^{-1}$, $2_4 1_5^{-1}1_9^{-1}0_8$, $3_5 2_6^{-1}2_8^{-1}1_7$, $4_6 3_7^{-2}2_6^2$, $4_6^{-1}4_8^{-1}3_5$, $3_5 2_6^{-1}1_9^{-1}0_8$, $4_8^{-1}3_7^{-1}2_6^2$, $4_6 3_7^{-1}2_6 2_8^{-1}1_7$, $4_6 3_7^{-1}2_6 1_9^{-1}0_8$, $4_8^{-1}2_6 2_8^{-1}1_7$, $4_6 2_8^{-2}1_7^2$, $4_8^{-1}2_6 1_9^{-1}0_8$, $4_6 2_8^{-1}1_7 1_9^{-1}0_8$, $4_8^{-1}3_7 2_8^{-2}1_7^2$, $4_8^{-1}3_7 2_8^{-1}1_7 1_9^{-1}0_8$, $4_6 1_9^{-2}0_8^2$, $3_9^{-1}1_7^2$, $3_9^{-1}2_8 1_7 1_9^{-1}0_8$, $4_8^{-1}3_7 1_9^{-2}0_8^2$, $2_{10}^{-1}1_7 0_8$, $3_9^{-1}2_8^2 1_9^{-2}0_8^2$, $2_8 2_{10}^{-1}1_9^{-1}0_8^2$, $3_9 2_{10}^{-2}0_8^2$, $4_{10}3_{11}^{-1}0_8^2$, $4_{12}^{-1}0_7 0_8^2$.

Let $M' = \tilde{e}_1\tilde{e}_2\tilde{e}_3\tilde{e}_2\tilde{e}_1\tilde{e}_0 M = Y_{1,-3}Y_{0,4}^{-1}Y_{0,-2}^{-1}$. The following 26 monomials appear in $\mathcal{M}_{I_0}(M')$:

$1_{-3}0_{-2}^{-1}0_4^{-1}$, $2_{-2}1_{-1}^{-1}0_4^{-1}$, $3_{-1}2_0^{-1}0_4^{-1}$, $4_0 3_1^{-1}2_0 0_4^{-1}$, $4_0 2_2^{-1}1_1 0_4^{-1}$, $4_2^{-1}2_0 0_4^{-1}$, $4_0 1_3^{-1}0_2 0_4^{-1}$, $4_2^{-1}3_1 2_2^{-1}1_1 0_4^{-1}$, $4_2^{-1}3_1 1_3^{-1}0_2 0_4^{-1}$, $3_3^{-1}2_2 1_1 0_4^{-1}$, $3_3^{-1}2_2^2 1_3^{-1}0_2 0_4^{-1}$, $2_4^{-1}1_1 1_3 0_4^{-1}$, $2_2 2_4^{-1}0_2 0_4^{-1}$, $1_1 1_5^{-1}$, $3_3 2_4^{-2}1_3 0_2 0_4^{-1}$, $2_2 1_3^{-1}1_5^{-1}0_2$, $4_4 3_5^{-1}1_3 0_2 0_4^{-1}$, $3_3 2_4^{-1}1_5^{-1}0_2$, $4_6^{-1}1_3 0_2 0_4^{-1}$, $4_4 3_5^{-1}2_4 1_5^{-1}0_2$, $4_6^{-1}2_4 1_5^{-1}0_2$, $4_4 2_6^{-1}0_2$, $4_6^{-1}3_5 2_6^{-1}0_2$, $3_7^{-1}2_6 0_2$, $2_8^{-1}1_7 0_2$, $1_9^{-1}0_2 0_8$.

Let $M'' = \tilde{e}_0 M' = Y_{0,-4}Y_{0,4}^{-1}$. We have $\mathcal{M}_{I_0}(M'') = \{M''\}$.

The above exhausts all monomials of $\mathcal{M}(M)/\tau_6$. As an application we have

$$\mathcal{B}(W(\varpi_4)) \simeq \mathcal{B}_{I_0}(\varpi_4) \sqcup \mathcal{B}_{I_0}(\varpi_1) \sqcup \mathcal{B}_{I_0}(0).$$

Remark 6.5. The authors do not find the description of the examples $\ell = 2$, 3, 4 in the literature.

6.5. Type $D_4^{(3)}$.

6.5.1. First we consider $\ell = 1$. Let $M = Y_{1,0}Y_{0,1}^{-1}Y_{0,3}^{-1}$. As $\tilde{f}_1 M = Y_{1,2}^{-2}Y_{0,3}^{-1}Y_{2,1}$ we see as in Proposition 3.4 that $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_\ell)$. The following 7 monomials appear in $\mathcal{M}_{I_0}(M)$:

$M = 1_0 0_1^{-1}0_3^{-1}$, $m_2 = 1_2^{-1}0_3^{-1}2_1$, $m_3 = 2_3^{-1}1_2^2 0_3^{-1}$, $m_4 = 1_2 1_4^{-1}$, $m_5 = 1_4^{-2}2_3 0_3$, $m_6 = 2_5^{-1}1_4 0_3$, $m_7 = 1_6^{-1}0_3 0_5$.

Let $M' = \tau_2(\tilde{e}_0 M) = Y_{0,1}Y_{0,5}^{-1}$. We have $\mathcal{M}_{I_0}(M') = \{M'\}$. The crystal graph of $\mathcal{M}(M)$ is given in Figure 10. We find that $z_l = \tau_{-2}$ and $\mathcal{M}(M)/\tau_2 = \mathcal{M}_{I_0}(M) \sqcup \mathcal{M}_{I_0}(M')$.

This crystal was described in [11].

6.5.2. Now we consider $\ell = 2$. Let $M = Y_{2,0}Y_{0,2}^{-3}$. It follows from Corollary 3.3 that $\mathcal{M}(M) \simeq \mathcal{B}(\varpi_\ell)$. As $\tilde{e}_2\tilde{e}_1^3\tilde{e}_2^2\tilde{e}_1^3\tilde{e}_0^3 M = \tau_{-4}(M)$, $\mathcal{M}(M)$ is preserved under τ_4, which is of weight $3\delta = d_2\delta$. Therefore $z_\ell = \tau_{-4}$ and so $\mathcal{M}(m)/\tau_4 \simeq \mathcal{B}(W(\varpi_\ell))$.

The following 14 monomials appear in $\mathcal{M}_{I_0}(M)$:

$$\begin{tikzcd}
 & & & M' \arrow[dlll, bend right, "0[2]"'] & & & \\
M \arrow[r, "1"] & m_2 \arrow[r, "2"] & m_3 \arrow[r, "1"] & m_4 \arrow[r, "1"] & m_5 \arrow[r, "2"] \arrow[ll, bend right, "0[2]"'] & m_6 \arrow[r, "1"] \arrow[lll, bend left, "0[2]"] & m_7 \arrow[ulll, bend right, "0[2]"']
\end{tikzcd}$$

FIGURE 10. (Type $D_4^{(3)}$) the crystal $\mathcal{B}(\varpi_1)$

$2_0 0_2^{-3}$, $2_2^{-1} 1_1^3 0_2^{-3}$, $1_1^2 1_3^{-1} 0_2^{-2}$, $2_2 1_1 1_3^{-2} 0_2^{-1}$, $1_3^{-3} 1_2^2$, $2_4^{-1} 1_1 1_3 0_2^{-1}$, $2_2 2_4^{-1}$, $1_1 1_5^{-1} 0_2^{-1} 0_4$, $2_4^{-2} 1_3^3$, $1_3^{-1} 1_5^{-1} 2_2 0_4$, $2_4^{-1} 1_3^2 1_5^{-1} 0_4$, $1_3 1_5^{-2} 0_4^2$, $1_5^{-3} 0_4^3 2_4$, $2_6^{-1} 0_4^3$.

Let $M_2 = \tilde{e}_1 \tilde{e}_0 M = Y_{0,2}^{-2} Y_{1,-1}$. The following 7 monomials appear in $\mathcal{M}_{I_0}(M_2)$:

$0_2^{-2} 1_{-1}$, $0_0 0_2^{-2} 1_1^{-1} 2_0$, $0_0 0_2^{-2} 1_1^2 2_2^{-1}$, $0_0 0_2^{-1} 1_1 1_3^{-1}$, $1_3^{-2} 2_2 0_0$, $2_4^{-1} 1_3 0_0$, $1_5^{-1} 0_0 0_4$.

Let $M_3 = \tilde{e}_1 \tilde{e}_2 \tilde{e}_1 \tilde{e}_0 M_2 = Y_{1,-3} Y_{0,-2}^{-1} Y_{0,2}^{-1}$. The following 7 monomials appear in $\mathcal{M}_{I_0}(M_3)$:

$0_{-2}^{-1} 0_2^{-1} 1_{-3}$, $0_2^{-1} 1_{-1}^{-1} 2_{-2}$, $0_2^{-1} 1_{-1}^2 2_0^{-1}$, $0_0 0_2^{-1} 1_{-1} 1_1^{-1}$, $1_1^{-2} 2_0 0_0^2 0_2^{-1}$, $2_2^{-1} 1_1 0_0^2 0_2^{-1}$, $1_3^{-1} 0_0^2$.

Let $M_4 = \tilde{e}_0 M_3 = Y_{0,-4} Y_{0,2}^{-1}$. We have $\mathcal{M}_{I_0}(M_4) = \{M_4\}$.

By direct calculation we can see that all monomials of $\mathcal{M}(M)/\tau_4$ are connected to either M or M_2 or M_3 or M_4 in the I_0-crystal. As an application we have

$$\mathcal{B}(W(\varpi_2)) \simeq \mathcal{B}_{I_0}(\varpi_2) \sqcup \mathcal{B}_{I_0}(\varpi_1) \sqcup \mathcal{B}_{I_0}(\varpi_1) \sqcup \mathcal{B}_{I_0}(0).$$

The authors do not find the description of this example in the literature.

7. DISCUSSIONS

(1) As we saw in the simply-laced type examples (except the last one in 6.1.4) in this paper, we can construct explicit bijections between monomial crystals $\mathcal{M}(m)$ and the set $\mathcal{C}(m_0)$ of monomials in q–characters counted with multiplicities. (Here m_0 is obtained from m by setting $Y_{0,*}$ as 1.) Their origin is combinatorial and we do not understand their representation theoretical meaning yet. In the example in 6.1.2, the global crystal base element corresponding to the exceptional monomial does not belong to a single l–weight subspace.

Also we can check that the bijection is compatible with the crystal structure in the following sense: Let $\mathcal{M}_{I_0}(m_0)$ be the component of the monomial crystal for $\mathfrak{g}_{I_0}$ containing m_0. Let $p\colon \mathcal{M}(m) \to \mathcal{M}_{I_0}(m_0)$ be the composition of the above mentioned bijection and the map obtained by forgetting multiplicities. Then p is a morphism of the crystal (but not strict). This is not true in general.

Counterexample: In the q–character of $W(\varpi_3)$ for E_6 we have monomials $m_1 = Y_{3,4} Y_{3,6}^{-1} Y_{4,3} Y_{2,5}^{-1} Y_{1,4}$ and $m_2 = Y_{3,4}^2 Y_{3,6}^{-1} Y_{4,5}^{-1} Y_{5,4} Y_{2,5}^{-1} Y_{1,4}$ with coefficients $1+2t^2+t^4$ and $1 + t^2 + t^4$. We have $\tilde{f}_4 m_1 = m_2$ in the monomial crystal. If we had a crystal morphism which preserves the weight, the 4 vectors corresponding to m_1 would necessarily satisfy $\varphi_4 \geq 1$, and each of them would be sent by $\tilde{f}_4$ to vectors corresponding to m_2. As there are only 3 of them, we have a contradiction.

(2) In [35] Naito-Sagaki proved that the crystal of Lakshmibai-Seshadri paths of shape ϖ_ℓ is isomorphic to $\mathcal{B}(\varpi_\ell)$. This result is better than Theorem 3.2 in the sense that they determine all paths, not in a recursive way as ours. Therefore it would be nice if we could give an explicit map from the path crystal to the monomial crystal.

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CNRS - UMR 8100 : Laboratoire de Mathématiques de Versailles, 45 avenue des Etats-Unis , Bat. Fermat, 78035 VERSAILLES, FRANCE

E-mail address: `hernandez @ math . cnrs . fr`

URL: `http://www.math.uvsq.fr/~hernandez`

Department of Mathematics, Kyoto University, KYOTO 606-8502, JAPAN

E-mail address: `nakajima@math.kyoto-u.ac.jp`

URL: `http://www.math.kyoto-u.ac.jp/~nakajima`