Global analytic geometry

Master course description

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The aim of this course is to explain in details the emerging subject of global analytic geometry and to give an overview of its applications. This geometry, which entails and generalizes complex analytic, p-adic and arithmetic algebraic geometry, is a modern and precise explanation of the analogy between number fields and Riemann surfaces (analogy which already dates of more than one century) transparent in Emil Artin's book [Art67]. A large audience overview of this subject due to Poineau can be found in the EMS newsletter [Poi08a] and more precise explanations of the various motivations underlying this geometric approach to number theory can be found in the introduction of the preprint [Pau08].

The main references for the course is the first chapters of Berkovich's book [Ber90] and Poineau's thesis [Poi08b]. We could also use a bit of Artin's book [Art67] or Neukirch's book [Neu92], and a bit of Hartshorne's book [Har94] as complements.

The plan of the beginning of the course is:

1. Banach algebras and seminorms over $\mathbb{Z}$.
2. The spectrum of a Banach algebra, uniform norms.
3. Examples: the spectrum of $\mathbb{Z}$, the complex analytic and p-adic affine line.
4. Geometric spaces as ringed spaces (spaces with functions).
5. Analytic functions: link with complex and p-adic analytic functions. All numbers are arithmetic analytic functions.
6. Compactifying the analytic spectrum of $\mathbb{Z}$. The picard group of $\mathbb{Z}$ is $\mathbb{R}_+^*$.
7. Idèles and adèles are germs of analytic functions.

If time permits, and depending on the interests of the audience, we could also treat the following specialized matters:

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1. An account of Poineau's results on the affine line over $\mathbb{Z}$ (coherence, stein spaces, noetherianity).

2. Bruhat-Tits buildings embed in p-adic analytic grassmanians.

3. An account of Durov's thesis [Dur07]: the field with one element and algebraic compactification of the spectrum of $\mathbb{Z}$.

4. An account of the relations with analytic number theory (spectral version of the functional equation of Riemann's zeta function).

This course can be seen as a self contained introduction to geometric number theory directed to people knowing a bit of complex analytic geometry (or differential geometry).

Prerequisites:

- basic analytic geometry (e.g. Riemann surfaces. This will be explained rapidly during the course).
- basics on banach spaces over $\mathbb{C}$.
- undergraduate knowledge on arithmetics (modular calculus in $\mathbb{Z}/p\mathbb{Z}$, polynomials).
- basic commutative algebra (definition of a ring, an ideal).

References

[Art67] Emil Artin. *Algebraic numbers and algebraic functions.* Gordon and Breach Science Publishers, New York, 1967.

[Ber90] Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1990.

[Dur07] Nikolai Durov. *New Approach to Arakelov Geometry.* arXiv.org:0704.2030, 2007.

[Har94] R. Hartshorne. *Algebraic Geometry.* Springer, 1994.

[Neu92] Jürgen Neukirch. *Algebraische Zahlentheorie.* Springer-Verlag, Berlin, 1992.

[Pau08] F Paugam. Global analytic geometry. *Paris (Submitted)*, pages 1–50, 2008.

[Poi08a] Jérôme Poineau. Global analytic geometry. pages 20–22, 2008.

[Poi08b] Jérôme Poineau. La droite de berkovich sur z. 2008.