Model categories

1. Let C be a category and $A \xrightarrow{i} B$, $X \xrightarrow{p} Y$ two arrows of C. We will say that i has the <u>left lifting property with respect to</u> p or that p has the <u>right lifting property with respect to</u> i if for every commutative (solid) square

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "b"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

there exists a <u>lifting</u> $h : B \to X$ such that

$$a = hi \quad \text{and} \quad b = ph$$

i.e. if the map

$$\mathrm{Hom}(B,X) \longrightarrow \mathrm{Hom}(A,X) \times_{\mathrm{Hom}(A,Y)} \mathrm{Hom}(B,Y)$$

$$h \longmapsto (hi,\ ph)$$

is surjective. If K is a class of arrows of C we denote by $\ell(K)$ (resp. $r(K)$) the class of arrows of C satisfying the left (resp. right) lifting property with respect to all arrows in K. The following definition is due to Quillen

A closed model category is a category C endowed with three classes of arrows W, Cof, Fib satisfying the following conditions:

CM1] the category C has finite limits and finite colimits;

CM2] the class of arrows W satisfy the two out of three property: in a commutative triangle of C, if two of the three arrows are in W, so is the third

CM3] W, Cof, Fib are stable under retracts

CM4] $Cof \subset l(Fib \cap W)$, $Fib \subset r(Cof \cap W)$

CM5] Every arrow f in C can be decomposed as $f = pj$ and $f = qi$ with $p \in Fib$, $j \in Cof \cap W$, $q \in Fib \cap W$, $i \in Cof$

Arrows in W are called weak equivalences, arrows in Cof, cofibrations, in $Cof \cap W$ trivial cofibrations, in Fib fibrations, in $Fib \cap W$ trivial fibrations. If we denote by ϕ (resp. $*$) an initial (resp. final) object of C (CM1), an object X of C is called cofibrant (resp. fibrant) if $\phi \longrightarrow X$ is a cofibration (resp. $X \to *$ a fibration)

It can be "easily" proved that in a closed model category the following relations hold

$$Cof = l(Fib \cap W) \quad , \quad Fib = r(Cof \cap W)$$

$$Cof \cap W = l(Fib) \quad , \quad Fib \cap W = r(Cof)$$

$W = (Fib \cap W) \circ (Cof \cap W)$ (an arrow of C is in W if and only if it can be written as $f = qj$ with $q \in Fib \cap W$ and $j \in Cof \cap W$)

In particular any two out of the three classes W, Cof, Fib determine the third one

Examples

1) $C = \mathrm{Top}$ = category of topological spaces and continuous maps

W = ordinary weak equivalences

Fib = Serre fibrations $= r(\mathbb{J})$

$$\mathbb{J} = \{ [0,1]^n \hookrightarrow [0,1]^{n+1} \mid n \geq 0 \}$$
$$(t_1, \dots, t_n) \mapsto (0, t_1, \dots, t_n)$$

$\mathrm{Cof} = \ell(\mathrm{Fib} \cap W)$

2) $C = \mathrm{SSets}$ = category of simplicial sets

Cof = monomorphisms

Fib = Kan fibrations

$W = r(\mathrm{Cof}) \circ \ell(\mathrm{Fib})$ (= maps of simplicial sets whose topological realization is an homotopy equivalence of CW-complexes)

3) $C = C(\mathcal{A})$ = category of unbounded cochain complexes of $\mathcal{A}$, $\mathcal{A}$ being a Grothendieck category, i.e. a cocomplete abelian category, with exact filtered colimits and a generator

W = quasi-isomorphisms

Cof = monomorphisms

$\mathrm{Fib} = r(\mathrm{Cof} \cap W)$

4) many others ...

2. The following "Ken Brown" lemma is very useful.

<u>Lemma</u> Let $(\mathcal{C}, W, Cof, Fib)$ be a closed model category, $(\mathcal{C}', W')$ a localizer such that W' satisfies the two out of three property, and $F : \mathcal{C} \to \mathcal{C}'$ a functor carrying trivial cofibrations between cofibrant objects of $\mathcal{C}$ to arrows in W'. Then F carries all weak equivalences between cofibrant objects of $\mathcal{C}$ to arrows in W'.

<u>Theorem</u> (Existence of derived functors) Let $(\mathcal{C}, W, Cof, Fib)$ be a closed model category, $\mathcal{D}$ a category and $F : \mathcal{C} \to \mathcal{D}$ a functor carrying trivial cofibrations between cofibrant objects of $\mathcal{C}$ to isomorphisms of $\mathcal{D}$. Then the functor F admits an absolute left derived functor

<u>Sketch of the construction of the derived functor</u>
(this can be skipped and go directly to corollary page 9)
For every object X of $\mathcal{C}$ choose by CM5 a decomposition

$$\phi \rightarrowtail LX \overset{p_X}{\underset{\sim}{\twoheadrightarrow}} X$$

of $\phi \to X$ to a cofibration followed by a trivial fibration p_X. The object LX is then cofibrant. For every arrow $f : X \to Y$ of $\mathcal{C}$ choose by CM4 a lifting Lf

$$\begin{tikzcd}
\phi \arrow[r] \arrow[d, tail] & LY \arrow[d, two heads, "\wr", "p_Y"'] \\
LX \arrow[r, "f p_X"'] \arrow[ur, dashed, "Lf"] & Y
\end{tikzcd}
\qquad p_Y Lf = f p_X$$

so that

$$\begin{tikzcd} LX \arrow[r, "p_X"] \arrow[d, "L(f)"'] & X \arrow[d, "f"] \\ LY \arrow[r, "p_Y"'] & Y \end{tikzcd} \quad (*)$$

commutes

Define a functor $C \xrightarrow{\mathcal{L}F} D$ by

$$X \longmapsto F(LX)$$
$$f \longmapsto F(Lf)$$

One has to verify

- $F(L1_X) = 1_{F(LX)}$
- $F(L(gf)) = F(Lg)\,F(Lf) \qquad X \xrightarrow{f} Y \xrightarrow{g} Z$

$$\begin{tikzcd} LX \arrow[r, "p_X"] \arrow[d, "L1_X"] \arrow[d, bend right=60, "1_{LX}"'] & X \arrow[d, "1_X"] \\ LX \arrow[r, "p_X"'] & X \end{tikzcd}$$

$p_X \cdot L1_X = 1_X \cdot p_X = p_X\, 1_{LX}$

$$\begin{tikzcd} LX \arrow[r, "p_X"] \arrow[d, "Lf"] \arrow[dd, bend right=60, "L(gf)"'] & X \arrow[d, "f"] \\ LY \arrow[r, "p_Y"] \arrow[d, "Lg"] & Y \arrow[d, "g"] \\ LZ \arrow[r, "p_Z"'] & Z \end{tikzcd}$$

$$p_Z\, Lg\, Lf = g\, p_Y\, Lf$$
$$= gf\, p_X = p_Z\, L(gf)$$

Therefore to conclude in both cases it's enough to prove

<u>Lemma</u> Let $A \underset{v}{\overset{u}{\rightrightarrows}} B \xrightarrow{p} T$ a diagram in C such that $pu = pv$, p is a trivial fibration and A cofibrant. Then $F(u) = F(v)$

Proof Choose by CM5 a decomposition

$$A \amalg A \xrightarrow{(\delta^0, \delta^1)} \tilde{A} \xrightarrow[\sim]{s} A$$

of the codiagonal $A \amalg A \to A$ to a cofibration followed by a trivial fibration. As A and $\tilde{A}$ are cofibrant and s a weak equivalence the Ken Brown lemma implies that $\mathbb{F}(s)$ is an isomorphism and as $s\delta^0 = 1_A = s\delta^1$ this implies that $F(\delta^0) = F(\delta^1)$. By CM4 there is a lifting h

$$\begin{tikzcd} A \amalg A \arrow[r, "{(u,v)}"] \arrow[d, tail, "{(\delta^0, \delta^1)}"'] & B \arrow[d, two heads, "p"', "\sim"] \\ \tilde{A} \arrow[r, "pus = pvs"'] \arrow[ur, dashed, "h"] & T \end{tikzcd}$$

so that $u = h\delta^0$, $v = h\delta^1$ which implies

$$F(u) = F(h)F(\delta^0) = F(h)F(\delta^1) = F(v)$$

$\square$

The commutativity of the square (*) and CM2 imply that if $f : X \to Y$ is a weak equivalence of $\mathcal{C}$, $Lf : LX \to LY$ is also a weak equivalence, and as LX and LY are cofibrant the Ken Brown lemma implies that $\mathcal{L}F(f) = F(Lf)$ is an isomorphism. So the functor $\mathcal{L}F : \mathcal{C} \to \mathcal{D}$ induces a functor $LF : W^{-1}\mathcal{C} \to \mathcal{D}$ such that $\mathcal{L}F = LF \circ p$

$$\begin{tikzcd} \mathcal{C} \arrow[d, "p"'] \arrow[drr, "\mathcal{L}F"] & & \\ \mathrm{Ho}\,\mathcal{C} := W^{-1}\mathcal{C} \arrow[rr, "LF = \mathcal{L}F"'] & & \mathcal{D} \end{tikzcd}$$

The commutativity of the squares (*) implies that if we define α by $\alpha_X = F(p_X)$ then $\alpha : LF \circ P \longrightarrow F$ is a natural transformation

$$\begin{tikzcd} C \arrow[d, "P"'] \arrow[dr, "F"] \\ W^{-1}C \arrow[r, "LF"'] \arrow[ur, phantom, "\alpha \nearrow"] & D \end{tikzcd} \qquad \begin{tikzcd} \alpha_X : LF \circ P X \arrow[r] \arrow[d, phantom, "\shortparallel"] & FX \\ \mathcal{L}FX \arrow[d, phantom, "\cong"] & \\ FLX \arrow[uur, "F(p_X) = \alpha_X"'] & \end{tikzcd}$$

Let us prove that (LF, α) is a left derived functor of F. First observe that by the Ken Brown lemma if X is cofibrant, then $\alpha_X = F(p_X)$ is an isomorphisme

Let $\gamma : G \circ P \longrightarrow F$ a natural transformation where $G : W^{-1}C \longrightarrow D$ is a functor. By the precise universal property of the localization, in order to prove the (LF, α) is a left derived functor, it is enough to prove that there exists a unique natural transformation $\delta : G \circ P \longrightarrow LF \circ P$ so that the triangle

$$\begin{tikzcd} G \circ P \arrow[rr, "\gamma"] \arrow[dr, "\delta"'] & & F \\ & LF \circ P \arrow[ur, "\alpha"'] & \end{tikzcd} \quad \text{commutes}$$

<u>Unicity</u> comes from the following commutative prism and the fact that α_{LX} and $GP(p_X)$ are invertible

$$\begin{tikzcd}
GP(LX) \arrow[rr, "\gamma_{LX}"] \arrow[dr, "\delta_{LX}"] \arrow[dd, "GP(\rho_X)"', "\wr"] & & F(LX) \arrow[dd, "F(\rho_X)"] \\
& LF\circ P(LX) \arrow[ur, "\sim", "\alpha_{LX}"'] \arrow[dd, "\wr", "LF\circ P(\rho_X)"' near start] & \\
GP(X) \arrow[rr] \arrow[dr, "\delta_X"'] & & F(X) \\
& LF\circ P(X) \arrow[ur, "\alpha_X"'] &
\end{tikzcd}$$

Existence follows by defining δ by this prism:

$$\delta_X = LFP(\rho_X)\ \alpha_{LX}^{-1}\ \gamma_{LX}\ (GP(\rho_X))^{-1}$$

and observing that the relation $\gamma = \alpha\delta$ is a consequence of the commutativity of the prism and naturality consequence of the following commutative diagram for $X \to Y$ an arrow in $\mathbb{C}$

$$\begin{tikzcd}
& GP(LX) \arrow[rr] \arrow[dl, "\sim"'] \arrow[dd] & & F(LX) \arrow[dl] \arrow[dd] & & LFP(LX) \arrow[ll, "\sim"'] \arrow[dl, "\sim"'] \arrow[dd] \\
GP(X) \arrow[rr] \arrow[dd] & & F(X) \arrow[dd] & & LFP(X) \arrow[ll] \arrow[dd] & \\
& GP(LY) \arrow[rr] \arrow[dl, "\sim"'] & & F(LY) \arrow[dl] & & LFP(LY) \arrow[ll, "\sim"'] \arrow[dl, "\sim"'] \\
GP(Y) \arrow[rr] & & F(Y) & & LFP(Y) \arrow[ll] &
\end{tikzcd}$$

(and the invertibility of the arrows $\xrightarrow{\sim}$)

It remains to prove that this derived functor is absolute. This is immediate if we observe that for any functor $H: D \to D'$

the functor $F' = HF : C \to D'$ satisfies the hypothesis of the theorem, and if we apply the same construction (with the same choice of LX and Lf), we obtain a left derived functor (LF', α') of F' such that $LF' = H \circ LF$ and $\alpha' = H * \alpha$.

<u>Corollary</u> Let (C, W, Cof, Fib) be a closed model category, (C', W') a localizer such that W' satisfies the two out of three property, and $F : C \to C'$ a functor carrying trivial cofibrations between cofibrant objects of C to arrows in W'. Then the functor F admits an absolute total left derived functor $(\underline{L}F, \alpha)$

$$\begin{tikzcd}
C \arrow[r, "F"] \arrow[d, "p"'] & C' \arrow[d, "p'"] \\
Ho\, C = W^{-1}C \arrow[r, "\underline{L}F"'] \arrow[ur, phantom, "\alpha"] & W'^{-1}C'
\end{tikzcd}$$

.

<u>Theorem</u> (Composition of derived functors) Let $(\mathcal{C}, W, Cof, Fib)$, $(\mathcal{C}', W', Cof', Fib')$ and $(\mathcal{C}'', W'', Cof'', Fib'')$ three closed model categories, $F : \mathcal{C} \to \mathcal{C}'$, $F' : \mathcal{C}' \to \mathcal{C}''$ two functors, and suppose that F and F' carries trivial cofibrations between cofibrant objects to weak equivalences <u>and</u> that F <u>carries cofibrant objects to cofibrant objects</u>. If $(\underline{L}F, \alpha)$, $(\underline{L}F', \alpha')$ are total left derived fuctors respectively of F, F' (which exist and are absolute by the existence theorem), then $(\underline{L}F' \circ \underline{L}F, (\alpha' * F)(\underline{L}F' * \alpha))$ is

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\alpha}{\Nearrow}"] & \mathcal{C}' \arrow[r, "F'"] \arrow[d, "p'"] \arrow[dr, phantom, "\overset{\alpha'}{\Nearrow}"] & \mathcal{C}'' \arrow[d, "p''"] \\
W^{-1}\mathcal{C} \arrow[r, "\underline{L}F"'] & W'^{-1}\mathcal{C}' \arrow[r, "\underline{L}F'"'] & W''^{-1}\mathcal{C}''
\end{tikzcd}$$

an absolute total left derived functor of $F'F$.

<u>Theorem</u> (Adjunction) Let $(\mathcal{C}, W, Cof, Fib)$ and $(\mathcal{C}', W', Cof', Fib')$ two closed model categories,

$$F : \mathcal{C} \longrightarrow \mathcal{C}' \ , \quad G : \mathcal{C}' \longrightarrow \mathcal{C}$$

a pair of adjoint functors, and suppose that F (resp. G) carries trivial cofibrations (resp. trivial fibrations) between cofibrant (resp. fibrant) objects to weak equivalences. If $(\underline{L}F, \alpha)$, (resp. $(\underline{R}G, \beta)$) is a total left (resp. right) derived functor of F (resp. G) (which exists and is absolute by the existence theorem (resp the dual existence theorem)) then $\underline{L}F, \underline{R}G$ is a pair of adjoint functors.

This theorem is now just an application of the abstract adjunction theorem. It is much stronger that its usual formulation in terms of Quillen's adjunctions (a notion that was introduced by Quillen)

A <u>Quillen's</u> <u>adjunction</u> is a pair of adjoint functors between closed model categories satisfying the equivalent conditions of the following proposition (and if (F, G) is a Quillen adjunction, F is called a <u>left Quillen functor</u> and G a <u>right Quillen functor</u>).

<u>Proposition</u> Let (F, G) be an adjoint pair of functors between closed model categories. The following conditions are equivalent:

i) F carries trivial cofibrations to trivial cofibrations and G carrie trivial fibrations to trivial fibrations;

ii) F carries trivial cofibrations to trivial cofibrations and cofibrations to cofibrations;

iii) G carries trivial fibrations to trivial fibrations and fibrations to fibrations;

iv) F carries cofibrations to cofibrations and G carries fibrations to fibrations.

<u>Corollary</u> (of the theorem) Let (F, G) a Quillen adjunction between closed model categories. Then $\underline{L}F$ and $\underline{R}G$ exist, are absolute and $(\underline{L}F, \underline{R}G)$ is a pair of adjoint functors.

A Quillen's adjunction is called a <u>Quillen's equivalence</u> if the equivalent conditions of the following proposition are satisfied

<u>Proposition</u> Let $(F : C \to C', G : C' \to C)$ a Quillen adjunction. The following conditions are equivalent:

i) $\underline{L}F$ is an equivalence of categories;

ii) $\underline{R}G$ is an equivalence of categories;

iii) for every cofibrant object X of C and every fibrant object X' of C' a map $FX \to X'$ of C' is a weak equivalence if and only if the associated map $X \to GX'$ of C is a weak equivalence

A model category $(\mathcal{C}, W, Cof, Fib)$ is called <u>I-projective</u> (<u>projectively I-exponential</u> or more simply), where I is a small category, if C^I endowed with pointwise weak equivalences, pointwise fibrations, and cofibrations defined by the left lifting property with respect to trivial fibrations is a closed model category. This structure is called the <u>projective model structure</u> on C^I. The model category is called <u>projective</u> (<u>projectively exponential</u> or more simply) if it is I-projective for every small category I. The notions of <u>I-injective</u> and <u>injective</u> closed model category are defined dually.

<u>Theorem</u> Let $(\mathcal{C}, W, Cof, Fib)$ a projective model category. If the category C is cocomplete and stable by arbitrary small products, then the localizer (C, W) is cocomplete.

The proof goes through several lemmas. In what follows, fix a closed model category (C, W, Cof, Fib) such that C is cocomplete.

<u>Lemma 1</u> If C is I-projective and C^I endowed with the projective model structure, then

$$C^I \xrightarrow{\varinjlim_I} C \,, \qquad C \xrightarrow{\Delta_I} C^I$$

$$F \longmapsto \varinjlim F \qquad X \longmapsto \text{constant functor of value } X$$

is a Quillen adjunction. In particular, the homotopy colimit functor $\underrightarrow{\mathrm{holim}}_I : W_I^{-1} C^I \to W^{-1} C$ exists and is the total left derived functor of $\varinjlim_I$ ($\underrightarrow{\mathrm{holim}}_I = L \varinjlim_I$)

Proof As weak equivalences and fibrations are defined pointwise in C^I, the functor Δ_I carries fibrations (resp. trivial fibrations) of C to fibrations (resp. trivial fibrations) of C^I.

More generally.

Lemma 2 Let $u : I \to J$ a functor between small categories. If C is I and J - projective and C^I, C^J endowed with the projective model structures, then

$$C^I \xrightarrow{u_!} C^J \,, \qquad C^J \xrightarrow{u^*} C^I$$

is a Quillen adjunction. In particular the homotopy left Kan - extension functor $u_{!\mathbb{R}} : W_I^{-1} C^I \longrightarrow W_J^{-1} C^J$ exists and is the total left derived functor of $u_!$ ($u_{!\mathbb{R}} = \underline{L}\, u_!$)

The proof is similar to the proof of lemma 1

Lemma 3 Let $u : I \to J$ be a local isomorphism between small categories (which means that for every object i of I the functor $I/i \longrightarrow J/u(i)$ induced by u is an isomorphism, or equivalently that u is a Grothendieck fibration with discrete fibers). If C admits arbitrary small products, if C is I and J - projective and if C^I and C^J are endowed with the projective model structure, then u^* is a left Quillen functor.

<u>Proof</u> As u^* carries weak equivalences to weak equivalences it's enough to prove that u^* carries cofibrations to cofibration. As u is a local isomorphisme, it can be easily verified that for every object j of J the functor

$$\coprod_{u(i_0)=j} i_0\backslash I \longrightarrow j\backslash I$$
$$(i, i_0 \xrightarrow{f} i) \longmapsto (i, j = u(i_0) \xrightarrow{u(f)} u(i))$$

is an isomorphism. Remark that $(i_0, i_0 \xrightarrow{1_{i_0}} i_0)$ is an initial object of $i_0\backslash I$. So, as C admits arbitrary small products, the <u>right</u> Kan extension functor $f_* : C^I \longrightarrow C^J$ exists and is defined by

$$u_*(F)(j) = \varprojlim_{\substack{j\backslash I \\ (i,\, j\to u(i))}} F(i) \simeq \varprojlim_{\substack{\coprod_{u(i_0)=j} i_0\backslash I \\ (i,\, i_0\to i)}} F(i) \simeq \prod_{u(i_0)=j} F(i_0)$$

As trivial fibrations are stable by products, u_* carries trivial fibrations to trivial fibration and as a consequence u^* carries cofibrations to cofibrations.

<u>Proof of the theorem</u> Suppose now that C is cocomplete, projective and stable by small products. By lemmas 1, 2 the localizer (C, W) admits homotopy colimits and more generally homotopy left Kan extensions. It remains to prove that every functor $u : I \to J$ between small categories, any $F : I \to C$ and any object j of J the canonical map

$$\underline{L}u_!(F)(j) \leftarrow \underrightarrow{\mathrm{holim}}_{I/j} (F \,|\, (I/j))$$

is an isomorphisme in $W^{-1}C$, i.e. that

$$\overline{j^*}\,\underline{L}u_! \xleftarrow{\sim} \underrightarrow{\mathrm{holim}}_{I/j}\,\overline{k^*} \qquad (1)$$

where $j : e \to J$ denotes also the functor from the point category e to J defined by the object j and $k : I/j \to I$ the forgetful functor $(i, u(i)\to j)\mapsto i$. By the classical formula for left Kan extension functors we have a canonical isomorphism

$$(*) \qquad j^* u_! \xleftarrow{\sim} \underrightarrow{\lim}_{I/j}\, k^*$$

Observe that $\overline{j^*} = \underline{L}\, j^*$ and that by lemma 2 $u_!$ is a left Quillen functor, so the pair $j^*, u_!$ satisfy the conditions of the theorem of composition of total left derived functors (it can be proved that j^* is also a left Quillen functor (cf. lemma 4), this is not needed here, as j^* carries weak equivalence to weak equivalences) and

$$\overline{j^*}\,\underline{L}u_! \simeq \underline{L}j^*\cdot\underline{L}u_! \simeq \underline{L}(j^*u_!)$$

On the other hand k is a local isomorphism and by lemmas 1 and 3 $\underrightarrow{\lim}_{I/j}$ and k^* are left Quillen functors, so the composition theorem applies again and

$$\underrightarrow{\mathrm{holim}}_{I/j}\,\overline{k^*} \simeq \underline{L}\,\underrightarrow{\lim}_{I/j}\cdot\underline{L}\,k^* \simeq L\left(\underrightarrow{\lim}_{I/j}\, k^*\right)$$

and $(*)$ implies the theorem

<u>Remark</u> In order to prove that (1) is an isomorphisme the only "projectivity" condition used is that C is I and I/j - projective

Lemma 4 Let $(\mathcal{C}, W, Cof, Fib)$ a closed model category stable under arbitrary small products and I a small category. Suppose that $\mathcal{C}^I$ is endowed with a closed model structure such that weak equivalences are defined pointwise and such that pointwise fibrations are fibrations. Then for every object i de I, if we denote also $i : e \to I$ the functor from the point category e to I defined by the object i, the functor i^* is a left Quillen functor. In particular any cofibration in $\mathcal{C}^I$ is a pointwise cofibration (but the converse is not true)

Proof As $\mathbb{C}$ is stable under arbitrary small products the right Kan extension functor i_*, right adjoint of i^* exists and is defined by

$$(i_* X)(i') = \varprojlim_{\substack{i' \backslash e \\ i' \to i}} X = \prod_{Hom(i',i)} X \qquad (*)$$

As i^* carries weak equivalence to weak equivalences, it is enough to prove that i^* carries cofibrations to cofibrations, or equivalently that i_* carries trivial fibrations to trivial fibrations. This is a consequence of the formula $(*)$, and the stability of trivial fibrations by products.

<u>Theorem</u> Every cofibrantly generated closed model category is projectively exponential.

We recall the definition of a cofibrantly generated closed model category.

Let α an infinite cardinal. A set E (resp. a category I) is called α-<u>small</u> if $\mathrm{card}(E) < \alpha$ (resp. if I is small and $\mathrm{card}\,\mathrm{Ar}\, I < \alpha$). An infinite cardinal α is called <u>regular</u> if α-small colimits (indexed by α-small categories) of α-small sets are α-small. Let α a regular cardinal and A a category. The category A is called α-<u>filtered</u> if for every α-small category I, any functor $F : I \to A$ has a cocone in A (there exists an object X in A and a natural transformation $F \to X$, where X denotes also the constant functor $I \to A$ of value X). Let C a category stable under small α-filtered colimits (indexed by a small α-filtered category). An object X of C is called α-<u>presentable</u> if the functor

$$\mathrm{Hom}_C(X, ?) : C \to \mathrm{Sets}, \quad T \mapsto \mathrm{Hom}_C(X, T)$$

preserves small α-filtered colimits. A object X of a category C is called <u>presentable</u> if there exists a regular cardinal α such that C is stable under small α-filtered colimits and X is α-presentable.

Let C a cocomplete category and $\mathfrak{I}$ a set of arrows of C. We say that $\mathfrak{I}$ allows the small object argument if the domains of arrows in $\mathfrak{I}$ are presentable

The small object argument implies then that every arrow f in C can be written as

$$f = qi \quad , \quad q \in r(\mathfrak{I}) \quad , \quad i \in \mathrm{cell}(\mathfrak{I})$$

where $\mathrm{cell}(\mathfrak{I})$ denotes the class of arrows in C which are transfinite compositions of pushouts of arrows in $\mathfrak{I}$

An easy formal lemma, known as the "retract lemma" implies then that $lr(\mathfrak{I})$ is the class of retracts of arrows in $\mathrm{cell}(\mathfrak{I})$.

We will say that a closed model category (C, W, Cof, Fib) is cofibrantly generated if C is cocomplete and if there exist sets $\mathfrak{I}, \mathfrak{J}$, allowing the small object argument such that

$$Fib = r(\mathfrak{J}) \quad \text{and} \quad Fib \cap W = r(\mathfrak{I})$$

or equivalently

$$Cof = lr(\mathfrak{I}) \quad , \quad Cof \cap W = lr(\mathfrak{J})$$

We will say that $\mathfrak{I}$ (resp. $\mathfrak{J}$) generates the cofibrations (resp the trivial cofibrations) and that $(\mathfrak{I}, \mathfrak{J})$ generates the cofibrantly generated closed model category C.

To prove the theorem on page 18, we need the following Lemma

<u>Lemma</u> (Crans?) Let $(\mathcal{C}, W, Cof, Fib)$ ~~be~~ cofibrantly generated closed model category, generated by a pair $(\mathcal{I}, \mathcal{J})$ and let

$$F : \mathcal{C} \longrightarrow \mathcal{C}', \quad G : \mathcal{C}' \longrightarrow \mathcal{C}$$

a pair of adjoint functors. Suppose:

i) $\mathcal{C}'$ is complete and cocomplete;

ii) the sets $F(\mathcal{I})$ and $F(\mathcal{J})$ allows the small object argument

iii) $G(\ell r(F\mathcal{J})) \subset W$

Define

$$W' = G^{-1}(W), \quad Fib' = G^{-1}(Fib), \quad Cof' = \ell(Fib' \cap W').$$

Then $(\mathcal{C}', W', Cof', Fib')$ is a cofibrantly generated closed model category generated by $(F(\mathcal{I}), F(\mathcal{J}))$

<u>Proof</u> Axioms CM1, CM2, CM3 are clear, and 1/2 of CM4 is by definition. Observe that

$$Fib' = G^{-1}(Fib) = G^{-1}(r(\mathcal{J})) = r(F(\mathcal{J}))$$

and

$$Fib' \cap W' = G^{-1}(Fib \cap W) = G^{-1}(r(\mathcal{I})) = r(F(\mathcal{I}))$$

In particular

$$Cof' = \ell r(F(\mathcal{I}))$$

and

$$\ell r(F(\mathcal{J})) = \ell(Fib') \subset \ell(Fib' \cap W') = Cof'.$$

As the condition (iii) implies that $\ell r(F(\mathcal{J})) \subset W'$ we have

$$\ell r(F(\mathcal{J})) \subset Cof' \cap W'$$

Let us prove the other inclusion. Let i in $Cof' \cap W'$.

As $F(\mathcal{J})$ allows the small object argument, factorize

$$i = qj \;,\quad j \in lr(F(\mathcal{J})) \;,\quad q \in r(F(\mathcal{J})) = Fib'$$

As i and j are in W' so is q, so $q \in F'ib \cap W'$ and i has the left lifting property with respect to q. The retract lemma implies that i is a retract of j and so $i \in lr(F(\mathcal{J}))$

$$\begin{tikzcd} \cdot \arrow[r, "j"] \arrow[d, "i"'] & \cdot \arrow[d, "q"] \\ \cdot \arrow[r, "="'] \arrow[ur, dashed, "s"] & \cdot \end{tikzcd} \qquad \begin{array}{l} qs = 1 \\ j = si \end{array} \qquad \begin{tikzcd} \cdot \arrow[r, "="] \arrow[d, "i"'] & \cdot \arrow[r, "="] \arrow[d, "j"] & \cdot \arrow[d, "i"] \\ \cdot \arrow[r, "s"] \arrow[rr, bend right, "="'] & \cdot \arrow[r, "q"] & \cdot \end{tikzcd}$$

We have proved that

$$Cof' \cap W' = lr(F(\mathcal{J})) = l(Fib')$$

and

$$Cof' = lr(F(\mathcal{I})) = l(F' \cap W')$$

and this ends the proof as CM5 is now a consequence of the fact that $F(\mathcal{I})$ and $F(\mathcal{J})$ allow the small object argument. $\square$

We can now prove a more precise version of the theorem on page 18:

<u>Proposition</u> Let $(\mathcal{C}, W, Cof, Fib)$ be a cofibrantly generated closed model category generated by the pair $(\mathcal{I}, \mathcal{J})$, and I a small category. Then $(\mathcal{C}^I, W_I, Cof_I, Fib_I)$, where W_I (resp. Fib_I) is the class of pointwise weak equivalences (resp. fibrations) and $Cof_I = l(Fib_I \cap W_I)$,

is a cofibrantly generated closed model category generated by the pair $(\mathcal{I}_I, \mathcal{J}_I)$ where

$$\mathcal{I}_I = \bigcup_{i \in \mathrm{Ob}\, I} i_!(\mathcal{I}), \quad \mathcal{J}_I = \bigcup_{i \in \mathrm{Ob}\, I} i_!(\mathcal{J}),$$

where $i_! : C \longrightarrow C^I$ is the left adjoint of the functor $i^* : C^I \longrightarrow C$, $X \longmapsto X(i)$.

<u>Proof</u> Observe that the natural bijection

$$\mathrm{Hom}_{C^I}(i_! X, Y) \simeq \mathrm{Hom}_C(X, Y(i))$$

implies that if X is a presentable object of C then $i_! X$ is a presentable object of C^I, so that $\mathcal{I}_I$ and $\mathcal{J}_I$ allow the small object argument

The particular case where I is a discrete category is an easy exercise left to the reader, and gives simply the product model structure where cofibrations are also defined pointwise

The general case follows from the discrete case by the preceding lemma. Let I_0 denote the discrete category whose objects are the objects ~~are the objects~~ of I and $k : I_0 \longrightarrow I$ the inclusion. In order to apply the preceding lemma and conclude the only thing one has to verify is the condition (iii) of this lemma for the adjunction

$$C^{I_0} \xrightarrow{k_!} C^I, \quad C^I \xrightarrow{k^*} C^{I_0}$$

Let i an object of I and $i_0 : e \to I_0$, $i : e \to I$ the functors defined by the object i.

$$\begin{tikzcd} e \arrow[r, "i_0"] \arrow[rr, bend right, "i"'] & I_0 \arrow[r, "k"] & I \end{tikzcd}$$

We have to prove that

$$k^* \, lr \, k_! \, i_{0!}(\mathcal{J}) \subset W_{I_0}$$

which simply means that

$$lr\,(i_!(\mathcal{J})) \subset W_I \, .$$

In order to prove this, it is enough to prove the stronger condition that $lr\,(i_!(\mathcal{J}))$ is contained in the class of pointwise trivial cofibrations. As this class is stable under pushout, transfinite composition and retracts it is enough (by the small object argument) to prove that $i_!(\mathcal{J})$ is contained in this class. But an explicit calculation shows that for every object X of C

$$(i_! X)(i') = \coprod_{\mathrm{Hom}(i,i')} X \quad , \; i' \in \mathrm{ob}\, I \, ,$$

and as trivial cofibrations of C are stable under small coproducts this finishes the proof

Reedy categories

A Reedy category is a triple (I, I_+, I_-), where I is a small category and I_+, I_- two (non full) subcategories of I satisfying the following conditions:

R1) Every arrow k of I can be uniquely decomposed as $k = k_+ k_-$ with k_+ (resp. k_-) in I_+ (resp. I_-).

R2) There exists a function d from the set of objects of I to a well ordered set such that if $k : i \to i'$ is a non identical map of I_+ (resp. I_-) then $d(i') > d(i)$ (resp. $d(i') < d(i)$).

Elementary properties

a) The subcategories I_+ and I_- have same set of objects as I.

b) The intersection of I_+ and I_- is reduced to the identities of I.

c) Every split monomorphism is in I_+; every split epimorphism is in I_-.

d) The only isomorphismes of I are identities.

Examples 1) The category Δ of simplices whose objects are the non-empty ordered sets

$$[n] = \{0 < 1 < \cdots < n\}, \quad n \geq 0,$$

and arrows the non decreasing maps, endowed with the subcategories Δ_+ and Δ_- where Δ_+ (resp. Δ_-) is the subcategory of Δ with same objects and whose arrows are the monomorphisms (resp. epimorphisms) of Δ, is a Reedy category.

2) If (I, I_+, I_-) is a Reedy category, then (I^o, I_-^o, I_+^o) is a Reedy category.

3) If (I, I_+, I_-) and (J, J_+, J_-) are two Reedy categories, then $(I \times J, I_+ \times J_+, I_- \times J_-)$ is a Reedy category

4) If (I, I_+, I_-) is a Reedy category and X a presheaf on I, then $(I/X, I_+/X, I_-/X)$ is a Reedy category

A <u>direct category</u> I is a small category I such that (I, I, I_0), where I_0 is the discrete subcategory of I with same objects as I and only identities as morphisms, is a Reedy category. This means simply that there exists a function d from the set of objects of I to a well ordered set such that if $k : i \to i'$ is a non identity arrow of I, then $d(i) < d(i')$. The notion of <u>inverse category</u> is defined dually.

If (I, I_+, I_-) is a Reedy category, then I_+ is a direct category and I_- an inverse category

If I is a <u>finite</u> category the following conditions are equivalent:

a) I is a direct category; b) I is an inverse category;

c) the free category generated by the graph of non identity arrows of I is finite;

d) the nerve of I is a finite simplicial set (having a finite set of non degenerate simplices)

Fix a Reedy category (I, I_+, I_-)

For an object i of I denote $\partial_i^+ I$ (resp $\partial_i^- I$) the full subcategory of I_+/i (resp. $i\backslash I_-$) with set of objects $Ob(I_+/i) - \{(i, 1_i)\}$ (resp. $Ob(i\backslash I_-) - \{(i,1_i)\}$)

Le C be a cocomplete category. For i an object of I define the <u>latching object functor</u>

$$L_i : C^I \longrightarrow C$$

by

$$L_i X = \underset{\substack{\partial_i^+ I \\ (i', i' \to i)}}{\varinjlim} X(i')$$

and the <u>latching object natural transformation</u>

$$L_i \longrightarrow i^*$$

by the universal property of colimits

$$L_i X = \underset{\substack{\partial_i^+ I \\ (i', i' \to i)}}{\varinjlim} X(i') \longrightarrow X(i) = i^*(X)$$

applied to the cocone

$$\left(X(i') \longrightarrow X(i)\right)_{i' \to i \,\in\, Ar(\partial_i^+ I)}$$

Dually, if C is a complete category, define the <u>matching object functor</u>

$$M_i : C^I \longrightarrow C \quad , \quad i \in Ob\, I,$$

by

$$M_i X = \varprojlim_{\substack{\partial_i^- I \\ (i', i' \to i)}} X(i')$$

and the <u>matching object natural transformation</u>

$$i^* \longrightarrow M_i$$

$$i^* X = X(i) \longrightarrow \varprojlim_{\substack{\partial_i^- I \\ (i', i \to i')}} X(i')$$

Fix now a complete and cocomplete closed model category (C, W, Cof, Fib). A <u>Reedy cofibration</u> (resp. a <u>Reedy fibration</u>) in C^I is a map $X \to Y$ of C^I such that for every object i of I the map

$X(i) \coprod_{L_i X} L_i Y \to Y(i)$ deduced from the commutative square

$$\begin{array}{ccc} L_i X & \longrightarrow & L_i Y \\ \downarrow & & \downarrow \\ X(i) & \longrightarrow & Y(i) \end{array}$$

(resp.

$X(i) \to Y(i) \times_{M_i Y} M_i(X)$ deduced from the commutative square

$$\begin{array}{ccc} X(i) & \longrightarrow & M_i X \\ \downarrow & & \downarrow \\ Y(i) & \longrightarrow & M_i Y \end{array}$$)

is a cofibration (resp a fibration) of C

A Reedy weak equivalence in C^I is a pointwise weak equivalence

<u>Theorem</u> The category C^I endowed with Reedy weak equivalences, Reedy cofibrations and Reedy fibrations is a closed model category

The closed model category structure on C^I of the preceeding theorem is known as the <u>Reedy model structure</u>

<u>Some properties of the Reedy model structure</u>

a) An arrow $X \to Y$ of C^I is a trivial cofibration (resp. a trivial fibration) if and only if for every object i of I the canonical map

$$X(i) \coprod_{L_i X} L_i Y \longrightarrow Y(i) \quad (\text{resp. } X(i) \longrightarrow Y(i) \times_{M_i Y} M_i X)$$

is a trivial cofibration (resp. a trivial fibration).

b) Reedy cofibrations, trivial cofibrations, fibrations, trivial fibrations are respectively pointwise cofibrations, trivial cofibrations, fibrations, trivial fibration the converse being false in general

c) If $X \to Y$ is an arrow in C^I which is a Reedy cofibration (resp. trivial cofibration) then for every object i of I, $L_i X \to L_i Y$ is a cofibration (resp. trivial cofibration). Dually if $X \to Y$ is a Reedy fibration (resp. trivial fibration) then $M_i X \to M_i Y$ is a fibration resp a trivial fibration.

A Reedy category is called <u>left connected</u> if for every object i of I the category $\partial_i^- I$ is connected (possibly empty).

<u>Proposition</u> Let (I, I_+, I_-) a left connected Reedy category and (C, W, Cof, Fib) a complete and cocomplete closed model category. If we endow C^I with the Reedy model structure then the pair of functors

$$C^I \xrightarrow{\varinjlim_I} C \qquad\qquad C \xrightarrow{\Delta_I} C^I$$

$$F \longmapsto \varinjlim F \qquad\qquad X \longmapsto \text{constant functor of value } X$$

is a Quillen adjunction and in particular $\varinjlim_I$ has a total left derived functor $\mathbb{L}\varinjlim_I \simeq \underrightarrow{holim}_I$

<u>Proof</u> Observe that if $X \to Y$ is an arrow in C and i an object of i, as the category $\partial_i^- I$ is connected, the morphism

$$\begin{array}{ccc} M_i(\Delta_I(X)) & \longrightarrow & M_i(\Delta_I(Y)) \\ \| & & \| \\ \varprojlim \Delta_I(X)|\partial_i^- I & & \varprojlim \Delta_I(X)|\partial_i^- I \end{array}$$

is either an isomorphism of final objects (if $\partial_i^- I$ is empty) or isomorphic to the arrow $X \to Y$, and the canonical map

$$\begin{array}{ccc} X & \longrightarrow & Y \times_{M_i(\Delta_I(X))} M_i(\Delta_I(X)) \\ \| & & \| \\ \Delta_I(X)(i) & & \Delta_I(Y)(i) \end{array}$$

either isomorphic to $X \to Y$ or an isomorphism

So if $X \to Y$ is a fibration (resp trivial fibration) of C, then $\Delta_I(X) \longrightarrow \Delta_I(Y)$ is a Reedy fibration (resp. trivial fibration) of C^I.

The notion of right connected Reedy category is defined dually.

If I is a direct category, by definition (I, I_+, I_-), where
$$I_+ = I \quad \text{and} \quad I_- = I_0 = \text{discrete category with same objects as } I,$$
is a Reedy category, and every object i de I, $\partial_i^- I$ is empty. In this case we have even better: For every model category C and any morphism $X \to Y$ of C^I, $M_i X \to M_i Y$ is an isomorphism of final objects and the canonical map
$$X(i) \longrightarrow Y(i) \times_{M_i Y} M_i X$$
is isomorphic to the map $X(i) \to Y(i)$. So in this case Reedy fibrations are exactly pointwise fibrations, so:

<u>Proposition</u> Let I a direct category and (C, W, Cof, Fib) a closed model category. If C is cocomplete then C is projectively I-exponential