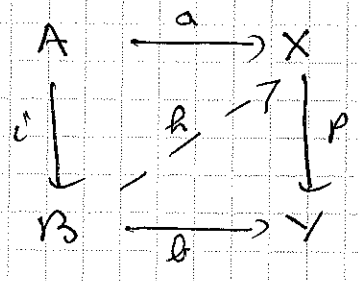


Model categories

1. Let C be a category and $A \xrightarrow{i} B, X \xrightarrow{p} Y$ two arrows of C . We will say that i has the left lifting property with respect to p or that p has the right lifting property with respect to i if for every commutative (solid) square



there exists a lifting $h: B \rightarrow X$ such that $a = h i$ and $\theta = p h$

i.e. if the map

$$\begin{array}{ccc}
 \text{Hom}(B, X) & \longrightarrow & \text{Hom}(A, X) \times_{\text{Hom}(A, Y)} \text{Hom}(B, Y) \\
 h & \longmapsto & (h i, p h)
 \end{array}$$

is surjective. If K is a class of arrows of C we denote by $L(K)$ (resp. $r(K)$) the class of arrows of C satisfying the left (resp. right) lifting property with respect to all arrows in K . The following definition is due to Quillen.

A closed model category is a category C endowed with three classes of arrows W, Cof, Fib satisfying the following conditions:

CM1] the category C has finite limits and finite colimits;

CM2] the class of arrows W satisfy the two out of three property: in a commutative triangle of C , if two of the three arrows are in W , so is the third

CM3] $W, \text{Cof}, \text{Fib}$ are stable under retracts

CM4] $\text{Cof} = \ell(\text{Fib} \cap W)$, $\text{Fib} = \mathcal{r}(\text{Cof} \cap W)$

CM5] Every arrow f in C can be decomposed as $f = pj$ and $f = qj$ with $p \in \text{Fib}$, $j \in \text{Cof} \cap W$, $q \in \text{Fib} \cap W$, $j \in \text{Cof}$

Arrows in W are called weak equivalences, arrows in Cof , cofibrations, in $\text{Cof} \cap W$ trivial cofibrations, in Fib fibrations, in $\text{Fib} \cap W$ trivial fibrations. If we denote by ϕ (resp. $*$) an initial (resp. final) object of C (CM1), an object X of C is called cofibrant (resp. fibrant) if $\phi \rightarrow X$ is a cofibration (resp. $X \rightarrow *$ a fibration)

It can be "easily" proved that in a closed model category the following relations hold

$$\text{Cof} = \ell(\text{Fib} \cap W), \quad \text{Fib} = \mathcal{r}(\text{Cof} \cap W)$$

$$\text{Cof} \cap W = \ell(\text{Fib}), \quad \text{Fib} \cap W = \mathcal{r}(\text{Cof})$$

$W = (\text{Fib} \cap W) \circ (\text{Cof} \cap W)$ (an arrow of C is in W if and only if it can be written as $f = qj$ with $q \in \text{Fib} \cap W$ and $j \in \text{Cof} \cap W$)

In particular any two out of the three classes $W, \text{Cof}, \text{Fib}$ determine the third one

Examples

1) $C = \text{Top} =$ category of topological spaces and continuous maps

$W =$ ordinary weak equivalences

$\text{Fib} =$ Serre fibrations $= \mathcal{r}(\mathbb{I})$

$$\mathbb{I} = \left\{ [0,1]^m \hookrightarrow [0,1]^{m+1} \mid m \geq 0 \right\}$$

$$(k_1, \dots, k_m) \mapsto (0, k_1, \dots, k_m)$$

$\text{Cof} = \mathcal{L}(\text{Fib} \cap W)$

2) $C = \text{S Sets} =$ category of simplicial sets

$\text{Cof} =$ monomorphisms

$\text{Fib} =$ Kan fibrations

$W = \mathcal{r}(\text{Cof}) \circ \mathcal{L}(\text{Fib})$ (= maps of simplicial sets whose topological realization is a homotopy equivalence of CW-complexes)

3) $C = C(\mathcal{A}) =$ category of unbounded cochain complexes of \mathcal{A} , \mathcal{A} being a Grothendieck category, i.e. a cocomplete abelian category, with exact filtered colimits and a generator

$W =$ quasi-isomorphisms

$\text{Cof} =$ monomorphisms

$\text{Fib} = \mathcal{r}(\text{Cof} \cap W)$

4) many others ...

2. The following "Ken Brown" lemma is very useful.

Lemma Let $(C, W, \text{Cof}, \text{Fib})$ be a closed model category, (C', W') a localizer such that W' satisfies the two out of three property, and $F: C \rightarrow C'$ a functor carrying trivial cofibrations between cofibrant objects of C to arrows in W' . Then F carries all weak equivalences between cofibrant objects of C to arrows in W' .

Theorem (Existence of derived functors) Let $(C, W, \text{Cof}, \text{Fib})$ be a closed model category, D a category and $F: C \rightarrow D$ a functor carrying trivial cofibrations between cofibrant objects of C to isomorphisms of D . Then the functor F admits an absolute left derived functor

Sketch of the construction of the derived functor (this can be skipped and go directly to corollary page 9)

For every object X of C choose by CM5 a decomposition

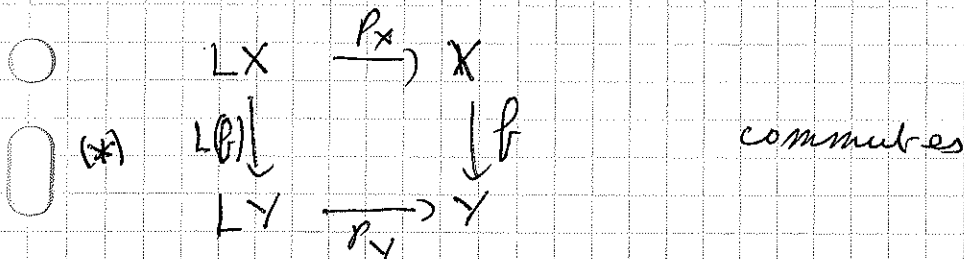
$$\emptyset \rightarrow LX \xrightarrow{p_X} X$$

of $\emptyset \rightarrow X$ to a cofibration followed by a trivial fibration p_X . The object LX is then cofibrant. For every arrow $f: X \rightarrow Y$ of C choose by CM4 a lifting Lf

$$\begin{array}{ccc}
 \emptyset & \longrightarrow & LY \\
 \downarrow & \searrow^{Lf} & \downarrow p_Y \\
 Y & \xrightarrow{f} & Y \\
 \downarrow & \searrow^{p_X} & \downarrow \\
 LX & \xrightarrow{p_X} & Y
 \end{array}$$

$$p_Y Lf = f p_X$$

so that



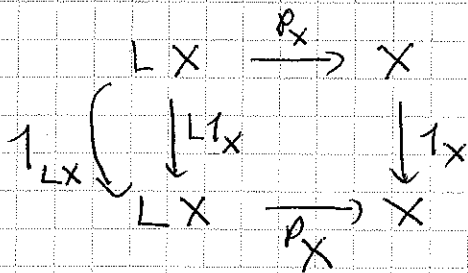
Define a functor $C \xrightarrow{LF} D$ by

$$\begin{aligned}
 X &\longmapsto F(LX) \\
 f &\longmapsto F(Lf)
 \end{aligned}$$

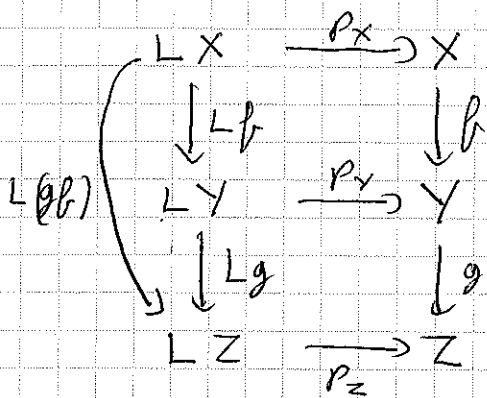
One has to verify

$$- F(L1_X) = 1_{F(LX)}$$

$$- F(Lg \circ Lf) = F(Lg) \circ F(Lf) \quad X \xrightarrow{p} Y \xrightarrow{q} Z$$



$$p_x \circ L1_X = 1_X \circ p_x = p_x \circ 1_{LX}$$



$$\begin{aligned}
 p_z \circ Lg \circ Lf &= g \circ p_y \circ Lf \\
 &= g \circ f \circ p_x = p_z \circ L(g \circ f)
 \end{aligned}$$

Therefore to conclude in both cases it's enough to prove

Lemma Let $A \begin{smallmatrix} \xrightarrow{u} \\ \xrightarrow{v} \end{smallmatrix} B \xrightarrow{p} T$ a diagram in C such that $pu = pv$, p is a trivial fibration and A cofibrant. Then $F(u) = F(v)$

Proof Choose by CM5 a decomposition

$$A \amalg A \xrightarrow{(\delta^0, \delta^1)} \tilde{A} \xrightarrow[\sim]{s} A$$

of the coequalizer $A \amalg A \rightarrow A$ to a cofibration followed by a trivial fibration. As A and \tilde{A} are cofibrant and s a weak equivalence the Ken Brown lemma implies that $F(s)$ is an isomorphism and as $s\delta^0 = 1_A = s\delta^1$ this implies that $F(\delta^0) = F(\delta^1)$. By CM4 there is a lifting h

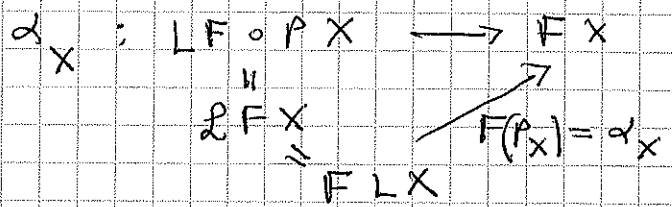
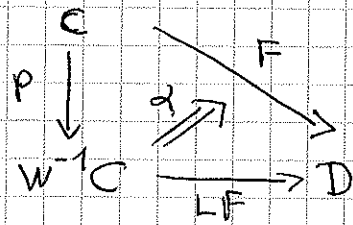
$$\begin{array}{ccc} A \amalg A & \xrightarrow{(u, v)} & B \\ \downarrow (\delta^0, \delta^1) & \searrow h & \downarrow p \\ \tilde{A} & \xrightarrow[pus = pvs]{} & T \end{array}$$

so that $u = h\delta^0, v = h\delta^1$ which implies $F(u) = F(h)F(\delta^0) = F(h)F(\delta^1) = F(v)$ \square

The commutativity of the square (*) and CM2 imply that if $f: X \rightarrow Y$ is a weak equivalence of C , $Lf: LX \rightarrow LY$ is also a weak equivalence, and as LX and LY are cofibrant the Ken Brown lemma implies that $L(F) = F(Lf)$ is an isomorphism. So the functor $Lf: C \rightarrow D$ induces a functor $LF: W^{-1}C \rightarrow D$ such

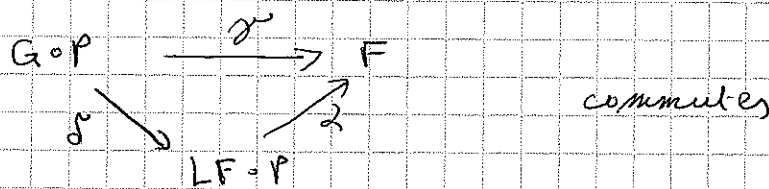
$$\begin{array}{ccc} C & & D \\ p \downarrow & \searrow Lf & \\ H_0 C = W^{-1}C & \xrightarrow{LF = Lf} & D \end{array} \quad \text{that } Lf = LF \circ p$$

The commutativity of the squares (*) implies that if we define α by $\alpha_x = F(p_x)$ then $\alpha: LF \circ P \rightarrow F$ is a natural transformation

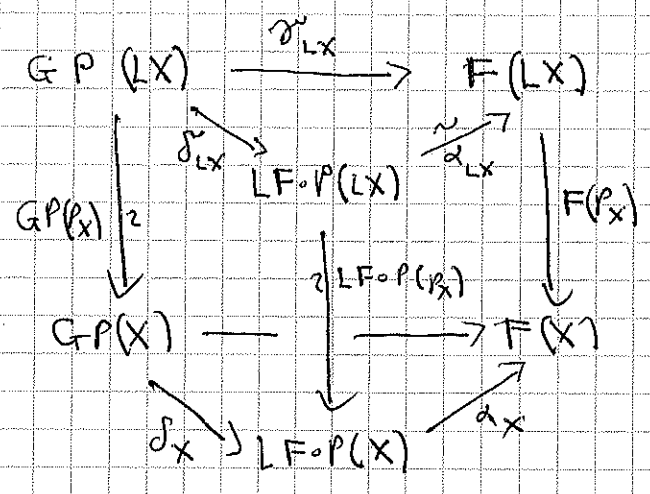


Let us prove that (LF, α) is a left derived functor of F . First observe that by the Ker Brown lemma if X is cofibrant, then $\alpha_x = F(p_x)$ is an isomorphism

Let $\gamma: G \circ P \rightarrow F$ be a natural transformation where $G: W^{-1}C \rightarrow D$ is a functor. By the precise universal property of the localization, in order to prove that (LF, α) is a left derived functor, it is enough to prove that there exists a unique natural transformation $\delta: G \circ P \rightarrow LF \circ P$ so that the triangle



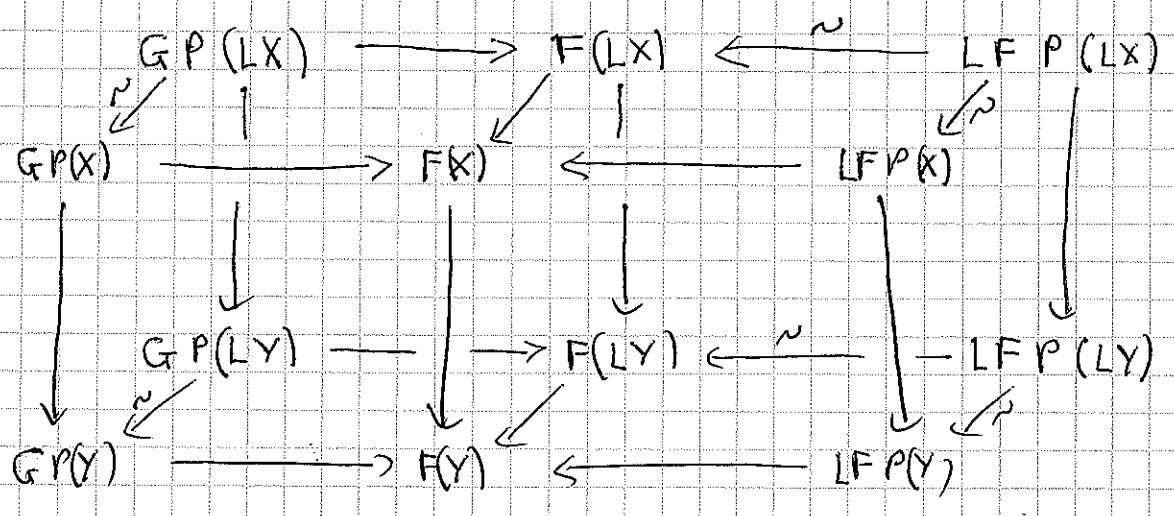
Unicity comes from the following commutative prism and the fact that α_x and $G(p_x)$ are invertible



Existence follows by defining δ by this prism.

$$\delta_X = LF.P(P_X) \alpha_{LX}^{-1} \alpha_{LX} (GP(P_X))^{-1}$$

and observing that the relation $\gamma = \alpha \delta$ is a consequence of the commutativity of the prism and naturality consequence of the following commutative diagram for $X \rightarrow Y$ an arrow in \mathcal{C}



(and the invertibility of the arrows $\xrightarrow{\sim}$)

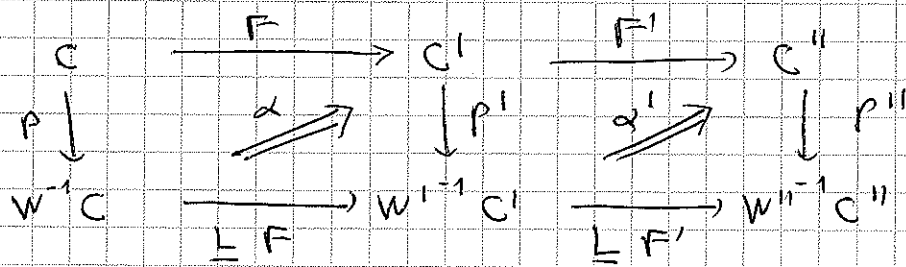
It remains to prove that this derived functor is absolute. This is immediate if we observe that for any functor $H: D \rightarrow D'$

The functor $F' = HF : C \rightarrow D'$ satisfies the hypotheses of the theorem, and if we apply the same construction (with the same choice of L_X and L_f), we obtain a ^{left} derived functor (LF', α') of F' such that $LF' = H \circ LF$ and $\alpha' = H \circ \alpha$.

Corollary Let $(C, W, \text{Cof}, \text{Fib})$ be a closed model category, (C', W') a localizer such that W' satisfies the two out of three property, and $F : C \rightarrow C'$ a functor carrying trivial cofibrations between cofibrant objects of C to arrows in W' . Then the functor F admits an absolute total left derived functor (LF, α)

$$\begin{array}{ccc}
 C & \xrightarrow{F} & C' \\
 p \downarrow & \nearrow \alpha & \downarrow p' \\
 \text{Ho } C = W^{-1}C & \xrightarrow{LF} & W'^{-1}C'
 \end{array}$$

Theorem (Composition of derived functors) Let $(C, W, \text{cof}, \text{Fib})$, $(C', W', \text{cof}', \text{Fib}')$ and $(C'', W'', \text{cof}'', \text{Fib}'')$ be three closed model categories, $F: C \rightarrow C'$, $F': C' \rightarrow C''$ two functors, and suppose that F and F' carries trivial cofibrations between cofibrant objects to weak equivalences and that F carries cofibrant objects to cofibrant objects. If $(\underline{L}F, \alpha)$, $(\underline{L}F', \alpha')$ are total left derived functors respectively of F, F' (which exist and are absolute by the existence theorem), then $(\underline{L}F' \circ \underline{L}F, \alpha' * F)(\underline{L}F' * \alpha)$ is



an absolute total left derived functor of $F'F$.

Theorem (Adjunction) Let $(C, W, \text{Cof}, F, \theta)$ and $(C', W', \text{Cof}', F', \theta')$ two closed model categories,

$$F: C \longrightarrow C', \quad G: C' \longrightarrow C$$

a pair of adjoint functors, and suppose that F (resp. G) carries trivial cofibrations (resp. trivial fibrations) between cofibrant (resp. fibrant) objects to weak equivalences. If $(\underline{L}F, \alpha)$, (resp. $(\underline{R}G, \beta)$) is a total left (resp. right) derived functor of F (resp. G) (which exists and is absolute by the existence theorem (resp. the dual existence theorem)) then $\underline{L}F, \underline{R}G$ is a pair of adjoint functors.

This theorem is now just an application of the abstract adjunction theorem. It is much stronger than its usual formulation in terms of Quillen's adjunctions (a notion that was introduced by Quillen).

A Quillen's adjunction is a pair of adjoint functors between closed model categories satisfying the equivalent conditions of the following proposition (and if (F, G) is a Quillen adjunction, F is called a left Quillen functor and G a right Quillen functor).

Proposition Let (F, G) be an adjoint pair of functors between closed model categories. The following conditions are equivalent:

- i) F carries trivial cofibrations to trivial cofibrations and G carries trivial fibrations to trivial fibrations;
- ii) F carries trivial cofibrations to trivial cofibrations and cofibrations to cofibrations;
- iii) G carries trivial fibrations to trivial fibrations and fibrations to fibrations;
- iv) F carries cofibrations to cofibrations and G carries fibrations to fibrations.

Corollary (of the theorem) Let (F, G) a Quillen adjunction between closed model categories.

Then $\underline{L}F$ and $\underline{R}G$ exist, are absolute and $(\underline{L}F, \underline{R}G)$ is a pair of adjoint functors.

A Quillen's adjunction is called a Quillen's equivalence if the equivalent conditions of the following proposition are satisfied

Proposition Let $(F: C \rightarrow C', G: C' \rightarrow C)$ a Quillen adjunction. The following conditions are equivalent:

- i) $\underline{L}F$ is an equivalence of categories;
- ii) $\underline{R}G$ is an equivalence of categories;
- iii) for every cofibrant object X of C and every fibrant object X' of C' a map $FX \rightarrow X'$ of C' is a weak equivalence if and only if the associated map $X \rightarrow GX'$ of C is a weak equivalence.

A model category $(\mathcal{C}, W, \text{Cof}, \text{Fib})$ is called projectively I-exponential or more simply I-projective, where I is a small category, if \mathcal{C}^I endowed with pointwise weak equivalences, pointwise fibrations, and cofibrations defined by the left lifting property with respect to trivial fibrations. This structure is called the projective model structure on \mathcal{C}^I . The model category is called projectively exponential or more simply, projective if it is I-projective for every small category I . The notions of I-injective and injective closed model category are defined dually.

Theorem Let $(\mathcal{C}, W, \text{Cof}, \text{Fib})$ a projective model category and stable by arbitrary small products, then the localizer (\mathcal{C}, W) is cocomplete.

The proof goes through several lemmas. In what follows, fix a closed model category $(\mathcal{C}, W, \text{Cof}, \text{Fib})$ such that \mathcal{C} is cocomplete.

Lemma 1 If \mathcal{C} is I-projective and \mathcal{C}^I endowed with the projective model structure, then

$$\begin{array}{ccc} \mathcal{C}^I & \xrightarrow{\text{colim}_{\rightarrow I}} & \mathcal{C} \\ \downarrow & & \downarrow \Delta_I \\ \mathcal{C} & \xrightarrow{\Delta_I} & \mathcal{C}^I \end{array}$$

$$F \mapsto \text{colim}_{\rightarrow} F \quad X \mapsto \text{constant functor of value } X$$

is a Quillen adjunction. In particular, the homotopy colimit functor $\text{holim}_{\rightarrow I} : W_I^{-1} \mathcal{C}^I \rightarrow W^{-1} \mathcal{C}$ exists and is the total left derived functor of $\text{colim}_{\rightarrow I}$ ($\text{holim}_{\rightarrow I} = \underline{L} \text{colim}_{\rightarrow I}$)

Proof As weak equivalences and fibrations are defined pointwise in C^I , the functor Δ_I carries fibrations (resp. trivial fibrations) of C to fibrations (resp. trivial fibrations) of C^I .

More generally,

Lemma 2 Let $u: I \rightarrow J$ a functor between small categories, If C is I and J -projective and C^I, C^J endowed with the projective model structures, then

$$C^I \xrightarrow{u_!} C^J, \quad C^J \xrightarrow{u^*} C^I$$

is a Quillen adjunction. In particular the homotopy left Kan-extension functor

$u_{!R}: W_I^{-1} C^I \rightarrow W_J^{-1} C^J$ exists and is the total left derived functor of $u_!$ ($u_{!R} = \underline{L} u_!$)

The proof is similar to the proof of lemma 1

Lemma 3 Let $u: I \rightarrow J$ be a local isomorphism between small categories (which means that for every object i of I the functor $I/i \rightarrow J/u(i)$ induced by u is an isomorphism, or equivalently that u is a Grothendieck fibration with discrete fibers). If C admits arbitrary small products, if C is I and J -projective and if C^I and C^J are endowed with the projective model structure, then u^* is a left Quillen functor.

Proof As u^* carries weak equivalences to weak equivalences it's enough to prove that u^* carries cofibrations to cofibrations. As u is a local isomorphism, it can be easily verified that for every object j of J the functor

$$\coprod_{u(d_0)=j} \mathcal{C}_{j \setminus I} \longrightarrow \mathcal{C}_{j \setminus I}$$

$$(i, u_0 \xrightarrow{f} d) \mapsto (i, u_0 \xrightarrow{u(f)} u(d))$$

is an isomorphism. Remark that $(i_0, i_0 \xrightarrow{1_{i_0}} i_0)$ is an initial object of $\mathcal{C}_{j \setminus I}$. So, as \mathcal{C} admits arbitrary small products, the right Kan extension functor $f_* : \mathcal{C}^I \rightarrow \mathcal{C}^J$ exists and is defined by

$$u_* (F)(j) = \lim_{\leftarrow \substack{j \setminus I \\ (i, j \rightarrow u(i))}} F(i) \simeq \lim_{\leftarrow \substack{\coprod_{u(d_0)=j} \mathcal{C}_{j \setminus I} \\ (i, u_0 \rightarrow d)}} F(d) \simeq \prod_{u(d_0)=j} F(d_0)$$

As trivial fibrations are stable by products, u_* carries trivial fibrations to trivial fibrations and as a consequence u^* carries cofibrations to cofibrations.

Proof of the theorem Suppose now that \mathcal{C} is cocomplete, projective and stable by small products. By lemmas 1, 2 the localizer (\mathcal{C}, w) admits homotopy colimits and more generally homotopy left Kan extensions. It remains to prove that every functor $u: I \rightarrow J$ between small categories, any $F: I \rightarrow \mathcal{C}$ and any object j of J the canonical map

$$\underline{L} u_*(F)(j) \xrightarrow{\text{holim}} \mathcal{C}_{j \setminus I} (F|_{(I/j)})$$

is an isomorphism in $W^{-1}C$, i.e. that

$$\bar{j}^* \underline{L} u_i \xleftarrow{\sim} \lim_{\rightarrow I/j} \underline{L} R^* \quad (1)$$

where $j: \mathcal{C} \rightarrow \mathcal{J}$ denotes also the functor from the point category \mathcal{C} to \mathcal{J} defined by the object j and $\alpha: I/j \rightarrow I$ the forgetful functor $(i, u(i) \rightarrow j) \mapsto i$

By the classical formula for left Kan extension functors we have a canonical isomorphism

$$(*) \quad j^* u_i \xleftarrow{\sim} \lim_{\rightarrow I/j} R^*$$

Observe that $\bar{j}^* = \underline{L} j^*$ and that by lemma 2 u_i is a left Quillen functor, so the pair \bar{j}^*, u_i satisfy the conditions of the theorem of composition of total left derived functors (it can be proved that j^* is also a left Quillen functor, ^(cf. lemma 4) but this is not needed here, as j^* carries weak equivalence to weak equivalence) and

$$\bar{j}^* \underline{L} u_i \simeq \underline{L} j^* \cdot \underline{L} u_i \simeq \underline{L} (j^* u_i)$$

On the other hand R is a local isomorphism and by lemmas 1 and 3 $\lim_{\rightarrow I/j}$ and R^* are left Quillen functors, so the composition theorem applies again and

$$\lim_{\rightarrow I/j} R^* \simeq \underline{L} \lim_{\rightarrow I/j} R^* \simeq \underline{L} (\lim_{\rightarrow I/j} R^*)$$

and (*) implies the theorem

Remark In order to prove that (1) is an isomorphism the only "projectivity" condition used is that \mathcal{C} is \mathbb{I} and \mathbb{I}/j -projective

Lemma 4 Let $(\mathcal{C}, \mathcal{W}, \text{Cof}, \text{Fib})$ a closed model category stable under arbitrary small products and I a small category. Suppose that \mathcal{C}^I is endowed with a closed model structure such that weak equivalences are defined pointwise and such that pointwise fibrations are fibrations. Then for every object i of I , if we denote also $i: e \rightarrow I$ the functor from the point category e to I defined by the object i , the functor i^* is a left Quillen functor. In particular any cofibration in \mathcal{C}^I is a pointwise cofibration (but the converse is not true)

Proof As \mathcal{C} is stable under arbitrary small products the right Kan extension functor i_* , right adjoint of i^* exists and is defined by

$$(i_* X)(i') = \lim_{\substack{i' \setminus e \\ i' \rightarrow i}} X = \prod_{\text{Hom}(i', e)} X \quad (*)$$

As i^* carries weak equivalences to weak equivalences, it is enough to prove that i^* carries cofibrations to cofibrations, or equivalently that i_* carries trivial fibrations to trivial fibrations. This is a consequence of the formula (*), and the stability of trivial fibrations by products.

Theorem Every cofibrantly generated closed model category is projectively exponential.

We recall the definition of a cofibrantly generated closed model category.

Let α an infinite cardinal. A set E (resp. a category I) is called α -small if $\text{card}(E) < \alpha$ (resp. if I is small and $\text{card } \text{Ar } I < \alpha$). An infinite cardinal α is called regular if α -small colimits (indexed by α -small categories) of α -small sets are α -small. Let α a regular cardinal and A a category. The category A is called α -filtered if for every α -small category I , any functor $F: I \rightarrow A$ has a cocone in A (where exists an object X in A and a natural transformation $F \rightarrow X$, where X denotes also the constant functor $I \rightarrow A$ of value X).

Let C a category stable under small α -filtered colimits (indexed by a small α -filtered category). An object X of C is called α -presentable if the functor $\text{Hom}_C(X, ?) : C \rightarrow \text{Sets}, T \mapsto \text{Hom}_C(X, T)$ preserves small α -filtered colimits. A object X of a category C is called presentable if there exists a regular cardinal α such that C is stable under small α -filtered colimits and X is α -presentable.

Let C a cocomplete category and \mathcal{J} a set of arrows of C . We say that \mathcal{J} allows the small object argument if the domains of arrows in \mathcal{J} are presentable

The small objects argument implies then that every arrow f in C can be written as

$$f = q_i, \quad q \in \text{rc}(\mathcal{J}), \quad i \in \text{cell}(\mathcal{J})$$

where $\text{cell}(\mathcal{J})$ denotes the class of arrows in C which are transfinite compositions of pushouts of arrows in \mathcal{J}

An easy formal lemma, known as the "retract lemma" implies then that $\text{rc}(\mathcal{J})$ is the class of retracts of arrows in $\text{cell}(\mathcal{J})$.

We will say that a closed model category $(C, W, \text{Cof}, \text{Fib})$ is cofibrantly generated if C is cocomplete and if there exist sets \mathcal{J}, \mathcal{I} , allowing the small object arguments such that

$$\text{Fib} = \text{rc}(\mathcal{I}) \quad \text{and} \quad \text{Fib} \cap W = \text{rc}(\mathcal{J})$$

or equivalently

$$\text{Cof} = \text{rc}(\mathcal{J}), \quad \text{Cof} \cap W = \text{rc}(\mathcal{I})$$

We will say that \mathcal{J} (resp. \mathcal{I}) generates the cofibrations (resp. the fibrations) and that $(\mathcal{J}, \mathcal{I})$ generates the cofibrantly generated closed model category C .

To prove the theorem on page 18, we need the following lemma

Lemma (Quans?) Let $(\mathcal{C}, W, \text{Cof}, \text{Fib})$ be cofibrantly generated closed model category, generated by a pair $(\mathcal{I}, \mathcal{J})$ and let

$$F: \mathcal{C} \rightarrow \mathcal{C}', \quad G: \mathcal{C}' \rightarrow \mathcal{C}$$

a pair of adjoint functors. Suppose:

i) \mathcal{C}' is complete and cocomplete;

ii) the sets $F(\mathcal{I})$ and $F(\mathcal{J})$ allows the small object argument

iii) $G(\text{Im}(F\mathcal{J})) \subset W$

Define

$$W' = G^{-1}(W), \quad \text{Fib}' = G^{-1}(\text{Fib}), \quad \text{Cof}' = \ell(\text{Fib}' \cap W')$$

Then $(\mathcal{C}', W', \text{Cof}', \text{Fib}')$ is a cofibrantly generated closed model category generated by $(F(\mathcal{I}), F(\mathcal{J}))$

Proof Axioms CM1, CM2, CM3 are clear, and 1/2 of CM4 is by definition. Observe that

$$\text{Fib}' = G^{-1}(\text{Fib}) = G^{-1}(\text{Im}(\mathcal{J})) = \text{Im}(F(\mathcal{J}))$$

and

$$\text{Fib}' \cap W' = G^{-1}(\text{Fib} \cap W) = G^{-1}(\text{Im}(\mathcal{J})) = \text{Im}(F(\mathcal{J}))$$

In particular

$$\text{Cof}' = \ell(\text{Im}(F(\mathcal{J})))$$

and

$$\ell(\text{Im}(F(\mathcal{J}))) = \ell(\text{Fib}') \subset \ell(\text{Fib}' \cap W') = \text{Cof}'.$$

As the condition (iii) implies that $\ell(\text{Im}(F(\mathcal{J}))) \subset W'$

we have

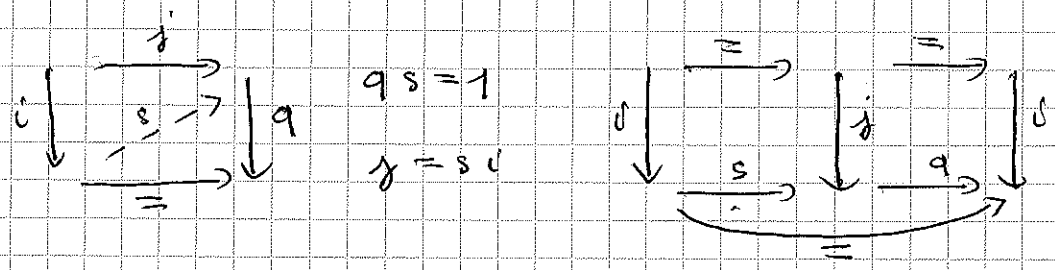
$$\ell(\text{Im}(F(\mathcal{J}))) \subset \text{Cof}' \cap W'$$

Let us prove the other inclusion. Let i in $\text{Cof}' \cap W'$

As $F(\mathcal{J})$ allows the small object argument, factorize

$$i = qj, \quad j \in \text{lr}(F(\mathcal{J})), \quad q \in \text{r}(F(\mathcal{J})) = F\mathcal{O}'$$

As i and j are in W' so is q , so $q \in F\mathcal{O}' \cap W'$ and i has the left lifting property with respect to q . The retract lemma implies that i is a retract of j and so $i \in \text{lr}(F(\mathcal{J}))$



We have proved that

$$\text{Cof}' \cap W' = \text{lr}(F(\mathcal{J})) = \ell(F\mathcal{O}')$$

and

$$\text{Cof}' = \text{lr}(F(\mathcal{J})) = \ell(F' \cap W')$$

and this ends the proof as CMS is now a consequence of the fact that $F(\mathcal{J})$ and $F(\mathcal{J}')$ allow the small object argument. \square

We can now prove a more precise version of the theorem on page 18:

Proposition Let $(C, W, \text{Cof}, \text{Fib})$ be a cofibrantly generated closed model category generated by the pair $(\mathcal{I}, \mathcal{J})$, and I a small category. Then $(C, W_I, \text{Cof}_I, \text{Fib}_I)$, where W_I (resp. Fib_I) is the class of pointwise weak equivalences (resp. fibrations) and $\text{Cof}_I = \ell(\text{Fib}_I \cap W_I)$

is a cofibrantly generated closed model category generated by the pair $(\mathcal{J}_I, \mathcal{F}_I)$ where

$$\mathcal{J}_I = \bigcup_{i \in \text{Ob } I} i_!(\mathcal{J}) \quad , \quad \mathcal{F}_I = \bigcup_{i \in \text{Ob } I} i_!(\mathcal{F}) \quad ,$$

where $i_! : \mathcal{C} \rightarrow \mathcal{C}^I$ is the left adjoint of the functor $i^* : \mathcal{C}^I \rightarrow \mathcal{C}$, $X \mapsto X(i)$.

Proof Observe that the natural bijection

$$\text{Hom}_{\mathcal{C}^I}(i_!X, Y) \simeq \text{Hom}_{\mathcal{C}}(X, Y(i))$$

implies that if X is a presentable object of \mathcal{C} then $i_!X$ is a presentable object of \mathcal{C}^I , so that \mathcal{J}_I and \mathcal{F}_I allow the small object argument

The particular case where I is a discrete category is an easy exercise left to the reader, and gives simply the product model structure where cofibrations are also defined pointwise

The general case follows from the discrete case by the preceding lemma. Let I_0 denote the discrete category whose objects are the objects ~~are~~ the objects of I and $R : I_0 \rightarrow I$ the inclusion. In order to apply the preceding lemma and conclude the only thing one has to verify is the condition (iii) of this lemma for the adjunction

$$\mathcal{C} \xrightarrow{R_!} \mathcal{C}^{I_0} \quad , \quad \mathcal{C}^{I_0} \xrightarrow{R^*} \mathcal{C}$$

Let i an object of I and $i_0: e \rightarrow I_0$, $d: e \rightarrow I$ the functors defined by the object i .

$$\begin{array}{ccc}
 e & \xrightarrow{i_0} & I_0 \xrightarrow{k} I \\
 & \searrow & \nearrow \\
 & & i
 \end{array}$$

We have to prove that

$$R^* \text{ or } R_! i_0!(\mathcal{F}) \in W_{I_0}$$

which simply means that

$$\text{or } (i_!(\mathcal{F})) \in W_I.$$

In order to prove this, it is enough to prove the stronger condition that $\text{or } (i_!(\mathcal{F}))$ is contained in the class of pointwise trivial fibrations. As this class is stable under pushout, transfinite composition and retracts it is enough (by the small object argument) to prove that $i_!(\mathcal{F})$ is contained in this class. But an explicit calculation shows that for every object X of C

$$(i_! X)(i') = \coprod_{\text{Hom}(i_0, d')} X, \quad i' \in \text{ob } I,$$

and as trivial fibrations of C are stable under small coproducts this finishes the proof.

Reedy categories

1

A Reedy category is a triple (I, I_+, I_-) , where I is a small category and I_+, I_- two (non full) subcategories of I satisfying the following conditions:

R1 Every arrow k of I can be uniquely decomposed as $k = k_+ k_-$ with k_+ (resp. k_-) in I_+ (resp. I_-).

R2 There exists a function λ from the set of objects of I to a well ordered set such that if $k: i \rightarrow i'$ is a non identical map of I_+ (resp. I_-) then $\lambda(i') > \lambda(i)$ (resp. $\lambda(i') < \lambda(i)$).

Elementary properties

a) The subcategories I_+ and I_- have same set of objects as I .

b) The intersection of I_+ and I_- is reduced to the identities of I .

c) Every split monomorphism is in I_+ ; every split epimorphism is in I_- .

d) The only isomorphisms of I are identities.

Examples 1) The category Δ of simplices whose objects are the non-empty ordered sets

$$[n] = \{0 < 1 < \dots < n\}, \quad n \geq 0,$$

and arrows the non decreasing maps, endowed with the subcategories Δ_+ and Δ_- where Δ_+ (resp. Δ_-) is the subcategory of Δ with same

objects and whose arrows are the monomorphisms (resp. epimorphisms) of Δ , is a Reedy category.

2) If (I, I_+, I_-) is a Reedy category, then $(I^{\circ}, I_-^{\circ}, I_+^{\circ})$ is a Reedy category.

3) If (I, I_+, I_-) and (J, J_+, J_-) are two Reedy categories, then $(I \times J, I_+ \times J_+, I_- \times J_-)$ is a Reedy category.

4) If (I, I_+, I_-) is a Reedy category and X a presheaf on I , then $(I/X, I_+/X, I_-/X)$ is a Reedy category.

A direct category I is a small category I such that (I, I, I_0) , where I_0 is the discrete subcategory of I with same objects as I and only identities as morphisms, is a Reedy category. This means simply that there exists a function λ from the set of objects of I to a well ordered set such that if $f: i \rightarrow i'$ is a non identity arrow of I , then $\lambda(i) < \lambda(i')$. The notion of inverse category is defined dually.

If (I, I_+, I_-) is a Reedy category, then I_+ is a direct category and I_- an inverse category.

If I is a finite category the following conditions are equivalent:

- I is a direct category; (b) I is an inverse category;
- the free category generated by the graph of non identity arrows of I is finite;
- the nerve of I is a finite simplicial set (having a finite set of non degenerate simplices).

Fix a Reedy category (I, I_+, I_-)

For an object i of I denote $\mathcal{D}_i^+ I$ (resp $\mathcal{D}_i^- I$) the full subcategory of I_+ / i (resp. $i \setminus I_-$) with set of objects $\text{Ob}(I_+ / i) = \{i, i_0\}$ (resp. $\text{Ob}(i \setminus I_-) = \{i, i_0\}$)

Let C be a cocomplete category. For i an object of I define the latching object functor

$$L_i : C^I \longrightarrow C$$

by
$$L_i X = \text{colim}_{\substack{\mathcal{D}_i^+ I \\ (i', i' \rightarrow i)}} X(i')$$

and the latching object natural transformation

$$L_i \longrightarrow i^*$$

by the universal property of colimits

$$L_i X = \text{colim}_{\substack{\mathcal{D}_i^+ I \\ (i', i' \rightarrow i)}} X(i') \longrightarrow X(i) = i^*(X)$$

applied to the cocone

$$(X(i') \longrightarrow X(i))_{i' \rightarrow i \in \text{Ar}(\mathcal{D}_i^+ I)}$$

Dually, if C is a complete category, define the matching object functor

$$M_i : C^I \longrightarrow C, \quad i \in \text{Ob} I,$$

by

$$M_i X = \lim_{\leftarrow \substack{\mathcal{I} \\ (i', d' \rightarrow i)}} X(i')$$

and the matching object natural transformation

$$i^* \longrightarrow M_i$$

$$i^* X = X(i) \longrightarrow \lim_{\leftarrow \substack{\mathcal{I} \\ (i', d' \rightarrow i)}} X(i')$$

Fix now a complete and cocomplete closed model category $(C, W, \text{Cof}, \text{Fib})$. A Reedy cofibration (resp. a Reedy fibration) in C^I is a map $X \rightarrow Y$ of C^I such that for every object i of I the map

$$\begin{array}{ccc} X(i) \amalg_{L_i X} L_i Y \rightarrow Y(i) & \text{deduced from the commutative square} & \begin{array}{ccc} L_i X & \longrightarrow & L_i Y \\ \downarrow & & \downarrow \\ X(i) & \longrightarrow & Y(i) \end{array} \end{array}$$

(resp.

$$\begin{array}{ccc} X(i) \rightarrow Y(i) \times_{M_i Y} M_i X & \text{deduced from the commutative square} & \begin{array}{ccc} X(i) & \longrightarrow & M_i X \\ \downarrow & & \downarrow \\ Y(i) & \longrightarrow & M_i Y \end{array} \end{array}$$

is a cofibration (resp a fibration) of C .
 A Reedy weak equivalence in C^I is a pointwise weak equivalence

Theorem The category C^I endowed with Reedy weak equivalences, Reedy cofibrations and Reedy fibrations is a closed model category

The closed model category structure on C^I of the preceding theorem is known as the Reedy model structure

Some properties of the Reedy model structure

a) An arrow $X \rightarrow Y$ of C^I is a trivial cofibration (resp. a trivial fibration) if and only if for every object i of I the canonical map

$$X(i) \amalg_{L_i X} L_i Y \rightarrow Y(i) \quad (\text{resp. } X(i) \rightarrow Y(i) \times_{M_i Y} M_i X)$$

is a trivial cofibration (resp. a trivial fibration).

b) Reedy cofibrations, trivial cofibrations, fibrations, trivial fibrations are respectively pointwise cofibrations, trivial cofibrations, fibrations, trivial fibration the converse being false in general

c) If $X \rightarrow Y$ is an arrow in C^I which is a Reedy cofibration (resp. trivial cofibration) then for every object i of I , $L_i X \rightarrow L_i Y$ is a cofibration (resp. trivial cofibration). Dually if $X \rightarrow Y$ is a Reedy fibration (resp. trivial fibration) then $M_i X \rightarrow M_i Y$ is a fibration (resp. a trivial fibration).

A Reedy category is called left connected if for every object i of I the category $\mathcal{D}_i^- I$ is connected (possibly empty).

Proposition Let (I, I_+, I_-) a left connected Reedy category and $(C, W, \text{Cof}, \text{Fib})$ a complete and cocomplete closed model category. If we endow C^I with the Reedy model structure then the pair of functors

$$\begin{array}{ccc} C^I & \xrightarrow{\text{lim}_{\rightarrow I}} & C & & C & \xrightarrow{\Delta_I} & C^I \\ F \dashv & & \text{lim}_{\rightarrow} F & & X \dashv & & \text{constant functor of value } X \end{array}$$

is a Quillen adjunction and in particular $\text{lim}_{\rightarrow I}$ has a total left derived functor

$$\underline{L} \text{lim}_{\rightarrow I} \simeq \underline{\text{holim}}_{\rightarrow I}$$

Proof Observe that if $X \rightarrow Y$ is an arrow in C and i an object of i , as the category $\mathcal{D}_i^- I$ is connected, the morphism

$$\begin{array}{ccc} M_i(\Delta_I(X)) & \longrightarrow & M_i(\Delta_I(Y)) \\ \text{lim}_{\leftarrow} \parallel & & \text{lim}_{\leftarrow} \parallel \\ \Delta_I(X) |_{\mathcal{D}_i^- I} & & \Delta_I(Y) |_{\mathcal{D}_i^- I} \end{array}$$

is either an isomorphism of final objects (if $\mathcal{D}_i^- I$ is empty) or isomorphic to the arrow $X \rightarrow Y$, and the canonical map

$$\begin{array}{ccc} X \longrightarrow Y & \times & M_i(\Delta_I(X)) \\ \parallel & & \parallel \\ \Delta_I(X)(i) & & \Delta_I(Y)(i) \end{array} \quad \begin{array}{l} \text{either isomorphic} \\ \text{to } X \rightarrow Y \text{ or} \\ \text{an isomorphism} \end{array}$$

So if $X \rightarrow Y$ is a fibration (resp. trivial fibration) of C , then $\Delta_I(X) \rightarrow \Delta_I(Y)$ is a Reedy fibration (resp. trivial fibration) of C^I .

The notion of right connected Reedy category is defined dually.

If I is a direct category, by definition (I, I_+, I_-) where

$$I_+ = I \text{ and } I_- = I_0 = \text{discrete category with same objects as } I$$

is a Reedy category, and every object i of I , $J_i I$ is empty. In this case we have even better.

For every model category C and any morphism $X \rightarrow Y$ of C^I , $M_i X \rightarrow M_i Y$ is an isomorphism of final objects and the canonical map

$$X(i) \rightarrow Y(i) \times_{M_i Y} M_i X$$

is isomorphic to the map $X(i) \rightarrow Y(i)$. So in this case Reedy fibrations are exactly pointwise fibrations, so:

Proposition Let I a direct category and (C, W, CoF, Fib) a closed model category. If C is cocomplete then C is projectively I -exponential