Pursuing Stacks

Section 69

Alexandre Grothendieck

Édité par G. Maltsiniotis

Note de l'éditeur

Ce texte, édité par G. Maltsiniotis, est extrait de *Pursuing Stacks*, tapuscrit d'Alexandre Grothendieck, long de plus de plus de 600 pages, qui sera publié dans la collection *Documents Mathématiques* de la Société Mathématique de France. Il paraîtra en deux volumes, le premier [13], édité par G. Maltsiniotis, comportant les cinq premiers chapitres, le second [14], édité par M. Künzer, G. Maltsiniotis et B. Toën, étant consacré aux deux derniers chapitres, ainsi qu'à la correspondance de Grothendieck avec R. Brown, T. Porter, H.-J. Baues, A. Joyal, et R. Thomason, autour des sujets traitées dans la « Poursuite ».

Dans l'extrait qui suit, toutes les notes de bas de page sont de l'éditeur, ainsi que les références bibliographiques. Le texte de Grothendieck est en anglais, mais le « métalangage » de l'édition est le français.

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69. Digression on six weeks' scratchwork: *derivators*, and integration of homotopy types.

Before resuming more technical work again, I would like to have a short retrospective of the last six weeks' scratchwork, now lying on my desk as a thick bunch of scratchnotes nobody but I could possibly make any sense of.

The first thing I had on my mind has been there now for nearly twenty years – ever since it had become clear, in the SGA 5 seminar on L-functions [12] and *a propos* the formalism of traces in terms of derived categories, that Verdier's set-up of derived categories [29] was insufficient for formulating adequately some rather evident situations and relationships, such as the addition formula for traces, or the multiplicative formula for determinants [(1)]. It then became apparant that the derived category of an abelian category (say) was too coarse an object in various respects, that it had to be complemented by similar "triangulated categories" (such as the derived category of a suitable category of "triangles" of complexes, or the whole bunch of derived categories of categories of filtered complexes of order n with variable n), closely connected to it. Deligne and Illusie had both set out, independently, to work out some set-up meeting the most urgent requirements [6], [20]. (Illusie's treatment in terms of filtered complexes was written down and published in his thesis six years later (Springer Lecture Notes N° 239) [20].) While adequate for the main tasks then at hand, neither treatment was really wholly satisfactory to my taste. One main feature I believe making me feel uncomfortable, was that the extra categories which had to be introduced, to round up somewhat a stripped-and-naked triangulated category, were triangulated categories in their own right, in Verdier's sense, but remaining nearly as stripped by themselves as the initial triangulated category they were intended to provide clothing for. In other words, there was a lack of inner stability in the formalism, making it appear as very much provisional still. Also, while interested in associating to an abelian category a handy sequence of "filtered derived categories", Illusie made no attempt to pin down what exactly the inner structure of the object he had arrived at was – unlike Verdier, who had introduced, alongside with the notion of a derived category of an abelian category, a general notion of triangulated categories, into which these derived categories would fit. The obvious idea which was in my head by then for avoiding such shortcomings, was that an abelian category $\mathcal{A}$ gave rise, not only to the single usual derived category $\mathbb{D}(\mathcal{A})$ of Verdier, but also, for every type of diagrams to the derived category of the abelian category of all $\mathcal{A}$-valued diagrams of this type. In precise terms, for any small category I, we get the category $\mathbb{D}(\underline{\mathrm{Hom}}(I, \mathcal{A}))$, depending functorially in a contravariant way on I. Rewriting

[1] N. Éd. Voir à ce propos commentaires dans [9].

this category $\mathbb{D}_{\mathcal{A}}(I)$ say, the idea was to consider

$$(*) \qquad I \mapsto \mathbb{D}_{\mathcal{A}}(I)\,,$$

possibly with I suitably restricted (for instance to finite categories, or to finite ordered sets, corresponding to finite *commutative* diagrams), as embodying the "full" triangulated structure defined by $\mathcal{A}$. This of course at once raises a number of questions, such as recovering the usual triangulated structure of $\mathbb{D}(\mathcal{A}) = \mathbb{D}_{\mathcal{A}}(e)$ (e the final object of (Cat)) in terms of $(*)$, and pinning down, too, the relevant formal properties (and possibly even extra structure) one had to assume on $(*)$. I had never so far taken the time to sit down and play around some and see how this goes through, expecting that surely someone else would do it some day and I would be informed – but apparently in the last eighteen years nobody ever was interested. Also, it had been rather clear from the start that Verdier's constructions could be adapted and did make sense for non-commutative homotopy set-ups, which was also apparent in between the lines in Gabriel-Zisman's book on the foundations of homotopy theory [11], and a lot more explicitly in Quillen's axiomatization of homotopical algebra [25]. This axiomatization I found very appealing indeed – and right now still his little book is my most congenial and main source of information on foundational matters of homotopical algebra. I remember though my being a little disappointed at Quillen's not caring either to pursue the matter of what exactly a "non-commutative triangulated category structure" (of the type he was getting from his model categories) was, just contenting himself to mumble a few words about existence of "higher structure" (than just the Dold-Puppe sequences), which (he implied) needed to be understood. I felt of course that presumably the variance formalism $(*)$ should furnish any kind of "higher" structure one was looking for, but it wasn't really my business to check.(2)

It still isn't, however I did some homework on $(*)$ – it was the first thing indeed I looked at in these six weeks, and some main features came out very readily indeed. It turns out that the main formal variance property to demand on $(*)$, presumably even the only one, is that for a given map $f : I \to J$ on the indexing categories of diagram-types I and J, the corresponding functor

$$f^* : \mathbb{D}(J) \to \mathbb{D}(I)$$

should have both a left and a right adjoint, say $f_!$ and f_*. In case $J = e$, the two functors we get from $\mathbb{D}(I)$ to $\mathbb{D}(e) = \mathbb{D}_0$ (the "stripped" triangulated category) can be viewed as *a substitute for taking, respectively, direct and inverse limits in* $\mathbb{D}_0$ (for a system of objects indexed by I), *which in the usual sense don't generally exist in* $\mathbb{D}_0$ (except just finite sums and products). These operations admit as important special cases, when I is either one of the two mutually dual

(2) N. Éd. Grothendieck a rédigé un manuscrit long de près de 2000 pages autour de ces questions, les « Dérivateurs » [15]. Des notions proches de celle des dérivateurs ont été explorées de façon indépendante par D. W. Anderson [1], A. Heller [16], [17], [18], [19], B. Keller [21] et J. Franke [10]. Ce dernier a montré comment on peut, en particulier, retrouver la structure de catégorie triangulée. Ce résultat s'adapte avec des modifications mineures dans le cadre proprement dit des dérivateurs [22], [24].

categories

$$(**)\qquad \begin{array}{ccc} & & b \\ & \nearrow & \\ a & & \\ & \searrow & \\ & & c \end{array} \qquad \text{or} \qquad \begin{array}{ccc} & & b \\ & \swarrow & \\ a & & \\ & \nwarrow & \\ & & c \end{array} \quad ,$$

the operation of (binary) amalgamated sums or fiber products, and hence also of taking "cofibers" and "fibers" of maps, in the sense introduced by Cartan-Serre in homotopy theory about thirty years ago. I also checked that the two mutually dual Dold-Puppe sequences follow quite formally from the set-up. One just has to fit in a suitable extra axiom to insure the usual exactness properties for these sequences.

Except in the commutative case when starting with an abelian category as above, I did not check however that there is indeed such "higher variance structure" in the usual cases, when a typical "triangulated category" in some sense or other turns up, for instance from a model category in Quillen's sense. What I did check though in this last case, under a mild additional assumption which seems verified in all practical cases, is the existence of the operation $f_! = \int_I$ ("integration") and $f_* = \Pi_I$ (cointegration) for the special case $f : I \to e$, when I is either one of the two categories $(**)$ above. I expect that working some more, one should get under the same assumptions at least the existence of $f_!$ and f_* for any map $f : I \to J$ between finite ordered sets.(3)

My main interest of course at present is in the category (Hot) itself, more generally in $\mathrm{Hot}(\mathcal{W}) = \mathcal{W}^{-1}(\mathrm{Cat})$, where $\mathcal{W}$ is a "basic localizer". More generally, if (M, W) is any modelizer say, the natural thing to do, paraphrasing $(*)$, is for any indexing category I to endow $\underline{\mathrm{Hom}}(I, M)$ with the set of arrows W_I defined by componentwise belonging to W, and to define

$$\mathbb{D}_{(M,W)}(I) = \mathbb{D}(I) \overset{\mathrm{def}}{=} W_I^{-1}\ \underline{\mathrm{Hom}}(I, M)\,,$$

with the obvious contravariant dependence on I, denoted by f^* for $f : I \to J$. The question then arises as to the existence of left and right adjoints, $f_!$ and f_*. In case we take $M = (\mathrm{Cat})$, the existence of $f_!$ goes through with amazing smoothness: interpreting a "model" object of $\underline{\mathrm{Hom}}(I, (\mathrm{Cat}))$ namely a functor

$$I \to (\mathrm{Cat})$$

in terms of a cofibered category X over I

$$p : X \to I\,,$$

and assuming for simplicity f cofibering too, $f_!(X)$ is just X itself, the total category of the cofibering, viewed as a (cofibered) category over J by using the

[3] N. Éd. D.-C. Cisinski a démontré l'existence de $f_!$ et de f_*, dans le cas d'une catégorie de modèles de Quillen, sans aucune condition restrictive [3]. Ce résultat a été, par la suite, redémontré de façon différente par W. G. Dwyer, P. S. Hirschhorn, D. M. Kan et J. H. Smith [8]. D.-C. Cisinski l'a généralisé pour une notion de catégorie de modèles bien plus faible que celle de Quillen, les catégories dérivables [5]. Cette généralisation est détaillée par A. Rădulescu-Banu [26]. L'existence de ces foncteurs adjoints est aussi longuement étudiée, dans divers contextes, par Grothendieck lui-même dans les « Dérivateurs » [15], et plus particulièrement dans les chapitres VI, XIII, XVII et XIX.

functor $q = f \circ p$! This applies for instance when J is the final category, and yields the operation of "integration of homotopy types" $\int_I$, in terms of the total category of a cofibered category over I.[(4)] If we want to rid ourselves from any extra assumption on f, we can describe $\mathbb{D}(I)$ (up to equivalence) in terms of the category $(\mathrm{Cat})_{/I}$ of categories X over I (not necessarily cofibered over I), $\mathcal{W}_I$ being replaced by the corresponding notion of *"$\mathcal{W}$-equivalence relative to I"* for maps $u : X \to Y$ of objects of (Cat) over I, by which we mean a map u such that the localized maps

$$u_{/i} : X_{/i} \to Y_{/i}$$

are in $\mathcal{W}$, for any i in I. Regarding now any category X over I as a category over J by means of $f \circ p$, this is clearly compatible with the relative weak equivalences $\mathcal{W}_I$ and $\mathcal{W}_J$, and yields by localization the looked-for functor $f_!$.[(5)]

This amazingly simple construction and interpretation of the basic $f_!$ and $\int_I$ operations is one main reward, it appears, for working with the "basic modelizer" (Cat), which in this occurrence, as in the whole test and asphericity story, quite evidently deserves his name. It has turned out since that in some other respects – for instance, paradoxically, when it comes to the question of relationship between this lofty integration operation, and true honest amalgamated sums – the modelizers $\widehat{A}$ associated to test categories A (namely, the so-called "elementary modelizers") are more convenient tools than (Cat). Thus, it appears very doubtful still that (Cat) is a "model category" in Quillen's sense, in any reasonable way (with $\mathcal{W}$ of course as the set of "weak equivalences")[(6)]. I finally got the feeling that a good mastery of the basic aspects of homotopy types and of basic relationships among these, will require mainly great "*aisance*" in playing around with a number of available descriptions of homotopy types by models, no one among which (not even by models in (Cat), and surely still less by semi-simplicial structures) being adequate for replacing all others.

As for the f_* and cointegration Π_I operations among the categories $\mathbb{D}(I)$, except in the very special case noted above (corresponding to fiber products), I did not hit upon any ready-to-use candidate for it, and I doubt there is any. I do believe these operations exist indeed, and I even have in mind a rather general condition on a pair (M, W) with $W \subset \mathrm{Fl}(M)$, for both basic operations $f_!$ and f_* to exist between the corresponding categories $\mathbb{D}_{(M,W)}(I)$ – but to establish this expectation may require a good amount of work. I'll come back upon these matters in due course.

There arises of course the question of giving a suitable name to the structure $I \mapsto \mathbb{D}(I)$ I arrived at, which seems to embody at least some main features of a satisfactory notion of a "triangulated category" (not necessarily commutative), gradually emerging from darkness. I have thought of calling such a structure a *"derivator"*, with the implication that its main function is to furnish us with a somehow "complete" bunch (in terms of a rounded-up self-contained formalism)

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[4] N. Éd. Cette construction est reprise par Thomason [27].

[5] N. Éd. Pour plus de détails, voir [15], chapitre VI, pages 134-147, ou [23, §3.1].

[6] N. Éd. Thomason a établi l'existence d'une telle structure de catégorie de modèles de Quillen sur (Cat), avec comme ensemble d'équivalences faibles $\mathcal{W} = W_{(\mathrm{Cat})}$ [28] (voir commentaires de Grothendieck, section 87). D.-C. Cisinski a généralisé ce résultat pour tout localisateur de base $\mathcal{W}$ satisfaisant à l'axiome des fibrations (L5), section 64, et *accessible* (engendré par un *petit* ensemble de flèches de (Cat)) [4, théorèmes 5.2.15 et 5.3.14].

of categories $\mathbb{D}(I)$, which are being looked at as *"derived categories"* in some sense or other. The only way I know of for constructing such derivator, is as above in terms of a pair (M, W), submitted to suitable conditions for ensuring existence of $f_!$ and f_*, at least when f is any map between finite ordered sets. We may look upon $\mathbb{D}(I)$ as a refinement and substitute for the notion of family of objects of $\mathbb{D}(e) = \mathbb{D}_0$ indexed by I, and the integration and cointegration operations from $\mathbb{D}(I)$ to $\mathbb{D}_0$ as substitutes (in terms of these finer objects) of direct and inverse limits in $\mathbb{D}_0$. When tempted to think of these latter operations (with values in $\mathbb{D}_0$) as the basic structure involved, one cannot help though looking for the same kind of structure on any one of these subsidiary categories $\mathbb{D}(I)$, as these are being thought of as derived categories in their own right. It then appears at once that the "more refined substitutes" for J-indexed systems of objects of $\mathbb{D}(I)$ are just the objects of $\mathbb{D}(I \times J)$, and the corresponding integration and cointegration operations

$$\mathbb{D}(I \times J) \to \mathbb{D}(I)$$

are nothing but $p_!$ and p_*, where $p : I \times J \to I$ is the projection. Thus one is inevitably conducted to look at operations $f_!$ and f_* instead of merely integration and cointegration – thus providing for the "inner stability" of the structure described, as I had been looking for from the very start.

The notion of integration of homotopy types appears here as a natural by-product of an attempt to grasp the "full structure" of a triangulated category. However, I had been feeling the need for such a notion of integration of homotopy types for about one or two years already (without any clear idea yet that this operation should be one out of two main ingredients of a (by then still very misty) notion of a triangulated category of sorts). This feeling arose from my ponderings on stratified structures and the "screwing together" of such structures in terms of simple building blocks (essentially, various types of "tubes" associated to such structures, related to each other by various proper maps which are either inclusions, or – in the equisingular case at any rate – fiber maps). This "screwing together operation" could be expressed as being a direct limit of a certain finite system of spaces. In the cases I was most interested in (namely the Mumford-Deligne compactifications $M^\wedge_{g,\nu}$ of the modular multiplicities $M_{g,\nu}$ [7]), these spaces or "tubes" have exceedingly simple homotopy types – they are just $K(\pi, 1)$-spaces, where each π is a Teichmüller-type discrete group (practically a product of usual Teichmüller groups). It then occurred to me that the whole homotopy type of $M^\wedge_{g,\nu}$, or of any locally closed union of strata, or (more generally still) of "the" tubular neighbourhood of such a union in any larger one, *etc.* – that all these homotopy types should be expressible in terms of the given system of spaces, and more accurately still, just in terms of the corresponding system of fundamental groupoids (embodying their homotopy types). In this situation, what I was mainly out for, was precisely an accurate and workable description of this direct system of groupoids (which could be viewed as just one section of the whole "Teichmüller tower" of Teichmüller groupoids . . .). Thus, it was a rewarding extra feature of the situation (by then just an expectation, as a matter of fact), that such a description should at the same time yield a "purely algebraic" description of the homotopy types of all the spaces (rather, multiplicities, to be wholly accurate) which I could think of in terms of the natural stratification of $M^\wedge_{g,\nu}$. There was an awareness that this operation on homotopy types could not be described simply in terms of a functor $I \to (\mathrm{Hot})$, where I is the indexing category, that a functor $i \mapsto X_i : I \to M$ (where M is some model category such as (Spaces) or (Cat)) should be available in order to define an "integrated homotopy type" $\int X_i$. This justified feeling got somewhat blurred lately, for a little while, by the definitely unreasonable expectation that finite limits should exist in (Hot) after all, why not! It's enough to have a look though (which probably I did years ago and then forgot in the meanwhile) to make sure they don't ...

Whether or not this notion of "integration of homotopy types" is more or less well known already under some name or other, isn't quite clear to me[(7)]. It isn't familiar to Ronnie Brown visibly, but it seems he heard about such a kind of thing, without his being specific about it. It was the episodic correspondence with him[(8)] which finally pushed me last January to sit down for an afternoon and try to figure out what there actually was, in a lengthy and somewhat rambling letter to Illusie (who doesn't seem to have heard at all about such operations). This preliminary reflection proved quite useful lately, I'll have to come back anyhow upon some of the specific features of integration of homotopy types later, and there is not much point dwelling on it any longer at present.

[7] N. Éd. Cette notion était bien sûr connue sous le nom de « limite directe homotopique », ou « colimite homotopique », ou encore « limite inductive homotopique », introduite pour la première fois en toute généralité par Bousfield et Kan [2].

[8] N. Éd. La correspondance avec Ronnie Brown sera publiée dans le deuxième volume de « Pursuing Stacks » [14].

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