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Markov-Krein transform

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Abstract The Markov-Krein transform maps a positive measure on the real line to a probability measure. It is implicitely defined through an identity linking two holomorphic functions. We propose an explicit formula whose proof is obtained by considering boundary values of holomorhic functions. This transform appears in several classical questions in analysis and probability theory: Markov moment problem, Dirichlet distributions, orbital measures. An asymptotic property for this transform involves Thorin-Bondesson distributions.

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1. Introduction. — The starting point of this work is an observation by Okounkov about the orbital measures for the action of the unitary group $U(n)$ on the space $Herm(n,\mathbb{C})$ of $n\times n$ Hermitian matrices. The projection of such a measure on the straight line generated by a rank one matrix is a probability measure on $\mathbb{R}$, the density of which is a spline function, i.e. a piecewise polynomial function (see [Olshanski-Vershik,1996], Proposition 8.2 p.172). More generally we consider the action of the unitary group $U(n,\mathbb{F})$ on the space $Herm(n,\mathbb{F})$, for $\mathbb{F}=\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, the skew field of quaternions, and the projection μ of an orbital measure. In general the density of μ is not a spline function. The measure

μ satisfies the remarkable formula

$$\int_{\mathbb{R}} \frac{1}{(z-t)^{n\frac{d}{2}}} \mu(dt) = \prod_{i=1}^{n} \frac{1}{(z-a_i)^{\frac{d}{2}}},$$

where $a_1, \ldots, a_n$ are the eigenvalues of an Hermitian matrix in the orbit, and $d = \dim_{\mathbb{R}} \mathbb{F} = 1, 2$ or 4. This formula is a special case of the Markov-Krein relation

$$\int_{\mathbb{R}} \frac{1}{(z-t)^{\kappa}} \mu(dt) = \exp\Big(-\int_{\mathbb{R}} \log(z-u)\nu(du)\Big),$$

where μ is a probability measure, ν a positive measure, and $\kappa = \nu(\mathbb{R})$, the total measure of ν. In fact, taking

$$\nu = \sum_{i=1}^{n} \frac{d}{2} \delta_{a_i},$$

one gets the first formula. In Section 3 it will be proven that given a positive measure ν with compact support, there is a unique probability measure μ with compact support satisfying the Markov-Krein relation. Hence we get a map: the Markov-Krein transform associates to the positive measure ν the probability measure μ. We will see in Section 2 that this transform is related to the Dirichlet distributions in case ν is a discrete measure. An explicit formula for this transform is given in Section 4 by using boundary values of holomorphic functions. This formula is related to the one obtained in [Cifarelli-Regazzini,1990]. In Section 6 we consider a sequence (ν_n) of positive measures and the sequence (μ_n) of the Markov-Krein transforms. We study the asymptotic of μ_n in case $\nu_n(\mathbb{R})$ goes to infinity. The result we will establish involves Thorin-Bondesson distributions (or extended generalized gamma convolutions, EGGC), a class of probability measures introduced by Thorin ([1977], [1978], see also [Bondesson,1992]). The Markov-Krein transform shows up in several questions of classical analysis. We have mentionned its relation to orbital measures. It appears in the solution of the Markov moment problem by Krein and Nudel'man [1977]. The problem is as follows: Given a sequence (c_n) of Hausdorff moments,

$$c_n = \int_{[a,b]} t^n \sigma(dt),$$

under which condition is the positive measure σ absolutely continuous with respect to the Lebesgue measure: $\sigma(dt) = \omega(t)dt$, with $0 \leq \omega(t) \leq 1$?

We revisit that problem in Section 7. Finally we consider in last section spline distributions with equidistant knots, and recall an example studied by Tricomi [1933]. A large part of the book by Kerov [2003] is devoted to the Markov-Krein correspondence in the framework of the asymptotic analysis for the representations of the symmetric group. It has been a source of inspiration for our work.

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2. The generalized spline distributions $M_n(a;\tau)$. — We recall definitions and results from [Fourati,2011a]. For $\tau=(\tau_1,\dots,\tau_n)\in(\mathbb{R}_+^*)^n$ ($n\geq 2$), the Dirichlet distribution $D_n^{(\tau)}$ is the probability measure on the simplex

$$\Delta_{n-1}=\{u=(u_1,\dots,u_n)\in\mathbb{R}^n \mid u_i\geq 0,\ u_1+\cdots+u_n=1\}$$

defined by

$$\int_{\Delta_{n-1}} f(u)D_n^{(\tau)}(du)=\frac{1}{C_n(\tau)}\int_{\Delta_{n-1}} f(u)u_1^{\tau_1-1}\dots u_n^{\tau_n-1}\alpha(du),$$

where α is the uniform probability measure on Δ_{n-1}, i.e. the normalized restriction to Δ_{n-1} of the Lebesgue measure on the hyperplane $u_1+\cdots+u_n=1$, and

$$C_n(\tau)=\int_{\Delta_{n-1}} u_1^{\tau_1-1}\dots u_n^{\tau_n-1}\alpha(du).$$

The evaluation of the constant $C_n(\tau)$ gives

$$C_n(\tau)=(n-1)!\frac{\Gamma(\tau_1)\dots\Gamma(\tau_n)}{\Gamma(|\tau|)},$$

where $|\tau|=\tau_1+\cdots+\tau_n$.

For $a=(a_1,\dots,a_n)\in\mathbb{R}^n$, with $a_1\leq\cdots\leq a_n$, the probability measure $M_n(a;\tau)$ on $\mathbb{R}$ is the image of the Dirichlet distribution $D_n^{(\tau)}$ by the map

$$\Delta_{n-1}\to\mathbb{R},\ u\mapsto a_1u_1+\cdots+a_nu_n,$$

i.e., for a continuous function F on $\mathbb{R}$,

$$\int_{\mathbb{R}} F(t)M_n(a;\tau;dt)=\int_{\Delta_{n-1}} F(a_1u_1+\cdots+a_nu_n)D_n^{(\tau)}(du).$$

The support of $M_n(a;\tau)$ is compact, $\mathrm{supp}\big(M_n(\tau;a)\big) \subset [a_1, a_n]$. If $\tau_1 = \dots = \tau_n = 1$, then $M_n(a,\tau)$ is a spline distribution (see [Curry-Schoenberg,1966]). For $\tau_i > 0$, we will say that $M_n(a;\tau)$ is a generalized spline distribution.

For instance, for $n = 2$,

$$\int_{\mathbb{R}} F(t) M_2(a;\tau;dt)$$
$$= \frac{\Gamma(\tau_1+\tau_2)}{\Gamma(\tau_1)\Gamma(\tau_2)} \int_0^1 F\big(a_1(1-u)+a_2 u\big)(1-u)^{\tau_1-1} u^{\tau_2-1} du.$$

By the change of variable $t = a_1(1-u) + a_2 u$ we get

$$\int_{\mathbb{R}} F(t) M_2(a;\tau;dt)$$
$$= \frac{(a_2-a_1)^{-(\tau_1+\tau_2-1)}}{B(\tau_1,\tau_2)} \int_{a_1}^{a_2} F(t)(t-a_1)^{\tau_2-1}(a_2-t)^{\tau_1-1} dt.$$

We define the function $\log z$ on $\mathbb{C}\backslash]-\infty, 0]$ and, for $\alpha \in \mathbb{C}$, the function z^α as follows : if $z = re^{i\theta}$, with $r > 0$, $-\pi < \theta < \pi$, then $\log z = \log r + i\theta$, and $z^\alpha = e^{\alpha \log z} = r^\alpha e^{i\alpha\theta}$.

THEOREM 2.1. — *The probability measure $M_n(a;\tau)$ satisfies the relation*

$$\int_{\mathbb{R}} \frac{1}{(z-t)^{|\tau|}} M_n(a;\tau;dt) = \prod_{i=1}^n \Big(\frac{1}{z-a_i}\Big)^{\tau_i},$$

for $z \in \mathbb{C}\backslash]-\infty, a_n]$.

This is a special case of the Markov-Krein relation we will consider in next Section.

Proof.

Assume first $\mathrm{Re}\, z > a_n$. We will evaluate in two ways the integral

$$I(a,z) = \int_{\mathbb{R}_+^n} \exp\Big(-\sum_{i=1}^n (z-a_i)x_i\Big) x_1^{\tau_1-1} \dots x_n^{\tau_n-1} dx_1 \dots dx_n.$$

First, by the theorem of Fubini,

$$I(a,z) = \prod_{i=1}^n \int_0^\infty e^{-x_i(z-a_i)} x_i^{\tau_i-1} dx_i = \prod_{i=1}^n \frac{\Gamma(\tau_i)}{(z-a_i)^{\tau_i}}.$$

Second, we will use the following integration formula: if the function f is integrable on $\mathbb{R}_+^n$, then

$$\int_{\mathbb{R}_+^n} f(x)dx_1 \dots dx_n = \frac{1}{(n-1)!}\int_0^\infty \Big(\int_{\Delta_{n-1}} f(ru)\alpha(du)\Big) r^{n-1}dr.$$

Hence we get

$$\begin{aligned}
I(a,z) &= \frac{1}{(n-1)!} \\
&\int_0^\infty \Big(\int_{\Delta_{n-1}} e^{-r(z-a_1u_1-\dots-a_nu_n)} u_1^{\tau_1-1}\dots u_n^{\tau_n-1}\alpha(du)\Big) r^{|\tau|-1}dr \\
&= \frac{\Gamma(|\tau|)}{(n-1)!}\int_{\Delta_{n-1}} \frac{u_1^{\tau_1-1}\dots u_n^{\tau_n-1}}{(z-a_1u_1-\dots-a_nu_n)^{|\tau|}} \\
&= C_n(\tau)\frac{\Gamma(|\tau|)}{(n-1)!}\int_{\Delta_{n-1}} \frac{D_n^{(\tau)}(du)}{(z-a_1-a_1-\cdots a_nu_n)^{|\tau|}} \\
&= \Gamma(\tau_1)\dots\Gamma(\tau_n)\int_{\mathbb{R}} \frac{1}{(z-t)^{|\tau|}} M_n(a;\tau;dt).
\end{aligned}$$

From both evaluations of $I(a,z)$ one gets the formula of Theorem 2.1. Since both handsides of the formula are holomorphic in $\mathbb{C}\backslash]-\infty,a_n]$, the formula holds for $z\in\mathbb{C}\backslash]-\infty,a_n]$ by analytic continuation. $\square$

The moments of the measure $M_n(a;\tau)$,

$$\mathfrak{M}_n(a;\tau;m) = \int_{\mathbb{R}} t^m M_n(a;\tau;dt),$$

are given by

$$\mathfrak{M}_n(a;\tau;m) = \frac{m!}{(\kappa)_m}\sum_{|\lambda|=m} \frac{(\tau_1)_{\lambda_1}\dots(\tau_n)_{\lambda_n}}{\lambda_1!\dots\lambda_n!} a_1^{\lambda_1}\dots a_n^{\lambda_n},$$

where $\lambda\in\mathbb{N}^n$, $|\lambda| = \lambda_1+\dots+\lambda_n$.

In case of $\tau_1=\dots=\tau_n=\theta$ we will write $M_n^\theta(a;dt)$ and $\mathfrak{M}_n^\theta(a;m)$, and then

$$\mathfrak{M}_n^\theta(a;m) = \frac{(\theta)_m}{(n\theta)_m} P_{[m]}(a_1,\dots,a_n;\theta),$$

where $P_{[m]}(a_1,\dots,a_n;\theta)$ is the Jack polynomial associated to the partition $[m]=(m,\dots,0)$ with parameter θ. In the special case of $\theta=1$,

$$\mathfrak{M}_n^1(a;m) = \frac{m!(n-1)!}{(n+m-1)!} h_m(a_1,\dots,a_n),$$

where h_m is the complete symmetric function. It can be written

$$\mathfrak{M}_n^1(a;m) =$$

$$\frac{m!(n-1)!}{(n+m-1)!}\frac{1}{V(a_1,\ldots,a_n)}\begin{vmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_n \\ \vdots & \vdots & \ldots & \vdots \\ a_1^{n-2} & a_2^{n-2} & \ldots & a_n^{n-2} \\ a_1^{m+n-1} & a_2^{m+n-1} & \ldots & a_n^{m+n-1} \end{vmatrix},$$

where $V(a_1,\ldots,a_n)$ is the Vandermonde polynomial:

$$V(a_1,\ldots,a_n) = \prod_{i<j}(a_j - a_i),$$

and the Fourier Laplace transform of $M_n^1(a;dt)$,

$$\widehat{M_n^1}(a;z) = \int_{\mathbb{R}} e^{zt} M_n^1(a;dt),$$

is given by

$$\widehat{M_n^1}(a;z) = (n-1)!\frac{1}{V(a_1,\ldots,a_n)}\begin{vmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_n \\ \vdots & \vdots & \ldots & \vdots \\ a_1^{n-2} & a_2^{n-2} & \ldots & a_n^{n-2} \\ e^{a_1 z} & e^{a_2 z} & \ldots & e^{a_n z} \end{vmatrix}.$$

3. The Markov-Krein transform. — Let ν be a nonzero positive measure on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \log(1+|u|)\nu(du) < \infty,$$

and μ a probability measure on $\mathbb{R}$. We say that the measures μ and ν are linked by the *Markov-Krein relation* if, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\int_{\mathbb{R}} \frac{1}{(z-t)^\kappa}\mu(dt) = \exp\Big(-\int_{\mathbb{R}} \log(z-u)\nu(du)\Big),$$

where $\kappa = \nu(\mathbb{R})$, the total measure of ν. By Theorem 2.1, the measures $\mu = M_n(\tau;a)$ and

$$\nu = \sum_{i=1}^n \tau_i \delta_{a_i}$$

are linked by the Markov-Krein relation. In fact, in this case, the Markov-Krein relation becomes

$$\int_{\mathbb{R}} \frac{1}{(z-t)^{\kappa}} \mu(dt) = \prod_{i=1}^{n} \Big(\frac{1}{z-a_i}\Big)^{\tau_i}, \quad \kappa = \tau_1 + \cdots + \tau_n.$$

Let us assume that the measures μ and ν are compactly supported, and denote by h_m and p_m the moments:

$$h_m = \int_{\mathbb{R}} t^m \mu(dt), \quad p_m = \int_{\mathbb{R}} t^m \nu(dt).$$

(Observe that $\kappa = \nu(\mathbb{R}) = p_0$.)

PROPOSITION 3.1. — *The measures μ and ν are linked by the Markov-Krein relation if and only if the moments h_m and p_m of μ and ν satisfy the relation, for sufficiently small z,*

$$\sum_{m=0}^{\infty} \frac{(\kappa)_m}{m!} h_m z^m = \exp\Big(\sum_{m=1}^{\infty} \frac{p_m}{m} z^m\Big).$$

It follows that h_m can be written as a polynomial in $p_1, \ldots, p_m$,

$$h_m = \frac{m!}{(\kappa)_m} \sum_{k=1}^{m} \frac{1}{k!} \sum_{\alpha_i \geq 1, \alpha_1 + \cdots + \alpha_k = m} \frac{p_{\alpha_1}}{\alpha_1} \cdots \frac{p_{\alpha_k}}{\alpha_k}.$$

THEOREM 3.2. — *For a given nonzero positive measure ν on $\mathbb{R}$ with compact support, there is a unique probability measure μ with compact support such that the measures ν and μ are linked by the Markov-Krein relation: for $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$\int_{\mathbb{R}} \frac{1}{(z-t)^{\kappa}} \mu(dt) = \exp\Big(-\int_{\mathbb{R}} \log(z-u) \nu(du)\Big),$$

where $\kappa = \nu(\mathbb{R})$.

By definition the *Markov-Krein transform* is the map which associates to the positive measure ν the probability measure μ.

Proof. If the measure μ exists, it is unique, since, by Proposition 3.1, the moments of μ are determined by those of ν.

Assume $\mathrm{supp}(\nu) \subset [a,b]$. There is a sequence $\nu^{(n)}$ of measures with finite support in $[a,b]$,

$$\nu^{(n)} = \sum_{i=1}^{n} \tau_i^{(n)} \delta_{a_i^{(n)}}$$

which converges weakly to ν. By Theorem 2.1 the measures $\nu^{(n)}$ and $\mu^{(n)} = M_n(\tau^{(n)}; a^{(n)})$ are linked by the Markov-Krein relation. The moment $p_m^{(n)}$ of ν_n converges to the corresponding moment p_m of ν. Observe that $h_m^{(0)} = 1$, and, for $m \geq 1$, by Proposition 3.1, the moments $h_m^{(n)}$ have limits h_m . The numbers h_m are moments of a probability measure μ, and μ is the weak limit of $\mu^{(n)}$. Furthermore the measures μ and ν are linked by the Markov-Krein relation. $\square$

4. An explicit formula for the Markov-Krein transform. — We recall first the definition of hyperfunctions of one variable and some of their elementary properties (see for instance [Morimoto,1993]). Let $U \subset \mathbb{R}$ be open and $W \subset \mathbb{C}$ a complex open neighborhood of U. The space $\mathcal{B}(U)$ of hyperfuncions on U is defined as

$$\mathcal{B}(U) = \mathcal{O}(W \setminus U)/\mathcal{O}(W),$$

where, for $V \subset \mathbb{C}$ open, $\mathcal{O}(V)$ is the space of holomorphic functions on V. For $F \in \mathcal{O}(W \setminus U)$, the equivalence class of F is denoted by $[F]$. Define

$$F^+ = \begin{cases} F & \text{on } W^+ \\ 0 & \text{on } W^- \end{cases}, \quad F^- = \begin{cases} 0 & \text{on } W^+ \\ -F & \text{on } W^-. \end{cases}$$

($W^\pm = \{z \in W \mid \pm \mathrm{Im}\, z > 0\}$.) The hyperfunctions $[F^+]$ and $[F^-]$ are denoted by $F(x+i0)$ and $F(x-i0)$, and called the boundary values of F. Hence

$$[F] = F(x+i0) - F(x-i0).$$

Intuitively $[F]$ is the jump of F along U. An hyperfunction $f \in \mathcal{B}(U)$ vanishes on an open set $U_0 \subset U$ if there is a representative F of f which is holomorphic on $(W \setminus U) \cup U_0$. The support $\mathrm{supp}(f)$ of the hyperfunction $f \in \mathcal{B}(U)$ is the smallest closed set $C \subset U$ such that f vanishes on $U \setminus C$. The space of hyperfunctions on U with support contained in C is denoted by $\mathcal{B}_C(U)$.

An analytic functional on a compact set $K \subset \mathbb{R}$ is a linear form on the space $\mathfrak{A}(K)$ of analytic functions in a neighborhood of K,

$$\mathfrak{A}(K) = \bigcap_{U \supset K} \mathcal{O}(U),$$

where U is a complex open neighborhood of K. The space of analytic functionals on K is denoted by $\mathfrak{A}'(K)$. The Cauchy transform G_T of $T \in \mathfrak{A}'(K)$ is defined by

$$G_T(z) = -\frac{1}{2i\pi}\langle T_t, \frac{1}{z-t}\rangle.$$

The function G_T is holomorphic on $\mathbb{C} \setminus K$, and defines an hyperfunction $[G_T]$. The map $\Phi : T \mapsto f = [G_T]$ is an isomorphism from $\mathfrak{A}'(K)$ onto $\mathcal{B}_K(\mathbb{R})$. It follows that the space $\mathcal{D}'_K$ of distributions supported in K can be seen as a subspace of $\mathcal{B}_K(\mathbb{R})$.

Let $U \subset \mathbb{R}$ be open, and $\varepsilon > 0$. A function F defined on

$$\{z = x + iy \mid x \in U,\ 0 < |y| < \varepsilon\}$$

is said to be of moderate growth along U if, for every $K \subset U$ compact, there is a constant $C > 0$ and an integer $N > 0$ such that

$$|F(x + iy)| \leq \frac{C}{|y|^N} \quad (x \in K,\ 0 < |y| < \varepsilon).$$

Let $T \in \mathfrak{A}'(K)$, $f \in \mathcal{B}_K(\mathbb{R})$ its image by the isomorphism Φ, and F a representative of f. Then T is a distribution if and only if F is of moderate growth along $\mathbb{R}$. In such a case, for $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\langle T, \varphi \rangle = \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \big(F(t + i\varepsilon) - F(t - i\varepsilon)\big)\varphi(t)dt.$$

Furthermore $\operatorname{supp}(T) = \operatorname{supp}(f)$.

For $\alpha \in \mathbb{C}$ the distribution Y_α is defined, for $\operatorname{Re}\alpha > 0$, by

$$\langle Y_\alpha, \varphi \rangle = \frac{1}{\Gamma(\alpha)} \int_0^\infty \varphi(t) t^{\alpha - 1} dt \quad \big(\varphi \in \mathcal{D}(\mathbb{R})\big),$$

and admits an analytic continuation for $\alpha \in \mathbb{C}$. These distributions Y_α satisfy

$$Y_\alpha * Y_\beta = Y_{\alpha+\beta},\ Y_0 = \delta,\ Y_{-m} = \delta^{(m)}\ (m \in \mathbb{N}).$$

In particular $Y_\alpha * Y_{-\alpha} = \delta$.

Recall that, for $\alpha \in \mathbb{C}$, the holomorphic function z^α in $\mathbb{C} \setminus]-\infty, 0]$ is defined as follows: if $z = re^{i\theta}$ with $r > 0$, $-\pi < \theta < \pi$, then $z^\alpha = r^\alpha e^{i\alpha\theta}$. The function z^α is of moderate growth along $\mathbb{R}$, and

$$[z^\alpha] = -2i\pi \frac{1}{\Gamma(-\alpha)} \check{Y}_{\alpha+1}.$$

In particular, for $m \in \mathbb{N}$, $[z^m] = 0$, and, for $m \geq 1$,

$$[z^{-m}] = -2i\pi \frac{1}{(m-1)!} \delta^{(m-1)}.$$

We will now give an explicit formula for the Markov-Krein transform. Let ν be a positive measure on $\mathbb{R}$ with compact support, $\kappa = \nu(\mathbb{R})$. Recall that the Markov-Krein transform μ of ν is the unique probability measure μ such that

$$\int_{\mathbb{R}} \frac{1}{(z-t)^\kappa} \mu(dt) = \exp\Big(-\int_{\mathbb{R}} \log(z-u)\nu(du)\Big)$$

(Theorem 3.2). Furthermore the support of μ is compact.

THEOREM 4.1. — *Let q be the holomorphic function defined on $\mathbb{C} \setminus \mathbb{R}$ by*

$$q(z) = \exp\Big(-\int_{\mathbb{R}} \log(z-u)\nu(du)\Big).$$

Then q is of moderate growth, and

$$\mu = -\frac{1}{2i\pi}\Gamma(\kappa)\check{Y}_{\kappa-1} * [q].$$

Observe that, if $\kappa = 1$, then $\mu = -\frac{1}{2i\pi}[q]$.

LEMMA 4.2. — *Let the function f be holomorphic on $\mathbb{C} \setminus \mathbb{R}$, and μ a measure on $\mathbb{R}$ with compact support. Then the function F, defined by*

$$F(z) = \int_{\mathbb{R}} f(z-t)\mu(dt),$$

is holomorphic on $\mathbb{C} \setminus \mathbb{R}$. If f is of moderate growth along $\mathbb{R}$, then F is of moderate growth as well and

$$[F] = [f] * \mu.$$

Proof of theorem 4.1.

The Markov-Krein relation can be written

$$\int_{\mathbb{R}} \frac{1}{(z-t)^\kappa} \mu(dt) = q(z).$$

By Lemma 4.2 the function q is of moderate growth along $\mathbb{R}$, and

$$[z^{-\kappa}] * \mu = [q].$$

We saw that

$$[z^{-\kappa}] = -2i\pi\frac{1}{\Gamma(\kappa)}\check{Y}_{1-\kappa}.$$

Therefore, since $\check{Y}_{\kappa-1} * \check{Y}_{1-\kappa} = \delta$,

$$\mu = -\frac{1}{2i\pi}\Gamma(\kappa)\check{Y}_{\kappa-1} * [q]. \qquad \square$$

The logarithmic potential of the measure ν is defined on $\mathbb{R}$ by

$$U^\nu(x) = \int_{\mathbb{R}} \log \frac{1}{|x-u|} \nu(du),$$

with values in $]-\infty, \infty]$.

THEOREM 4.3. — *If* $\exp U^\nu$ *is locally integrable and* $\kappa = \nu(\mathbb{R}) \geq 1$, *then the probability measure* μ *has a density* h. *Define*

$$g(x) = \frac{1}{\pi} \sin\bigl(\pi\nu(]x,\infty[) \exp U^\nu(x).$$

(i) *If* $\kappa = 1$, *then* $h(x) = g(x)$.
(ii) *If* $\kappa > 1$, *then*

$$h(x) = (\kappa - 1) \int_x^\infty (s-x)^{\kappa-2} g(s) ds.$$

This formula is related to a formula given in [Cifarelli-Regazzini, 1990] (Part (ii) of Theorem 1, with $\tau = \infty$, $A(\tau) = 0$). The proof is there obtained by using results of Widder and Hirschman about generalized Stieltjes transforms.

Proof.
By Theorem 4.1 it amounts to show that the distribution $-\frac{1}{2i\pi}[q]$ is defined by the locally integrable function g. Define

$$H(z) = \int_{\mathbb{R}} \log \frac{1}{z-u} \nu(du).$$

The function $\log z$ can be written

$$\log z = \log |z| + i \mathrm{Arg}\,(z),$$

and

$$\mathrm{Arg}\,(x \pm i0) = \begin{cases} 0 & \text{if } x > 0, \\ \pm\pi & \text{if } x < 0. \end{cases}$$

It follows that

$$H(x \pm i0) = U^\nu(x) \mp i\pi\nu([x,\infty[),$$

and

$$\begin{aligned} -\frac{1}{2i\pi}[q](x) &= -\frac{1}{2i\pi}\bigl(\exp H(x+i0) - \exp H(x-i0)\bigr) \\ &= -\frac{1}{2i\pi} \exp U^\nu(x) \bigl(e^{-i\pi\nu([x,\infty[)} - e^{i\pi\nu([x,\infty[)}\bigr) \\ &= \frac{1}{\pi} \exp U^\nu(x) \sin\bigl(\pi\nu([x,\infty[)\bigr) = g(x). \end{aligned}$$

□

Examples

1) Assume the measure ν to be discrete

$$\nu = \sum_{i=1}^{n} \tau_i \delta_{a_i} \quad (a_1 < \cdots < a_n,\ n \geq 3).$$

Then its Markov-Krein transform is the probability measure $M_n(a_1, \ldots, a_n; \tau_1, \ldots, \tau_n)$. In that case

$$q(z) = \prod_{i=1}^{n} \Big(\frac{1}{z - a_i}\Big)^{\tau_i}.$$

a) Assume $\tau_1 = \cdots = \tau_n = 1$. Then q is a rational function which can be written

$$q(z) = \sum_{i=1}^{n} c_i \frac{1}{z - a_i}, \text{ with } c_i = \prod_{j \neq i} \frac{1}{a_j - a_i}.$$

Therefore

$$[q] = -2i\pi \sum_{j=1}^{n} c_i \delta_{a_i}.$$

Since

$$\check{Y}_{n-1} * \delta_a = \frac{1}{(n-2)!}(a - x)_+^{n-2},$$

the measure μ has a density h given by

$$h(x) = (n-1) \sum_{a_i > x} c_i (a_i - x)^{n-2}.$$

This density is a spline function with knots $a_1, \ldots, a_n$: the function h is of class $\mathcal{C}^{n-3}$, and its restriction to each interval $[a_j, a_{j+1}]$ is a polynomial of degree $\leq n-2$. In this case $M_n(a;\tau)$ is a spline distribution.

b) Assume $0 < \tau_i < 1$ $(1 \leq i \leq n)$, $\kappa = \tau_1 + \cdots + \tau_n \geq 1$. Then the function

$$\exp U^{\nu}(x) = \prod_{i=1}^{n} |x - a_i|^{-\tau_i}$$

is locally integrable and

$$g(x) = \frac{1}{\pi} \sin\big(\pi \sum_{a_i > x} \tau_i\big) \prod_{i=1}^{n} |x - a_i|^{-\tau_i}.$$

If $\kappa = 1$, then the density h of μ is equal to g. For $\kappa > 1$, the density h of μ is given by

$$h(x) = (\kappa - 1) \int_0^x (s - x)^{\kappa - 2} g(s) ds.$$

We have assumed the measures ν and μ to be compactly supported. In fact it is possible to define the Markov-Krein transform of a positive measure ν on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \log(1 + |u|) \nu(du) < \infty.$$

As an example let us consider the Cauchy measure

$$\nu(du) = \frac{1}{\pi} \frac{1}{1 + u^2} du.$$

In [Yamato,1984] it is shown that the Markov-Krein transform μ of ν is equal to ν (See also [Cifarelli-Regazzini, 1990]). In fact, by residue Theorem, one gets the following formula for the Cauchy-Stieltjes transform of the Cauchy measure:

$$G_\nu(z) = \int_{\mathbb{R}} \frac{1}{z - t} \frac{1}{\pi} \frac{dt}{1 + t^2} = \begin{cases} \frac{1}{z+i} & \text{if } \operatorname{Im} z > 0, \\ \frac{1}{z-i} & \text{if } \operatorname{Im} z < 0\ . \end{cases}$$

Similarly one gets also

$$\int_{\mathbb{R}} \log(z - u) \frac{1}{\pi} \frac{du}{1 + u^2} = \begin{cases} \log(z + i) & \text{if } \operatorname{Im} z > 0, \\ \log(z - i) & \text{if } \operatorname{Im} z < 0\ . \end{cases}$$

Therefore, for $z \in \mathbb{C} \setminus \mathbb{R}$

$$\int_{\mathbb{R}} \frac{1}{z - t} \frac{1}{\pi} \frac{dt}{1 + t^2} = \exp\Bigl(- \int_{\mathbb{R}} \log(z - u) \frac{1}{\pi} \frac{du}{1 + u^2} \Bigr).$$

It is possible to establish this result by using the formula of Theorem 4.3. The logarithmic potential U^ν of ν is given by

$$U^\nu(x) = \int_{\mathbb{R}} \log \frac{1}{|x - u|} \frac{1}{\pi} \frac{du}{1 + u^2} = -\frac{1}{2} \log(1 + x^2).$$

Furthermore

$$\nu(]x, \infty[) = \frac{1}{\pi} \int_x^\infty \frac{du}{1 + u^2} = \frac{1}{\pi} \Bigl(\frac{\pi}{2} - \operatorname{Arctg} x \Bigr),$$

and

$$\sin \pi\nu(]x,\infty[) = \sin\Big(\frac{\pi}{2} - \operatorname{Arctg} x\Big) = \cos(\operatorname{Arctg} x) = \frac{1}{\sqrt{1+x^2}}.$$

By Theorem 4.3 the density of the Markov-Krein transform μ of ν is given by

$$g(x) = \frac{1}{\pi}\exp U^\nu(x)\sin \pi\nu(]x,\infty[) = \frac{1}{\pi}\frac{1}{1+x^2}.$$

Moreover let us consider, for $\kappa > 1$, the measure

$$\nu_\kappa(du) = \frac{\kappa}{\pi}\frac{du}{1+u^2}.$$

One gets, by residue Theorem,

$$\int_{\mathbb{R}} \frac{1}{(z-t)^\kappa}\frac{1}{\pi}\frac{dt}{1+t^2} = \begin{cases} \frac{1}{(z+i)^\kappa} & \text{if } \operatorname{Im} z > 0, \\ \frac{1}{(z-i)^\kappa} & \text{if } \operatorname{Im} z < 0\ . \end{cases}$$

Therefore the Markov-Krein transform of ν_κ is equal to ν_1:

$$\int_{\mathbb{R}} \frac{1}{(z-t)^\kappa}\frac{1}{\pi}\frac{dt}{1+t^2} = \exp\Big(-\int_{\mathbb{R}} \log(z-u)\frac{\kappa}{\pi}\frac{du}{1+u^2}\Big).$$

5. Thorin-Bondesson distributions. — For $\xi \in \mathbb{R}^*$, $\tau > 0$, let $\gamma(\xi,\tau)$ denote the gamma distribution on $\mathbb{R}$ with density

$$Y(\xi u)\frac{|\xi|^\tau}{\Gamma(\tau)}e^{-\xi u}|u|^{\tau-1}.$$

The Fourier-Laplace transform φ of $\gamma(\xi,\tau)$ is given by

$$\varphi(z) = \int_{\mathbb{R}} e^{zt}\gamma(\xi,\tau;dt) = \Big(\frac{\xi}{\xi - z}\Big)^\tau.$$

It is defined for $\operatorname{Re} z < \xi$ if $\xi > 0$, and for $\operatorname{Rez} z > \xi$ if $\xi < 0$, and admits a holomorphic extension to $\mathbb{C}\setminus[\xi,\infty[$ if $\xi > 0$, and to $\mathbb{C}\setminus]-\infty,\xi]$ if $\xi < 0$.

A Thorin-Bondesson distribution (or extended generalized gamma convolution, EGGC) is a probability measure μ on $\mathbb{R}$ which is a limit for the tight topology of convolution products of gamma distributions

$$\mu = \lim_{n\to\infty}\Big(\prod_{i=1}^n\Big)^*\gamma(\xi_i^{(n)},\tau_i^{(n)}).$$

(See [Thorin,1977,1978], [Bondesson,1992]). The set $\mathcal{T}_e$ of Thorin-Bondesson distributions is closed for the tight topology and a semi-group for the convolution. In [Schilling-Song-Vondraček,2012], Chapter 9 is devoted to the measures in the Bondesson class, denoted BO. These measures are sub-probabilities supported by $[0,\infty[$. The probability measures in the Bondesson class are precisely the Thorin-Bondesson distributions (in our terminology) which are supported by $[0,\infty[$.

The Fourier-Laplace transform φ of

$$\gamma(\xi_1,\ldots,\xi_n;\tau_1,\ldots,\tau_n) := \gamma(\xi_1,\tau_1) * \cdots * \gamma(\xi_n,\tau_n)$$

is given by

$$\varphi(z) = \int_{\mathbb{R}} e^{zt}\gamma(\xi_1,\ldots,\xi_n;\tau_1,\ldots,\tau_n;dt) = \prod_{i=1}^{n}\Big(\frac{\xi_i}{\xi_i - z}\Big)^{\tau_i}.$$

It is defined for $|\mathrm{Re}\, z| < \sigma$, with $\sigma = \inf|\xi_i|$, and admits a holomorphic continuation to $\mathbb{C}\setminus]-\infty,-\sigma]\cup[\sigma,\infty[$. Let us observe that the function φ can be written

$$\varphi(z) = \exp\Big(\int_{\mathbb{R}} \log\Big(\frac{\xi}{\xi - z}\Big)\nu(d\xi)\Big),$$

with

$$\nu = \sum_{i=1}^{n}\tau_i\delta_{\xi_i}.$$

Its logarithmic derivative

$$\Phi(z) = \frac{\varphi'(z)}{\varphi(z)} = \sum_{i=1}^{n}\tau_i\frac{1}{\xi_i - z}$$

is a Pick function. In fact

$$\mathrm{Im}\,\Phi(z) = \mathrm{Im}\, z\sum_{i=1}^{n}\frac{\tau_i}{|\xi_i - z|^2} = \mathrm{Im}\, z\int_{\mathbb{R}}\frac{1}{|u - z|^2}\nu(du).$$

Recall that a Pick function is a holomorphic function Φ defined in $\mathbb{C}\setminus\mathbb{R}$ such that $\Phi(\bar z) = \overline{\Phi(z)}$, and $\mathrm{Im}\,\Phi(z) \geq 0$ if $\mathrm{Im}\, z > 0$. By a theorem of Nevanlinna a Pick function admits the following representation

$$\Phi(z) = \beta + \gamma z + \int_{\mathbb{R}}\frac{1 + z\xi}{\xi - z}\eta(d\xi),$$

with $\beta \in \mathbb{R}$, $\gamma \geq 0$ and η is a bounded positive measure on $\mathbb{R}$. Furthermore

$$\beta = \operatorname{Re}\Phi(i), \ \gamma = \lim_{y\to\infty} \frac{1}{y} \operatorname{Im}\Phi(iy), \ \eta = \frac{1}{2i\pi}\frac{1}{1+\xi^2}[\Phi].$$

Let us observe that this representation can be written

$$\Phi(z) = \beta + \gamma z + \int_{\mathbb{R}} \Big(\frac{1}{\xi - z} - \frac{\xi}{1+\xi^2}\Big)\nu(d\xi),$$

with $\nu(d\xi) = (1+\xi^2)\eta(d\xi)$.

The measure $\gamma(\xi_1, \ldots, \xi_n; \tau_1, \ldots, \tau_n)$ is infinitely divisible. In fact, for $t > 0$, the measures

$$\mu_t = \gamma(\xi_1, \ldots, \xi_n; t\tau_1, \ldots, t\tau_n)$$

form a continuous semi-group of probability measures. Since a limit of infinitely divisible probability measures is infinitely divisible as well, every measure μ in $\mathcal{T}_e$ is infinitely divisible. Its Fourier-Laplace transform is of the form

$$\varphi(z) = \int_{\mathbb{R}} e^{zt}\mu(dt) = e^{\psi(z)},$$

where ψ is a continuous function on $i\mathbb{R}$. Let $\mathcal{B}_e$ denote the set of continuous functions $\psi(z)$ on $i\mathbb{R}$ such that $e^{\psi(z)}$ is the Fourier-Laplace transform of a measure μ in $\mathcal{T}_e$. The Fourier-Laplace transform of the gamma distribution $\gamma(\xi, \tau)$ is

$$\varphi(z) = \big(\frac{\xi}{\xi - z}\big)^{\tau}.$$

Hence the function

$$\psi(z) = \log\frac{\xi}{\xi - z}$$

belongs to $\mathcal{B}_e$. Observe that, for $\beta \in \mathbb{R}^*$,

$$\lim_{n\to\infty}\left(\frac{\frac{n}{\beta}}{\frac{n}{\beta} - z}\right)^n = \lim_{n\to\infty}\Big(1 - \frac{\beta}{n}z\Big)^{-n} = e^{\beta z}$$

is the Fourier-Laplace transform of

$$\lim_{n\to\infty}\gamma\Big(\frac{n}{\beta}; n\Big) = \delta_\beta.$$

Hence $\delta_\beta \in \mathcal{T}_e$, and the function $\psi(z) = \beta z$ belongs to $\mathcal{B}_e$. Similarly, for $\alpha > 0$,

$$\lim_{n\to\infty}\Bigl(\frac{\sqrt{\frac{2n}{\alpha}}}{\sqrt{\frac{2n}{\alpha}}-z}\Bigr)^n\Bigl(\frac{-\sqrt{\frac{2n}{\alpha}}}{-\sqrt{\frac{2n}{\alpha}}-z}\Bigr)^n = \lim_{n\to\infty}\Bigl(1-\frac{1}{n}\frac{\alpha}{2}z^2\Bigr)^{-n} = e^{\alpha\frac{z^2}{2}}$$

is the Fourier-Laplace transform of

$$\lim_{n\to\infty}\Bigl(\gamma\Bigl(\sqrt{\frac{2n}{\alpha}},n\Bigr)*\gamma\Bigl(-\sqrt{\frac{2n}{\alpha}},n\Bigr)\Bigr).$$

Hence the function $\frac{z^2}{2}$ belongs to $\mathcal{B}_e$.

THEOREM 5.1. — *Let ψ be a continuous function on $i\mathbb{R}$, with $\psi(0) = 0$. The following properties are equivalent.*

(i) *The function ψ belongs to $\mathcal{B}_e$: For every $t > 0$, the function $e^{t\psi}$ is the Fourier-Laplace transform of a probability measure in $\mathcal{T}_e$.*

(ii) *The restriction of ψ to $i\mathbb{R}^*$ admits a holomorphic extension to $\mathbb{C}\backslash\mathbb{R}$, the derivative of which is a Pick function.*

(iii) *The function ψ admits the representation*

$$\psi(z) = \beta z + \gamma\frac{z^2}{2} + \int_{\mathbb{R}^*}\Bigl(\log\frac{\xi}{\xi - z} - \frac{\xi z}{1+\xi^2}\Bigr)\nu(d\xi),$$

with $\beta \in \mathbb{R}$, $\gamma \geq 0$, and ν is a positive measure on $\mathbb{R}^*$ such that

$$\int_{0<|\xi|\leq 1}\log\frac{1}{|\xi|}\nu(d\xi) < \infty, \quad \int_{|\xi|\geq 1}\frac{1}{\xi^2}\nu(d\xi) < \infty,$$

or, equivalently

$$\int_{\mathbb{R}^*}\log\Bigl(1+\frac{1}{\xi^2}\Bigr)\nu(d\xi) < \infty.$$

Furthermore

$$\beta = \mathrm{Re}\,\psi'(i), \gamma = \lim_{y\to\infty}\frac{1}{y}\mathrm{Im}\,\psi'(iy),\ \ \nu = \frac{1}{2i\pi}[\psi'].$$

This is a reformulation of results in [Bondesson,1992], Section 7. By the change of variable $\xi \mapsto u = \frac{1}{\xi}$, we get the representation

$$\psi(z) = \beta z + \gamma\frac{z^2}{2} - \int_{\mathbb{R}^*}\bigl(\log(1-uz) + \frac{uz}{u^2+1}\bigr)\nu_0(du),$$

where the measure ν_0, image of the measure ν by this map, satisfies

$$\int_{\mathbb{R}^*} \log(1+u^2)\nu_0(du) < \infty.$$

Observe that

$$\operatorname{Re}\psi(i) = -\frac{1}{2}\Big(\gamma + \int_{\mathbb{R}^*} \log(1+u^2)\nu_0(du)\Big).$$

To the measure ν_0 on $\mathbb{R}^*$ we associate the bounded positive measure $\tilde{\nu}$ on $\mathbb{R}$ defined by, for a bounded continuous function on $\mathbb{R}$,

$$\int_{\mathbb{R}} f(u)\tilde{\nu}(du) = \gamma f(0) + \int_{\mathbb{R}^*} f(u)\log(1+u^2)\nu_0(du).$$

Noticing that

$$\lim_{u\to 0} \frac{1}{u^2}\Big(\log(1-uz) + \frac{uz}{u^2+1}\Big) = -\frac{1}{2}z^2,$$

we obtain the following representation

$$\psi(z) = \beta z - \int_{\mathbb{R}} \Big(\log(1-uz) + \frac{uz}{1+u^2}\Big)\frac{\tilde{\nu}(du)}{\log(1+u^2)}.$$

By modifying slightly the statement of Theorem 7.1.1 in [Bondesson,1992], one gets the following one. On the set $\mathcal{B}_e$ we consider the topology of uniform convergence on compact sets in $i\mathbb{R}$, and on the set $\mathcal{M}(\mathbb{R})$ of positive bounded measures, the tight topology.

THEOREM 5.2. — *The map*

$$\mathcal{B}_e \to \mathbb{R}\times\mathcal{M}(\mathbb{R}), \quad \psi \mapsto (\beta,\tilde{\nu})$$

is a homeomorphism.

Example: Symmetric stable laws

For $0<\alpha\leq 2$, the function ψ defined on $i\mathbb{R}$ by $\psi(iy) = -|y|^\alpha$ belongs to $\mathcal{B}_e$. It extension to $\mathbb{C}\setminus\mathbb{R}$ is given by

$$\begin{aligned}\psi(z) &= -(-iz)^\alpha, \text{ if } \operatorname{Im} z > 0,\\ &= -(iz)^\alpha, \text{ if } \operatorname{Im} z < 0\end{aligned},$$

which is a Pick function. If $0<\alpha<2$, the function ψ admits the following representation

$$\psi(z) = \frac{\alpha}{\pi}\cos(\alpha-1)\frac{\pi}{2}\int_{\mathbb{R}^*}\Big(\log\frac{\xi}{\xi-z} - \frac{\xi z}{1+\xi^2}\Big)|\xi|^{\alpha-1}d\xi.$$

If $\alpha = 2$, then $\psi(z) = z^2$. In that case $\beta = 0$, $\gamma = 2$, and $\nu = 0$.

6. An asymptotic property for the Markov-Krein transform. In this section we consider a sequence (ν_n) in $\mathcal{M}_c(\mathbb{R})$ and the sequence (μ_n) of the Markov-Krein transforms: for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\int_{\mathbb{R}} (1 - zt)^{-\kappa_n} \mu_n(dt) = \exp\Big(\int_{\mathbb{R}} -\log(1 - zu)\nu_n(du)\Big),$$

where $\kappa_n = \nu_n(\mathbb{R})$. We will study the convergence of the sequence (μ_n) assuming that $\kappa_n = \nu_n(\mathbb{R})$ goes to infinity.

We consider first a simple example. Recall that
$M_n(a_1, \ldots, a_n; \tau_1, \ldots, \tau_n)$ is the Markov-Krein transform of the discrete measure $\nu = \sum_{i=1}^n \tau_i \delta_{a_i}$.

Proposition 6.1. — *Fix $\xi \in \mathbb{R}^*$ and $\tau > 0$. For the tight topology*

$$\lim_{n\to\infty} M_2(0, \frac{n}{\xi}; n, \tau) = \gamma(\xi; \tau).$$

Proof.
Assume $\xi > 0$. For a bounded continuous function f on $\mathbb{R}$,

$$\begin{aligned}
&\int_{\mathbb{R}} f(t) M_2(0, \frac{n}{\xi}; n, \tau; dt) \\
&= \frac{\left(\frac{n}{\xi}\right)^{-(n+\tau-1)}}{B(n,\tau)} \int_0^{\frac{n}{\xi}} f(t)\Big(\frac{n}{\xi} - t\Big)^{n-1} t^{\tau-1} dt \\
&= \frac{\xi^\tau}{n^\tau \frac{\Gamma(n)\Gamma(\tau)}{\Gamma(n+\tau)}} \int_0^{\frac{n}{\xi}} f(t)\Big(1 - \frac{t\xi}{n}\Big)^{n-1} t^{\tau-1} dt.
\end{aligned}$$

Hence

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(t) M_2(0, \frac{n}{\xi}; n, \tau); dt = \frac{\xi^\tau}{\Gamma(\tau)} \int_0^\infty f(t) e^{-\xi t} t^{\tau-1} dt. \qquad \square$$

More generally

Proposition 6.2. — *Fix $\xi_1, \ldots, \xi_k \in \mathbb{R}^*$ and $\tau_1, \ldots, \tau_k > 0$. For the tight topology*

$$\lim_{n\to\infty} M_{k+1}\big(0, \frac{n}{\xi_1}, \ldots, \frac{n}{\xi_k}; n, \tau_1, \ldots, \tau_k\big) = \gamma(\xi_1, \ldots, \xi_k; \tau_1, \ldots, \tau_k).$$

Proof.

Put

$$\nu_n = n\delta_0 + \sum_{i=1}^{k} \tau_i \delta_{(\frac{n}{\xi_i})},$$
$$\mu_n = M_{k+1}\big(0, \frac{n}{\xi_1}, \ldots, \frac{n}{\xi_k}; n, \tau_1, \ldots, \tau_n\big).$$

By Theorem 2.1

$$\int_{\mathbb{R}} \frac{1}{(z-t)^{\kappa_n}} \mu_n(dt) = z^{-n} \prod_{i=1}^{k} \frac{1}{\Big(z - \frac{n}{\xi_i}\Big)^{\tau_i}},$$

with $\kappa_n = \tau_1 + \cdots + \tau_k + n$. This relation can also be written

$$\int_{\mathbb{R}} \frac{1}{\Big(1 - \frac{tz}{n}\Big)^{\kappa_n}} \mu_n(dt) = \prod_{i=1}^{k} \Big(\frac{\xi_i}{\xi_i - z}\Big)^{\tau_i}.$$

The two first moments of ν_n are given by

$$p_1^{(n)} = \sum_{i=1}^{k} \tau_i \big(\frac{n}{\xi_i}\big) = n \sum_{i=1}^{n} \frac{\tau_i}{\xi},$$
$$p_2^{(n)} = \sum_{i=1}^{k} \tau_i^2 \big(\frac{n}{\xi}\big)^2 = n^2 \sum_{i=1}^{k} \big(\frac{\tau_i}{\xi_i}\big)^2.$$

Therefore the second moment of μ_n, given by

$$h_2^{(n)} = \frac{2}{\kappa_n(\kappa_n+1)} \big((p_1^{(n)})^2 + p_2^{(n)}\big),$$

is bounded. It follows that the sequence (μ_n) is relatively compact.

LEMMA 6.3. — *Let (μ_n) be a sequence in $\mathcal{M}(\mathbb{R})$ which converges for the tight topology to a measure μ, and let (κ_n) be a sequence of positive numbers going to infinity. Then*

$$\lim_{n\to\infty} \int_{\mathbb{R}} \Big(1 - i\frac{yt}{\kappa_n}\Big)^{-\kappa_n} \mu_n(dt) = \int_{\mathbb{R}} e^{iyt} \mu(dt),$$

uniformly on compact sets.

(See [Curry-Schoenberg,1966], Lemma 3, p.92.)

We continue the proof of Proposition 6.2. Let μ_0 be the limit of a converging subsequence (μ_{n_j}). Then, by Lemma 6.3, for $z \in i\mathbb{R}$,

$$\int_{\mathbb{R}} e^{zt} \mu_0(dt) = \prod_{i=1}^{k} \Big(\frac{\xi_i}{\xi_i - z}\Big)^{\tau_i}.$$

It follows that $\mu_0 = \gamma(\xi_1, \ldots, \xi_k; \tau_1, \ldots, \tau_k)$, and it is the only possible limit for a converging subsequence. This proves that the sequence (μ_n) converges with the limit $\gamma(\xi_1, \ldots, \xi_k; \tau_1, \ldots, \tau_k)$. ∎

PROPOSITION 6.4. — *Assume that $\lim_{n\to\infty} \kappa_n = \infty$, and that the sequence (μ_n) converges to a probability measure μ for the tight topology. Then μ is a Thorin-Bondesson distribution. Moreover, every Thorin-Bondesson distribution is obtained in that way.*

Proof.

Define

$$F_n(z) = \int_{\mathbb{R}} \Big(1 - \frac{zt}{\kappa_n}\Big)^{-\kappa_n} \mu_n(dt).$$

Then, by Lemma 6.3,

$$\lim_{n\to\infty} F_n(iy) = F(iy) := \int_{\mathbb{R}} e^{ity} \mu(dt),$$

uniformly on compact sets in $\mathbb{R}$. On the other hand

$$\begin{aligned} F_n(z) &= \exp\Big(\int_{\mathbb{R}} -\log\big(1 - \frac{zu}{\kappa_n}\big)\nu_n(dt)\Big) \\ &= \exp\Big(\int_{\mathbb{R}} -\log(1 - zu)\widetilde{\nu}_n(du)\Big), \end{aligned}$$

where $\widetilde{\nu}_n$ is the image of ν_n by the dilation of ratio $\frac{1}{\kappa_n}$. By Theorem 5.1 there are Thorin-Bondesson distributions $\widetilde{\mu}_n$ such that, for $z \in i\mathbb{R}$,

$$F_n(z) = \int_{\mathbb{R}} e^{zt} \widetilde{\mu}_n(dt).$$

By Lévy-Cramer Theorem,

$$\lim_{n\to\infty} \widetilde{\mu}_n = \mu$$

for the tight topology. Since the set $\mathcal{T}_e$ of Thorin-Bondesson distributions is closed for the tight topology, it follows that μ is a Thorin-Bondesson distribution.

The set of such limits is closed. On the other hand, by Proposition 6.2, this set contains the gamma convolutions $\gamma(\xi_1, \ldots, \xi_k; \tau_1, \ldots, \tau_k)$. Hence this set is dense in $\mathcal{T}_e$. Being closed and dense it is equal to $\mathcal{T}_e$. $\square$

The following theorem describes the representation for the Fourier-Laplace transform of the Thorin-Bondesson distribution μ, limit of the sequence (μ_n). Define

$$\beta_n = \int_{\mathbb{R}} u\widetilde{\nu}_n, \quad \sigma_n(du) = u^2\widetilde{\nu}_n(du),$$

where $\widetilde{\nu}_n$ is, as before, the image of ν_n by the dilation of ratio $\frac{1}{\kappa_n}$.

THEOREM 6.5. — *Assume that β_n and σ_n have limits,*

$$\lim_{n\to\infty} \beta_n = \beta, \quad \lim_{n\to\infty} \sigma_n = \sigma$$

(for the tight topology). Then μ_n has a limit μ whose Fourier-Laplace transform is given by

$$\int_{\mathbb{R}} e^{zt}\mu(dt) = \exp\Big(\beta z - \int_{\mathbb{R}} \frac{\log(1-zu)+zu}{u^2}\sigma(du)\Big).$$

Observe that

$$\lim_{u\to 0} \frac{\log(1-zu)+zu}{u^2} = -\frac{z^2}{2}.$$

Therefore the function

$$u \mapsto \frac{\log(1-zu)+zu}{u^2}$$

has a continuous extension to $\mathbb{R}$, and the formula in the theorem can be written

$$\int_{\mathbb{R}} e^{zt}\mu(dt) = \exp\Big(\beta z + \frac{1}{2}\gamma z^2 - \int_{\mathbb{R}^*} \big(\log(1-zu)+zu\big)\tau(du)\Big),$$

with $\gamma = \sigma(\{0\})$, and τ is the measure on $\mathbb{R}^*$ given by $\tau(du) = \frac{1}{u^2}\sigma(du)$.

Proof.

Let us prove that the sequence (μ_n) is relatively compact. For that we will show that the second moments $h_2^{(n)}$ of the measures μ_n are bounded. We know that

$$h_2^{(n)} = \frac{2}{\kappa_n(\kappa_n+1)}\big((p_1^{(n)})^2 + p_2^{(n)}\big),$$

where $p_m^{(n)}$ are the moments of order m of the measures ν_n. Since

$$p_1^{(n)} = \kappa_n \beta_n, \; p_2^{(n)} = \kappa_n^2 \sigma_n(\mathbb{R}),$$

we get

$$h_2^{(n)} = \frac{2\kappa_n}{\kappa_n + 1}\big(\beta_n^2 + \sigma_n(\mathbb{R})\big).$$

The sequences $\big(\sigma_n(\mathbb{R})\big)$ and (β_n) are converging, and hence the sequence $\big(h_2^{(n)}\big)$ is bounded. Therefore the sequence (μ_n) is relatively compact. Let μ_0 be the limit of a converging subsequence of (μ_n). We get

$$\int_{\mathbb{R}} e^{zt} \mu_0(dt) = \exp\Big(\beta z - \int_{\mathbb{R}} \frac{\log(1 - zu) + zu}{u^2} \sigma(du)\Big).$$

This shows that there exists only one possible limit for a converging subsequence. Therefore the sequence (μ_n) converges. $\square$

Let us consider the case where

$$\nu_n = \sum_{k=1}^{n} \tau_i^{(n)} \delta_{a_i^{(n)}},$$

where $a^{(n)} = (a_1^{(n)}, \dots a_n^{(n)})$ and $\tau^{(n)} = (\tau_1^{(n)}, \dots, \tau_n^{(n)})$ are n-uples of real numbers. Then $\mu_n = M_n(\tau^{(n)}; a^{(n)})$, and

$$\kappa_n = \sum_{i=1}^{n} \tau_i^{(n)}, \; \beta_n = \sum_{i=1}^{n} \tau_i^{(n)} \alpha_i^{(n)}, \; \sigma_n = \sum_{i=1}^{n} \kappa_i^{(n)} (\alpha_i^{(n)})^2 \delta_{\alpha_i^{(n)}},$$

with $\alpha_i^{(n)} = \frac{1}{\kappa_n} a_i^{(n)}$.

THEOREM 6.6. — *Assume that the numbers $\tau_i^{(n)}$ are bounded from below: $\tau_i^{(n)} \geq \tau$ with $\tau > 0$. Assume that the measure σ_n converges to a measure σ for the tight topology.*

(i) *Then σ has the form*

$$\sigma = \sum_{j=1}^{\infty} \tau_j \alpha_j^2 \delta_{\alpha_j} + \gamma \delta_0,$$

where (α_j) is a sequence of real numbers, $\tau_j \geq \tau$, and $\gamma \geq 0$.

(ii) *Assume moreover that* $\lim_{n\to\infty}\beta_n=\beta$. *Then the measure* $\mu_n = M_n(\tau^{(n)},a^{(n)})$ *converges to a Thorin-Bondesson distribution* μ *such that*

$$\int_{\mathbb{R}} e^{zt}\mu(dt) = e^{\frac{1}{2}\gamma z^2}e^{\beta z}\prod_{j=1}^{\infty}\Big(\frac{e^{-\alpha_j z}}{1-z\alpha_j}\Big)^{\kappa_j}.$$

Proof.

Part (i) follows from

LEMMA 6.7. — *Let* (μ_n) *be a sequence of discrete measures of the form*

$$\mu_n = \sum_{i=1}^{n}\kappa_i^{(n)}\delta_{\alpha_i^{(n)}},$$

where $\alpha_i^{(n)}$ *and* $\kappa_i^{(n)}$ *are real numbers. Assume that* $\kappa_i^{(n)} \geq \kappa > 0$ *for all* n *and* i, *and that* μ_n *converges to* μ *for the vague topology. Then* μ *is of the form*

$$\mu = \sum_{j=1}^{\infty}\kappa_i\delta_{\alpha_j},$$

where (α_j) *is a sequence of real numbers and* $\kappa_i \geq \kappa$.

Part (ii) follows from Theorem 6.5. ∎

For $a_1 < a_2 < \cdots < a_n$, $\tau_1 = \ldots = \tau_n = 1$, the probability measure $M_n(a_1,\ldots,a_n;1,\ldots,1)$ is a spline distribution. In that special case the following theorem has been established by Schoenberg and Curry:

THEOREM 6.8. — *Assume that a sequence* $\mu_n = M_n(a_1^{(n)},\ldots,a_n^{(n)};1,\ldots,1)$ *converges to a measure* μ. *Then* μ *is a Pólya distribution: its Fourier-Laplace transform is a Pólya function,*

$$\Phi(z) = \int e^{zt}\mu(dt), = e^{\frac{1}{2}\gamma z^2}e^{\beta z}\prod_{j=1}^{\infty}\frac{e^{-\alpha_j z}}{1-z\alpha_j},$$

with

$$\gamma \geq 0,\ \beta\in\mathbb{R},\ \alpha_j\in\mathbb{R},\ \sum_{j=1}^{\infty}\alpha_j^2 < \infty.$$

Conversely every Pólya distribution is the limit of such a sequence of spline distributions.

([Curry-Schoenberg,1996], Theorem 6, p.93.)

7. The Markov-Krein transform and the Markov moment problem. — The map

$$\nu \mapsto (\mu, \kappa), \quad \mathcal{M}_c(\mathbb{R}) \to \mathcal{M}_c^1(\mathbb{R}) \times \mathbb{R}_+,$$

where μ is the Markov-Krein transform of ν and $\kappa = \nu(\mathbb{R})$, is injective, but not surjective. It is an open question to determine the image of this map. We will present a result by Kerov which is related to that question. Kerov made the following definition: a continuous diagram supported by a compact interval $[a, b]$ is a real function ω defined on $\mathbb{R}$ satisfying

$$|\omega(u_1) - \omega(u_2)| \leq |u_1 - u_2| \quad (u_1, u_2 \in \mathbb{R}),$$

and there is $c \in \mathbb{R}$ such that $\omega(u) = |u - c|$ for $u \notin [a, b]$ ([Kerov,2003], p.48 and p.150).

The terminology comes from the representation theory of the symmetric group. By the theorem of Ascoli-Arzela, the set $\mathcal{D}[a, b]$ of continuous diagrams supported by $[a, b]$ is compact for the topology of uniform convergence.

To a continuous diagram $\omega \in \mathcal{D}[a, b]$ we associate the distribution $\nu_\omega = \frac{1}{2}\omega''$ (the second derivative is taken in the distribution sense). Then $\langle \nu_\omega, 1 \rangle = 1$ and ν_ω is a probability measure if and only if ω is convex. The map $\omega \mapsto \nu_\omega''$ is injective, and, if ν_ω is a measure, then

$$\omega(u) = \int_{\mathbb{R}} |u - x| \nu_\omega(dx).$$

THEOREM 7.1. — *The map which associates to a continuous diagram $\omega \in \mathcal{D}[a, b]$ the Markov transform μ of ν_ω is a homeomorphism from $\mathcal{D}[a, b]$ onto the set $\mathcal{M}^1[a, b]$ of probability measures on $[a, b]$.*

([Kerov,2003], p.152.)

The Markov-Krein transform μ of ν_ω is determined by the relation

$$\int_{[a,b]} \frac{1}{z - t} \mu(dt) = \exp\Big(-\langle \nu_\omega, \log(z - u) \rangle\Big).$$

Proof.

The main step in the proof is as follows. Consider interlacing sequences

$$a_1 < b_1 < a_2 < \cdots < b_{n-1} < a_n.$$

Then there is a continuous diagram ω such that

$$\nu_\omega = \frac{1}{2}\omega'' = \sum_{i=1}^{n} \delta_{a_i} - \sum_{i=1}^{n-1} \delta_{b_i}.$$

It is called a rectangular diagram: a piecewise affine function, each affine segment has slope ± 1. Then

$$F(z) := \exp\Big(-\langle \nu_\omega, \log(z-u)\rangle\Big) = \frac{\prod_{i=1}^{n-1}(z-b_i)}{\prod_{i=1}^{n}(z-a_i)}.$$

is a rational function with simple poles at $a_1, \dots, a_n$ which can be written

$$F(z) = \sum_{i=1}^{n} \frac{\alpha_i}{z-a_i}.$$

From the interlacing property it follows that the numbers α_i are positive, and the probability measure

$$\mu = \sum_{i=1}^{n} \alpha_i \delta_{a_i}$$

is the Markov-Krein transform of ν_ω. One can see that the Markov-Krein transform is a bijection from the set of measures

$$\nu = \sum_{i=1}^{n} \delta_{a_i} - \sum_{i=1}^{n-1} \delta_{b_i},$$

with interlacing sequences: $a_1 < b_1 < a_2 < \cdots < b_{n-1} < a_n$, onto the set of probability measures

$$\mu = \sum_{i=1}^{n} \alpha_i \delta_{a_i}.$$

Then one shows that this map extends continuously from $\mathcal{D}[a,b]$ onto $\mathcal{M}^1[a,b]$. $\square$

In general the distribution ν_ω is not a measure. In fact let us produce an example of a continuous diagram such that ν_ω is not a measure. Take $[a,b] = [0,1]$ and consider the sequence of measures

$$\nu_n = \sum_{k=1}^{2n+1} (-1)^{k-1} \delta_{\frac{1}{k}},$$

and the sequence of rectangular diagrams

$$\omega_n(u) = \int_{\mathbb{R}} |u-x| \nu_n(dx).$$

Then

$$\nu_{n+1} - \nu_n = -\delta_{\frac{1}{2n+2}} + \delta_{\frac{1}{2n+3}},$$

and

$$\sup |\omega_{n+1}(u) - \omega_n(u)| \leq \frac{1}{2n+2} - \frac{1}{2n+3}.$$

Hence the sequence (ω_n) converges uniformly to a continuous diagram ω. In the distribution sense

$$\frac{1}{2}\omega'' = \lim_{n\to\infty} \nu_n.$$

Since

$$\|\nu_n\| = \sum_{k=1}^{2n+1} \frac{1}{k}$$

is unbounded, ω'' is not a measure by the theorem of Banach-Steinhaus.

Consider the first derivative ω' of a continuous diagram. It is a measurable function f on $\mathbb{R}$ such that

$$-1 \leq f(u) \leq 1, \ f(u) = -1 \text{ for } u < a, \ f(u) = 1 \text{ for } u > b,$$

and the function $h(u) = \frac{1}{2}(\omega' + 1)$ satisfies

$$0 \leq h(u) \leq 1, \ h(u) = 0 \text{ if } u < a, \ h(u) = 1 \text{ if } u > b.$$

The function $h(u) - Y(u-b)$ has compact support and derivative $\nu_\omega - \delta_b$. Therefore

$$\langle \nu_\omega - \delta_b, \log(z-u)\rangle = \langle h - Y(u-b), \frac{1}{z-u}\rangle = \int_a^b \frac{1}{z-u} h(u)du.$$

If μ is the Markov-Krein transform of ν_ω,

$$\begin{aligned}\int_{\mathbb{R}} \frac{1}{z-t}\mu(dt) &= \exp\Big(-\int_a^b \frac{h(u)}{z-u}du - \log(z-b)\Big) \\ &= \frac{1}{z-b}\exp\Big(-\int_a^b \frac{h(u)}{z-u}du\Big).\end{aligned}$$

Corollary 7.2 (Krein-Nudel'man). — *The map which associates to the function h the probability measure μ such that*

$$\int_{[a,b]} \frac{1}{z-t}\mu(dt) = \frac{1}{z-b}\exp\Big(-\int_a^b \frac{h(u)}{z-u}du\Big)$$

is a bijection form the set of measurable functions h on $[a,b]$, satisfying $0 \leq h(u) \leq 1$, onto $\mathcal{M}^1([a,b])$.

[Krein-Nudel'man,1977], p.395,396.

Recall the Markov moment problem. Consider a sequence (c_m) of Hausdorff moments:

$$c_m = \int_{[a,b]} u^m d\sigma(du),$$

with $\sigma \in \mathcal{M}^1([a,b])$. The problem is to determine under which condition the measure σ is absolutely continuous with respect to the Lebesgue measure: $\sigma(du) = h(u)du$, with $0 \leq h(u) \leq 1$.

Theorem 7.3 (Krein-Nudel'man). — *The sequence (c_m) is a Markov moment sequence if and only if the sequence (a_m), defined by*

$$\sum_{m=0}^{\infty} \frac{a_m}{z^{m+1}} = \frac{1}{z-b}\exp\Big(-\sum_{m=0}^{\infty} \frac{c_m}{z^{m+1}}\Big),$$

is a Hausdorff moment sequence: there is $\mu \in \mathcal{M}^1([a,b])$ such that

$$a_m = \int_{[a,b]} t^m \mu(dt).$$

[Krein-Nudel'man,1977], p.243.

8. Example of Tricomi. — In this last section we revisit an example studied by Tricomi [1933] (see also [Schoenberg,1946], [Curry-Schoenberg,1966], Example 4 p.104). We consider spline distributions with equidistant knots: $a_j = a + ju$, with $a \in \mathbb{R}$, $u > 0$, $j = 0, \ldots, n$, and

$$\mu_n = M_{n+1}(a, a+u, \ldots a+nu; 1, \ldots, 1).$$

It is the Markov-Krein transform of the measure

$$\mu_n = \sum_{j=0}^{n} \delta_{a_j}.$$

From the formula

$$c_j = \prod_{i \neq j} \frac{1}{a_i - a_j},$$

in Section 4, Example 1, we get

$$c_j = (-1)^j \frac{1}{u^n} \frac{1}{j!(n-j)!}.$$

Therefore the density h_n of the measure μ_n is given by

$$h_n(t) = \frac{1}{u^n} \frac{1}{(n-1)!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (t - a - ju)_+^{n-1}.$$

This formula is also given in a slightly different form in [Uspensky,1937] (Example 3 p.277), and is essentielly due to Laplace (*Mémoire sur les probabililtés*, 1778, 1781 , § IX, p.404 [Laplace,1893]). One can check that $\mathrm{supp}(h_n) = [a, a+n]$. In fact, for $t \geq a + n$,

$$h_n(t) = \frac{1}{h^n} \frac{1}{(n-1)!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (t - a - ju)^{n-1}.$$

and, for any polynomial p with $\deg(p) \leq n - 1$,

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} p(x - j) \equiv 0.$$

Recall that the divided differences are defined as follows: for $a_1 < a_2 < \cdots < a_n$, and a function f on $\mathbb{R}$,

$$\begin{aligned} f[a_1, a_2] &= \frac{f(a_2) - f(a_1)}{a_2 - a_1}, \\ f[a_1, a_2, \ldots, a_n] &= \frac{f[a_2, \ldots, a_n] - f[a_1, \ldots a_{n-1}]}{a_n - a_1}, \end{aligned}$$

and the Hermite-Genocchi formula: for a function f of class $\mathcal{C}^{n-1}$,

$$f[a_1, \ldots, a_n] = \frac{1}{(n-1)!} \int_{\mathbb{R}} f^{(n-1)}(t) M_n(a_1, \ldots, a_n; 1, \ldots, 1; dt).$$

(See for instance [Faraut,2005], Theorem 1.1.) In the present case,

$$f[a, a+u, a+2u, \ldots, a+nu] = \frac{1}{n!} (\Delta_u^n f)(a),$$

where Δ_u is defined by

$$(\Delta_u f)(t) = \frac{f(t+u) - f(t)}{u}.$$

The Hermite-Genocchi formula can be written in this special case, for a function f of class $\mathcal{C}^n$,

$$\big(\Delta_u^n f\big)(u) = \int_{\mathbb{R}} f^{(n)}(t)\mu_n(dt).$$

For the special case $f(t) = e^{tz}$, we get

$$\Big(\frac{e^{uz}-1}{u}\Big)^n e^{az} = z^n \int_{\mathbb{R}} e^{tz}\mu_n(dt).$$

Hence the Fourier-Laplace transform of μ_n is given by

$$\widehat{\mu_n}(z) = e^{za}\Big(\frac{z^{uz}-1}{uz}\Big)^n.$$

Therefore the measure μ_n equals the following convolution product:

$$\mu_n = \delta_a * \mu^{*n},$$

with

$$\int_{\mathbb{R}} f(t)\mu(dt) = \frac{1}{u}\int_0^u f(t)dt.$$

Taking $a = -n\frac{u}{2}$, we get

$$\mu_n = M_{n+1}\big(-n\frac{u}{2}, -(n-2)\frac{u}{2}, \ldots, (n-2)\frac{u}{2}, n\frac{u}{2}; 1, \ldots, 1\big).$$

The density $h_n(t)$ of μ_n is given by

$$h_n(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \Big(\frac{\sin\frac{ux}{2}}{\frac{ux}{2}}\Big)^n e^{itx}dx.$$

The measure μ_n is the Markov-Krein transform of

$$\mu_n = \sum_{k=0}^{n} \delta_{a_k}, \text{ with } a_k = (2k-n)\frac{u}{2},$$

and $\kappa_n = \nu_n(\mathbb{R}) = n + 1$. Then, with the notation of Section 6,

$$\sigma_n = \sum_{k=0}^{n} (\alpha_k^{(n)})^2 \delta_{\alpha_k^{(n)}}, \text{ with } \alpha_k^{(n)} = \frac{(2k-n)}{n+1}\frac{u}{2},$$

and

$$\operatorname{supp}(\sigma_n) \subset \Big[-\frac{n}{n+1}\frac{u}{2}, \frac{n}{n+1}\frac{u}{2}\Big],$$

$$\sigma_n(\mathbb{R}) = \frac{1}{(n+1)^2}\frac{u^2}{4}\sum_{k=0}^{n}(2k-n)^2 = \frac{1}{(n+1)^2}\frac{u^2}{4}\frac{n(n+1)(n+2)}{3}.$$

We take now $u = 2\sqrt{\frac{3}{n}}$. Then

$$\operatorname{supp}(\sigma_n) \subset \Big[-\frac{\sqrt{3n}}{n+1}, \frac{\sqrt{3n}}{n+1}\Big],$$

and

$$\sigma_n(\mathbb{R}) = \frac{n+2}{n+1}.$$

Hence

$$\lim_{n\to\infty} \sigma_n = \delta_0.$$

By Theorem 6.3, the measure μ_n converges to the normal Gaussian measure:

$$\lim_{n\to\infty} \mu_n(dt) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt.$$

From the proof of Theorem VI.1 it follows that

$$\lim_{n\to\infty} \prod_{k=0}^{n}\Big(1 - z\frac{a_k}{\kappa_n}\Big) = \lim_{n\to\infty}\prod_{k=0}^{n}\Big(1 - z\frac{2k-n}{n+1}\sqrt{\frac{3}{n}}\Big) = e^{-\frac{z^2}{2}}.$$

Observe that the Fourier-Laplace transform of μ_n is given by

$$\widehat{\mu_n}(z) = \int_{\mathbb{R}} e^{zt}\mu_n(dt) = \left(\frac{\sinh z\sqrt{\frac{3}{n}}}{z\sqrt{\frac{3}{n}}}\right)^n,$$

and that

$$\lim_{n\to\infty}\left(\frac{\sinh z\sqrt{\frac{3}{n}}}{z\sqrt{\frac{3}{n}}}\right)^n = e^{\frac{z^2}{2}}.$$

Since

$$\mu_n = \mu_1^{*n},$$

where μ_1 is the measure given by

$$\int_{\mathbb{R}} f(t)\mu_1(dt) = \frac{1}{u}\int_{-\frac{u}{2}}^{\frac{u}{2}} f(t)dt,$$

the convergence of μ_n to the normal Gaussian measure also follows from the central limit theorem.

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