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RANDOM MATRICES AND ORTHOGONAL POLYNOMIALS

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The central question of the theory of random matrices is to determine the asymptotic behavior of the eigenvalues of large random symmetric or Hermitian matrices. In the case of the unitary Gaussian ensemble, i.e. the space of Hermitian matrices equipped with a unitarily invariant Gaussian probability, Mehta's formulae express the eigenvalue density in terms of the Christoffel-Darboux kernel of the Hermite polynomials. In fact orthogonal polynomials are a powerful tool in this theory. We will present in this course methods in the theory of random matrices which are using orthogonal polynomials.

I. INTRODUCTION

For $\mathbb{F}=\mathbb{R},\mathbb{C}$ or $\mathbb{H}$, let $H_n = Herm(n,\mathbb{F})$ be the space of $n\times n$ Hermitian matrices with entries in $\mathbb{F}$. On H_n one considers the probability law defined by

$$\mathbb{P}_n(dx)=\frac{1}{C_n}\exp\bigl(-\gamma tr(x^2)\bigr)m_n(dx),$$

where γ is a positive parameter, m_n is the Euclidean measure associated with the inner product

$$(x|y)=tr(xy),$$

and

$$C_n=\int_{H_n}\exp\bigl(-\gamma tr(x^2)\bigr)m_n(dx)=\Bigl(\sqrt{\frac{\pi}{\gamma}}\Bigr)^N,$$

where

$$N=\dim_{\mathbf{R}}H_n=n+\frac{\beta}{2}n(n-1),\quad \beta=\dim_{\mathbb{R}}\mathbb{F}=1,2,4.$$

This probability is invariant under the group $U_n = U(n;\mathbb{F})$ of $n\times n$ unitary matrices with entries in $\mathbb{F}$, acting on H_n by the transformations

$$x \mapsto uxu^* \quad (u \in U_n).$$

For $\mathbb{F}=\mathbb{R}$, it is the orthogonal group $O(n)$, for $\mathbb{F}=\mathbb{C}$ it is the unitary group $U(n)$, and for $\mathbb{F}=\mathbb{H}$, it is isomorphic to the symplectic group $Sp(n)$, maximal compact subgroup of the complex symplectic group $Sp(n,\mathbb{C})$.

The probability space $(H_n, \mathbb{P}_n)$ is called *Gaussian orthogonal ensemble* for $\mathbb{F}=\mathbb{R}$, *Gaussian unitary ensemble* for $\mathbb{F}=\mathbb{C}$, and *Gaussian symplectic ensemble* for $\mathbb{F}=\mathbb{H}$.

The general problem in the theory of random matrices is to study asymtotics of probabilities related to the eigenvalues of a random matrix for large n.

a) *Statistical distribution of the eigenvalues*

If $B \subset \mathbb{R}$ is a Borel set, one denotes by $\xi_{n,B}$ the random variable defined by

$$\xi_{n,B}(x) = \frac{1}{n}\#\{\text{eigenvalues of } x \text{ in } B\}.$$

Let $\mu_n(B)$ be its expectation,

$$\mu_n(B) = \mathbb{E}_n(\xi_{n,B}).$$

Then μ_n is a probability measure on $\mathbb{R}$, it is the statistical distribution of the eigenvalues. If χ_B is the characteristic function of the set B, then

$$\xi_{n,B}(x) = \frac{1}{n}\big(\chi_B(\lambda_1) + \cdots + \chi_B(\lambda_n)\big),$$

if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of x. In the sense of symbolic calculus this can be written

$$\xi_{n,B}(x) = \frac{1}{n}\operatorname{tr}\chi_B(x).$$

Therefore

$$\mu_n(B) = \frac{1}{n}\int_{H_n} \operatorname{tr}\chi_B(x)\mathbb{P}_n(x).$$

More generally, if φ is a bounded measurable function on $\mathbb{R}$,

$$\int_{\mathbb{R}} \varphi(t)\mu_n(dt) = \frac{1}{n}\int_{H_n} \operatorname{tr}\big(\varphi(x)\big)\mathbb{P}_n(dx).$$

Question : what can be said about the asymptotics of μ_n as n goes to infinity ? The answer is given by the following theorem of Wigner.

The semi-circle law σ_a of radius a is the probability measure defined on $\mathbb{R}$ by

$$\int_{\mathbb{R}} \varphi(t)\sigma_a(dt) = \frac{2}{\pi a^2}\int_{-a}^{a} \varphi(t)\sqrt{a^2-t^2}dt.$$

The theorem of Wigner says that, after scaling, the measure μ_n converges to the semi-circle law σ_a of radius

$$a = \sqrt{\frac{\beta}{\gamma}}.$$

THEOREM (WIGNER). — *Let φ be a bounded continuous function on $\mathbb{R}$. Then*

$$\lim_{n\to\infty}\int_{\mathbb{R}} \varphi\Big(\frac{t}{\sqrt{n}}\Big)\mu_n(dt) = \frac{2}{\pi a^2}\int_{-a}^{a}\varphi(u)\sqrt{a^2-u^2}du.$$

This means that, for large n, the density of eigenvalues is approximatively

$$\frac{2}{\pi a^2}\sqrt{na^2-\lambda^2},$$

if $|\lambda| \leq a\sqrt{n}$, and 0 if $|\lambda| \geq a\sqrt{n}$.

In the original proof Wigner considers the moments of the measure μ_n:

$$\mathfrak{M}_k(\mu_n) = \int_{\mathbb{R}} t^k\mu_n(dt) = \frac{1}{n}\int_{H_n}\mathrm{tr}(x^k)\mathbb{P}_n(dx),$$

and by combinatorial computations determines the asymptotics of $\mathfrak{M}_k(\mu_n)$ as n goes to infinity: for k fixed,

$$\mathfrak{M}_{2k}(\mu_n) \sim \Big(\frac{\beta}{4\gamma}\Big)^k\frac{(2k)!}{k!(k+1)!}n^k.$$

Note that the moments of odd order vanish. On the other hand it is easy to compute the moments of the semi-circle law:

$$\mathfrak{M}_{2k}(\sigma_a) = \Big(\frac{a^2}{4}\Big)^k\frac{(2k)!}{k!(k+1)!}.$$

In fact

$$\begin{aligned}\mathfrak{M}_{2k}(\sigma_a) &= \frac{2}{\pi a^2}\int_{-a}^{a} t^{2k}\sqrt{a^2-t^2}dt = \frac{2a^{2k}}{\pi}\int_0^1 u^{k-\frac{1}{2}}\sqrt{1-u}du \\ &= \frac{2a^{2k}}{\pi}B\Big(k+\frac{1}{2},\frac{3}{2}\Big) = \frac{2a^{2k}}{\pi}\frac{\Gamma\big(k+\frac{1}{2}\big)\Gamma\big(\frac{3}{2}\big)}{\Gamma(k+2)} = \frac{a^{2k}}{2^{2k}}\frac{(2k)!}{k!(k+1)!}.\end{aligned}$$

The proof by Pastur uses the Cauchy transform. Recall that the Cauchy transform of a probability measure μ on $\mathbb{R}$ is the function G_μ defined on $\mathbb{C} \setminus \mathbb{R}$ by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t} \mu(dt).$$

For $\mu = \mu_n$, writing $G_{\mu_n} = G_n$,

$$G_n(z) = \frac{1}{n} \int_{H_n} tr\big((zI - x)^{-1}\big) \mathbb{P}_n(dx).$$

After scaling one has to look at the functions

$$\tilde{G}_n(z) = \sqrt{n} G_n(\sqrt{n} z).$$

The proof amounts to showing that the functions $\tilde{G}_n$ converge,

$$\lim_{n\to\infty} \tilde{G}_n(z) = f(z),$$

and that the limit f is a holomorphic function satisfying

$$f(z)^2 - \frac{4}{a^2} z f(z) + \frac{4}{a^2} = 0.$$

Since $\Im G_n(z) < 0$ and hence $\Im f(z) < 0$ for $\Im z > 0$, necessarily

$$f(z) = \frac{2}{a^2}\big(z - \sqrt{z^2 - a^2}\big),$$

which is the Cauchy transform of the semi-circle law σ_a.

The proof we will present uses the Fourier transform,

$$\widehat{\mu_n}(\tau) = \int_{\mathbb{R}} e^{-it\tau} \mu_n(t) = \frac{1}{n} \int_{H_n} \operatorname{tr}\big(\exp(-i\tau x)\big) \mathbb{P}_n(dx).$$

We will see that it can be computed in terms of Laguerre polynomials. The convergence to the semi-circle law will follow by using the classical Lévy-Cramér theorem.

More general results are obtained by using logarithmic potential theory. One defines the energy of a probability measure μ by

$$I(\mu) = \int_{\mathbb{R}^2} \log \frac{1}{|s-t|} \mu(ds)\mu(dt) + \int_{\mathbb{R}} V(t)\mu(dt).$$

For $V(t) = \gamma t^2$, the semi-circle law appears as equilibrium measure: measure which realizes the minimum of the energy.

b) *Local behaviour : the probabilities* $A_n(m, \theta)$

For $\theta > 0$, and $0 \leq m \leq n$, one denotes by $A_n(m, \theta)$ the probability that a matrix $x \in H_n$ has m eigenvalues in the interval $[-\theta, \theta]$. By using orthogonal polynomials one can evaluate the probability $A_n(m, \theta)$ in terms of Fredholm determinants, and its behaviour as $n \to \infty$. In particular we will see that, for $m = 0$,

$$\lim_{n \to \infty} A_n\Big(0, \frac{\theta}{\sqrt{2n}}\Big) = \mathrm{Det}_{[-\theta,\theta]}(I - \mathcal{K}),$$

where Det is the Fredholm determinant, and $\mathcal{K}$ is the kernel

$$\mathcal{K}(\xi, \eta) = \frac{1}{\pi} \frac{\sin(\xi - \eta)}{\xi - \eta},$$

restricted to the square $[-\theta, \theta] \times [-\theta, \theta]$.

c) In the last chapter we consider the *Wishart unitary ensemble*. In that case there is an analogue Wigner Theorem: It is Marchenko-Pastur Theorem which describes the asymptotic of the statistical distribution of the eigenvalues for a Wishart random matrix.

II ORTHOGONAL POLYNOMIALS

1. Heine's formulae. — Let μ be a positive measure on $\mathbb{R}$. We assume that the support of μ is infinite, and that, for all $m \geq 0$,

$$\int_{\mathbb{R}} |t|^m \mu(dt) < \infty.$$

Hence, for all $j \in \mathbb{N}$, the moment of order j,

$$m_j = \int_{\mathbb{R}} t^j \mu(dt),$$

is defined. On the space $\mathcal{P}$ of polynomials in one variable with real coefficients one considers the inner product

$$(p|q) = \int_{\mathbb{R}} p(t)q(t)\mu(dt),$$

for which $\mathcal{P}$ is a pre-Hilbert space. The monomials $1, t, \ldots, t^m, \ldots$ are independent, and, by the Gram-Schmidt orthogonalization, one gets a sequence $\{p_m\}$ of orthogonal polynomials: p_m is of degree m, and

$$\int_{\mathbb{R}} p_m(t)p_n(t)\mu(dt) = 0 \text{ if } m \neq n.$$

If $\{p_m\}$ is a sequence of orthogonal polynomials we will write

$$p_m(t) = a_m t^m + \cdots,$$
$$d_m = \int_{\mathbb{R}} p_m(t)^2 \mu(dt).$$

Example: Hermite polynomials. The measure μ is Gaussian :

$$\mu(dt) = e^{-t^2} dt.$$

The Hermite polynomial H_m is defined by

$$H_m(t) = (-1)^m e^{t^2} \left(\frac{d}{dt}\right)^m e^{-t^2}.$$

Notice that $a_m = 2^m$. By integrating by parts one shows that

$$d_m = 2^m m! \sqrt{\pi}.$$

In fact, for any polynomial p,

$$\int_{\mathbb{R}} H_m(t)p(t)e^{-t^2}dt = \int_{\mathbb{R}} p^{(m)}(t)e^{-t^2}dt,$$

and

$$\int_{\mathbb{R}} e^{-t^2}dt = \sqrt{\pi}.$$

Let us consider the matrix of the moments of the measure μ:

$$M_{ij} = m_{i+j} = \int_{\mathbb{R}} t^{i+j}\mu(dt).$$

It is the matrix of the quadratic form $p \mapsto \|p\|^2$ with respect to the basis $\{1, t, \dots, t^m, \dots\}$. One defines

$$D_n = \det\Big((M_{ij})_{0\le i,j\le n-1}\Big).$$

PROPOSITION II.1.1.

$$D_n = \frac{1}{n!}\int_{\mathbb{R}^n} \prod_{1\le i<j\le n} (x_j - x_i)^2 \mu(dx_1)\dots\mu(dx_n).$$

Proof. The determinant D_n can be written as an integral on $\mathbb{R}^n$:

$$D_n = \int_{\mathbb{R}^n} \begin{vmatrix} x_1^0 & x_2^1 & \dots & x_n^{n-1} \\ x_1^1 & x_2^2 & \dots & x_n^n \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^n & \dots & x_n^{2n-2} \end{vmatrix} \mu(x_1)\dots\mu(dx_n)$$

$$= \int_{\mathbb{R}^n} \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix} x_1^0 \cdot x_2^1 \cdot x_3^2 \cdots x_n^{n-1}\mu(dx_1)\dots\mu(dx_n).$$

This integral does not change under a permutation $\sigma \in \mathfrak{S}_n$ of $\{1, \dots, n\}$. Therefore

$$D_n = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)$$

$$\int_{\mathbb{R}^n} \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix} x_{\sigma(1)}^0 x_{\sigma(2)}^1 \dots x_{\sigma(n)}^{n-1}\mu(dx_1)\dots\mu(dx_n).$$

By the classical evaluation of the Vandermonde determinant

$$\Delta(x) = \prod_{1\leq i<j\leq n} (x_j - x_i) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix}$$
$$= \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma) x^0_{\sigma(1)} x^1_{\sigma(2)} \dots x^{n-1}_{\sigma(n)},$$

the result is established. ∎

We assume that the orthogonal polynomials are normalized by the condition

$$p_m(t) = t^m + \cdots,$$

i.e. $a_m = 1$.

Proposition II.1.2.

$$D_n = d_0 d_1 \dots d_{n-1}.$$

Proof. Consider the polynomials in n variables $p_{\mathbf{m}}$ defined, for $\mathbf{m} = (m_1, m_2, \dots, m_n) \in \mathbb{N}^n$, $x = (x_1, x_2, \dots, x_n)$, by

$$p_{\mathbf{m}}(x) = p_{m_1}(x_1) p_{m_2}(x_2) \dots p_{m_n}(x_n).$$

They are orthogonal for the inner product

$$(p|q) = \int_{\mathbb{R}^n} p(x)q(x)\mu(dx_1)\dots\mu(dx_n),$$

and

$$\|p_{\mathbf{m}}\|^2 = d_{m_1} d_{m_2} \dots d_{m_n}.$$

Consider the expansion of the Vandermonde polynomial in this basis:

$$\Delta(x) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix}$$
$$= \begin{vmatrix} p_0(x_1) & p_0(x_2) & \dots & p_0(x_n) \\ p_1(x_1) & p_1(x_2) & \dots & p_1(x_n) \\ \vdots & \vdots & & \vdots \\ p_{n-1}(x_1) & p_{n-1}(x_2) & \dots & p_{n-1}(x_n) \end{vmatrix}$$
$$= \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma) p_0(x_{\sigma(1)} \dots p_{n-1}(x_{\sigma(n)}).$$

The second equality comes from the fact that the value of a determinant does not change if one adds to a row a linear combination of the other ones. Hence

$$\Delta(x) = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) p_{\sigma \cdot \delta}(x),$$

where

$$\sigma \cdot \mathbf{m} = (m_{\sigma(0)}, m_{\sigma(1)}, \dots, m_{\sigma(n)}),$$

and $\delta = (0, 1, \dots, n-1)$. From the orthogonality of the polynomials $p_{\mathbf{m}}$ it follows that

$$\int_{\mathbb{R}^n} \Delta(x)^2 \mu(dx_1) \dots \mu(dx_n) = n! d_0 \dots d_{n-1}. \qquad \square$$

This gives a way to evaluate the constant Z_n which will appear in Section 2 of Chapter III:

$$Z_n = \int_{\mathbb{R}^n} e^{-(\lambda_1^2 + \dots + \lambda_n^2)} \Delta(\lambda)^2 d\lambda_1 \dots d\lambda_n.$$

Corollary II.1.3.

$$Z_n = \pi^{\frac{n}{2}} 2^{-\frac{n(n-1)}{2}} \prod_{j=2}^{n} j!.$$

Proof. Take

$$\mu(dt) = e^{-t^2} dt.$$

The polynomials are then proportional to the Hermite polynomials:

$$p_m(t) = 2^{-m} H_m(t),$$

and

$$d_m = \|p_m\|^2 = 2^{-m} m! \sqrt{\pi}.$$

Therefore

$$\begin{aligned} c_n &= n! D_n = n! d_0 \dots d_{n-1} \\ &= n! \pi^{\frac{n}{2}} \prod_{j=0}^{n-1} 2^{-j} j! = \pi^{\frac{n}{2}} 2^{-\frac{n(n-1)}{2}} \prod_{j=0}^{n} j!. \end{aligned} \qquad \square$$

Let us consider the polynomials p_n defined by

$$p_n(t) = \begin{vmatrix} M_{00} & M_{01} & \dots & M_{0,n-1} & 1 \\ M_{1,0} & M_{1,1} & \dots & M_{1,n-1} & t \\ \vdots & \vdots & & \vdots & \vdots \\ M_{n,0} & M_{n,1} & \dots & M_{n,n-1} & t^n \end{vmatrix}.$$

This is a sequence of orthogonal polynomials in $L^2(\mathbb{R}, \mu)$ for which

$$a_n = D_n, \ d_n = D_n D_{n+1}.$$

In fact one sees that the integral

$$\int_{\mathbb{R}} t^j p_n(t) \mu(dt)$$

is zero if $j < n$, and equals D_n if $j = n$. One shows also

$$p_n(t) = \frac{1}{n!} \int_{\mathbb{R}^n} \prod_{i=1}^{n} (t - x_i) \Delta(x)^2 \mu(dx_0) \dots \mu(dx_{n-1}).$$

2. Christoffel-Darboux kernel. — Let S_n be the orthogonal projection of $L^2(\mathbb{R}, \mu)$ onto the space of polynomials of degree $\leq n - 1$. If $\{p_k\}$ is a sequence of orthogonal polynomials, this projection can be written, for $f \in L^2(\mathbb{R}, \mu)$,

$$S_n f(x) = \sum_{k=0}^{n-1} \frac{1}{d_k} (f|p_k) p_k(x) = \int_{\mathbb{R}} K_n(x, y) f(y) \mu(dy),$$

where K_n is the following kernel, called the Christoffel-Darboux kernel,

$$K_n(x, y) = \sum_{n=0}^{n-1} \frac{1}{d_k} p_k(x) p_k(y).$$

In order to get a simpler form for this kernel we will use a recurrence relation satisfied by the polynomials p_n. We will use the following notation

$$p_n(x) = a_n x^n + b_n x^{n-1} + \cdots$$
$$d_n = \int_{\mathbb{R}} p_n(x)^2 \mu(dx).$$

Proposition II.2.1.

$$xp_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x),$$

where

$$\alpha_n = \frac{a_n}{a_{n+1}},\ \beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}},\ \gamma_n = \frac{a_{n-1}}{a_n}\frac{d_n}{d_{n-1}}.$$

Proof. The polynomial $xp_n(x)$ is a linear combination of the polynomials $p_0, \dots, p_{n+1}$:

$$xp_n(x) = \sum_{k=0}^{n+1} c_{nk} p_k(x),$$

where

$$c_{nk} = \frac{1}{d_k}\int_{\mathbb{R}} xp_n(x)p_k(x)\mu(dx).$$

Notice that $c_{nk} = 0$ if $k > n+1$. Furthermore $d_k c_{nk} = d_n c_{kn}$, hence $c_{nk} = 0$ if $k < n-1$. Therefore

$$xp_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x),$$

with

$$\alpha_n = c_{n,n+1},\ \beta_n = c_{n,n},\ \gamma_n = c_{n,n-1}.$$

Identifying the coefficients of x_{n+1} and x^n we get

$$a_n = \alpha_n a_{n+1},\ b_n = \alpha_n b_{n+1} + \beta_n a_n.$$

From these relations, and taking into account that $d_{n-1}\gamma_n = d_n \alpha_{n-1}$, we get the stated formulas. $\square$

Example

Recall that the Hermite polynomials are defined by

$$H_n(x) = (-1)^n e^{x^2}\left(\frac{d}{dx}\right)^n e^{-x^2}.$$

One gets from this the generating function

$$w(x,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = e^{2xt-t^2}.$$

In fact the Taylor expansion of the function $f(x) = e^{-x^2}$ can be written

$$f(x-t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(-t)^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) e^{-x^2}.$$

The generating function $w(x,t)$ satisfies

$$\frac{\partial w}{\partial t} - (2x - 2t)w = 0.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} H_n(x) - 2x \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) + 2 \sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} H_n(x) = 0.$$

By looking at the coefficients of t^n one gets

$$xH_n(x) = \tfrac{1}{2}\, H_{n+1}(x) + nH_{n-1}(x).$$

From the recurrence relation one gets the following formulas for the Christoffel-Darboux kernel

PROPOSITION II.2.2.

$$K_n(x,y) = \frac{\alpha_{n-1}}{d_{n-1}} \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x-y},$$

and

$$K_n(x,x) = \frac{\alpha_{n-1}}{d_{n-1}} \big(p_n'(x)p_{n-1}(x) - p_n(x)p_{n-1}'(x)\big).$$

Proof. From the recurrence relation one obtains

$$\begin{aligned} &\frac{1}{d_k}(x-y)p_k(x)p_k(y) \\ &= \frac{\alpha_k}{d_k} p_{k+1}(x)p_k(y) + \frac{\beta_k}{d_k} p_k(x)p_k(y) + \frac{\gamma_k}{d_k} p_{k-1}(x)p_k(y) \\ &- \frac{\alpha_k}{d_k} p_k(x)p_{k+1}(y) - \frac{\beta_k}{d_k} p_k(x)p_k(y) - \frac{\gamma_k}{d_k} p_k(x)p_{k-1}(y). \end{aligned}$$

Since

$$\frac{\gamma_k}{d_k} = \frac{\alpha_{k-1}}{d_{k-1}},$$

this can be written

$$\begin{aligned} \frac{1}{d_k}(x-y)p_k(x)p_k(y) = {} & \frac{\alpha_k}{d_k}\big(p_{k+1}(x)p_k(y) - p_k(x)p_{k+1}(y)\big) \\ & - \frac{\alpha_{k-1}}{d_{k-1}}\big(p_k(x)p_{k-1}(y) - p_{k-1}(x)p_k(y)\big). \end{aligned}$$

Therefore

$$
\begin{aligned}
&(x-y)K_n(x,y) = \sum_{k=0}^{n-1} \frac{1}{d_k}(x-y)p_k(x)p_k(y) \\
&= \frac{\alpha_{n-1}}{d_{n-1}}\big(p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)\big) \\
&- \frac{\alpha_0}{d_0}\big(p_1(x)p_0(y) - p_0(x)p_1(y)\big) + \frac{1}{d_0}(x-y)p_0(x)p_0(y).
\end{aligned}
$$

The last line vanishes since

$$
p_0(x) = a_0,\ p_1(x) - p_1(y) = a_1(x-y),\ \alpha_0 = \frac{a_0}{a_1}.
$$

One obtains $K_n(x,x)$ as a limit. In fact

$$
K_n(x,y) = \frac{\alpha_{n-1}}{d_{n-1}}\Big(\frac{p_n(x)-p_n(y)}{x-y}p_{n-1}(y) - p_n(y)\frac{p_{n-1}(x)-p_{n-1}(y)}{x-y}\Big),
$$

and, as $y \to x$,

$$
K_n(x,x) = \frac{\alpha_{n-1}}{d_{n-1}}\big(p_n'(x)p_{n-1}(x) - p_n(x)p_{n-1}'(x)\big). \qquad \square
$$

III. SEMI-CIRCLE LAW AND WIGNER THEOREM

1. Weyl integration formula. — We recall the notation: $H_n = Herm(n, \mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, $U_n = U(n, \mathbb{F})$. By the spectral theorem every matrix $x \in H_n$ can be diagonalized in an orthogonal basis. The eigenvalues are real. This can be said as follows: The map

$$U_n \times D_n \to H_n, \quad (u, a) \mapsto uau^*,$$

is surjective, where D_n denote the space of real diagonal matrices.

Theorem III.1.1 (Weyl integration formula). — *If f is an integrable function on H_n, then*

$$\int_{H_n} f(x) m_n(dx) = c_n \int_{D_n} \int_{U_n} f(uau^*) \alpha_n(du) |\Delta(a)|^\beta da_1 \dots da_n,$$

where $a = \mathrm{diag}(a_1, \dots, a_n)$,

$$\Delta(a) = \prod_{j<k} (a_k - a_j)$$

is the Vandermonde determinant, α_n is the normalized Haar measure of the compact group U_n, c_n is a positive constant, and $\beta = \dim_{\mathbb{R}} \mathbb{F} = 1, 2,$ or 4.

If the function f is U_n-invariant,

$$f(uxu^*) = f(x) \quad (u \in U_n),$$

then f only depends on the eigenvalues $\lambda_1, \dots, \lambda_n$ of x,

$$f(x) = F(\lambda_1, \dots, \lambda_n),$$

where the function F is defined on $\mathbb{R}^n$, and is symmetric,

$$F(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)}) = F(\lambda_1, \dots, \lambda_n),$$

for $\sigma \in \mathfrak{S}_{\mathfrak{n}}$, the symmetric group. In that case the Weyl integration formula simplifies:

$$\int_{H_n} f(x) m_n(dx) = c_n \int_{\mathbb{R}^n} F(\lambda_1, \dots, \lambda_n) |\Delta(\lambda)|^\beta d\lambda_1 \dots d\lambda_n.$$

2. The density of the statistical distribution of the eigenvalues. Let V be a continuous real function on $\mathbb{R}$ such that, for all $m \geq 0$,

$$\int_{\mathbb{R}} |t|^m e^{-V(t)} dt < \infty.$$

The main example will be $V(t) = \gamma t^2$ ($\gamma > 0$). One considers on the space H_n the probability measure

$$\mathbb{P}_n(dx) = \frac{1}{C_n} e^{-\operatorname{tr}\left(V(x)\right)} m_n(dx),$$

with

$$C_n = \int_{H_n} e^{-\operatorname{tr}\left(V(x)\right)} m_n(dx).$$

If the function f is U_n-invariant,

$$f(uxu^*) = f(x) \quad (u \in U_n),$$

it only depends on the eigenvalues $\lambda_1, \ldots, \lambda_n$ of x,

$$f(x) = F(\lambda_1, \ldots, \lambda_n),$$

where F is a symmetric function. From the Weyl integration formula it follows that

$$\int_{H_n} f(x) \mathbb{P}_n(dx) = \int_{\mathbb{R}^n} F(\lambda) q_n(\lambda) d\lambda_1, \ldots d\lambda_n,$$

with

$$q_n(\lambda) = \frac{1}{Z_n} e^{-\left(V(\lambda_1)+\cdots+V(\lambda_n)\right)} |\Delta(\lambda|^\beta,$$

and

$$Z_n = \int_{\mathbb{R}^n} e^{-\left(V(\lambda_1)+\cdots+V(\lambda_n)\right)} |\Delta(\lambda)|^\beta d\lambda_1 \ldots d\lambda_n.$$

In particular, if

$$f(x) = \frac{1}{n} \operatorname{tr}\left(\varphi(x)\right),$$

where φ is a bounded measurable function on $\mathbb{R}$, then

$$f(x) = \frac{1}{n}\left(\varphi(\lambda_1) + \cdots + \varphi(\lambda_n)\right),$$

and

$$\int_{H_n} f(x)\mathbb{P}_n(dx) = \frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{R}^n}\varphi(\lambda_i)q_n(\lambda)d\lambda_1\ldots d\lambda_n$$
$$= \int_{\mathbb{R}^n}\varphi(\lambda_1)q_n(\lambda)d\lambda_1\ldots d\lambda_n$$
$$= \int_{\mathbb{R}}\varphi(t)w_n(t)dt,$$

with

$$w_n(t) = \int_{\mathbb{R}^{n-1}} q_n(t,\lambda_2,\ldots,\lambda_n)d\lambda_2\ldots d\lambda_n.$$

In particular, if $\varphi = \chi_B$, the characteristic function of the Borel set B,

$$f(x) = \frac{1}{n}\big(\chi_B(\lambda_1)+\cdots+\chi_B(\lambda_n)\big)$$
$$= \frac{1}{n}\#\{\text{eigenvalues of } x \in B\} = \xi_{n,B}(x),$$

and

$$\mu_n(B) = \mathbb{E}_n(\xi_{n,B}) = \int_B w_n(t)dt.$$

This means that the measure μ_n is absolutely continuous with respect to the Lebesgue measure, with density w_n.

3. Mehta's formulae. — From now on we assume that $\mathbb{F}=\mathbb{C}$, hence $H_n = Herm(n,\mathbb{C})$, $\beta = 2$. Let us consider the orthogonal polynomials p_m with respect to the weight $e^{-V(t)}dt$:

$$\int_{\mathbb{R}} p_k(t)p_m(t)e^{-V(t)}dt = 0 \text{ if } k \neq m,$$

normalized by the condition

$$p_m(t) = t^m + \cdots$$

Let d_m denote the square of the norm of p_m,

$$d_m = \int_{\mathbb{R}} |p_m(t)|^2 e^{-V(t)}dt.$$

Recall that Δ denotes the Vandermonde polynomial. Since the value of a determinant does not change if one adds to a row a linear combination of the other ones,

$$\Delta(\lambda) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{vmatrix}$$

$$= \begin{vmatrix} p_0(\lambda_1) & p_0(\lambda_2) & \dots & p_0(\lambda_n) \\ p_1(\lambda_1) & p_1(\lambda_2) & \dots & p_1(\lambda_n) \\ \vdots & \vdots & & \vdots \\ p_{n-1}(\lambda_1) & p_{n-1}(\lambda_2) & \dots & p_{n-1}(\lambda_n) \end{vmatrix}.$$

Therefore

$$\Delta(\lambda) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sign}(\sigma) p_0(\lambda_{\sigma(1)}) p_1(\lambda_{\sigma(2)}) \dots p_{n-1}(\lambda_{\sigma(n)}).$$

The terms of this sum are orthogonal in the space

$$L^2\big(\mathbb{R}^n, \otimes_{i=0}^n e^{-V(\lambda_i)} d\lambda_i\big),$$

hence

$$Z_n = \int_{\mathbb{R}^n} e^{-\big(V(\lambda_1)+\cdots V(\lambda_n)\big)} \Delta(\lambda)^2 d\lambda_1 \dots d\lambda_n = n! d_0 d_1 \dots d_{n-1}.$$

Define

$$\varphi_m(t) = \frac{1}{\sqrt{d_m}} e^{-\frac{1}{2}V(t)} p_m(t).$$

The functions φ_m are orthonormal in $L^2(\mathbb{R})$. Define also

$$K_n(s,t) = \sum_{k=0}^{n-1} \varphi_k(s) \varphi_k(t).$$

Up to the exponential factor $e^{-\frac{1}{2}\big(V(s)+V(t)\big)}$ it is the Christoffel-Darboux kernel for the orthogonal polynomials p_m. It is also the kernel of the orthogonal projection of $L^2(\mathbb{R})$ onto the subspace generated by $\varphi_0, \dots, \varphi_{n-1}$. We will use the following notation introduced by Fredholm: for a kernel $K(s,t)$,

$$K\begin{pmatrix} s_1 & s_2 & \dots & s_m \\ t_1 & t_2 & \dots & t_m \end{pmatrix} = \det\big(K(s_i,t_j)\big)_{1 \le i,j \le m}.$$

Proposition III.3.1 (Mehta's formula 1).

$$q_n(\lambda_1, \dots, \lambda_n) = \frac{1}{n!} K_n \begin{pmatrix} \lambda_1 & \dots & \lambda_n \\ \lambda_1 & \dots & \lambda_n \end{pmatrix}.$$

Proof. Recall that

$$\begin{aligned} q_n(\lambda_1, \dots, \lambda_n) &= \frac{1}{Z_n} e^{-\left(V(\lambda_1)+\dots+V(\lambda_n)\right)} \Delta(\lambda)^2, \\ Z_n &= n! d_0 \dots d_{n-1}, \\ \Delta(\lambda) &= \det\bigl(p_i(\lambda_j)\bigr)_{0\le i\le n-1, 1\le j\le n}, \\ \varphi_m(t) &= \frac{1}{\sqrt{d_m}} e^{-\frac{1}{2}V(t)} p_m(t). \end{aligned}$$

Putting everything together one obtains

$$q_n(\lambda_1, \dots, \lambda_n) = \frac{1}{n!} \det\bigl(\varphi_i(\lambda_j)\bigr)^2.$$

Consider the matrix $A = \bigl(\varphi_i(\lambda_j)\bigr)$. The entries b_{ij} of the matrix $B = A^T A$ are given by

$$b_{ij} = \sum_{k=0}^{n-1} \varphi_k(\lambda_i)\varphi_k(\lambda_j) = K_n(\lambda_i, \lambda_j),$$

hence

$$\det B = K_n \begin{pmatrix} \lambda_1 & \dots & \lambda_n \\ \lambda_1 & \dots & \lambda_n \end{pmatrix}. \qquad \square$$

Proposition III.3.2 (Mehta's formula 2). — *The density w_n of the measure μ_n, the statistical distribution of the eigenvalues, is given by*

$$w_n(t) = \frac{1}{n} K_n(t, t).$$

Proof. The correlation function R_m $(0 \le m \le n)$ is the function in m variables defined by

$$\begin{aligned} &R_m(\lambda_1, \dots, \lambda_m) \\ &= \frac{n!}{(n-m)!} \int_{\mathbb{R}^{n-m}} q_n(\lambda_1, \dots, \lambda_m, \lambda_{m+1}, \dots, \lambda_n) d\lambda_{m+1} \dots d\lambda_n. \end{aligned}$$

In particular, for $m = n$,

$$R_n(\lambda_1, \dots, \lambda_n) = n! q_n(\lambda_1, \dots, \lambda_n),$$

and, for $m = 1$,

$$R_1(\lambda_1) = n \int_{\mathbb{R}^{n-1}} q_n(\lambda_1, \lambda_2, \ldots, \lambda_n) d\lambda_2 \ldots d\lambda_n = n w_n(\lambda_1).$$

By a backwards recursion on m we will prove that

$$R_m(\lambda_1, \ldots, \lambda_m) = K_n \begin{pmatrix} \lambda_1 & \ldots & \lambda_m \\ \lambda_1 & \ldots & \lambda_m \end{pmatrix}.$$

For $m = n$ it is Formula 1, and, for $m = 1$, it is Formula 2. It will follow from the next lemma.

Lemma III.3.3. — *Let K be the kernel of the orthogonal projection P of $L^2(\mathbb{R})$ onto a subspace of dimension n. Then*

$$\int_{\mathbb{R}} K \begin{pmatrix} t_1 & \ldots & t_m \\ t_1 & \ldots & t_m \end{pmatrix} dt_m = (n - m + 1) K \begin{pmatrix} t_1 & \ldots & t_{m-1} \\ t_1 & \ldots & t_{m-1} \end{pmatrix}.$$

Proof. The kernel K satisfies
- $K(t,s) = \overline{K(s,t)}$ since $P^* = P$,
- $\int_{\mathbb{R}} K(s,u)K(u,t)du = K(s,t)$, since $P \circ P = P$,
- $\int_{\mathbb{R}} K(t,t)dt = n$, since $\operatorname{tr} P = n$.

Let A_m be the $m \times m$ Hermitian matrix with entries

$$a_{ij} = K(t_i, t_j) \quad (1 \leq i, j \leq m).$$

We write it as

$$A_m = \begin{pmatrix} A_{m-1} & \alpha \\ \alpha^* & \gamma \end{pmatrix},$$

with

$$\alpha = \big(K(t_i, t_m)\big) \ (1 \leq i \leq m-1), \quad \gamma = K(t_m, t_m).$$

The determinant of A_m can be evaluated as follows

$$\det A_m = \det A_{m-1} \cdot \gamma - \alpha^* \tilde{A}_{m-1} \alpha,$$

where $\tilde{A}_{m-1}$ is the matrix of the cofactors $\tilde{a}_{ij}$ of A_{m-1}. By integrating with respect to t_m we obtain

$$\begin{aligned} \int_{\mathbb{R}} K \begin{pmatrix} t_1 & \ldots & t_m \\ t_1 & \ldots & t_m \end{pmatrix} dt_m =& K \begin{pmatrix} t_1 & \ldots & t_{m-1} \\ t_1 & \ldots & t_{m-1} \end{pmatrix} \int_{\mathbb{R}} K(t_m, t_m) dt_m \\ & - \sum_{i,j=1}^{m-1} \tilde{a}_{i,j} \int_{\mathbb{R}} K(t_j, t_m) K(t_m, t_i) dt_m \\ &= nK \begin{pmatrix} t_1 & \ldots & t_{m-1} \\ t_1 & \ldots & t_{m-1} \end{pmatrix} - \sum_{i,j=1} \tilde{a}_{ij} K(t_j, t_i). \end{aligned}$$

Since

$$\sum_{j=1}^{m-1} \tilde{a}_{ij} a_{ji} = \det A_{m-1},$$

we obtain finally

$$\int_{\mathbb{R}} K \begin{pmatrix} t_1 & \dots & t_m \\ t_1 & \dots & t_m \end{pmatrix} dt_m = (n-m+1) K \begin{pmatrix} t_1 & \dots & t_{m-1} \\ t_1 & \dots & t_{m-1} \end{pmatrix}. \qquad \square$$

4. Fourier transform of the statistical distribution of the eigenvalues. — We assume now that $V(t) = \gamma t^2$ ($\gamma > 0$). In that case

$$p_m(t) = 2^{-m} \gamma^{-\frac{m}{2}} H_m(\sqrt{\gamma} t),$$

where H_m is the Hermite polynomial of degree m:

$$H_m(x) = (-1)^m e^{x^2} \left(\frac{d}{dx}\right)^m (e^{-x^2}),$$

and

$$d_m = \int_{\mathbb{R}} |p_m(t)|^2 e^{-\gamma t^2} dt = 2^{-m} \gamma^{-m-\frac{1}{2}} m! \sqrt{\pi}.$$

The Hermite functions

$$\varphi_m(t) = \frac{1}{\sqrt{d_m}} e^{-\frac{1}{2}\gamma t^2} p_m(t)$$

constitute a Hilbert basis of $L^2(\mathbb{R})$.

Recall that the density w_n of the measure μ_n, the statistical distribution of the eigenvalues of a random matrix, is given by

$$w_n(t) = \frac{1}{n} K_n(t,t) = \frac{1}{n} \sum_{k=0}^{n-1} \varphi_k(t)^2.$$

We will compute its Fourier transform. In fact we will determine first the Fourier transform of

$$W_r(t) = \sum_{k=0}^{\infty} r^k \varphi_k(t)^2 \quad (|r| < 1).$$

For that we will use the following classical formula of Mehler:

$$\sum_{k=0}^{\infty} \frac{1}{2^k k!} H_k(x) H_k(y) r^k = \frac{1}{\sqrt{1-r^2}} e^{\frac{2xyr - (x^2+y^2)r^2}{1-r^2}},$$

(see for instance [Lebedev,1972] p.65-66) which gives, for $y = x$,

$$\sum_{k=0}^{\infty} \frac{1}{2^k k!} H_k(x)^2 r^k = \frac{1}{\sqrt{1-r^2}} e^{2x^2 \frac{r}{1+r}}.$$

From this formula we get

$$W_r(t) = \sqrt{\frac{\gamma}{\pi}} \frac{1}{\sqrt{1-r^2}} e^{-\gamma \frac{1-r}{1+r} t^2}.$$

This is a Gauss function. Recall that

$$\int_{\mathbb{R}} e^{-it\tau} e^{-\alpha t^2} dt = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\tau^2}{4\alpha}} \quad (\alpha > 0).$$

Here $\alpha = \gamma \frac{1-r}{1+r}$. Therefore

$$\begin{aligned} \widehat{W_r}(\tau) &= \int_{\mathbb{R}} e^{-it\tau} W_r(t) dt \\ &= \frac{1}{1-r} e^{-\frac{1+r}{1-r} \frac{\tau^2}{4\gamma}} = e^{-\frac{\tau^2}{4\gamma}} \frac{1}{1-r} e^{-\frac{r}{1-r} \frac{\tau^2}{2\gamma}}. \end{aligned}$$

If one is familiar with classical orthogonal polynomials, one recognizes the generating function for the Laguerre polynomials

$$L_m^\alpha(x) = e^x \frac{x^{-\alpha}}{n!} \Big(\frac{d}{dx}\Big)^m (e^{-x} x^{m+\alpha}).$$

In fact

$$\sum_{k=0}^{\infty} L_k^\alpha(x) r^k = \frac{1}{(1-r)^{\alpha+1}} e^{-\frac{r}{1-r} x} \quad (|r| < 1)$$

(see for instance [Lebedev, 1972], p.77), and we obtain

$$\widehat{W_r}(\tau) = e^{-\frac{\tau^2}{4\gamma}} \sum_{k=0}^{\infty} r^k L_k^0\Big(\frac{\tau^2}{2\gamma}\Big).$$

Since W_r has been defined as

$$W_r(t) = \sum_{k=0}^{\infty} r^k \varphi_k(t)^2,$$

it follows that:

Proposition III.4.1.

$$\int_{\mathbb{R}} e^{-it\tau}\varphi_k(t)^2 dt = e^{-\frac{\tau^2}{4\gamma}} L_k^0\Big(\frac{\tau^2}{2\gamma}\Big).$$

But we want to compute the Fourier transform of

$$K_n(t,t) = \sum_{k=0}^{n-1} \varphi_k(t)^2.$$

Let us consider the product of the two Taylor series

$$\begin{aligned}
&\Big(\sum_{k=0}^{\infty} r^k\Big)\Big(\sum_{k=0}^{\infty} \varphi_k(t)^2 r^k\Big)\\
&= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} \varphi_k(t)^2\Big) r^n = \sum_{n=0}^{\infty} K_{n+1}(t,t) r^n,
\end{aligned}$$

or

$$\frac{1}{1-r} W_r(t) = \sum_{n=0}^{\infty} K_{n+1}(t,t) r^n.$$

On the other hand

$$\begin{aligned}
\frac{1}{1-r}\widehat{W_r}(\tau) &= e^{-\frac{\tau^2}{4\gamma}} \frac{1}{(1-r)^2} e^{-\frac{r}{1-r}\frac{\tau^2}{2\gamma}}\\
&= e^{-\frac{\tau^2}{4\gamma}} \sum_{n=0}^{\infty} r^n L_n^1\Big(\frac{\tau^2}{2\gamma}\Big).
\end{aligned}$$

□

Theorem III.4.2. — *The Fourier transform of the measure μ_n, the statistical distribution of the eigenvalues, is given by*

$$\widehat{\mu_n}(\tau) = \widehat{w_n}(\tau) = \frac{1}{n} e^{-\frac{\tau^2}{4\gamma}} L_{n-1}^1\Big(\frac{\tau^2}{2\gamma}\Big).$$

5. Tight topology and Lévy-Cramér Theorem. — On the set $\mathfrak{M}(\mathbb{R})$ of bounded positive measure on $\mathbb{R}$ we will consider the tight topology. It corresponds to the pointwise convergence on the space $\mathcal{C}_b(\mathbb{R})$ of bounded continuous functions on $\mathbb{R}$. For that topology a sequence μ_n of measures converges to the measure μ if, for every $f \in \mathcal{C}_b(\mathbb{R})$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(t)\mu_n(dt) = \int_{\mathbb{R}} f(t)\mu(dt).$$

The following sets form a basis for the neighborhoods of the measure μ_0: for $f_1, \dots, f_N \in \mathcal{C}_b(\mathbb{R})$, $\varepsilon > 0$,

$$\mathcal{V}(f_1, \dots, f_N; \varepsilon)$$
$$= \Big\{ \mu \in \mathfrak{M}(\mathbb{R}) \mid \Big| \int_{\mathbb{R}} f_k(t)\mu(dt) - \int_{\mathbb{R}} f_k(t)\mu_0(dt) \Big| < \varepsilon \ (k = 1, \dots, N) \Big\}.$$

One can show that this topology is metrizable.

A sequence μ_n converges to μ if and only if

- for every $f \in \mathcal{C}_c(\mathbb{R})$, the space of continuous functions on $\mathbb{R}$ with compact support,

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(t)\mu_n(dt) = \int_{\mathbb{R}} f(t)\mu(dt),$$

- $\lim_{n\to\infty} \mu_n(\mathbb{R}) = \mu(\mathbb{R})$.

The Fourier transform of a measure $\mu \in \mathfrak{M}(\mathbb{R})$ is defined by

$$\hat{\mu}(\tau) = \int_{\mathbb{R}} e^{-it\tau}\mu(dt).$$

The function $\hat{\mu}$ is bounded,

$$|\hat{\mu}(\tau)| \leq \hat{\mu}(0) = \mu(\mathbb{R}),$$

and uniformly continuous. If the sequence μ_n converges to μ, then, for every $\tau \in \mathbb{R}$,

$$\lim_{n\to\infty} \hat{\mu}_n(\tau) = \hat{\mu}(\tau),$$

and the convergence is uniform on compact sets.

THEOREM III.5.1(LÉVY-CRAMÉR). — *Let μ_n be a sequence in $\mathfrak{M}(\mathbb{R})$ such that, for every $\tau \in \mathbb{R}$,*

$$\lim_{n\to\infty} \hat{\mu}_n(\tau) = \varphi(\tau),$$

the function φ being continuous at 0. Then the sequence μ_n converges to a measure $\mu \in \mathfrak{M}(\mathbb{R})$ whose Fourier transform is equal to φ.

Proof. Put

$$C = \sup_n \mu_n(\mathbb{R}) = \sup_n \hat{\mu}_n(0),$$

then

$$|\hat{\mu}_n(\tau)| \leq \hat{\mu}_n(0) \leq C, \ |\varphi(\tau)| \leq \varphi(0) \leq C.$$

Consider the linear forms T_n defined on the space $\mathcal{C}_0(\mathbb{R})$ of continuous functions on $\mathbb{R}$ vanishing at infinity by

$$T_n(f) = \int_{\mathbb{R}} f(t)\mu_n(dt).$$

Then

$$|T_n(f)| \leq C\|f\|_\infty.$$

The Fourier transform of a function $g \in L^1(\mathbb{R})$ belongs to $\mathcal{C}_0(\mathbb{R})$ (it is the Riemann-Lebesgue property), and

$$T_n(\hat{g}) = \int_{\mathbb{R}} \hat{g}(t)\mu_n(dt) = \int_{\mathbb{R}} g(\tau)\hat{\mu}_n(\tau)d\tau.$$

By the Lebesgue dominated convergence theorem

$$\lim_{n\to\infty} T_n(\hat{g}) = \int_{\mathbb{R}} g(\tau)\varphi(\tau)d\tau.$$

Since the space $\mathfrak{F}\big(L^1(\mathbb{R})\big)$ is dense in $\mathcal{C}_0(\mathbb{R})$, it follows that $T_n(f)$ converges for all $f \in \mathcal{C}_0(\mathbb{R})$. The limit $T(f)$ is a positive linear form on $\mathcal{C}_0(\mathbb{R})$. By the Riesz theorem there exists a positive measure μ on $\mathbb{R}$ such that, for all $f \in \mathcal{C}_c(\mathbb{R})$,

$$T(f) = \int_{\mathbb{R}} f(t)\mu(dt).$$

The functions in $\mathcal{C}_0(\mathbb{R})$ are integrable with respect to μ, and for $f \in \mathcal{C}_0(\mathbb{R})$,

$$T(f) = \int_{\mathbb{R}} f(t)\mu(dt).$$

Let us consider the Poisson approximation of unity:

$$p_k(\tau) = \frac{1}{\pi}\frac{1}{1+k^2\tau^2}, \quad \hat{p}_k(t) = e^{-\frac{|t|}{k}}.$$

We get

$$T(\hat{p}_k) = \int_{\mathbb{R}} \hat{p}_k(t)\mu(dt) = \int_{\mathbb{R}} p_k(\tau)\varphi(\tau)d\tau.$$

By the Lebesgue monotone convergence theorem,

$$\lim_{k\to\infty} \int_{\mathbb{R}} \hat{p}_k(t)\mu(dt) = \int_{\mathbb{R}} \mu(dt) \ (\leq \infty).$$

On the other hand, since φ is continuous at 0,

$$\lim_{k\to\infty}\int_{\mathbb{R}} p_k(\tau)\varphi(\tau)d\tau = \varphi(0).$$

Therefore the measure μ is bounded and

$$\int_{\mathbb{R}} \mu(dt) = \varphi(0) = \lim_{n\to\infty} \hat{\mu}_n(0) = \lim_{n\to\infty}\int_{\mathbb{R}} \mu_n(dt).$$

Finally μ_n converges to μ, and φ is the Fourier transform of μ. ∎

6. Convergence to the semi-circle law. — Let us introduce the function

$$F_\nu(\tau) = \frac{\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\int_{-1}^{1} e^{-it\tau}(1-t^2)^{\nu-\frac{1}{2}}dt.$$

Up to a simple factor it is a Bessel function:

$$J_\nu(\tau) = \frac{1}{\Gamma(\nu+1)}\left(\frac{\tau}{2}\right)^\nu F_\nu(\tau).$$

(see for instance [Lebedev,1972], p. 114). The power series expansion of F_ν is as follows

$$F_\nu(\tau) = \sum_{k=0}^{\infty}(-1)^k \frac{\Gamma(\nu+1)}{\Gamma(k+\nu+1)}\frac{1}{k!}\left(\frac{\tau}{2}\right)^{2k}.$$

The Fourier transform of the semi-circle law σ_a of radius a equals

$$\hat{\sigma}_a(\tau) = \frac{2}{\pi a^2}\int_{-a}^{a} e^{-it\tau}\sqrt{a^2-t^2}dt = F_1(a\tau).$$

Theorem III.6.1 (Wigner). — *After scaling, the measure μ_n, the statistical distribution of the eigenvalues, converges to the semi-circle law σ_a of radius*

$$a = \sqrt{\frac{2}{\gamma}},$$

for the tight topology. Precisely, for every $f \in \mathcal{C}_b(\mathbb{R})$,

$$\lim_{n\to\infty}\int_{\mathbb{R}} f\Big(\frac{t}{\sqrt{n}}\Big)\mu_n(dt) = \frac{2}{\pi a^2}\int_{-a}^{a} f(u)\sqrt{a^2-u^2}du.$$

Proof. By the Lévy-Cramér theorem it amounts to showing that

$$\lim_{n\to\infty} \hat{\mu}_n\Big(\frac{\tau}{\sqrt{n}}\Big) = \hat{\sigma}_a(\tau).$$

We computed $\hat{\mu}_n$ in Section 4:

$$\hat{\mu}_n(\tau) = \frac{1}{n} e^{-\frac{\tau^2}{4\gamma}} L^1_{n-1}\Big(\frac{\tau^2}{2\gamma}\Big).$$

The expansion of the Laguerre polynomial L^α_n is given by

$$L^\alpha_n(x) = \sum_{k=0}^{n} \frac{(n+\alpha)!}{(k+\alpha)!} \frac{(-x)^k}{k!(n-k)!}$$

(see for instance [Lebedev, 1972], p.77). Hence we obtain

$$\hat{\mu}_n(\frac{\tau}{\sqrt{n}}) = e^{-\frac{\tau^2}{4\gamma n}} \sum_{k=0}^{n-1} (-1)^k c_k(n) \frac{1}{k!(k+1)!} \Big(\sqrt{\frac{2}{\gamma}} \frac{\tau}{2}\Big)^{2k},$$

with

$$c_k(n) = \frac{(n-1)(n-2)\ldots(n-k)}{n^k}.$$

Notice that $k \le n-1$, and

$$\lim_{n\to\infty} c_k(n) = 1, \; 0 \le c_k(n) \le 1.$$

By the convergence of the majorant series

$$\sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} R^{2k},$$

one obtains the limit:

$$\begin{aligned} \lim_{n\to\infty} \hat{\mu}_n\Big(\frac{\tau}{\sqrt{n}}\Big) &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!(k+1)!} \Big(\sqrt{\frac{2}{\gamma}} \frac{\tau}{2}\Big)^{2k} \\ &= F_1\Big(\sqrt{\frac{2}{\gamma}}\tau\Big) = \hat{\sigma}_a(\tau). \end{aligned}$$

∎

IV. THE PROBABILITIES $A_n(m,B)$

1. Fredholm determinant. — Let (X,μ) be a measured space such that $\mu(X)<\infty$. One considers the following integral equation

$$\varphi(x)-\lambda\int_X K(x,y)\varphi(y)\mu(dy)=f(x).$$

One assumes that K is a bounded measurable kernel on $X\times X$, and that f is measurable and bounded. One looks for a measurable bounded solution φ. For small λ one can solve the equation by iteration. For that one defines the sequence of functions: $u_0(x)=f(x)$,

$$u_{n+1}(x)=\int_X K(x,y)u_n(y)\mu(dy).$$

Then

$$|u_n(x)|\le \bigl(M\mu(X)\bigr)^n\|f\|_\infty,$$

where $M=\sup|K(x,y)|$. Therefore, if $|\lambda|<r=1/\bigl(M\mu(X)\bigr)$, then the series

$$\varphi(x)=\sum_{n=0}^{\infty}\lambda^n u_n(x)$$

converges uniformly on X. It is the unique solution of the integral equation. One defines the iterated kernels $K^{(n)}$ by $K^{(1)}=K$, and

$$K^{(n)}(x,y)=\int_X K^{(n-1)}(x,z)K(z,y)\mu(dz).$$

The series

$$\Gamma(x,y;\lambda)=\sum_{n=1}^{\infty}\lambda^{n-1}K^{(n)}(x,y)$$

converges uniformly on $X\times X$ for $|\lambda|<r$. Its sum $\Gamma(x,y;\lambda)$ is called the resolvent kernel because

$$\varphi(x)=f(x)+\lambda\int_X \Gamma(x,y;\lambda)f(y)\mu(dy).$$

As a function of λ, $\Gamma(x,y;\lambda)$ is holomorphic for $|\lambda|<r$.

The Fredholm determinant has been introduced in order to prove that the resolvent kernel $\Gamma(x, y; \lambda)$ admits a meromorphic continuation to $\mathbb{C}$. It is defined by the following series

$$\begin{aligned}
D(\lambda) &= \mathrm{Det}(I - \lambda K) \\
&= 1 - \lambda \int_X K(x, x)\mu(dx) + \cdots \\
&+ \frac{(-\lambda)^n}{n!} \int_{X^n} K\begin{pmatrix} x_1 & \dots & x_n \\ x_1 & \dots & x_n \end{pmatrix} \mu(dx_1)\dots\mu(dx_n) + \cdots
\end{aligned}$$

Proposition IV.1.1. — *The series converges for all $\lambda \in \mathbb{C}$ and $D(\lambda)$ is an entire function.*

Proof. To prove the convergence one uses the Hadamard inequality: let A be a $n \times n$ complex matrix, and let $A_1, \dots, A_n$ denote the columns, then

$$|\det A| \leq \|A_1\| \dots \|A_n\|.$$

($||A_j||$ denotes the Euclidean norm of A_j.) It follows that

$$\left|K\begin{pmatrix} x_1 & \dots & x_n \\ x_1 & \dots & x_n \end{pmatrix}\right| \leq (\sqrt{nM^2})^n = n^{\frac{n}{2}} M^n.$$

If a_n denotes the coefficient of λ^n in the series defining $D(\lambda)$,

$$|a_n| \leq u_n = \frac{1}{n!} n^{\frac{n}{2}} M^n \mu(X)^n,$$

and

$$\frac{u_{n+1}}{u_n} = \frac{1}{\sqrt{n+1}}\Big(1 + \frac{1}{n}\Big)^{\frac{n}{2}} M\mu(X)$$

has limit 0. It follows that the radius of convergence is infinite. ∎

One defines also

$$\begin{aligned}
&D(x, y; \lambda) \\
&= K(x, y) + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_{X^n} K\begin{pmatrix} x & x_1 & \dots & x_n \\ y & x_1 & \dots & x_n \end{pmatrix} \mu(dx_1)\dots\mu(dx_n).
\end{aligned}$$

As $D(\lambda)$ does, this series converges for all λ.

Theorem IV.1.2 (Fredholm). — *For $|\lambda| < r$,*

$$\Gamma(x, y; \lambda) = \frac{D(x, y; \lambda)}{D(\lambda)}.$$

Therefore the resolvent kernel has a meromorphic continuation to $\mathbb{C}$.

Proof. Put

$$D_0(x, y; \lambda) = D(\lambda)\Gamma(x, y; \lambda).$$

It is well defined for small λ, and satisfies

$$D_0(x, y; \lambda) = K(x, y)D(\lambda) + \lambda \int_X K(x, z)D_0(z, y; \lambda)\mu(dz).$$

Put also

$$D(\lambda) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} a_n,$$
$$D_0(x, y; \lambda) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} A_n(x, y).$$

Notice that $a_0 = 1$, $A_0(x, y) = K(x, y)$. By identifying the coefficients of λ^n we get

$$A_n(x, y) = K(x, y)a_n - n \int_X K(x, z)A_{n-1}(z, y)\mu(dz).$$

Define also

$$B_n(x, y) = \int_{X^n} K\begin{pmatrix} x & x_1 & \dots & x_n \\ y & x_1 & \dots & x_n \end{pmatrix} \mu(dx_1)\dots\mu(dx_n).$$

We will see that the sequences A_n and B_n of kernels satisfy the same recursion relation. Since

$$A_0(x, y) = K(x, y),\ B_0(x, y) = K(x, y),$$

it will follow that, for every n, $A_n(x, y) = B_n(x, y)$, and

$$D_0(x, y; \lambda) = D(x, y; \lambda).$$

Let us expand the determinant

$$K\begin{pmatrix} x & x_1 & \dots & x_n \\ y & x_1 & \dots & x_n \end{pmatrix} = \begin{vmatrix} K(x, y) & K(x, x_1) & \dots & K(x, x_n) \\ K(x_1, y) & K(x_1, x_1) & \dots & K(x_1, x_n) \\ \vdots & \vdots & & \vdots \\ K(x_n, y) & K(x_n, x_1) & \dots & K(x_n, x_n) \end{vmatrix}$$

with respect to the entries of the first row:

$$
\begin{aligned}
&= K(x,y)K\begin{pmatrix} x_1 & \dots & x_n \\ x_1 & \dots & x_n \end{pmatrix} - K(x,x_1)K\begin{pmatrix} x_1 & x_2 & \dots & x_n \\ y & x_2 & \dots & x_n \end{pmatrix} \\
&+ \dots + (-1)^k K(x,x_k)K\begin{pmatrix} x_1 & x_2 & \dots & x_k & x_{k+1} & \dots & x_n \\ y & x_1 & \dots & x_{k-1} & x_{k+1} & \dots & x_n \end{pmatrix} \\
&+ \dots + (-1)^n K(x,x_n)K\begin{pmatrix} x_1 & x_2 & \dots & x_n \\ y & x_1 & \dots & x_{n-1} \end{pmatrix}.
\end{aligned}
$$

Integrating with respect to $x_1, \dots, x_n$, and noticing that

$$
\begin{aligned}
&\int_{X^n} K(x,x_k)K\begin{pmatrix} x_1 & x_2 & \dots & x_k & x_{k+1} & \dots & x_n \\ y & x_1 & \dots & x_{k-1} & x_{k+1} & \dots & x_n \end{pmatrix} \\
&\mu(dx_1)\dots\mu(dx_n) \\
&= (-1)^{k-1}\int_X K(x,z)B_{n-1}(z,y)\mu(dz),
\end{aligned}
$$

we obtain

$$
B_n(x,y) = K(x,y)a_n - n\int_X K(x,z)B_{n-1}(z,y)\mu(dz). \qquad \square
$$

We introduce the following notation, for a kernel K,

$$
\begin{aligned}
S_n(K) &= \frac{1}{n!}\int_{X^n} K\begin{pmatrix} x_1 & \dots & x_n \\ x_1 & \dots & x_n \end{pmatrix}\mu(dx_1)\dots\mu(dx_n), \\
T_n(K) &= \int_X K^{(n)}(x,x)\mu(dx) \\
&= \int_{X^n} K(x_1,x_2)K(x_2,x_3)\dots K(x_n,x_1)\mu(dx_1)\dots\mu(dx_n).
\end{aligned}
$$

By definition

$$
D(\lambda) = \mathrm{Det}(I-\lambda K) = \sum_{n=0}^{\infty}(-\lambda)^n S_n(K).
$$

Proposition IV.1.3. — *For $|\lambda| < r$,*

$$
\frac{D'(\lambda)}{D(\lambda)} = -\sum_{n=0}^{\infty} T_{n+1}(K)\lambda^n.
$$

Proof. By definition

$$\Gamma(x,y;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K^{(n)}(x,y) \quad (|\lambda| < r),$$

therefore

$$\int_X \Gamma(x,x;\lambda)\mu(dx) = \sum_{n=1}^{\infty} \lambda^{n-1} T_n(K).$$

By Theorem IV.1.2

$$\Gamma(x,y;\lambda) = \frac{D(x,y;\lambda)}{D(\lambda)}.$$

Recall that

$$\begin{aligned} &D(x,y;\lambda) \\ &= K(x,y) + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_{X^n} K\begin{pmatrix} x & x_1 & \dots & x_n \\ y & x_1 & \dots & x_n \end{pmatrix} \mu(dx_1)\dots\mu(dx_n). \end{aligned}$$

Then

$$\begin{aligned} \int_X D(x,x;\lambda)\mu(dx) &= S_1(K) + \sum_{n=1}^{\infty} (n+1) S_{n+1}(K)(-\lambda)^n \\ &= -D'(\lambda). \end{aligned}$$

□

We will need two further properties.

Proposition IV.1.4. — *Let K_j be a sequence of bounded measurable kernels on X such that, for every j, x, y,*

$$\lim_{j\to\infty} K_j(x,y) = K(x,y) \ (\forall x,y \in X),$$
$$|K_j(x,y)| \le M.$$

Then, for every $\lambda \in \mathbb{C}$,

$$\lim_{j\to\infty} \mathrm{Det}(I - \lambda K_j) = \mathrm{Det}(I - \lambda K),$$

and the convergence is uniform in λ on compact sets.

Let (X,μ), (Y,ν) be two measured spaces such that $\mu(X) < \infty$, $\nu(Y) < \infty$, and $\varphi : X \to Y$ a measurable map. One assumes that there is a bounded measurable function h on X such that, for $f \in L^1(Y,\nu)$,

$$\int_Y f(y)\nu(dy) = \int_X f\big(\varphi(x)\big) h(x)\mu(dx).$$

Proposition IV.1.5. — *For a bounded measurable kernel K on Y, let us denote by $\tilde{K}$ the kernel defined on X as*

$$\tilde{K}(x,x') = K\big(\varphi(x),\varphi(x')\big)h(x').$$

Then

$$\operatorname{Det}(I-\lambda\tilde{K}) = \operatorname{Det}(I-\lambda K).$$

2. Finite rank kernels. — A finite rank kernel is of the form

$$K(x,y) = \sum_{i=1}^{n} f_i(x)g_i(y).$$

We assume that the functions f_i are linearly independent. To the kernel K one associates the integral operator L defined by

$$\tilde{L}f(x) = \int_X K(x,y)f(y)\mu(dy).$$

The space E generated by the functions f_i is invariant under $\tilde{L}$. Let L denote its restriction to E. The matrix $A=(a_{ij})$ of L with respect to the basis $\{f_i\}$ is

$$a_{ij} = \int_X f_j(y)g_i(y)\mu(dy).$$

Further

$$\operatorname{tr}(L) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n}\int_X f_i(x)g_i(x)\mu(dx) = \int_X K(x,x)\mu(dx),$$
$$\operatorname{tr}(L^m) = \int_X K^{(m)}(x,x)\mu(dx) = T_m(K).$$

Theorem IV.2.1.

$$\operatorname{Det}(I-\lambda K) = \det(I-\lambda L).$$

The left hand side denotes the Fredholm determinant, the right hand side the usual one.

Proof. Put $d(\lambda)=\det(I-\lambda L)$. Then

$$\frac{d'(\lambda)}{d(\lambda)} = -\sum_{m=0}^{\infty}\operatorname{tr}(L^{m+1})\lambda^m.$$

In fact, if $\alpha_1, \dots, \alpha_n$ are the eigenvalues of L, then

$$d(\lambda) = \det(I - \lambda L) = \prod_{j=1}^{n} (1 - \lambda\alpha_j),$$

and, for small λ,

$$\begin{aligned} \frac{d'(\lambda)}{d(\lambda)} &= -\sum_{j=1}^{n} \frac{\alpha_j}{1 - \lambda\alpha_j} = -\sum_{j=1}^{n} \Big(\sum_{m=0}^{\infty} \alpha_j^{m+1} \lambda^m \Big) \\ &= -\sum_{m=0}^{\infty} \Big(\sum_{j=1}^{n} \alpha_j^{m+1} \Big) \lambda^m = -\sum_{m=0}^{\infty} \operatorname{tr}(L^{m+1}) \lambda^m. \end{aligned}$$

Therefore, since $T_{m+1}(K) = \operatorname{tr}(L^{m+1})$,

$$\frac{D'(\lambda)}{D(\lambda)} = \frac{d'(\lambda)}{d(\lambda)}.$$

Furthermore, since $D(0) = 1$, $d(0) = 1$, it follows that

$$D(\lambda) = d(\lambda). \qquad \square$$

Let $\Lambda^m(L)$ be the operator on the exterior power ΛE of E such that

$$\Lambda^m(L)(v_1 \wedge \dots \wedge v_m) = (Lv_1) \wedge \dots \wedge (Lv_m) \quad (v_1, \dots, v_m \in E).$$

The eigenvalues of $\Lambda^m(L)$ are the numbers $\alpha_{j_1}\alpha_{j_2} \dots \alpha_{j_m}$ $(j_1 < j_2 < \dots < j_m)$, and its trace is

$$\operatorname{tr}\big(\Lambda^m(L)\big) = \sum_{j_1 < \dots < j_m} \alpha_{j_1} \dots \alpha_{j_m} = \sigma_m(\alpha_1, \dots, \alpha_n),$$

where σ_m is the m-th elementary symmetric function. Hence

$$\det(I - \lambda L) = \sum_{m=0}^{n} (-1)^m \operatorname{tr}\big(\Lambda^m(L)\big) \lambda^m.$$

Corollary IV.2.2.

$$\operatorname{tr}\big(\Lambda^m(L)\big) = S_m(K) = \frac{1}{m!} \int_{X^m} K \begin{pmatrix} x_1 & \dots & x_m \\ x_1 & \dots & x_m \end{pmatrix} \mu(dx_1) \dots \mu(dx_m).$$

The number $\operatorname{tr}\big(\Lambda(L)\big)$ can be expressed as a sum of determinants of order m extracted from the matrix A of L:

$$\operatorname{tr}\big(\Lambda(L)\big) = \sum_{\#I=m} \Delta_I(A),$$

where $I \subset \{1,\dots,n\}$ has m elements, and $\Delta_I(A)$ is the associated determinant: if $I = \{j_1,\dots,j_m\}$, then

$$\Delta_I(A) = \det\big(a_{j_k j_\ell}\big)_{1\le k,\ell<m}.$$

It is possible to prove Corollary IV.2.2 by showing directly that

$$\frac{1}{m!}\int_{X^m} K\begin{pmatrix} x_1 & \dots & x_m \\ x_1 & \dots & x_m \end{pmatrix} \mu(dx_1)\dots\mu(dx_m) = \sum_{\#I=m} \Delta_I(A).$$

(See [Katz-Sarnak,1999] p.142-143.)

Notice that $\operatorname{tr}\big(\Lambda^m(L)\big) = 0$ if $m > n$.

Exercise

If K is the kernel of the orthogonal projection P on a linear subspace $E \subset L^2(X,\mu)$ of dimension n, then

$$\int_{X^m} K\begin{pmatrix} x_1 & \dots & x_m \\ x_1 & \dots & x_m \end{pmatrix} \mu(dx_1)\dots\mu(dx_m) = \begin{cases} \frac{n!}{(n-m)!} & \text{if } m \le n, \\ 0 & \text{if } m > n. \end{cases}$$

3. The probabilities $A_n(m,B)$. — We consider on $H_n = Herm(m,\mathbb{C})$ the probability measure

$$\mathbb{P}_n(dx) = \frac{1}{C_n} e^{-\operatorname{tr}\left(V(x)\right)} m_n(dx),$$

where V is a continuous function on $\mathbb{R}$ such that

$$\forall m \ge 0,\ \int_{\mathbb{R}} |t|^m e^{-V(t)} dt < \infty.$$

Recall that (see Section III.2), if f is a $U(n)$-invariant function, then

$$f(x) = F(\lambda_1,\dots,\lambda_n),$$

where F is a symmetric function, $\lambda_1,\dots,\lambda_n$ are the eigenvalues of x, and

$$\int_{H_n} f(x)\mathbb{P}_n(dx) = \int_{\mathbb{R}^n} F(\lambda_1,\dots,\lambda_n) q_n(\lambda_1,\dots,\lambda_n) d\lambda_1 \dots d\lambda_n.$$

For a Borel set $B \subset \mathbb{R}$, $A_n(m, B)$ denotes the probability that a random matrix x has exactly m eigenvalues in B. For $m = 0$, $A_n(0, B)$ is the probability for B to be a *hole* in the spectrum. Let $\lambda_{\max}$ denote the largest eigenvalue. Then

$$\mathbb{P}_n\big(\{\lambda_{\max} \leq \alpha\}\big) = A_n(0,]\alpha, \infty[).$$

We will see that the probability $A_n(0, B)$ can be expressed as a Fredholm determinant. Recall that the kernel

$$K_n(s,t) = \sum_{k=0}^{n-1} \varphi_k(s)\varphi_k(t)$$

has been introduced in Section III.3.

PROPOSITION IV.3.1. — *Assume that the Borel set B is of finite Lebesgue measure. Then*

$$A_n(0, B) = \mathrm{Det}_B(I - K_n).$$

The index B means that the kernel $K_n(s,t)$ is restricted to B.

Proof. Let χ be the characteristic function of the set B. Then the characteristic function of the set $\{\forall j,\ \lambda_j \notin B\}$ is

$$\prod_{j=1}^{n} \big(1 - \chi(\lambda_j)\big).$$

Therefore

$$A_n(0, B) = \int_{\mathbb{R}^n} \prod_{j=1}^{n} \big(1 - \chi(\lambda_j)\big) q_n(\lambda_1, \ldots, \lambda_n) d\lambda_1 \ldots d\lambda_n.$$

More generally we will compute

$$A(z) = \int_{\mathbb{R}^n} \prod_{j=1}^{n} \big(1 - z\chi(\lambda_j)\big) q_n(\lambda_1, \ldots, \lambda_n) d\lambda_1 \ldots d\lambda_n.$$

Recall the formulae for the elementary symmetric functions:

$$\begin{aligned}
\sigma_1(\alpha_1, \ldots, \alpha_n) &= \alpha_1 + \cdots + \alpha_n, \\
\sigma_2(\alpha_1, \ldots, \alpha_n) &= \sum_{i<j} \alpha_i \alpha_j, \\
&\ \vdots \\
\sigma_n(\alpha_1, \ldots, \alpha_n) &= \alpha_1 \ldots \alpha_n,
\end{aligned}$$

and

$$\prod_{j=1}^{n}(1 - z\alpha_j) = 1 - \sigma_1 z + \sigma_2 z^2 - \cdots + (-1)^n \sigma_n z^n.$$

Therefore

$$\prod_{j=1}^{n}\bigl(1 - z\chi(\lambda_j)\bigr) = \sum_{k=0}^{n}(-1)^k z^k \sigma_k\bigl(\chi(\lambda_1), \dots, \chi(\lambda_n)\bigr).$$

We compute now the integral of each term. By using the symmetry of the function q_n we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sigma_k\bigl(\chi(\lambda_1), \dots, \chi(\lambda_n)\bigr) q_n(\lambda_1, \dots, \lambda_n) d\lambda_1 \dots d\lambda_n \\
&= \binom{n}{k} \int_{\mathbb{R}^n} \chi(\lambda_1) \dots \chi(\lambda_k) q_n(\lambda_1, \dots, \lambda_n) d\lambda_1 \dots d\lambda_n \\
&= \frac{1}{k!} \int_{B^k} R_k(\lambda_1, \dots, \lambda_k) d\lambda_1 \dots d\lambda_k,
\end{aligned}$$

where R_k is the k-th correlation function (see Section III.3). We get finally

$$A(z) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} z^k \int_{B^k} R_k(\lambda_1, \dots, \lambda_k) d\lambda_1 \dots d\lambda_k.$$

As we saw in the proof of Proposition III.3.2 (Mehta's formula 2), this is also equal to

$$A(z) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} z^k \int_{B^k} K_n \begin{pmatrix} \lambda_1 & \dots & \lambda_k \\ \lambda_1 & \dots & \lambda_k \end{pmatrix} d\lambda_1 \dots d\lambda_k,$$

and this is precisely the definition of the Fredholm determinant for the restriction of the kernel K_n to B:

$$A(z) = \mathrm{Det}_B(I - zK_n). \qquad \square$$

PROPOSITION IV.3.2.

$$A_n(m, B) = \frac{1}{m!}\Bigl(-\frac{d}{dz}\Bigr)^m \mathrm{Det}_B(I - zK_n)\big|_{z=1}.$$

Proof. The probability $A_n(m, B)$ can be written

$$A_n(m, B) = \int_{\mathbb{R}^n} \sum_{\#I=m} \prod_{i \in I} \chi(\lambda_i) \prod_{j \notin I} \bigl(1 - \chi(\lambda_j)\bigr) q_n(\lambda_1, \dots, \lambda_n) d\lambda_1 \dots d\lambda_n,$$

where the summation is taken over all subsets $I \subset \{1,\dots,n\}$ with m elements. On the other hand one can establish the following formula

$$\Big(-\frac{d}{dz}\Big)^m \prod_{i=1}^{n}(1-z\alpha_i) = m! \sum_{\#I=m} \prod_{i\in I}\alpha_i \prod_{j\notin I}(1-z\alpha_j). \qquad \square$$

Notice that

$$A(0) = \sum_{m=0}^{n} A_n(m,B) = 1,$$

and

$$A'(0) = \sum_{m=0}^{n} mA_n(m,B) = \mu_n(B)$$

is the expectation of the number of eigenvalues in the set B.

V. ASYMPTOTICS OF THE PROBABILITIES $A_n(m,B)$

1. Hermite polynomials and functions. — The Hermite polynomials are defined by

$$H_n(x) = (-1)^n e^{x^2} \Big(\frac{d}{dx}\Big)^n e^{-x^2} = 2^n x^n + \cdots$$

They are orthogonal with respect to the Gaussian measure $\mu(dx) = e^{-x^2} dx$:

$$\int_{\mathbb{R}} H_m(x) H_n(x) e^{-x^2} dx = 0 \text{ if } m \neq n,$$

and

$$d_n = \int_{\mathbb{R}} H_n(x)^2 e^{-x^2} dx = 2^n n! \sqrt{\pi}.$$

In Section 2 of Chapter II we saw the following formula for the generating function:

$$w(x,t) := \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = e^{2xt - t^2}.$$

Therefore

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = \Big(\sum_{j=0}^{\infty} \frac{(2x)^j}{j!}\Big)\Big(\sum_{k=0}^{\infty} \frac{(-t^2)^k}{k!}\Big),$$

and

$$H_n(x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{n!}{k!(n-2k)!} (2x)^{n-2k}.$$

From this one deduces that

$$H'(x) = 2n H_{n-1}(x).$$

Notice that

$$H_{2n}(0) = (-1)^n \frac{(2n)!}{n!},\ H_{2n+1}(0) = 0,$$
$$H'_{2n}(0) = 0,\ H'_{2n+1}(0) = 2(-1)^n \frac{(2n+1)!}{n!}.$$

The Hermite function φ_n is defined by

$$\varphi_n(x) = \frac{1}{\sqrt{d_n}} e^{-\frac{x^2}{2}} H_n(x).$$

The system $\{\varphi_n\}$ is a Hilbert basis of $L^2(\mathbb{R})$. Let us recall the Christoffel-Darboux kernel as defined in Section 3 of Chapter III:

$$K_n(x,y) = \sum_{k=0}^{n-1} \varphi_k(x)\varphi_k(y) = e^{-\frac{x^2+y^2}{2}} \sum_{k=0}^{n-1} \frac{1}{d_k} H_k(x)H_k(y).$$

From Proposition II.2.2 it follows that

Proposition V.1.1. — *For $x \neq y$,*

$$K_n(x,y) = \sqrt{\frac{n}{2}} \frac{\varphi_n(x)\varphi_{n-1}(y) - \varphi_{n-1}(x)\varphi_n(y)}{x-y},$$

and

$$K_n(x,x) = n\varphi_{n-1}(x)^2 - \sqrt{n(n-1)}\varphi_n(x)\varphi_{n-2}(x).$$

2. Asymptotics of the Hermite functions. — The Hermite function $u = \varphi_n$ is an eigenfunction of the oscillator operator:

$$u'' - x^2 u = -(2n+1)u \qquad (E).$$

In fact, it follows from the recursion formula

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0,$$

and the relation $H_n'(x) = 2nH_{n-1}(x)$. For x small one considers the equation

$$u'' + (2n+1)u = x^2 u \qquad (E)$$

as a perturbation of the equation

$$u'' + (2n+1)u = 0. \qquad (E_0)$$

The solutions of (E_0) are

$$A\cos(\sqrt{2n+1}x) + B\sin(\sqrt{2n+1}x).$$

Solving the differential equation

$$u'' + (2n+1)u = g(x)$$

by using the Lagrange variation of constants method, one obtains

$$u(x) = u(0)\cos(\sqrt{2n+1}x) + u'(0)\frac{\sin(\sqrt{2n+1}x)}{\sqrt{2n+1}} + \int_0^x \frac{\sin\big(\sqrt{2n+1}(x-y)\big)}{\sqrt{2n+1}} g(y)dy.$$

Let $r(x)$ denote this last integral. For $g(x) = x^2u(x)$, by the Schwarz inequality

$$|r(x)| \leq \frac{1}{\sqrt{2n+1}} \Big(\int_0^x y^4 dy\Big)^{\frac{1}{2}} \Big(\int_0^x u(y)^2 dy\Big)^{\frac{1}{2}},$$

and, if u is square integrable,

$$|r(x)| \leq \frac{1}{\sqrt{5}} \frac{1}{\sqrt{2n+1}} \Big(\int_0^\infty u(y)^2 dy\Big)^{\frac{1}{2}} |x|^{\frac{5}{2}}.$$

One establishes finally:

Proposition V.2.1.

$$\varphi_n(x) = \alpha_n \cos\Big(\sqrt{2n+1}x - n\frac{\pi}{2}\Big) + r_n(x),$$

with

$$|r_n(x)| \leq \frac{1}{2\sqrt{5}} \frac{1}{\sqrt{2n+1}} |x|^{\frac{5}{2}}.$$

For $n = 2m$,

$$\alpha_{2m} = \varphi_{2m}(0) = \frac{(2m)!}{m!} \frac{1}{\sqrt{d_{2m}}},$$

and, if $n = 2m+1$,

$$\alpha_{2m+1} = \varphi'_{2m+1}(0) \frac{1}{\sqrt{4m+3}} = 2\frac{(2m+1)!}{m!} \frac{1}{\sqrt{d_{2m+1}}} \frac{1}{\sqrt{4m+3}}.$$

As $n \to \infty$,

$$\alpha_n \sim \frac{1}{\sqrt{\pi}} \Big(\frac{2}{n}\Big)^{\frac{1}{4}}.$$

The last equivalence is obtained by using the Stirling formula

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}.$$

3. Asymptotics of the probabilities $A_n(m, B)$. — One considers on $H_n = Herm(n, \mathbb{C})$ the Gaussian probability measure

$$\mathbb{P}_n(dx) = \frac{1}{C_n} e^{-\operatorname{tr}(x^2)} m_n(dx).$$

i.e., from now on, $V(t) = t^2$ with our previous notation. Recall that, for a Borel set $B \subset \mathbb{R}$, $A_n(m, B)$ is the probability that a Hermitian matrix x has m eigenvalues in B. In Section IV.3 we saw that

$$A_n(0, B) = \mathrm{Det}_B(I - K_n),$$

where K_n is the Christoffel-Darboux kernel:

$$K_n(s, t) = \frac{1}{n} \sum_{k=0}^{n-1} \varphi_k(s)\varphi_k(t),$$

and

$$A_n(m, B) = \frac{1}{m!}\Big(-\frac{d}{dz}\Big)^m \mathrm{Det}_B(I - zK_n)\big|_{z=1}.$$

Let $\mathcal{K}$ be the kernel

$$\mathcal{K}(\xi, \eta) = \frac{1}{\pi} \frac{\sin(\xi - \eta)}{\xi - \eta}.$$

Theorem V.3.1. — *Let $B \subset \mathbb{R}$ be a bounded Borel set. Then*

$$\lim_{n\to\infty} A_n\Big(0, \frac{1}{\sqrt{2n}}B\Big) = \mathrm{Det}_B(I - \mathcal{K}).$$

Proof. Using results of Section IV.1 we can write

$$A_n\Big(0, \frac{1}{\sqrt{2n}}B\Big) = \mathrm{Det}_{\frac{1}{\sqrt{2n}}B}(I - K_n) = \mathrm{Det}_B(I - \tilde{K}_n),$$

where

$$\begin{aligned}\tilde{K}(\xi, \eta) &= K_n\Big(\frac{1}{\sqrt{2n}}\xi, \frac{1}{\sqrt{2n}}\eta\Big)\frac{1}{\sqrt{2n}} \\ &= \sqrt{\frac{n}{2}}\frac{1}{\xi - \eta}\bigg(\varphi_n\Big(\frac{1}{\sqrt{2n}}\xi\Big)\varphi_{n-1}\Big(\frac{1}{\sqrt{2n}}\eta\Big) - \varphi_{n-1}\Big(\frac{1}{\sqrt{2n}}\xi\Big)\varphi_n\Big(\frac{1}{\sqrt{2n}}\eta\Big)\bigg).\end{aligned}$$

By using the asymptotics of the Hermite functions φ_n which have been established in Section 2 one shows that

$$\lim_{n\to\infty} \tilde{K}_n(\xi, \eta) = \mathcal{K}(\xi, \eta),$$

and that there exists a constant $M > 0$ such that, for $\xi, \eta \in B$,

$$\forall n, \ |\tilde{K}_n(\xi, \eta)| \leq M.$$

It follows that

$$\lim_{n\to\infty} \mathrm{Det}_B(I - \tilde{K}_n) = \mathrm{Det}_B(I - \mathcal{K}). \qquad \square$$

COROLLARY V.3.2. — *Let $B \subset \mathbb{R}$ be a bounded Borel set. Then*

$$\lim_{n\to\infty} A_n\Big(m, \frac{1}{\sqrt{2n}}B\Big) = \frac{1}{m!}\Big(-\frac{d}{dz}\Big)^m \mathrm{Det}_B(I - z\mathcal{K})\big|_{z=1}.$$

Proof. We saw that

$$A_n\Big(m, \frac{1}{\sqrt{2n}}B\Big) = \frac{1}{m!}\Big(-\frac{d}{dz}\Big)^m Det_{\frac{1}{\sqrt{2n}}B}(I - zK_n)\big|_{z=1},$$

and this can be written

$$= \frac{1}{m!}\Big(-\frac{d}{dz}\Big)^m \mathrm{Det}_B(I - z\tilde{K}_n)\big|_{z=1}.$$

Since

$$\lim_{n\to\infty} \mathrm{Det}_B(I - z\tilde{K}_n) = \mathrm{Det}_B(I - z\mathcal{K})$$

uniformly in z on compact sets in $\mathbb{C}$,

$$\lim_{n\to\infty} \Big(-\frac{d}{dz}\Big)^m \mathrm{Det}_B(I - z\tilde{K}_n) = \Big(-\frac{d}{dz}\Big)^m \mathrm{Det}_B(I - z\mathcal{K}). \qquad \square$$

Remark

The convergence to the semi-circle law we saw in Section III.6 corresponds to asymptotics of $\varphi_n(x\sqrt{n})$ as $n \to \infty$. It is a convergence of global character. The convergence of the probabilities $A_n(m, B)$ has a local character. It corresponds to asymptotics of $\varphi_n\big(\frac{x}{\sqrt{n}}\big)$.

4. Asymptotics of the probabilities $A_n(0, B)$ in terms of the eigenvalues of a nuclear operator. — An operator A on a Banach space E is said to be nuclear (or of trace class) if it can be written

$$Av = \sum_{n=1}^{\infty} \langle f_n, v\rangle e_n,$$

with $e_n \in E$, $f_n \in E'$, and

$$\sum_{n=1}^{\infty} \|e_n\| \, \|f_n\| < \infty.$$

Assume now that $E = \mathcal{H}$ is a Hilbert space. Let $\{e_n\}$ be a Hilbert basis of $\mathcal{H}$. If the operator A is nuclear, then the series

$$\sum_{n=1}^{\infty}(Ae_n|e_n)$$

is absolutely convergent, and the sum does not depend on the Hilbert basis. By definition it is the trace of A:

$$\mathrm{tr}(A) = \sum_{n=1}^{\infty}(Ae_n|e_n).$$

A nuclear operator is compact. Conversely let A be a compact operator. Then A^*A is compact and selfadjoint ≥ 0. Let α_n be the non zero eigenvalues of A^*A. The numbers $\mu_n = \sqrt{\alpha_n}$ are called the characteristic values (or singular values) of A. One shows that the operator is nuclear if and only if

$$\|A\|_1 := \sum_{n=1}^{\infty}\mu_n < \infty,$$

and $\|\cdot\|_1$ is a norm on the space $\mathcal{L}_1(\mathcal{H})$ of nuclear operators on $\mathcal{H}$, and

$$|\,\mathrm{tr}(A)| \leq \|A\|_1.$$

If A is a nuclear operator, then $\Lambda^m(A)$ acting on the m-th exterior power $\Lambda^m(\mathcal{H})$ of $\mathcal{H}$ is nuclear too, and

$$\|\Lambda^m(A)\|_1 \leq \frac{\|A\|_1^m}{m!}.$$

The Fredholm determinant of $I - \lambda A$ is defined by

$$d(\lambda) = \det(I - \lambda A) = 1 + \sum_{m=1}^{\infty}(-1)^m\,\mathrm{tr}\big(\Lambda^m(A)\big)\lambda^m.$$

It is an entire function of λ, and, for small λ,

$$\frac{d'(\lambda)}{d(\lambda)} = -\sum_{m=0}^{\infty}\mathrm{tr}(A^{m+1})\lambda^m.$$

By the inequality above

$$|\det(I + A)| \leq \exp(\|A\|_1).$$

One shows that, for two nuclear operators A and B,

$$|\det(I+A)-\det(I+B)| \leq \|A-B\|_1 \exp(\|A\|_1+\|B\|_1+1).$$

Therefore the function $A \mapsto \det(I+A)$ is continuous on the space $\mathcal{L}_1(\mathcal{H})$ of nuclear operators.

Let the operator A be nuclear and selfadjoint, and let α_k be the non zero eigenvalues of A, each being repeated according to the dimension of the corresponding eigenspace. Then

$$\begin{aligned}\|A\|_1 &= \sum_k |\alpha_k|,\\ \operatorname{tr}(A) &= \sum_k \alpha_k,\\ \det(I-\lambda A) &= \prod_k (1-\lambda\alpha_k).\end{aligned}$$

An operator A on the Hilbert space $\mathcal{H}$ is said to be Hilbert-Schmidt if, for a Hilbert basis $\{e_n\}$,

$$(\|A\|_2)^2 := \sum_{m,n} |(Ae_n|e_m)|^2 < \infty.$$

This number does not depend on the basis, and $\|A\|_2$ is the Hilbert-Schmidt norm of A. A Hilbert-Schmidt operator is compact. Conversely let A be a compact operator, with characteristic values μ_n. Then A is Hilbert-Schmidt if and only if

$$\sum_{n=1}^{\infty} \mu_n^2 < \infty,$$

and this sum is equal to $(||A||_2)^2$. If the operator A is Hilbert-Schmidt and selfadjoint with non zero eigenvalues λ_n, then

$$(\|A\|_2)^2 = \sum_{n=1}^{\infty} \lambda_n^2.$$

The space $\mathcal{L}_2(\mathcal{H})$ of Hilbert-Schmidt operators is a Hilbert space for the inner product

$$(A|B)_2 = \sum_{n,m} (Ae_n|e_m)\overline{(Be_n|e_m)} = \sum_n (Ae_n|Be_n) = \sum_n (B^*Ae_n|e_n).$$

The product of two Hilbert-Schmidt operators is nuclear, and

$$\|AB\|_1 \leq \|A\|_2 \, \|B\|_2, \quad \mathrm{tr}(AB) = \mathrm{tr}(BA) = (A|B^*)_2.$$

Assume now that $\mathcal{H} = L^2(X, \mu)$, where (X, μ) is a measured space. Then a Hilbert-Schmidt operator A is an integral operator:

$$Af(x) = \int_X K(x,y) f(y) \mu(dy),$$

where $K(x,y)$ is a square integrable kernel: $K \in L^2(X \times X, \mu \otimes \mu)$, and

$$\begin{aligned} &\mathcal{L}_2(\mathcal{H}) \simeq L^2(X \times X, \mu \otimes \mu), \\ &(\|A\|_2)^2 = \int_{X \times X} |K(x,y)|^2 \mu(dx)\mu(dy). \end{aligned}$$

If A and B are Hilbert-Schmidt operators with kernels H and K, then $C = AB$ is an integral operator

$$Cf(x) = \int_X L(x,y) f(y) \mu(dy),$$

with kernel

$$L(x,y) = \int_X H(x,z) K(z,y) \mu(dz).$$

The operator C is nuclear and

$$\mathrm{tr}(C) = \int_X L(x,x) \mu(dx).$$

Assume furthermore that X is a compact topological space, that the measure μ is bounded with $\mathrm{supp}(\mu) = X$. Let the kernel K be continuous and Hermitian:

$$K(y,x) = \overline{K(x,y)},$$

and of positive type: for any $x_1, \ldots, x_N \in X$, and $c_1, \ldots, c_N \in \mathbb{C}$,

$$\sum_{i,j=1}^N K(x_i, x_j) c_i \bar{c}_j \geq 0.$$

The operator A on $L^2(X, \mu)$ associated to K:

$$Af(x) = \int_X K(x,y) f(y) \mu(dy),$$

is positive selfadjoint and compact. Let α_k be the non zero eigenvalues of A, and ψ_k the corresponding normalized eigenfunctions:

$$\int_X K(x,y)\psi_k(y)\mu(dy) = \alpha_k\psi_k(x),$$
$$\int_X |\psi_k(x)|^2\mu(dx) = 1.$$

THEOREM V.4.1 (MERCER). — *Let the kernel K be continuous, Hermitian, and of positive type. Then*
a) For $x, y \in X$,

$$K(x,y) = \sum_{k=1}^{\infty} \alpha_k\psi_k(x)\overline{\psi_k(y)}.$$

The convergence is uniform on $X \times X$.
b) The operator A is nuclear and

$$\mathrm{tr}(A) = \int_X K(x,x)\mu(dx).$$

For such an operator both definitions of Fredholm determinant agree:

$$\mathrm{Det}(I - \lambda K) = \det(I - \lambda A).$$

Let us come back to the kernel $\mathcal{K}$:

$$\mathcal{K}(x,y) = \frac{\sin(x-y)}{x-y}.$$

It is continuous on $\mathbb{R}$ and of positive type. In fact it is the limit of the kernels K_n which are of positive type. One can see it also directly:

$$\frac{\sin x}{x} = \frac{1}{2}\int_{-1}^{1} e^{itx}dt,$$

therefore

$$\sum_{j.k=1}^{N} \mathcal{K}(x_j,x_k)c_j\bar{c}_k = \frac{1}{2\pi}\int_{-1}^{1} \Big|\sum_{j=1}^{N} e^{itx_j}c_j\Big|^2 dt \geq 0.$$

The operator P on $L^2(\mathbb{R})$ with kernel $\mathcal{K}$ is the projection on the subspace of the functions whose Fourier transform support is $\subset [-1,1]$. In fact

$$\widehat{Pf} = \chi_{[-1,1]}\hat{f}.$$

Take now $B = [-\theta, \theta]$ ($\theta > 0$), and let A be the operator defined on $L^2([-\theta,\theta])$ by

$$Af(x) = \int_{-\theta}^{\theta} \mathcal{K}(x,y)f(y)dy.$$

It is positive selfadjoint and nuclear by Mercer's theorem. Let α_k be its eigenvalues (they are all positive). We can write $A = Q_B P Q_B$, where Q_B is the projection given by

$$Q_B f(x) = \chi_B(x)f(x).$$

It follows that, as a selfadjoint operator, $0 \leq A \leq I$, $0 \leq \alpha_k \leq 1$, and

$$\det(I - A) = \prod_k (1 - \alpha_k) \leq 1.$$

Finally

$$\lim_{n\to\infty} A_n\Big(0, [-\frac{1}{\sqrt{2n}}\theta, \frac{1}{\sqrt{2n}}\theta]\Big) = \prod_k (1 - \alpha_k).$$

If we were able to evaluate the infinite product $\prod_k (1 - \alpha_k)$ as a function of θ, it should give information about the asymptotic spacing of the small eigenvalues.

Exercise

Define

$$f(z) = \prod_k (1 - z\alpha_k).$$

Prove that

$$\frac{1}{m!}\Big(-\frac{d}{dz}\Big)^m f(z) = \sum_{j_1 < \cdots < j_m} \frac{\alpha_{j_1}}{1 - z\alpha_{j_1}} \cdots \frac{\alpha_{j_m}}{1 - z\alpha_{j_m}}.$$

VI WISHART UNITARY ENSEMBLE

1. The Wishart unitary ensemble. — Let Ω_n be the cone of positive definite $n \times n$ Hermitian matrices in the vector space $H_n = Herm(n, \mathbb{C})$. For $p > n-1$, the Wishart law W_n^p is the probability measure on Ω_n defined by

$$\int_{\Omega_n} f(x) W_n^p(dx) = \frac{1}{\Gamma_n(p)} \int_{\Omega_n} f(x) e^{-\operatorname{tr}(x)} (\det x)^{p-n} m_n(dx),$$

for a bounded measurable function f, where m_n is the Euclidean measure associated to the inner product $(x|y) = \operatorname{tr}(xy)$ on H_n, and Γ_n is the gamma function of the cone Ω_n:

$$\Gamma_n(p) = \int_{\Omega_n} e^{-\operatorname{tr}(x)} (\det x)^{p-n} m_n(dx).$$

The probability space (Ω_n, W_n^p) is called the Wishart unitary ensemble. In fact the Wishart law W_n^p is invariant for the action of the unitary group $U(n)$ given by the transformations

$$x \mapsto uxu^* \quad (u \in U(n).$$

Proposition VI.1.1.

$$\Gamma_n(p) = (2\pi)^{\frac{n(n-1)}{2}} \prod_{j=1}^{n} \Gamma(p - j + 1).$$

Proof. Let $T_n \subset GL(n; \mathbb{C})$ be the group of upper triangular matrices with positive diagonal entries. The map

$$T_n \to \Omega_n, \; t \mapsto x = tt^*,$$

is a diffeomorphism. If f is an integrable function on Ω_n with respect to m_n,

$$\int_{\Omega_n} f(x) m_n(dx) = 2^{\frac{n(n-1)}{2}} \int_{T_n} f(tt^*) \prod_{j=1}^{n} t_{jj}^{2j-1} \prod_{j=1}^{n} dt_{jj} \prod_{j<k} d(\Re t_{jk}) d(\Im t_{jk}).$$

Therefore

$$\Gamma_n(p) = 2^{\frac{n(n-1)}{2}} \int_{T_n} e^{-\left(\sum_{j=1}^{n} t_{jj}^2 + \sum_{j<k} |t_{jk}|^2\right)} \prod_{j=1}^{n} t_{jj}^{2(p-n)} \prod_{j=1}^{n} t_{jj}^{2j-1}$$
$$\prod_{j=1}^{n} dt_{jj} \prod_{j<k} d(\Re t_{jk}) d(\Im t_{jk}).$$

By using the classical formulae

$$\int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}, \qquad \int_0^{\infty} e^{-t^2}t^{\alpha}dt = \tfrac{1}{2}\,\Gamma\Big(\frac{\alpha+1}{2}\Big),$$

one obtains

$$\Gamma_n(p) = (2\pi)^{\frac{n(n-1)}{2}} \prod_{j=1}^{n} \Gamma(p-j+1). \qquad \square$$

The Laplace transform of the Wishart law has a simple expression:

Proposition VI.1.2. — *For $\zeta = \xi + i\eta \in H_n + iH_n \simeq M(n,\mathbb{C})$, with $\xi + I \in \Omega_n$,*

$$\mathcal{L}W_n^p(\zeta) = \int_{\Omega_n} e^{-\operatorname{tr}(\zeta x)} W_n^p(dx) = \det(I+\zeta)^{-p}.$$

Proof. One starts from the formula

$$\int_{\Omega_n} e^{-\operatorname{tr}(x)} (\det x)^{p-n} m_n(dx) = \Gamma_n(p),$$

and changes the variable: one puts $x = gx'g^*$ with $g \in GL(n,\mathbb{C})$. Then

$$m_n(dx) = |\det g|^{2n} m_n(dx'),$$

and

$$\begin{aligned}&\int_{\Omega_n} e^{-\operatorname{tr}(x)} (\det x)^{p-n} m_n(dx)\\ &= |\det g|^{2p} \int_{\Omega_n} e^{-\operatorname{tr}(gx'g^*)} (\det x')^{p-n} m_n(dx').\end{aligned}$$

Therefore, for $y = g^*g$,

$$\int_{\Omega_n} e^{-\operatorname{tr}(x'y)} (\det x')^{p-n} m_n(dx') = \Gamma_n(p)(\det y)^{-p}.$$

Since, for $y \in \Omega_n$, there exists $g \in GL(n,\mathbb{C})$ such that $y = g^*g$, the proposition is proven for $\Im(\zeta) = \eta = 0$.

The two functions $\zeta \mapsto \mathcal{L}W_n^p(\zeta)$ and $\zeta \mapsto \det(I+\zeta)^{-p}$ are holomorphic in the open set

$$\{\zeta = \xi + i\eta \mid \xi + I \in \Omega_n\} = (\Omega_n - I) + iH_n,$$

and are equal for $\zeta = \xi$, $\xi + I \in \Omega_n$. Hence they are equal in $(\Omega_n - I) + iH_n$. ∎

On the space $M(n, p; \mathbb{C})$ of $n \times p$ complexes matrices let us denote by $\mathbb{P}$ the Gaussian probability measure

$$\mathbb{P}(d\xi) = \frac{1}{\pi^{np}} e^{-\operatorname{tr}(\xi\xi^*)} m(d\xi).$$

We consider the map

$$Q : M(n, p; \mathbb{C}) \to \overline{\Omega_n}, \quad \xi \mapsto \xi\xi^*.$$

Proposition VI.1.3. — *If $p \geq n$, then the image by the map Q of the Gaussian probability $\mathbb{P}$ is the Wishart law W_n^p.*

This means that, for a function f on $\overline{\Omega_n}$ which is integrable with respect to W_n^p,

$$\int_{M(n,p;\mathbb{C})} f(x) W_n^p(dx) = \int_{\overline{\Omega_n}} f(\xi\xi^*) \mathbb{P}(d\xi).$$

Proof. The measure $\mu = Q(\mathbb{P})$ is the measure on $\overline{\Omega_n}$ such that, for a function f on $\overline{\Omega_n}$, measurable and bounded,

$$\int_{\overline{\Omega_n}} f(x)\mu(dx) = \int_{M(n,p;\mathbb{C})} f\big(Q(\xi)\big) \mathbb{P}(d\xi).$$

Let us compute the Laplace transform of the image $\mu = Q(\mathbb{P}$. By taking

$$f(x) = e^{-\operatorname{tr}(x\zeta)},$$

with $\zeta = \xi + i\eta \in H_n + iH_n$, $\xi + I \in \Omega_n$, we obtain

$$\begin{aligned} \mathcal{L}\mu(\zeta) &= \frac{1}{\pi^{np}} \int_{M(n,p;\mathbb{C})} e^{-\operatorname{tr}(\zeta\xi\xi^*)} e^{-\operatorname{tr}(\xi\xi^*)} m(d\xi) \\ &= \frac{1}{\pi^{np}} \int_{M(n,p;\mathbb{C})} e^{-\operatorname{tr}\left((I+\zeta)\xi\xi^*\right)} m(d\xi) \\ &= \det(I + \zeta)^{-p}. \end{aligned}$$

By the injectivity of the Laplace transform, this proves the proposition. ∎

If $p < n$, then the image of $\mathbb{P}$ is a well defined probability measure supported on the boundary $\partial\Omega_n$ of Ω_n. It is singular with respect to the Euclidean measure. We will denote it also by W_n^p. In fact it can be obtained by analytic continuation from W_n^p, $p > n-1$, with respect to p. Therefore we obtain a family of probability measures W_n^p for p in the so called Wallach set

$$\{0, 1, \dots, n-1\} \cup]n-1, \infty[.$$

2. The statistical distribution of the eigenvalues. — Assume $p > n-1$, and let f be a $U(n)$-invariant function on Ω_n:

$$f(uxu^*) = f(x).$$

The function f only depends on the eigenvalues of x,

$$f(x) = F(\lambda_1, \dots, \lambda_n),$$

where F is a symmetric function on $\mathbb{R}_+^n$. If f is integrable with respect to W_n^p, it follows from the Weyl integration formula (Theorem III.1.1) that

$$\int_{\Omega_n} f(x) W_n^p(dx) = \int_{\mathbb{R}_+^n} F(\lambda_1, \dots, \lambda_n) q_n^p(\lambda_1, \dots, \lambda_n) d\lambda_1 \dots d\lambda_n,$$

with

$$q_n^p(\lambda_1, \dots, \lambda_n) = \frac{1}{Z_n^p} e^{-(\lambda_1 + \cdots + \lambda_1)} \Delta(\lambda)^2 \prod_{j=1}^n \lambda_j^{p-n},$$

where

$$Z_n^p = \int_{\mathbb{R}_+^n} e^{-(\lambda_1 + \cdots + \lambda_1)} \Delta(\lambda)^2 \prod_{j=1}^n \lambda_j^{p-n} d\lambda_1 \dots \lambda_n.$$

As we did for the Gaussian unitary ensemble we will study the asymptotics of the statistical distribution of the eigenvalues, i.e. we will study, as n and p go to infinity, the asymptotics of the probability measure μ_n^p defined on $[0, \infty[$ by, if f is a bounded measurable function,

$$\int_{[0,\infty[} f(t) \mu_n^p(dt) = \int_{\Omega_n} \frac{1}{n} \operatorname{tr}\big(f(x)\big) W_n^p(dx).$$

For $p > n-1$ this measure is absolutely continuous with respect to the Lebesgue measure,

$$\mu_n^p(dt) = w_m^p(t) dt,$$

with

$$w_n^p(t) = \int_{\mathbb{R}_+} q(t, \lambda_2, \dots, \lambda_n) d\lambda_1 \dots d\lambda_n.$$

We will use Mehta's formulae to express this density w_n^p in terms of the Christoffel-Darboux kernel for the Laguerre polynomials. Recall that the Laguerre polynomials L_n^α ($\alpha > -1$) are defined by

$$L_n^\alpha(x) = \frac{1}{n!} e^x x^{-\alpha} \Big(\frac{d}{dx}\Big)^n (e^{-x} x^{n+\alpha}).$$

The Laguerre polynomials L_n^α are orthogonal with respect to the inner product

$$(p|q) = \int_0^\infty p(x) q(x) e^{-x} x^\alpha dx,$$

and

$$d_n^\alpha = \int_0^\infty \big(L_n^\alpha(x)\big)^2 e^{-x} x^\alpha dx = \frac{\Gamma(n+\alpha+1)}{n!}.$$

We define the Laguerre functions as

$$\varphi_n^\alpha(x) = \frac{1}{\sqrt{d_n^\alpha}} L_n^\alpha(x) e^{-\frac{x}{2}} x^{\frac{\alpha}{2}}.$$

They constitute a Hilbert basis of $L^2(\mathbb{R}_+)$. We define also the Christoffel-Darboux kernel

$$K_n^\alpha(x,y) = \sum_{k=0}^{n-1} \varphi_k^\alpha(x) \varphi_k^\alpha(y).$$

Proposition VI.2.1. — *For $p > n-1$, the density of the measure μ_n^p, the statistical distribution of the eigenvalues, is given by*

$$w_n^p(t) = \frac{1}{n} K_n^{p-n}(t,t).$$

This is a special case of Proposition III.3.2.

Assume $p \in \{0, 1, \dots, n-1\}$. A matrix $\xi \in M(n,p;\mathbb{C})$ can be decomposed as

$$\xi = u \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_p \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} v,$$

with $\alpha_1 \geq 0, \ldots, \alpha_p \geq 0$, $u \in U(n)$, $v \in U(p)$. The p eigenvalues of the $p \times p$ Hermitian matrix $\xi^*\xi$ are $\lambda_1 = \alpha_1^2, \ldots, \lambda_p = \alpha_p^2$, and the n eigenvalues of the $n \times n$ Hermitian matrix $\xi\xi^*$ are $\lambda_1, \ldots, \lambda_p, 0, \ldots, 0$. Hence, for $x = \xi\xi^*$,

$$\operatorname{tr}\bigl(\varphi(x)\bigr) = \varphi(\lambda_1) + \cdots + \varphi(\lambda_p) + (n-p)\varphi(0).$$

Therefore

PROPOSITION VI.2.2. — *For $p \in \{0, 1, \ldots, n-1\}$, the measure μ_n^p is given by*

$$\int_{[0,\infty[} \varphi(t)\mu_n^p(dt) = \Bigl(1 - \frac{p}{n}\Bigr)\varphi(0) + \frac{1}{n}\int_0^\infty \varphi(t)K_p^{n-p}(t,t)dt.$$

3. Convergence to the Marchenko-Pastur law. — The Marchenko-Pastur law μ_c ($c > 0$) is the probability measure on $[0, \infty[$ given by

$$\int_{[0,\infty[} \varphi(t)\mu_c(dt) = \max\{1-c, 0\}\varphi(0) + \frac{1}{2\pi}\int_a^b \varphi(t)\sqrt{(t-a)(b-t)}\frac{dt}{t},$$

where $a = (\sqrt{c} - 1)^2$, $b = (\sqrt{c} + 1)^2$.

Remark

It is possible to check that the measure μ_c depends continuously on c with respect to the tight topology.

Assuming that p depends on n: $p = p(n)$, in such a way that

$$\lim_{n\to\infty} \frac{p(n)}{n} = c,$$

we will see that, after scaling, the measure μ_n^p converges to the Marchenko-Pastur law μ_c as n goes to infinity.

THEOREM V.3.1(MARCHENKO-PASTUR). — *Assume that*

$$\lim_{n\to\infty} \frac{p(n)}{n} = c.$$

Then, for a bounded continuous function on $\mathbb{R}_+$,

$$\lim_{n\to\infty} \int_{[0,\infty[} \varphi\Bigl(\frac{t}{n}\Bigr)\mu_n^p(dt) = \int_{[0,\infty[} \varphi(t)\mu_c(dt).$$

We will present a proof due to H. Haagerup and S. Thorbjørnsen (Random matrices with complex Gaussian entries, Exp. Math. 21 (2003),293-337.) The method amounts to computing the Laplace transform of the measure μ_n^p, and to studying the asymptotic of this Laplace transform.

LEMMA VI.3.2.

$$\frac{d}{dt}\big(tK_n^\alpha(t,t)\big) = \sqrt{n(n+\alpha)}\varphi_{n-1}^\alpha(t)\varphi_n^\alpha(t).$$

Proof. Define

$$\mathcal{K}_n^\alpha(s,t) = \sum_{k=1}^{n-1} \frac{1}{d_k^\alpha} L_k^\alpha(s) L_k^\alpha(t).$$

By Proposition II.2.2,

$$\mathcal{K}_n^\alpha(t,t) = \frac{n!}{\Gamma(n+\alpha)}\big((L_{n-1}^\alpha)'(t)L_n^\alpha(t) - (L_n^\alpha)'(t)L_{n-1}^\alpha(t)\big),$$

and

$$\frac{d}{dt}\mathcal{K}_n^\alpha(t,t) = \frac{n!}{\Gamma(n+\alpha)}\big((L_{n-1}^\alpha)''(t)L_n^\alpha(t) - (L_n^\alpha)''(t)L_{n-1}^\alpha(t)\big),$$

By using that $u = L_n^\alpha$ is solution of the differential equation

$$tu'' + (\alpha+1-x)u' + nu = 0,$$

one obtains

$$t\frac{d}{dt}\mathcal{K}_n^\alpha(t,t) + (\alpha+1-t)\mathcal{K}_n^\alpha(t,t) = \frac{n!}{\Gamma(n+\alpha)} L_{n-1}^\alpha(t)L_n^\alpha(t).$$

Finally, since

$$K_n^\alpha(s,t) = \mathcal{K}_n^\alpha(s,t)s^{\frac{\alpha}{2}}t^{\frac{\alpha}{2}}e^{-\frac{s+t}{2}},$$

we obtain

$$\begin{aligned}
\frac{d}{dt}\big(tK_n^\alpha(t,t)\big) &= \frac{d}{dt}\big(\mathcal{K}_n^\alpha(t,t)t^{\alpha+1}e^{-t}\big) \\
&= \big(t\frac{d}{dt}\mathcal{K}_n^\alpha(t,t) + (\alpha+1-t)\mathcal{K}_n^\alpha(t,t)\big)t^\alpha e^{-t} \\
&= \frac{n!}{\Gamma(n+\alpha)} L_{n-1}^\alpha(t)L_n^\alpha(t)t^\alpha e^{-t} \\
&= \sqrt{n(n+\alpha)}\varphi_{n-1}^\alpha(t)\varphi_n^\alpha(t).
\end{aligned}$$

∎

Lemma VI.3.3. — *For $p > n-1$, $\Re\lambda > -1$,*

$$\int_0^\infty te^{-\lambda t}\mu_n^p(dt) = np\frac{1}{(1+\lambda)^{p+n}}{}_2F_1(1-p, 1-n, 2; \lambda^2).$$

Notice that ${}_2F_1(1-p, 1-n, 2; \lambda^2)$ is a polynomial in λ. In fact, since $1-n$ is a negative integer,

$$\begin{aligned}
&{}_2F_1(1-p, 1-n, 2; \lambda^2)\\
&= \sum_{j=0}^\infty \frac{(1-n)_j(1-p)_j}{(2)_j j!}\lambda^{2j}\\
&= \sum_{j=0}^{n-1} \frac{(n-1)(n-2)\dots(n-j)(p-1)(p-2)\dots(p-j)}{j!(j+1)!}\lambda^{2j}.
\end{aligned}$$

Proof. We start from the following result, we will not prove: for $\Re\lambda > -1$,

$$\begin{aligned}
&\int_0^\infty L_j^\alpha(t)L_k^\alpha(t)e^{-\lambda t}t^\alpha e^{-t}dt\\
&= \frac{d_j^\alpha d_k^\alpha}{\Gamma(\alpha+1)}\frac{\lambda^{j+k}}{(1+\lambda)^{\alpha+j+k+1}}{}_2F_1(-j, -k, \alpha+1; \frac{1}{\lambda^2}).
\end{aligned}$$

(See [Haagerup-Thorbjørnsen, 2003] p.317.) Taking $j = n-1$, $k = n$, we obtain

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t}\varphi_{n-1}^\alpha(t)\varphi_n^\alpha(t)dt\\
&= \frac{\sqrt{d_{n-1}^\alpha d_n^\alpha}}{\Gamma(\alpha+1)}\frac{\lambda^{2n-1}}{(1+\lambda)^{2n+\alpha}}{}_2F_1(-n+1, -n, \alpha+1; \frac{1}{\lambda^2}).
\end{aligned}$$

By using Lemma VI.3.2, and classical properties of the hypergeometric function ${}_2F_1$, Lemma VI.3.3 follows. ∎

Proof of Theorem VI.3.1 a) We assume first that $p > n-1$. By using Lemma VI.3.3 we can compute

$$\int_{\mathbb{R}_+}\frac{t}{n}e^{-\lambda\frac{t}{n}}\mu_n^p(dt) = \frac{n}{p}(1+\lambda)^{-(p+n)}{}_2F_1(1-p, 1-n, 2; \frac{\lambda^2}{n^2}).$$

First

$$\lim_{n\to\infty}\Big(1+\frac{\lambda}{n}\Big)^{-\big(p(n)+n\big)} = e^{-(c+1)\lambda}.$$

Now

$$\begin{aligned}
&{}_2F_1(1-p, 1-n; 2; \frac{\lambda^2}{n^2})\\
&= \sum_{j=0}^{\infty} \frac{(1-p)_j (1-n)_j}{(2)_j j!} \frac{\lambda^{2j}}{n^{2j}}\\
&= \sum_{j=0}^{\infty} a_j(n) \frac{1}{j!(j+1)!} \lambda^{2j},
\end{aligned}$$

with

$$a_j(n) = \frac{(n-1)(n-2)\dots(n-j)(p-1)(p-2)\dots(p-j)}{n^{2j}}.$$

Since

$$\lim_{n\to\infty} a_j(n) = 1, \quad |a_j(n)| \leq \gamma^j,$$

with

$$\gamma = \sup \frac{p(n)}{n},$$

it follows that

$$\lim_{n\to\infty} {}_2F_1(1-p, 1-n; 2; \frac{\lambda^2}{n^2}) = \sum_{j=1}^{\infty} \frac{c^j}{j!(j+1)!} \lambda^{2j} = F_1(2i\sqrt{c}\lambda).$$

with the notation of Section III.6. We saw that $F_1(r\tau)$ is the Fourier transform of the semi-circle law σ_r. The factor $e^{-(c+1)\lambda}$ corresponds to a shift: $e^{-i(c+1)\tau} F_1(2\sqrt{c}\tau)$ is the Fourier transform of the probability measure ν on $\mathbb{R}$ defined by

$$\int_{\mathbb{R}} f(t)\nu(dt) = \frac{1}{2\pi} \int_a^b f(t)\sqrt{(t-a)(b-t)}dt,$$

with

$$a = -2\sqrt{c} + c + 1 = (\sqrt{c} - 1)^2, \quad b = 2\sqrt{c} + c + 1 = (\sqrt{c} + 1)^2.$$

By Lévy-Kramér Theorem (Theorem III.5.1), this shows that, for every $\varphi \in \mathcal{C}_c([0,\infty[)$,

$$\lim_{n\to\infty} \int_{[0,\infty[} \frac{t}{n} \varphi\Big(\frac{t}{n}\Big) \mu_n^p(dt) = \frac{1}{2\pi} \int_a^b f(t)\sqrt{(t-a)(b-t)}dt.$$

It follows that there is a constant $A \geq 0$ such that, for $\psi \in \mathcal{C}_c([0,\infty[)$,

$$\lim_{n\to\infty} \int_{[0,\infty[} \psi\Big(\frac{t}{n}\Big)\mu_n^p(dt) = A\psi(0) + \frac{1}{2\pi}\int_a^b \psi(t)\sqrt{(t-a)(b-t)}\frac{dt}{t}.$$

The integral

$$I(c) = \frac{1}{2\pi}\int_a^b f(t)\sqrt{(t-a)(b-t)}\frac{dt}{t}$$

can be evaluated:

$$I(c) = \begin{cases} 1, & \text{if } c > 1, \\ c, & \text{if } c < 1. \end{cases}$$

Since the limit measure is a probability measure, it follows that $A = 0$

b) For $p \in \{0, 1, \dots, n-1\}$, by Proposition VI.2.2:

$$\int_{[0,\infty[} \varphi(t)\mu_n^p(dt) = \Big(1 - \frac{p}{n}\Big)\varphi(0) + \frac{1}{n}\int_0^\infty \varphi(t)K_p^{n-p}(t,t)dt.$$

One shows as in the case $p > n-1$ that, for $\varphi \in \mathcal{C}_c([0,\infty[)$,

$$\lim_{n\to\infty} \int_{[0,\infty[} \frac{t}{n}\varphi\Big(\frac{t}{n}\Big)\mu_n^p(dt) = \frac{1}{2\pi}\int_a^b \varphi(t)\sqrt{(t-a)(b-t)}dt,$$

and that there exists a constant $A \geq 0$ such that, for $\psi \in \mathcal{C}_c([0,\infty[)$,

$$\lim_{n\to\infty} \int_{[0,\infty[} \psi\Big(\frac{t}{n}\Big)\mu_n^p(dt) = A\psi(0) + \frac{1}{2\pi}\int_a^b \psi(t)\sqrt{(t-a)(b-t)}\frac{dt}{t}.$$

We saw that $I(c) = c$ for $c < 1$. Since the limit measure is a probability measure, it follows that $A = 1 - c$. $\square$

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