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PROJECTIONS OF ORBITAL MEASURES FOR THE ACTION OF A PSEUDO-UNITARY GROUP

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Abstract. The pseudo-unitary group $U(p,q)$ acts on the space $\mathrm{Herm}(n,\mathbb{C})$ of $n\times n$ Hermitian matrices ($n=p+q$). For an orbit of convex type we study the projection of the orbit on the subspace $\mathrm{Herm}(n-1,\mathbb{C})$ and the projection of the associated orbital measure. By using an explicit formula for the Fourier–Laplace transform of such an orbital measure due to Ben Saïd and Ørsted (2005), we prove an analogue of a formula due to Baryshnikov (2001), which is related to the action of the unitary group $U(n)$.

1. Introduction. For a Hermitian matrix $X\in\mathrm{Herm}(n,\mathbb{C})$, the classical spectral theorem says that the eigenvalues of X are real and the corresponding eigenspaces are orthogonal. The unitary group $U(n)$ acts on the space $\mathrm{Herm}(n,\mathbb{C})$ by the transformations $X\mapsto uXu^*$ ($u\in U(n)$). For this action every orbit contains a diagonal matrix and can be described as

$$\mathcal{O}_A=\{uAu^*\mid u\in U(n)\},\quad A=\mathrm{diag}(a_1,\ldots,a_n).$$

By the spectral theorem

$$\mathcal{O}_A=\big\{X\in\mathrm{Herm}(n,\mathbb{C})\mid \mathrm{spectrum}(X)=\{a_1,\ldots,a_n\}\big\}.$$

Let p denote the projection of $\mathrm{Herm}(n,\mathbb{C})$ onto $\mathrm{Herm}(n-1,\mathbb{C})$ which maps the matrix X to the $(n-1)\times(n-1)$ left corner Y of X. The Cauchy interlacing theorem, which is also called Rayleigh's theorem, says that the sequence of the eigenvalues $\mu_1\geq\ldots\geq\mu_{n-1}$ of Y interlace the sequence of the eigenvalues $\lambda_1\geq\ldots\geq\lambda_n$ of X:

$$\lambda_1\geq\mu_1\geq\lambda_2\geq\ldots\geq\mu_{n-1}\geq\lambda_n.$$

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In other words the set $p(\mathcal{O}_A)$ is equal to the set of matrices $Y \in \mathrm{Herm}(n-1,\mathbb{C})$ whose eigenvalues $\mu_1 \geq \ldots \geq \mu_{n-1}$ satisfy

$$a_1 \geq \mu_1 \geq a_2 \geq \ldots \geq \mu_{n-1} \geq a_n.$$

(In case of the orthogonal group $O(n)$ acting on the space $Sym(n,\mathbb{R})$ of real symmetric matrices, we have the same results.) The orbit $\mathcal{O}_A$ carries a $U(n)$-invariant measure: the orbital measure μ_A. The projection $p(\mu_A)$ of this measure is supported by the compact set $p(\mathcal{O}_A)$, and the density of $p(\mu_A)$ is given by Baryshnikov's formula.

The purpose of this note is to extend these results to the case of the action of the pseudo-unitary group $U(p,q)$ on the space $\mathrm{Herm}(n,\mathbb{C})$ $(n=p+q)$. We will establish an analogue of the Cauchy interlacing theorem, and of Baryshnikov's formula. In order to explain our method we will give in the second section a proof of the Cauchy interlacing theorem, and of Baryshnikov's formula.

One can also consider, for the action of $U(n)$ on $\mathrm{Herm}(n,\mathbb{C})$, the projections of the orbital measure μ_A on the subspaces $\mathrm{Herm}(k,\mathbb{C})$ for $k \leq n-2$. Formulas for these projections have been obtained by Olshanski [8] (see also [5]). Observe that the action of $U(n)$ on $\mathrm{Herm}(n,\mathbb{C})$ is nothing but the adjoint representation of the compact Lie group $U(n)$ on its Lie algebra $\mathfrak{u}(n) = i\mathrm{Herm}(n,\mathbb{C})$. In [9] Zubov considers the action of a classical compact Lie group on its Lie algebra, and similar results are obtained. In a more general setting Heckman [7] considers a compact Lie group acting on its Lie algebra and the projection onto the Lie algebra of a closed subgroup.

In [3] projections of orbital measures are studied in the context of stochastic processes.

The method of proof by Baryshnikov, which is different from the one we present, is inspired by the computation by Gelfand and Naimark [6] of the spherical functions for the Riemannian symmetric space $GL(n,\mathbb{C})/U(n)$. In [4], following the method of Gelfand and Naimark, we computed the spherical functions for the ordered symmetric space $GL(n,\mathbb{C})/U(p,q)$.

2. Cauchy interlacing theorem, and Baryshnikov's formula. The unitary group $U(n)$ acts on the space $\mathrm{Herm}(n,\mathbb{C})$ of Hermitian matrices by the transformations $X \mapsto uXu^*$. This action is nothing but the adjoint representation of the compact Lie group $U(n)$ on its Lie algebra $\mathfrak{u}(n) = i\mathrm{Herm}(n,\mathbb{C})$. Let p be the projection of $\mathrm{Herm}(n,\mathbb{C})$ onto the subspace of matrices with zero entries on the last row and the last column, identified with $\mathrm{Herm}(n-1,\mathbb{C})$.

THEOREM 2.1 (Interlacing theorem). *Let $X \in \mathrm{Herm}(n,\mathbb{C})$ with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$, and its projection $Y = p(X) \in \mathrm{Herm}(n-1,\mathbb{C})$ with eigenvalues $\mu_1 \geq \ldots \geq \mu_{n-1}$. The eigenvalues of Y interlace those of X:*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \ldots \geq \mu_{n-1} \geq \lambda_n.$$

Proof. We first assume that the eigenvalues of X are distinct: $\lambda_1 > \ldots > \lambda_n$. We can write $X = uAu^*$ with $u \in U(n)$, and $A = \mathrm{diag}(a_1,\ldots,a_n)$, $a_i = \lambda_i$. We will evaluate in two ways the rational function

$$R(z) = [(zI - X)^{-1}]_{n,n},$$

the lower right entry of the inverse matrix $(zI_n - X)^{-1}$.

On one hand, by Cramer's formulas,

$$R(z) = \frac{\det^{(n-1)}(zI_{n-1} - Y)}{\det^{(n)}(zI_n - X)} = \frac{\prod_{i=1}^{n-1}(z-\mu_i)}{\prod_{i=1}^{n}(z-a_i)} .$$

The poles of R are the eigenvalues a_i of X, and the zeros are the eigenvalues μ_i of Y. On the other hand, since

$$(zI_n - X)^{-1} = u(zI_n - A)^{-1}u^*,$$

we get

$$R(z) = [u(zI_n - A)^{-1}u^*]_{n,n} = \sum_{i=1}^{n} |u_{ni}|^2 \frac{1}{z-a_i} .$$

The residues are the numbers $w_i = |u_{ni}|^2$, hence nonnegative. We assume further that $w_i > 0$ for all i. Then on each interval $]a_{i+1}, a_i[$ the function R decreases from $+\infty$ to $-\infty$, hence each interval $]a_{i+1}, a_i[$ contains a unique zero of R and:

$$a_1 > \mu_1 > a_2 > \mu_2 > \ldots > \mu_{n-1} > a_n.$$

In case the eigenvalues are not distinct or some w_i vanish, the result is obtained from the generic case by using continuity arguments. ∎

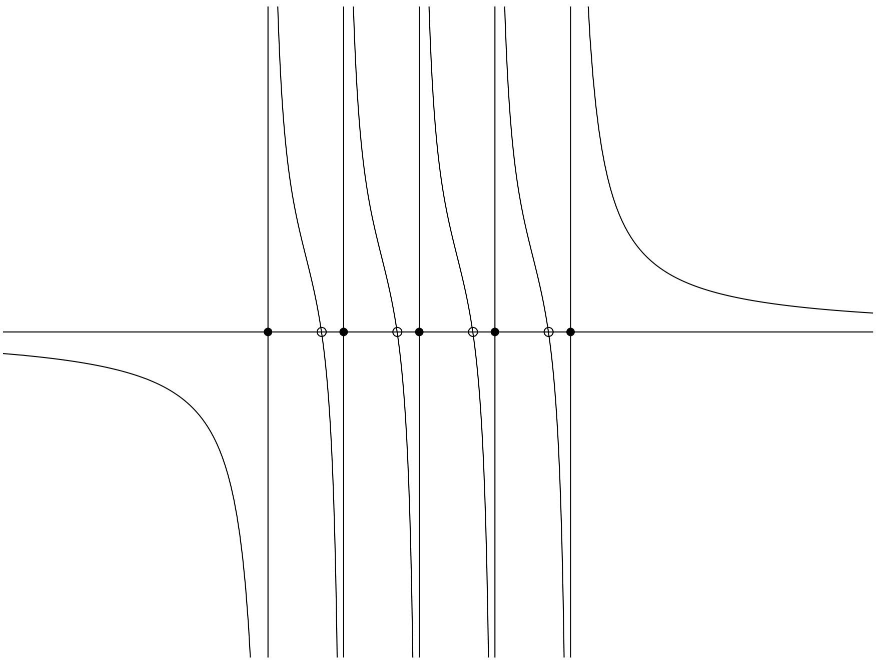

Fig. 1. Graph of the rational function $R(z)$, $n = 5$

The orbit $\mathcal{O}_A$ carries a natural $U(n)$-invariant probability measure, the orbital measure μ_A, which is the image under the map

$$U(n) \to \mathcal{O}_A, \quad u \mapsto uAu^*,$$

of the normalized Haar measure α_n of the compact group $U(n)$: for a function f on $\mathrm{Herm}(n,\mathbb{C})$,

$$\int_{\mathcal{O}_A} f(X)\,\mu_A(dX) = \int_{U(n)} f(uAu^*)\,\alpha_n(du).$$

We are interested in the image $\mu_A^{(n-1)} = p(\mu_A)$ of the orbital measure μ_A under the projection $p : \mathrm{Herm}(n,\mathbb{C}) \to \mathrm{Herm}(n-1,\mathbb{C})$. For a function f defined on $\mathrm{Herm}(n-1,\mathbb{C})$,

$$\int_{\mathrm{Herm}(n-1,\mathbb{C})} f(Y)\,\mu_A^{(n-1)}(dY) = \int_{\mathcal{O}_A} f\big(p(X)\big)\,\mu_A(dX).$$

The measure $\mu_A^{(n-1)}$ is given by a formula due to Baryshnikov [1]. More precisely it is a formula for the radial part $\nu_A^{(n-1)}$ of $\mu_A^{(n-1)}$. Let us recall the definition of the radial part of a measure μ on $\mathrm{Herm}(n,\mathbb{C})$ which is $U(n)$-invariant. The integral of a function f defined on $\mathrm{Herm}(n,\mathbb{C})$ can be written as

$$\int_{\mathrm{Herm}(n,\mathbb{C})} f(X)\,\mu(dX) = \int_{(\mathbb{R}^n)_+} \Big(\int_{U(n)} f\big(u\,\mathrm{diag}(t_1,\dots,t_n)u^*\big)\alpha_n(du)\Big)\,\nu(dt).$$

where ν is a measure on

$$(\mathbb{R}^n)_+ = \{t \in \mathbb{R}^n \,|\, t_1 \geq t_2 \geq \ldots \geq t_n\},$$

called the radial part of μ.

Theorem 2.2 (Baryshnikov's formula). *The radial part $\nu_A^{(n-1)}$ of the projection $\mu_A^{(n-1)}$ of the orbital measure μ_A is a probability measure on $(\mathbb{R}^{n-1})_+$ supported by*

$$\{t \in (\mathbb{R}^{n-1})_+ \,|\, a_1 \geq t_1 \geq a_2 \geq t_2 \geq \ldots \geq t_{n-1} \geq a_n\},$$

with density

$$(n-1)!\,\frac{V_{n-1}(t_1,\dots,t_{n-1})}{V_n(a_1,\dots,a_n)}\,.$$

This means that, for a function f defined on $(\mathbb{R}^{n-1})_+$,

$$\int_{(\mathbb{R}^{n-1})_+} f(t)\,\nu_A^{(n-1)}(dt) = \frac{(n-1)!}{V_n(a)} \int_{a_2}^{a_1} dt_1 \int_{a_3}^{a_2} dt_2 \ldots \int_{a_n}^{a_{n-1}} dt_{n-1}\,V_{n-1}(t)f(t).$$

In this formula V_n denotes the Vandermonde polynomial in n variables:

$$V_n(x_1,\dots,x_n) = \begin{vmatrix} x_1^{n-1} & \dots & x_n^{n-1} \\ \vdots & & \vdots \\ x_1 & \dots & x_n \\ 1 & \dots & 1 \end{vmatrix} = \prod_{1\leq i<j\leq n} (x_i - x_j).$$

The proof we will give is due to Olshanski [8]. The idea of the proof is the following observation: for a measure μ on $\mathrm{Herm}(n,\mathbb{C})$, the Fourier–Laplace transform of the projection $p(\mu)$ of the measure μ on $\mathrm{Herm}(n-1,\mathbb{C})$ is equal to the restriction to $\mathrm{Herm}(n-1,\mathbb{C})$ of the Fourier–Laplace transform of μ. The Fourier–Laplace transform of the orbital measure μ_A is explicitly known. This is the Itzykson–Zuber–Harish–Chandra formula:

Proposition 2.3. *For $Z = \operatorname{diag}(z_1, \dots, z_n)$, the Fourier–Laplace transform of the orbital measure μ_A is given by*

$$\widehat{\mu_A}(Z) = \int_{\mathcal{O}_A} e^{\operatorname{tr} ZX}\,\mu_A(dX) = \int_{U(n)} e^{\operatorname{tr}(ZuAu^*)} = \mathcal{E}_n(a; z),$$

where

$$\mathcal{E}_n(z; a) := \delta_n!\,\frac{1}{V_n(a)V_n(z)}\det(e^{z_i a_j})_{1\le i,j\le n},$$

and

$$\delta_n = (n-1, n-2, \dots, 1, 0), \quad \delta_n! = (n-1)!(n-2)!\dots 2!.$$

Corollary 2.4. *Let μ be a bounded measure on $\operatorname{Herm}(n,\mathbb{C})$ which is $U(n)$-invariant, and ν its radial part. The Fourier–Laplace transform of μ, for $Z = \operatorname{diag}(z_1, \dots, z_n)$, is given by*

$$\int_{\operatorname{Herm}(n,\mathbb{C})} e^{\operatorname{tr} ZX}\,\mu(dX) = \int_{(\mathbb{R}^n)_+} \mathcal{E}_n(z; t)\,\nu(dt).$$

Proof of Theorem 2.2. We first evaluate the function

$$F(z) = \frac{1}{\prod_{i=1}^{n-1}(z_i - z_n)}\det(e^{z_i a_j})_{1\le i,j\le n},$$

for $z_n = 0$:

$$F(z_1, \dots, z_{n-1}, 0) = \frac{1}{z_1 \cdots z_{n-1}}\begin{vmatrix} e^{a_1 z_1} & \dots & e^{a_n z_1} \\ \vdots & & \vdots \\ e^{a_1 z_{n-1}} & \dots & e^{a_n z_{n-1}} \\ 1 & \dots & 1 \end{vmatrix}.$$

By subtracting the i-th column from the $(i-1)$-th column, $i = n, \dots, 2$, one gets

$$F(z_1, \dots, z_{n-1}, 0) = \begin{vmatrix} (e^{a_1 z_1} - e^{a_2 z_1})/z_1 & \dots & (e^{a_{n-1} z_1} - e^{a_n z_1})/z_1 \\ \vdots & & \vdots \\ (e^{a_1 z_{n-1}} - e^{a_2 z_{n-1}})/z_{n-1} & \dots & (e^{a_{n-1} z_{n-1}} - e^{a_n z_{n-1}})/z_{n-1} \end{vmatrix}.$$

Since, for $a > b$,

$$\frac{e^{az} - e^{bz}}{z} = \int_b^a e^{tz}\,dt,$$

we get

$$F(z_1, \dots, z_{n-1}, 0) = \int_{a_2}^{a_1} dt_1 \int_{a_3}^{a_2} dt_2 \dots \int_{a_n}^{a_{n-1}} dt_{n-1} \det(e^{a_i t_j})_{1\le i,l\le n-1}.$$

The function $\mathcal{E}_n(z; a)$ can be written as

$$\mathcal{E}_n(z_1, \dots, z_n; a) = \delta_n!\,\frac{1}{V_n(a)}\,\frac{1}{V_{n-1}(z_1, \dots, z_{n-1})}F(z_1, \dots, z_n).$$

Hence we obtain

$$\begin{aligned}\mathcal{E}_n(z_1, \dots, z_{n-1}, 0; a) &= \delta_n!\,\frac{1}{V_n(a)}\,\frac{1}{V_{n-1}(z_1, \dots, z_{n-1})}\,F(z_1, \dots, z_{n-1}, 0) \\ &= \frac{(n-1)!}{V_n(a)}\int_{a_2}^{a_1} dt_1 \int_{a_3}^{a_2} dt_2 \dots \int_{a_n}^{a_{n-1}} dt_{n-1}\, V_{n-1}(t)\mathcal{E}_{n-1}(z_1, \dots, z_{n-1}; t).\end{aligned}$$

By Corollary 2.4, this implies Theorem 2.2. ∎

3. An analogue of the Cauchy interlacing theorem for the action of the pseudo-unitary group $U(p,q)$ on the space $\mathrm{Herm}(n,\mathbb{C})$ of Hermitian matrices. Recall that a matrix u belongs to the pseudo-unitary group $U(p,q)$ if

$$uI_{p,q}u^* = I_{p,q}, \quad I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

The pseudo-unitary group $U(p,q)$ acts on the space $\mathrm{Herm}(n,\mathbb{C})$ $(n = p+q)$ by the transformations

$$X \mapsto uXu^*.$$

Note that this action is equivalent to the adjoint action of the Lie group $U(p,q)$ on its Lie algebra. In this section we assume $q \geq 1$. Let $\Omega_n \subset \mathrm{Herm}(n,\mathbb{C})$ be the cone of positive definite Hermitian matrices. We will consider orbits of $U(p,q)$ which are contained in Ω_n.

Proposition 3.1. *Every orbit which is contained in Ω_n is of the form*

$$\mathcal{O}_A = \{uAu^* \mid u \in U(p,q)\},$$

where A is a diagonal matrix with positive diagonal entries.

Proof. It follows from the following decomposition of $G = GL(n,\mathbb{C})$. Every $g \in G$ can be written $g = udv$, where $u \in U(p,q)$, $v \in U(n)$, d is a diagonal matrix with nonzero diagonal elements. Since every $X \in \Omega_n$ can be written $X = gg^*$, with $g \in G$, we get

$$X = udvv^*du^* = ud^2u^*. \ \blacksquare$$

For $X \in \mathrm{Herm}(n,\mathbb{C})$, a number $\lambda \in \mathbb{C}$ will be said to be a pseudo-eigenvalue of X if there exists a nonzero vector $v \in \mathbb{C}^n$ such that

$$Xv = \lambda I_{p,q} v$$

or, equivalently, if λ is an eigenvalue of $I_{p,q}X$. Two Hermitian matrices X and Y in the same $U(p,q)$-orbit have the same pseudo-eigenvalues. In fact, if $Y = uXu^*$, with $u \in U(p,q)$, then

$$Y - \lambda I_{p,q} = u(X - \lambda I_{p,q})u^*.$$

A diagonal matrix $X = \mathrm{diag}(x_1, \ldots, x_n)$ has pseudo-eigenvalues

$$\lambda_1 = x_1, \ \ldots, \ \lambda_p = x_p, \ \lambda_{p+1} = -x_{p+1}, \ \ldots, \ \lambda_{p+q} = -x_{p+q}.$$

Therefore, for $X \in \Omega_n$, the pseudo-eigenvalues of X are real, p pseudo-eigenvalues are positive, and q ones are negative.

Consider a diagonal matrix $A \in \Omega_n$, with pseudo-eigenvalues $a_1, \ldots, a_n$,

$$a_1 > 0, \ \ldots, \ a_p > 0, \quad a_{p+1} < 0, \ \ldots, \ a_{p+q} < 0,$$
$$A = \mathrm{diag}(a_1, \ldots, a_p, -a_{p+1}, \ldots, -a_{p+q}).$$

Then the orbit

$$\mathcal{O}_A = \{X \in uAu^* \mid u \in U(p,q)\},$$

is determined by

$$\mathcal{O}_A = \big\{X \in \Omega_n \mid \text{pseudo-spectrum}(X) = \{a_1, \ldots, a_n\}\big\}.$$

We fix a diagonal matrix

$$A = \mathrm{diag}(a_1, \ldots, a_p, -a_{p+1}, \ldots, -a_{p+q}),$$

with $a_1 \geq \ldots \geq a_p > 0 > a_{p+1} \geq \ldots \geq a_{p+q}$. The orbit $\mathcal{O}_A$ is contained in Ω_n. For $X \in \mathcal{O}_A$, $X = uAu^*$ with $u \in U(p,q)$, consider the projection $Y = p(X)$ of X on the subspace of Hermitian matrices with zeros on the last row and the last column, identified with $\mathrm{Herm}(n-1, \mathbb{C})$. The pseudo-eigenvalues of X are the numbers a_i. Restricted to $\mathbb{C}^{n-1}$, the matrix Y is positive definite: $Y \in \Omega_{n-1}$. We order its pseudo-eigenvalues as follows:

$$\mu_1 \geq \ldots \geq \mu_p > 0 > \mu_{p+1} \geq \ldots \geq \mu_{p+q-1}.$$

THEOREM 3.2. *The pseudo-eigenvalues of Y interlace the pseudo-eigenvalues of X in the following way:*

$$\begin{aligned}\mu_1 \geq a_1 \geq \mu_2 \geq a_2 \geq \ldots \geq \mu_p \geq a_p > 0 \\ > a_{p+1} \geq \mu_{p+1} \geq \ldots \geq a_{p+q-1} \geq \mu_{p+q-1} \geq a_{p+q}.\end{aligned}$$

Proof. We will evaluate in two different ways the rational function

$$R(z) = \big[(zI_{p,q} - X)^{-1}\big]_{n,n},$$

the lower right entry of the inverse $(zI_{p,q} - X)^{-1}$.

Observe that

$$zI_{p,q} - X = u(zI_{p,q} - A)u^*.$$

On one hand, by Cramer's formulas

$$R(z) = \frac{\det^{(n-1)}(zI_{p,q-1} - Y)}{\det^{(n)}(zI_{p,q} - X)} = -\frac{\prod_{i=1}^{n-1}(z - \mu_i)}{\prod_{i=1}^{n}(z - a_i)}.$$

On the other hand, since $zI_{p,q} - X = u(zI_{p,q} - A)u^*$, we get

$$R(z) = \big[u(zI_{p,q} - A)^{-1}u^*\big]_{n,n} = \sum_{i=1}^{p} \frac{|u_{ni}|^2}{z - a_i} - \sum_{i=p+1}^{p+q} \frac{|u_{ni}|^2}{z - a_i}.$$

The poles of the rational function R are the numbers $a_1, \ldots, a_n$, with residues

$$|u_{n1}|^2, \ldots, |u_{n,p}|^2, -|u_{n,p+1}|^2, \ldots, -|u_{n,p+q}|^2.$$

Observe that

$$\sum_{i=1}^{p} |u_{ni}|^2 - \sum_{i=p+1}^{p+q} |u_{ni}|^2 = -1,$$

and

$$R(z) \sim -\frac{1}{z} \quad (z \to \infty).$$

Moreover, the pseudo-eigenvalues of Y are the zeros of R. Inspecting the values of R near $\pm\infty$, and near the poles, one gets Theorem 3.2. ∎

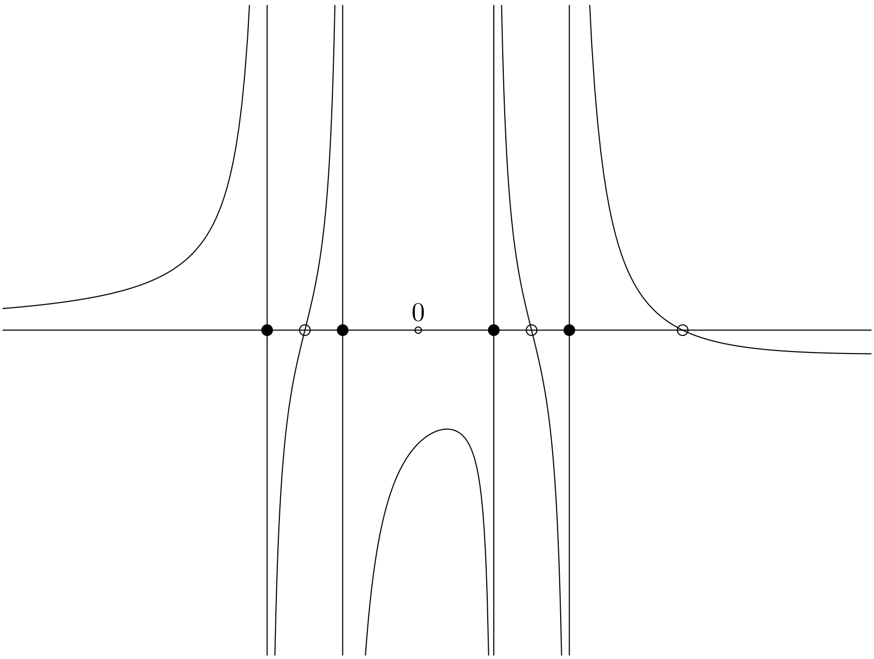

Fig. 2. Graph of the rational function $R(z)$, $p = 2$, $q = 2$

4. An analogue of Baryshnikov's formula. As in Section 3 we assume $q \geq 1$ in the present one. The orbit $\mathcal{O}_A$ of a diagonal matrix A in Ω_n carries an unbounded positive measure which is $U(p,q)$-invariant. The isotropic subgroup

$$\{u \in U(p,q) \mid uAu^* = A\}$$

is compact, hence such a measure can be defined as the image of a Haar measure α on $U(p,q)$

$$\int_{\mathcal{O}_A} f(X)\,\mu_A(dX) = \int_{U(p,q)} f(uAu^*)\,\alpha(du).$$

In this section we will determine the projection $\mu_A^{(n-1)} = p(\mu_A)$ of the orbital measure μ_A on the subspace $\mathrm{Herm}(n-1,\mathbb{C})$. Since the measure μ_A is unbounded, we will have to prove that the projection exists.

Every matrix $X \in \Omega_n$ can be written as $X = uTu^*$, where T is a diagonal matrix, $T = \mathrm{diag}(t_1,\ldots,t_p,-t_{p+1},\ldots,-t_{p+q})$ where the numbers t_i are the pseudo-eigenvalues of X, and $u \in U(p,q)$. Let μ be a positive measure on Ω_n which is $U(p,q)$-invariant. There is a positive measure ν on $\mathbb{R}^n$, the pseudo-radial part of μ, such that

$$\int_{\Omega_n} f(X)\,\mu(dX) = \int_{\mathbb{R}^n} \Big(\int_{U(p,q)} f(uTu^*)\,\alpha(du)\Big)\,\nu(dt).$$

The measure ν is supported by $(\mathbb{R}_+)^p \times (\mathbb{R}_-)^q$.

We will determine the pseudo-radial part $\nu_A^{(n-1)}$ of the projection $\mu_A^{(n-1)}$ of the orbital measure μ_A. This is an analogue of Baryshnikov's formula.

THEOREM 4.1. *Let $a_1, \dots, a_n$ be the pseudo-eigenvalues of the diagonal matrix A:*

$$A = \mathrm{diag}(a_1, \dots, a_p, -a_{p+1}, \dots, -a_{p+q}).$$

We assume

$$a_1 > \dots > a_p > 0 > a_{p+1} > \dots > a_{p+q}.$$

Then the projection $\mu_A^{(n-1)}$ of the orbital measure μ_A exists. The pseudo-radial part $\nu_A^{(n-1)}$ of $\mu_A^{(n-1)}$ is the measure on $\mathbb{R}^{n-1}$ supported by the set

$$\{t \in \mathbb{R}^{n-1} \mid t_1 \geq a_1 \geq t_2 \geq \dots \geq a_p > 0 > a_{p+1} \geq t_{p+1} \dots \geq t_{p+q-1} \geq a_{p+q}\},$$

with density

$$C \, \frac{V_{n-1}(t_1, \dots, t_{n-1})}{V_n(a_1, \dots, a_n)} \, .$$

For a function f defined on $\mathbb{R}^{n-1}$,

$$\int_{\mathbb{R}^{n-1}} f(t) \, \nu_A^{(n-1)}(dt) = \frac{C}{V_n(a_1, \dots, a_n)} \int_{a_1}^{\infty} dt_1 \int_{a_2}^{a_1} dt_2 \dots \int_{a_p}^{a_{p-1}} dt_p$$
$$\int_{a_{p+2}}^{a_{p+1}} dt_{p+1} \dots \int_{a_n}^{a_{n-1}} dt_{n-1} \, V_{n-1}(t) f(t).$$

We will prove this theorem by using an analogue of the Harish–Chandra–Itzykson–Zuber integral, i.e. an explicit formula for the Fourier–Laplace transform of the orbital measure μ_A:

PROPOSITION 4.2. *For $Z = \mathrm{diag}(z_1, \dots, z_n)$, with $\mathrm{Re}\, z_i < 0$ if $1 \leq i \leq p$, and $\mathrm{Re}\, z_i > 0$ if $p+1 \leq i \leq n$,*

$$\int_{\mathcal{O}_A} e^{\mathrm{tr}\, ZX} \, \mu_A(dX) = \int_{U(p,q)} e^{\mathrm{tr}(ZuAu^*)} \, \alpha(du) = C \mathcal{E}_{p,q}(z, a),$$

where

$$\mathcal{E}_{p,q}(z, a) := \frac{1}{V_n(a) V_n(z)} \det(e^{a_i z_j})_{1 \leq i,j \leq p} \det(e^{a_i z_j})_{p+1 \leq i,j \leq n}.$$

This is a special case of a formula obtained by Ben Saïd and Ørsted for reductive groups G such that $G^{\mathbb{C}}/G$ is an ordered symmetric space [2].

COROLLARY 4.3. *Let μ be a $U(p,q)$-invariant measure on Ω_n. The Laplace transform of μ, if it exists, is given, for $Z = \mathrm{diag}(z_1, \dots, z_n)$, by*

$$\int_{\Omega_n} e^{\mathrm{tr}\, ZX} \, \mu(dX) = \int_{\mathbb{R}^n} \mathcal{E}_{p,q}(z, t) \, \nu(dt),$$

where ν is the pseudo-radial part of μ.

Proof of Theorem 4.1. The idea is the same as for the proof of Theorem 2.2. The Fourier–Laplace transform of the projection $\mu_A^{(n-1)}$ is the restriction to $\mathrm{Herm}(n-1, \mathbb{C})$ of the Fourier–Laplace transform of μ_A. The Vandermonde polynomial $V_n(z_1, \dots, z_n)$, restricted to $z_n = 0$, can be written

$$V_n(z_1, \dots, z_{n-1}, 0)$$
$$= V_p(z_1, \dots, z_p) V_{q-1}(z_{p+1}, \dots, z_{p+q-1}) \prod_{1 \leq i \leq p < j \leq p+q-1} (z_i - z_j) \prod_{i=1}^{p+q-1} z_i.$$

a) Consider first

$$F_1(z_1,\dots,z_p)=\frac{1}{z_1\cdots z_p}\begin{vmatrix} e^{a_1z_1} & \dots & e^{a_pz_1}\\ \vdots & & \vdots\\ e^{a_1z_p} & \dots & e^{a_pz_p}\end{vmatrix}.$$

By subtracting the $(i-1)$-th column from the i-th column, $i=p,p-1,\dots,2$, one gets

$$F_1(z_1,\dots,z_p)=\frac{1}{z_1\cdots z_p}\begin{vmatrix} e^{a_1z_1} & e^{a_2z_1}-e^{a_1z_1} & \dots & e^{a_pz_1}-e^{a_{p-1}z_1}\\ \vdots & \vdots & & \vdots\\ e^{a_1z_p} & e^{a_2z_p}-e^{a_1z_p} & \dots & e^{a_pz_p}-e^{a_{p-1}z_p}\end{vmatrix}.$$

By using the formulas

$$\int_a^\infty e^{zt}\,dt=-\frac{1}{z}e^{za}\ (\operatorname{Re} z<0),\qquad \int_a^b e^{zt}\,dt=\frac{1}{z}(e^{zb}-e^{za}),$$

we get

$$F_1(z_1,\dots,z_p)=-\int_{a_1}^\infty dt_1\int_{a_2}^{a_1}dt_2\dots\int_{a_p}^{a_{p-1}}dt_p\det(e^{z_it_j})_{1\le i,j\le p}.$$

b) Then consider the second factor

$$F_2(z_{p+1},\dots,z_{p+q})=\frac{1}{\prod_{p+1\le i\le p+q-1}(z_i-z_{p+q})}\det(e^{z_ia_j})_{p+1\le i,j\le p+q}.$$

As we saw in the proof of Theorem 2.2,

$$F_2(z_{p+1},\dots,z_{p+q-1},0)=\int_{a_{p+2}}^{a_{p+1}}dt_{p+1}\dots\int_{a_{p+q}}^{a_{p+q-1}}dt_{p+q-1}\det(e^{z_it_j})_{p+1\le i,j\le p+q-1}.$$

c) By the results of a) and b) the function $\mathcal{E}_{p,q}(z_1,\dots,z_n;a)$ has an analytic continuation for $\operatorname{Re} z_i<0$ if $1\le i\le p$, $\operatorname{Re} z_i>0$ for $p+1\le i\le p+q-1$, and $\operatorname{Re} z_n\ge 0$. For $z_n=0$,

$$\mathcal{E}_{p,q}(z_1,\dots,z_{n-1},0;a)=\frac{1}{V_n(a)}\int_{a_1}^\infty dt_1\int_{a_2}^{a_1}dt_2\dots\int_{a_p}^{a_{p-1}}dt_p$$
$$\int_{a_{p+2}}^{a_{p+1}}dt_{p+1}\dots\int_{a_n}^{a_{n-1}}dt_{n-1}\,V_{n-1}(t)\mathcal{E}_{p,q-1}(z_1,\dots,z_{n-1};t).$$

d) To prove the existence of the projection $\mu_A^{(n-1)}$ we have to prove that if f is a function defined on $\operatorname{Herm}(n-1,\mathbb{C})$, measurable, bounded, positive, with compact support, then

$$\int_{\mathcal{O}_A} f\big(p(X)\big)\,\mu_A(dX)<\infty.$$

By c), for $Z=\operatorname{diag}(z_1,\dots,z_{n-1},0)$ with $\operatorname{Re} z_i<0$ if $1\le i\le p$, $\operatorname{Re} z_i>0$ for $p+1\le i\le p+q-1$,

$$\lim_{z_n\to 0,\operatorname{Re} z_n>0}\int_{\mathcal{O}_A} e^{\operatorname{tr} ZX}\,\mu_A(dX)=\mathcal{E}_{p,q}(z_1,\dots,z_{n-1},0;a).$$

By Fatou's Lemma it follows that if $Z_0=\operatorname{diag}(z_1,\dots,z_{n-1},0)$ with $z_i<0$ for $1\le i\le p$, $z_i>0$ for $p+1\le i\le p+q-1$,

$$\int_{\mathcal{O}_A} e^{\operatorname{tr} Z_0X}\,\mu_A(dX)<\infty.$$

This proves that the projection $\mu_A^{(n-1)}$ exists. In fact we can write $e^{\operatorname{tr} Z_0 X} = f(p(X))$, where f is a positive continuous function on $\mathrm{Herm}(n-1,\mathbb{C})$. Furthermore, by using the Lebesgue dominated convergence theorem, one shows that the Fourier–Laplace transform of the projection $\mu_A^{(n-1)}$ is given by

$$\widehat{\mu_A^{(n-1)}}(Z) = \int_{\mathrm{Herm}(n-1,\mathbb{C})} e^{\operatorname{tr} ZY}\, \mu_A^{(n-1)}(dY) = C\mathcal{E}_{p,q}(z_1,\ldots,z_{n-1},0;a)$$

for $Z = \operatorname{diag}(z_1,\ldots,z_{n-1})$, with $\operatorname{Re} z_i < 0$ for $1 \le i \le p$, $\operatorname{Re} z_i > 0$ for $p+1 \le i \le p+q-1$. Finally, by c) and Corollary 4.3, this finishes the proof of Theorem 4.1. ∎

Let $p^{(k)}$ be the projection which maps a matrix $X \in \mathrm{Herm}(n,\mathbb{C})$ to the $k \times k$ upper left corner $Y \in \mathrm{Herm}(k,\mathbb{C})$. One could consider the projection $\mu_A^{(k)} = p^{(k)}(\mu_A)$ of an orbital measure μ_A, and look for the existence and for an explicit formula for it. So far I know this is an open question.

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