Lectures on finite reductive groups and their representations

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Abstract

These lectures were given in a period of 7 weeks in Beijing by Olivier Dudas and Jean Michel. Sections 8 to 12 were written by Olivier Dudas, the rest by Jean Michel.

Contents

1 Prerequisite: affine algebraic groups

For complements consult [Geck].

An algebraic group is an algebraic variety such that the multiplication and inverse are continuous maps for the Zariski topology. We consider affine algebraic groups $\mathbf{G}$ over an algebraically closed field k, that is $\mathbf{G} = \operatorname{Spec} A$ where A is a k-algebra. The group structure gives a coalgebra structure on A (actually, a Hopf algebra structure).

Example 1.1. We will denote $\mathbb{G}_a = \operatorname{Spec} k[X]$ the additive group k^+ seen as an algebraic group. The comultiplication $k[X] \to k[X] \otimes k[X] \simeq k[X, Y]$ is given by $X \mapsto X + Y$.

Example 1.2. We will denote $\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$ the multiplicative group $k^\times$ seen as an algebraic group. Comultiplication is given by $X \mapsto XY$.

A *torus* is an algebraic group $\mathbf{T}$ isomorphic to $\mathbb{G}_m^r$; r is the *rank* of $\mathbf{T}$. The *group of characters* is $X(\mathbf{T}) = \operatorname{Hom}(\mathbf{T}, \mathbb{G}_m)$; we have $X(\mathbf{T}) \simeq \mathbb{Z}^r$ since a morphism $k[t, t^{-1}] \to k[t_1, \ldots, t_r, t_1^{-1}, \ldots, t_r^{-1}]$ is defined by the image of t, which must be invertible, thus be a monomial, unitary to get a coalgebra morphism. We define the *group of cocharacters* $Y(\mathbf{T}) = \mathrm{Hom}(\mathbb{G}_m, \mathbf{T})$, which is canonically dual to $X(\mathbf{T})$: for $\alpha \in X(\mathbf{T}), \alpha^\vee \in Y(\mathbf{T})$ we have $\alpha \circ \alpha^\vee \in \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z}$.

We define the *rank* of an affine algebraic group to be the rank of a maximal subtorus.

$\mathbf{G}$ is affine if and only if it is linear, that is it embeds as a closed subgroup of some $\mathrm{GL}_n(k)$. This allows to define *semi-simple* (resp. *unipotent*) elements of $\mathbf{G}$ as those whose image is such by the embedding (this is independent of the embedding). Any $g \in \mathbf{G}$ has a unique *Jordan decomposition* $g = g_s g_u$ where g_s is semi-simple, g_u unipotent and they commute.

Proposition 1.3. *Let $\mathbf{G}$ be a linear algebraic group over $\overline{\mathbb{F}}_p$. Then every element has finite order, the semi-simple elements are the p'-elements and the unipotent elements are the p-elements.*

Proof. This results from the fact that the above result holds for $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$. □

Example 1.4. A torus of rank n embeds as the diagonal matrices in $\mathrm{GL}_n(k)$ (thus $\mathrm{GL}_n(k)$ is of rank n). It consists of semisimple elements.

A connected semisimple group is a torus.

Example 1.5. A unipotent group embeds as a subgroup of the upper unitriangular matrices in some $\mathrm{GL}_n(k)$.

A unipotent group is nilpotent; in characteristic 0 it is necessarily connected; a connected unipotent group is isomorphic to an affine space as an algebraic variety.

The *Borel subgroups* are the maximal closed connected solvable subgroups. They embed as a subgroup of the upper triangular matrices in some $\mathrm{GL}_n(k)$.

The *radical* $\mathrm{Rad}(\mathbf{G})$ is the maximal normal closed connected solvable subgroup; $\mathbf{G}$ is *semisimple* if $\mathrm{Rad}(\mathbf{G}) = 1$. The *unipotent radical* $\mathrm{R_u}(\mathbf{G})$ is the maximal normal closed connected unipotent subgroup. $\mathbf{G}$ is *reductive* if $\mathrm{R_u}(\mathbf{G}) = 1$. We will write "reductive group" for "reductive algebraic linear group".

The quotient by a closed subgroup is a variety, and by a closed normal subgroup is an affine algebraic group; $\mathbf{G}/\mathrm{R_u}(\mathbf{G})$ is reductive. Its radical is a central torus.

Proposition 1.6 (Assumed)**.** *Let $\mathbf{B}$ be a connected solvable algebraic group, and let $\mathbf{T}$ be a maximal torus. Then $\mathbf{B} = \mathbf{T} \ltimes \mathrm{R_u}(\mathbf{B})$; every semisimple element is conjugate to an element of $\mathbf{T}$.*

Corollary 1.7. *Let $\mathbf{B}$ be as above and $\mathbf{S}$ be a subtorus. Then $N_{\mathbf{B}}(\mathbf{S}) = C_{\mathbf{B}}(\mathbf{S})$.*

Proof. If $n \in N_{\mathbf{B}}(\mathbf{S}), s \in \mathbf{S}$ then $[n, s] \in [\mathbf{B}, \mathbf{B}] \cap \mathbf{S} \subset \mathrm{R_u}(\mathbf{B}) \cap \mathbf{S} = 1$. □

Proposition 1.8 (Assumed)**.** *In a connected algebraic group the Borel subgroups are conjugate and self-normalizing; every element lies in some Borel subgroup.*

The last two points come from the fact that in GL_n, the normalizer of the upper unitriangular matrices are the upper triangular matrices, and from the fact that any matrix is triangularizable.

Corollary 1.9. $\mathrm{Rad}\,\mathbf{G}$ *is the connected component of the intersection of all Borel subgroups.*

Proof. Indeed $\mathrm{Rad}\,\mathbf{G}$ is in at least one Borel subgroup. Since it is normal and all Borel subgroups are conjugate, it is in their intersection. Since it is connected, it is in the connected component. Conversely this component is solvable and normal. □

Examples of reductive groups

Example 1.10. $\mathrm{GL}_n = \mathrm{Spec}\, k[t_{i,j}, \det(t_{i,j})^{-1}], i,j \in [1 \dots n]$. We have seen that the upper triangular matrices form a Borel subgroup. The lower triangular, conjugate to the upper triangular by the matrix of the permutation $(1,n)(2,n-1)\dots$, form another, whose intersection with the first is the group of diagonal matrices, a maximal torus. Thus GL_n is reductive by 1.9.

Example 1.11. $\mathrm{SL}_n = \mathrm{Spec}\, k[t_{i,j}]/(\det(t_{i,j}) - 1)$. The diagonal (resp. upper triangular) matrices are still a maximal torus (resp. Borel subgroup).

Example 1.12. PGL_n is the quotient of GL_n by $\mathbb{G}_m$ imbedded diagonally. To see it is an affine variety, we identify it to the subgroup of $g \in \mathrm{GL}(M_n(k))$ which are algebra automorphisms, that is such that $g(E_{i,j})g(E_{k,l}) = \delta_{j,k} g(E_{i,l})$ where $E_{i,j}$ is the *elementary matrix* defined by $\{E_{i,j}\}_{k,l} = \delta_{i,j}\delta_{k,l}$. The image of a maximal torus (resp. a Borel subgroup) of GL_n is a maximal torus (resp. a Borel subgroup).

If $\mathrm{char}\, k = p$ the center $Z\,\mathrm{SL}_p$ is $\mathrm{Spec}\, k[t]/(t^p - 1) = \mathrm{Spec}\, k[t]/(t-1)^p$ which as a variety has a single point thus is the trivial group, but is not trivial as a scheme! SL_p and PGL_p have the same points over k but are not isomorphic as group schemes.

Example 1.13. $\mathrm{Sp}_{2n}(k)$. On $V = k^{2n}$ with basis $(e_1, \dots, e_n, e'_n, \dots, e'_1)$, we define the *symplectic* bilinear form $\langle e_i, e_j \rangle = \langle e'_i, e'_j \rangle = 0$, $\langle e_i, e'_j \rangle = -\langle e'_j, e_i \rangle = \delta_{i,j}$. The group Sp_{2n} is the subgroup of $g \in \mathrm{GL}_{2n}$ which preserve this form. If $J' = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$ and $J = \begin{pmatrix} & J' \\ -J' & \end{pmatrix}$, we have $\langle v, v' \rangle = {}^t v J v'$ and g is symplectic if ${}^t g J g = J$. The matrices $\mathrm{diag}(t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1})$ form a maximal torus. The symplectic upper triangular matrices are a Borel subgroup; they consist of the matrices $\begin{pmatrix} B & BjS \\ 0 & J'\,{}^t B^{-1} J' \end{pmatrix}$ where B is upper triangular and S is symmetric. This group is indeed connected since it is the product of the connected varieties of the upper triangular and of the symmetric matrices; solvable as a subgroup of a solvable group; and maximal since it is in a single Borel subgroup of GL_{2n} (it stabilizes a single complete flag).

Proposition 1.14. *A* parabolic subgroup *of a connected algebraic group is a subgroup which contains a Borel subgroup.*

- *A parabolic subgroup* $\mathbf{P}$ *is connected and* $N_{\mathbf{G}}(\mathbf{P}) = \mathbf{P}$.
- *To distinct parabolic subgroups containing the same Borel subgroup are not conjugate.*

Proof. As the Borel subgroups are connected, $\mathbf{P}^\circ$ contains a Borel subgroup $\mathbf{B}$. As another Borel subgroup of $\mathbf{G}$ in $\mathbf{P}^\circ$ is $\mathbf{P}^\circ$-conjugate to $\mathbf{B}$, we have $N_{\mathbf{G}}(\mathbf{P}^\circ) = \mathbf{P}^\circ N_{\mathbf{G}}(\mathbf{B}) = \mathbf{P}^\circ\mathbf{B} = \mathbf{P}^\circ$. As $\mathbf{P} \subset N_{\mathbf{G}}(\mathbf{P}^\circ)$ we have $\mathbf{P} = \mathbf{P}^\circ$. Finally, using again that Borel subgroups of $\mathbf{P}$ are $\mathbf{P}$-conjugate, we get that two conjugate parabolic subgroups containing the same Borel subgroup are $N_{\mathbf{G}}(\mathbf{B})$-conjugate, thus are equal since $N_{\mathbf{G}}(\mathbf{B}) = \mathbf{B}$. $\square$

Proposition 1.15 (Assumed)**.** *Let* $\mathbf{T}$ *be a torus of a connected algebraic group* $\mathbf{G}$. *Then* $C_{\mathbf{G}}(\mathbf{T}) = N_{\mathbf{G}}(\mathbf{T})^\circ$*; the Borel subgroups of* $C_{\mathbf{G}}(\mathbf{T})$ *are the* $\mathbf{B} \cap C_{\mathbf{G}}(\mathbf{T})$ *where* $\mathbf{B}$ *is a Borel subgroup of* $\mathbf{G}$ *containing* $\mathbf{T}$.

The *Weyl group* $W_{\mathbf{G}}(\mathbf{T}) = N_{\mathbf{G}}(\mathbf{T})/C_{\mathbf{G}}(\mathbf{T})$ is thus finite; it can be identified to a finite subgroup of $\mathrm{GL}(X(\mathbf{T})) = \mathrm{GL}_{\mathrm{rank}\,\mathbf{G}}(\mathbb{Z})$.

Proposition 1.16 (Assumed)**.** *In a connected algebraic group, maximal tori are conjugate. If* $\mathbf{T}$ *is a maximal torus,* $C_{\mathbf{G}}(\mathbf{T})$ *is nilpotent.*

If $\mathbf{T}$ is a maximal torus, by 1.15 and 1.16, $C_{\mathbf{G}}(\mathbf{T})$ is in the intersection of Borel subgroups containing $\mathbf{T}$. Since by 1.7 for such a Borel subgroup $\mathbf{B}$ we have $N_{\mathbf{B}}(\mathbf{T}) = C_{\mathbf{B}}(\mathbf{T})$, we get that $w \longmapsto {}^w\mathbf{B}$ induces a bijection between $W_{\mathbf{G}}(\mathbf{T})$ and the set of Borel subgroups containing $\mathbf{T}$.

2 Prerequisites: Coxeter groups, root systems

2.1 Coxeter groups

Let W be a group generated by a set S of elements stable by taking inverses. Let $\{w_i, w'_i\}_{i\in I}$ be words in the elements of S (finite sequences of elements of S; the set of all words on S is denoted S^* and called the free monoid on S). We say that $\langle S \mid w_i = w'_i \text{ for } i \in I\rangle$ is a *presentation* of W is W is the "most general group" where the relation $w_i = w'_i$ holds. Formally, we take for W the quotient of S^* by the congruence relation on words generated by the relations $w_i = w'_i$.

Let $w \in W$ be the image of $s_1 \cdots s_k \in S^*$. Then this word is called a *reduced expression* for $w \in W$ if it is a word of minimal length representing w; we then write $l(w) = k$.

We assume now the set S which generates W consists of involutions, that is each element of S is its own inverse. Notice that reversing words is then equivalent to taking inverses in W. For $s, s' \in S$ we will denote $\Delta^{(m)}_{s,s'}$ the word $\underbrace{ss'ss'\cdots}_{m\text{ terms}}$. If the product ss' has finite order m, we will just denote $\Delta_{s,s'}$

for $\Delta^{(m)}_{s,s'}$; then the relation $\Delta_{s,s'} = \Delta_{s',s}$ holds in W. Writing the relation $(ss')^m = 1$ this way has the advantage that transforming a word by the use of this relation does not change the length — this will be useful. This kind of relation is called a *braid relation* because it is the kind of relations which defines the braid groups, groups related to the Coxeter groups which have a topological definition.

Definition 2.1. *A pair (W, S) where S is a set of involutions generating the group W is a* Coxeter system *if*

$$\langle s \in S \mid s^2 = 1, \Delta_{s,s'} = \Delta_{s',s} \text{ for pairs } s, s' \in S \text{ such that } ss' \text{ has finite order}\rangle$$

is a presentation of W.

We may ask if a presentation of the above kind defines always a Coxeter system. That is, given a presentation with relations $\Delta^{(m)}_{s,s'} = \Delta^{(m)}_{s',s}$, is m the order of ss' in the defined group? This is always the case, but it is not obvious.

If W contains a set S such that (W, S) is a Coxeter system we say that W is a *Coxeter group* and that S is a *Coxeter generating set*. Considering that W has a faithful reflection representation we will also sometimes call S the *generating reflections* of W, and the set R of W-conjugates of elements of S the *reflections* of W.

Characterizations of Coxeter groups

Theorem 2.2. *Let W be a group generated by the set S of involutions. Then the following are equivalent:*

(i) (W, S) is a Coxeter system.

(ii) There exists a map N from W to the set of subsets of R, the set of W-conjugates of S, such that $N(s) = \{s\}$ for $s \in S$ and for $x, y \in W$ we have $N(xy) = N(y)\dot{+}y^{-1}N(x)y$, where $\dot{+}$ denotes the symmetric difference of two sets (the sum $\pmod 2$ of the characteristic functions).

(iii) (Exchange condition) If $s_1 \cdots s_k$ is a reduced expression for $w \in W$ and $s \in S$ is such that $l(sw) \leq l(w)$, then there exists i such that $sw = s_1 \cdots \hat{s}_i \cdots s_k$.

(iv) W satisfies $l(sw) \neq l(w)$ for $s \in S$, $w \in W$, and (Matsumoto's lemma) two reduced expressions of the same word can be transformed one into the other by using just the braid relations. Formally, given any monoid M and any morphism $f : S^* \to M$ such that $f(\Delta_{s,s'}) = f(\Delta_{s',s})$ when ss' has finite order, then f is constant on the reduced expressions of a given $w \in W$.

Note that (iii) could be called the "left exchange condition". By symmetry there is a right exchange condition where sw is replaced by ws.

Proof. We first show that (i)⇒(ii). The definition of N may look technical and mysterious, but the intuition is that W has a reflection representation where it acts on a set of roots stable under the action of W (there are two opposite roots attached to each reflection), that these roots are divided into positive and negative by a linear form which does not vanish on any root, and that $N(w)$ records the reflections whose roots change sign by the action of w.

Computing step by step $N(s_1\cdots s_k)$ by the two formulas of (ii), we find

$$N(s_1\cdots s_k) = \{s_k\}\dot{+}\{^{s_k}s_{k-1}\}\dot{+}\cdots\dot{+}\{^{s_k s_{k-1}\cdots s_2}s_1\}. \tag{1}$$

Let us show that the function thus defined on S^* factors through W which will show (ii). To do that we need N to be compatible with the relations defining W, that is $N(ss)=\emptyset$ and $N(\Delta_{s,s'})=N(\Delta_{s',s})$. This is straightforward.

We now show (ii)⇒(iii). We will actually check the right exchange condition; by symmetry if (i) implies this condition it also implies the left condition. We first show that if $s_1\cdots s_k$ is a reduced expression for w, then $|N(w)|=k$, that is all the elements of R which appear on the right-hand side of (1) are distinct. Otherwise, there would exist $i<j$ such that $s_k\cdots s_i\cdots s_k = s_k\cdots s_j\cdots s_k$; then $s_i s_{i+1}\cdots s_j = s_{i+1}s_{i+2}\cdots s_{j-1}$ which contradicts that the expression is reduced.

We next observe that $l(ws)\leq l(w)$ implies $l(ws)<l(w)$. Indeed $N(ws)=\{s\}\dot{+}s^{-1}N(w)s$ thus by the properties of $\dot{+}$ we have $l(ws)=l(w)\pm 1$. Also, if $l(ws)<l(w)$, we must have $s\in s^{-1}N(w)s$ or equivalently $s\in N(w)$. It follows that there exists i such that $s=s_k\cdots s_i\cdots s_k$, which multiplying on left by w gives $ws=s_1\cdots\hat{s}_i\cdots s_k$ q.e.d.

We now show (iii)⇒(iv). The exchange condition implies $l(sw)\neq l(w)$ because if $l(sw)\leq l(w)$ it gives $l(sw)<l(w)$. Given $f:S^*\to M$ as in (iv) we use induction on $l(w)$ to show that f is constant on reduced expressions. Otherwise, let $s_1\cdots s_k$ and $s'_1\cdots s'_k$ be two reduced expressions for the same element w whose image by f differ. By the exchange condition there exists i such that $s'_1 s_1\cdots s_k = s_1\cdots\hat{s}_i\cdots s_k$ in W, thus $s'_1 s_1\cdots\hat{s}_i\cdots s_k$ is another reduced expression for w. If $i\neq k$ we may apply induction to deduce that $f(s_1\cdots s_k)=f(s'_1 s_1\cdots\hat{s}_i\cdots s_k)$ and similarly apply induction to deduce that $f(s'_1\cdots s'_k)=f(s'_1 s_1\cdots\hat{s}_i\cdots s_k)$, a contradiction. Thus $i=k$ and $s'_1 s_1\cdots s_{k-1}$ is a reduced expression for w such that $f(s'_1 s_1\cdots s_{k-1})\neq f(s_1\cdots s_k)$.

Arguing the same way, starting this time from the pair of expressions $s_1\cdots s_k$ and $s'_1 s_1\cdots s_{k-1}$, we get that $s_1 s'_1 s_1\cdots s_{k-2}$ is a reduced expression for w such that

$$f(s_1 s'_1 s_1\cdots s_{k-2})\neq f(s'_1 s_1\cdots s_{k-1});$$

Going on this process will stop when we get two reduced expressions of the form $\Delta^{(m)}_{s_1,s'_1}$, $\Delta^{(m)}_{s'_1,s_1}$, such that $f(\Delta^{(m)}_{s_1,s'_1})\neq f(\Delta^{(m)}_{s'_1,s_1})$. We cannot have m greater that the order of $s_1 s'_1$ since the expressions are reduced, nor less than that order, because the order would be smaller. And we cannot have m equal to the order of $s_1 s'_1$ because this contradicts the assumption.

We finally show (iv)⇒(i). (i) can be stated as: given any group G and a morphism of monoids $f:S^*\to G$ such that $f(s)^2=1$ and $f(\Delta_{s,s'})=f(\Delta_{s',s})$ then f factors through a morphism $g:W\to G$. Let us define g by $g(w)=f(s_1\cdots s_k)$ when $s_1\cdots s_k$ is a reduced expression for w. By (iv) the map g is well-defined. To see that g factors f we need to show that for any expression $w = s_1 \cdots s_k$ we have $g(w) = f(s_1 \cdots s_k)$. This will follow by induction on the length of the expression if we show that $f(s)g(w) = g(sw)$ for $s \in S, w \in W$. If $l(sw) > l(w)$ this equality is immediate from the definition of g. If $l(sw) < l(w)$ we use $f(s)^2 = 1$ to rewrite the equality $g(w) = f(s)g(sw)$ and we apply the reasoning of the first case. Finally $l(sw) = l(w)$ is excluded by assumption. $\square$

Finite Coxeter groups: the longest element

Proposition 2.3. *Let (W, S) be a Coxeter system. Then the following properties are equivalent for an element $w_0 \in W$:*

(i) $l(w_0 s) < l(w_0)$ *for all* $s \in S$.

(ii) $l(w_0 w) = l(w_0) - l(w)$ *for all* $w \in W$.

(iii) w_0 *has maximal length amongst elements of* W.

If such an element exists, it is unique and it is an involution, and W is finite.

Proof. It is clear that (ii) implies (iii) and that (iii) implies (i).

To see that (i) implies (ii), we will show by induction on $l(w)$ that w_0 as in (i) has a reduced a expression ending by a reduced expression for w^{-1}. Write $w^{-1} = vs$ where $l(v) + l(s) = l(w)$. By induction we may write $w_0 = yv$ where $l(w_0) = l(y) + l(v)$. The (right) exchange condition, using that $l(w_0 s) < l(w_0)$ but vs is reduced, shows that $w_0 s = \hat{y} v$ where $\hat{y}$ represents y with a letter omitted. It follows that $\hat{y} vs$ is a reduced expression for w_0.

An element satisfying (ii) is an involution since $l(w_0^2) = l(w_0) - l(w_0) = 0$ and is unique since another w_1 has same length by (iii) and $l(w_0 w_1) = l(w_0) - l(w_1) = 0$ thus $w_1 = w_0^{-1} = w_0$.

If w_0 as in (i) exists then S is finite since $S \subset N(w_0)$ and W is then finite by (iii). $\square$

Yet another characterization of Coxeter groups

Lemma 2.4. *Let W be group generated by the set S of involutions and let $\{D_s\}_{s \in S}$ be a set of subsets of W such that:*

- $D_s \ni 1$.
- $D_s \cap s D_s = \emptyset$.
- *If for* $s, s' \in S$ *we have* $w \in D_s, ws' \notin D_s$ *then* $ws' = sw$.

Then (W, S) is a Coxeter system, and $D_s = \{w \in W \mid l(sw) > l(w)\}$.

Proof. We will show the exchange condition. Let $s_1 \cdots s_k$ be a reduced expression for $w \notin D_s$ and let i be minimal such that $s_1 \cdots s_i \notin D_s$; we have $i > 0$ since $1 \in D_s$. From $s_1 \cdots s_{i-1} \in D_s$ and $s_1 \cdots s_i \notin D_s$ we get $s s_1 \cdots s_{i-1} = s_1 \ldots s_i$, whence $sw = s_1 \cdots \hat{s}_i \cdots s_k$ thus $l(sw) < l(w)$ and we have checked the exchange condition in this case. If $w \in D_s$ then $sw \notin D_s$ by the first part $l(w) < l(sw)$ so we have nothing to check. $\square$

Parabolic subgroups

Lemma-Definition 2.5. *Let (W,S) be a Coxeter system, let I be a subset of S, and let W_I be the subgroup of W generated by I. Then (W_I,I) is a Coxeter system. An element $w\in W$ is said* I-reduced *if it satisfies one of the equivalent conditions:*

(i) For any $v\in W_I$, we have $l(vw)=l(v)+l(w)$.

(ii) For any $s\in I$, we have $l(sw)>l(w)$.

(iii) w is of minimal length in the coset $W_I w$.
There is a unique I-reduced element in $W_I w$.

Proof. It is clear that (W_I,I) satisfies the exchange condition (a reduced expression in W_I is reduced in W by the exchange condition, and then satisfies the exchange condition in W_I) thus is a Coxeter system.

It is clear that (iii)⇒(ii) since (iii) implies $l(sw)\geq l(w)$ when $s\in I$. Let us show that not (iii)⇒ not (ii). If w' does not have minimal length in $W_I w'$, then $w'=vw$ with $v\in W_I$ and $l(w)<l(w')$; adding one by one the terms of a reduced expression for v to w, applying at each stage the exchange condition, we find that w' has a reduced expression of the shape $\hat v\hat w$ where $\hat v$ (resp. $\hat w$) denotes a subsequence of the chosen reduced expression. As $l(\hat w)\leq l(w)<l(w')$, we have $l(\hat v)>0$, thus w' has a reduced expression starting by an element of I, thus w' does not satisfy (ii).

(i)⇒(iii) is clear. Let us show not (i)⇒ not (iii). If $l(vw)<l(v)+l(w)$ then a reduced expression for vw has the shape $\hat v\hat w$ where $l(\hat w)<l(w)$. Then $\hat w\in W_I w$ and has a length smaller than that of w.

Finally, an element satisfying (i) is clearly unique in $W_I w$. □

Let us note that by exchanging left and right we have the notion of reduced-I element which satisfies the mirror lemma.

Fixed points under automorphisms

Proposition 2.6. *Let Γ be a group of automorphisms of the Coxeter system (W,S), that is of automorphisms of W preserving S. Let $(S/\Gamma)_{<\infty}$ the set of orbits I of Γ on S such that the subgroup W_I is finite. Then $(W^\Gamma,\{w_I\}_{I\in(S/\Gamma)_{<\infty}})$ is a Coxeter system, where W^Γ is the subgroup of Γ-fixed elements of W, and where w_I denotes the longest element of W_I, see 2.3. Further, if $w_{I_1}\cdots w_{I_k}$ is a reduced expression of some $w\in W^\Gamma$ in the above Coxeter system, we have $l(w)=\sum_{i=1}^{i=k} l(w_{I_i})$, where l is the length function of (W,S).*

Proof. We first show

Lemma 2.7. *If for $w\in W^\Gamma$ and $s\in S$ we have $l(ws)<l(w)$ then the Γ-orbit I of s is finite, and there exists w' such that $w=w'w_I$ with $l(w)=l(w')+l(w_I)$.*

Proof. Since Γ is an automorphism of (W,S) for any element $t\in I$ we will have $l(wt)<l(w)$. Write $w=w'v$ where w' is reduced-I; then $l(vt)<l(v)$ for any $t\in I$ which is possible only if W_I is finite and $v=w_I$, see 2.3. □

Let S_Γ be the set $\{w_I \mid I \in (S/\Gamma)_{<\infty}\}$; applying the lemma at each step starting from the right, we find that any $w \in W^\Gamma$ can be written $w = w_{I_1} \cdots w_{I_k}$ where $I_j \in S_\Gamma$ and $l(w) = \sum_{i=1}^{i=k} l(w_{I_i})$, in particular S_Γ generates W^Γ.

We will use the characterization 2.4 to see that (W^Γ, S_Γ) is a Coxeter system, but inverting right and left. For $w_I \in S_\Gamma$, let $D_{w_I} = \{w \in W^\Gamma \mid w \text{ is reduced-}I\}$. We clearly have $D_{w_I} \ni 1$ and $D_{w_I} \cap D_{w_I} w_I = \emptyset$. It remains to show that if $w \in D_{w_I}$ and $w_J w \notin D_{w_I}$, then $w_J w = w w_I$. We will use the function N of 2.2(ii). We have

Lemma 2.8. *For any $r \in N(w)$ we have $l(wr) < l(w)$.*

Proof. Indeed if $w = s_1 \cdots s_n$ is a reduced expression there exists i such that $r = s_n \cdots s_i \cdots s_n$ whence $wr = s_1 \cdots \hat{s}_i \cdots s_n$. □

It follows from the lemma that if, for $r \in W_I, w \in W^\Gamma$ we have $r \in N(w)$, then $N(w_I) \subset N(w)$. Indeed $l(wr) < l(w)$ thus w cannot be I-reduced thus by 2.7 it can be written $w_{I_1} \cdots w_{I_k}$ with the lengths adding and $I_k = I$.

In our situation $w \in D_{w_I}$ implies $N(w) \cap N(w_I) = \emptyset$ and $w_J w \notin D_{w_I}$ implies $N(w_J w) \supset N(w_I)$; since $N(w_J w) = N(w_J)^w \dot{+} N(w)$ it follows that $N(w_J)^w \supset N(w_I)$. If we show that this implies $w_J^w = w_I$ we are done. We remark first that we may assume that w is J-reduced; otherwise we may replace it by $w_J w$ since $w_J^{w_J w} = w_J^w$. We write then $N(w_I w^{-1}) = {}^w N(w_I) + N(w^{-1})$. Since w is J-reduced, $N(w^{-1}) \cap N(w_J) = \emptyset$, and since ${}^w N(w_I)$ meets $N(w_J)$, all of $N(w_J)$ must be contained in $N(w_I w^{-1})$ thus in ${}^w N(w_I)$. Thus we have $N(w_I) = w^{-1} N(w_J) w$ whence $W_I = w^{-1} W_J w$ whence $w_I = w^{-1} w_J w$ since $w^{-1} w_J w$ is the only non-trivial Γ-stable element of w_I.

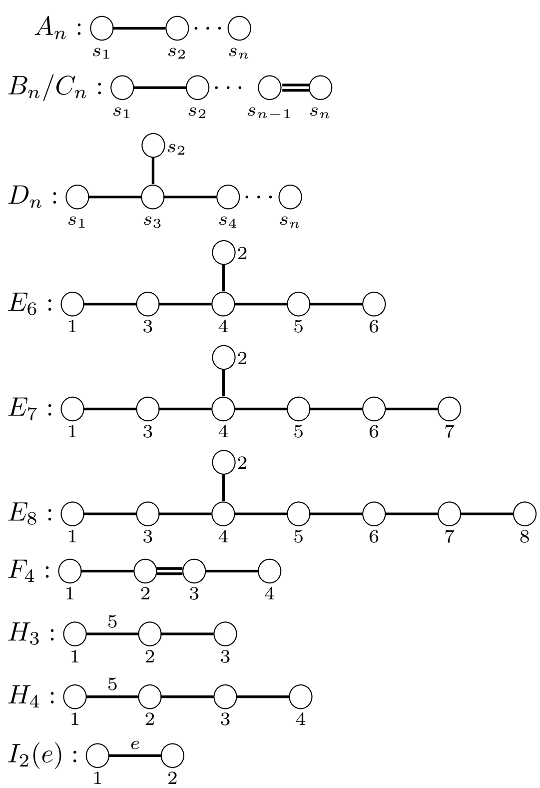
We now prove the final remark in the statement. If $w_{I_1} \cdots w_{I_k}$ is a reduced expression in (W^Γ, S_Γ) then by the property of D_{w_k} (see 2.4) we have $w \notin D_{w_k}$; thus by lemma 2.7 we have $w = w' w_{I_k}$ where the lengths add. This proves the result by induction on k. □

We have seen in the proof that 1 and w_I are the only Γ-stable elements of W_I.

Coxeter Diagrams

Coxeter systems are encoded by a graph with vertices S and edges encoding the order of ss' when it is greater than 2. This order is encoded by a single edge when 3, a double edge when 4, a triple edge when 6, and an edge decorated by the order when 5 or greater than 6.

The diagrams for finite irreducible (here it means connected diagrams) Coxeter groups are



A finite Coxeter group is called a *Weyl group* if it is a reflection group over $\mathbb{Q}$. This selects in the above list exactly the diagrams where the order of ss' is always in $\{2, 3, 4, 6\}$. The group $I_2(6)$ is also denoted G_2.

2.2 Root systems

In this section V is a finite dimensional real vector space and V^* is its dual.

Notation 2.9. A *reflection* $s \in \mathrm{GL}(V)$ is an element of order 2 such that $\mathrm{Ker}(s - \mathrm{Id})$ is an hyperplane. It follows that s has an eigenvalue -1 with multiplicity 1, and that if $\alpha \in V$ is an eigenvector for -1 and $\alpha^\vee \in V^*$ is a linear form for $\mathrm{Ker}(s - \mathrm{Id})$, chosen such that $\alpha^\vee(\alpha) = 2$, then $s(x) = x - \alpha^\vee(x)\alpha$.

We call α a *root* attached to the reflection s and $\alpha^\vee$ the corresponding *coroot*. They are unique up to inverse to each other scalings. Conversely any pair of non-zero vectors $\alpha \in V, \alpha^\vee \in V^*$ such that $\alpha^\vee(\alpha) = 2$ define a reflection.

Definition 2.10. • *A* root system *is a finite set $\Phi \subset V$ with a bijection $\alpha \mapsto \alpha^\vee : \Phi \to \Phi^\vee \subset V^*$ such that Φ generates V, and for any $\alpha \in \Phi$ we have $\alpha^\vee(\alpha) = 2$ and Φ is stabilized by the reflection s_α of root α and coroot $\alpha^\vee$.*

- *The system is* crystallographic *if* $\alpha^\vee(\beta) \in \mathbb{Z}$ *for all* $\alpha, \beta \in \Phi$.
- *The system is* reduced *if for any* α *we have* $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$.

If the system is crystallographic, Φ and $\Phi^\vee$ generate dual lattices.

Some other authors reserve the name "root system" for the systems we call crystallographic; any finite Coxeter groups has a root system in our sense, but only the Weyl groups have crystallographic ones.

In the following we fix a root system Φ and denote W the group generated by $\{s_\alpha\}_{\alpha\in\Phi}$. It is finite since its elements are determined by the permutation of Φ they induce. Thus there exists a W-invariant scalar product $(,)$.

Lemma 2.11. *Identifying* V *to* V^* *by* $(,)$ *we have* $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$.

Proof. Using the invariance of $(,)$ we get for all $v \in V$ that $(\alpha, v) = (s_\alpha\alpha, s_\alpha v) = (-\alpha, v - \alpha^\vee(v)\alpha)$ which gives $\alpha^\vee(v) = \frac{2(\alpha,v)}{(\alpha,\alpha)}$. $\square$

Using the identification of lemma 2.11 allows to work in an Euclidean space and forget $\Phi^\vee$; but keeping $V^\vee$ allows to extend the theory to infinite root systems.

In the following we assume Φ reduced, in order to simplify somewhat the statements and proofs — a non-reduced system BC_n occurs in certain parts of reductive group theory that we will not cover.

Theorem 2.12. *Given an order on* V *such that every root is positive or negative (equivalently, given a linear form which does not vanish on* Φ*), denote* Φ^+ *the set of positive roots. Then there exists a unique basis* $\Pi \subset \Phi^+$ *of* V *such that* $\Phi^+ = \Phi \cap \mathbb{R}_{\geq 0}\Pi$.

Proof. Note first that there exists a minimal subset $\Pi \subset \Phi^+$ such that $\Phi^+ = \Phi \cap \mathbb{R}_{\geq 0}\Pi$: to obtain such a subset, starting from Φ^+, just iteratively remove elements which are a positive linear combination of others in the considered subset.

Lemma 2.13. *For a minimal* Π *as above* $(\alpha, \beta) \leq 0$ *for* $\alpha, \beta \in \Pi, \alpha \neq \beta$.

Proof. Assume by contradiction that $(\alpha, \beta) > 0$. Then $s_\alpha(\beta) = \beta - c\alpha$ where $c = \frac{2(\alpha,\beta)}{\alpha,\alpha} > 0$. Either $s_\alpha(\beta) \in \Phi^+$ or $-s_\alpha(\beta) \in \Phi^+$.

In the first case by assumption $s_\alpha(\beta) = \sum_{\gamma\in\Pi} c_\gamma\gamma$ with $c_\gamma \geq 0$; we rewrite this $\sum_{\gamma\in\Pi-\{\beta\}} c_\gamma\gamma + c\alpha + (c_\beta - 1)\beta = 0$. We cannot have $c_\beta - 1 \geq 0$ since a non-zero sum of positive vectors cannot be zero. Thus we expressed β as an element of $\mathbb{R}_{\geq 0}(\Pi - \{\beta\})$ which contradicts the minimality of Π.

In the second case we similarly rewrite $-s_\alpha(\beta) = \sum_{\gamma\in\Pi} c_\gamma\gamma$ with $c_\gamma \geq 0$ to $\sum_{\gamma\in\Pi-\{\alpha\}} c_\gamma\gamma + \beta + (c_\alpha - c)\alpha = 0$, and similarly we must have $c_\alpha - c < 0$ giving an expression of α as an element of $\mathbb{R}_{\geq 0}(\Pi - \{\alpha\})$ which again contradicts the minimality of Π. $\square$

Let us see now that Π is a basis. We know it generates V since Φ does. We have to exclude a linear dependence amongst its elements. Such a relation can we written $v = \sum_{\alpha\in\Pi_1} c_\alpha \alpha = \sum_{\beta\in\Pi_2} c_\beta \beta$ where v is a nonzero vector, where $c_\alpha, c_\beta \geq 0$ and where $\Pi = \Pi_1 \sqcup \Pi_2$. But then we have $0 < (v, v) = (\sum_{\alpha\in\Pi_1} c_\alpha \alpha, \sum_{\beta\in\Pi_2} c_\beta \beta)$ which contradicts lemma 2.13.

We finally show that Π is unique: if there are two such bases $\Pi \neq \Pi'$ let us consider $\alpha \in \Pi - \Pi'$; express it on Π' as $\alpha = \sum_{\beta\in\Pi'} c_\beta \beta$ then express each involved β on Π: since $\beta \neq \alpha$ these expressions will involve a root in $\Pi - \alpha$ (we use here that the system is reduced) and this root will remain when doing the sum, since the coefficients are positive; this is a contradiction. $\square$

A Φ^+ as above is called a *positive subsystem* and a Π as above a *simple subsystem*.

Note that in the basis Π the coefficients of the matrix s_α are 1 or $-\alpha^\vee(\beta)$, thus in this basis we have $W \subset \mathrm{GL}_n(\mathbb{Z})$ if the root system is crystallographic, where $n = |\Pi|$.

Proposition 2.14. *Two positive (resp. simple) subsystems are W-conjugate.*

Proof. It is enough to consider positive subsystems since they determine simple subsystems.

Lemma 2.15. *For $\alpha \in \Pi$ and any $\beta \in \Phi^+ - \{\alpha\}$ we have $s_\alpha(\beta) \in \Phi^+$.*

Proof. If $\beta \in \Phi^+ - \{\alpha\}$ then $\beta = \sum_{\gamma\in\Pi} c_\gamma \gamma$ where at least one $c_\gamma > 0$ with $\gamma \neq \alpha$, otherwise $\beta \in \Phi^+ \cap \mathbb{R}_{\geq 0}\alpha = \{\alpha\}$. But then $s_\alpha(\beta) = \beta - \alpha^\vee(\beta)\alpha$ has the same coefficient on γ, and as any root has all nonzero coefficients on Π of the same sign, the root $s_\alpha(\beta)$ is positive. $\square$

We use the lemma to conjugate another positive subsystem Φ' on Φ^+, using induction on $|\Phi^+ \cap -\Phi'|$. If this number is positive then $\Pi \cap -\Phi' \neq \emptyset$, otherwise $\Pi \subset \Phi'$ which implies $\Phi^+ \subset \Phi'$ which implies $\Phi^+ = \Phi'$ since all positive subsystems have same cardinality $|\Phi|/2$. Choose thus $\alpha \in \Pi \cap -\Phi'$; since $s_\alpha(\Phi^+) = (\Phi^+ - \{\alpha\}) \coprod \{-\alpha\}$, the set $s_\alpha(\Phi^+)$ is a positive subsystem such that $|s_\alpha(\Phi^+) \cap -\Phi'| = |\Phi^+ \cap -\Phi'| - 1$. $\square$

Corollary 2.16. *Every root is in the W-orbit of* Π.

Proof. It is enough to show it for every positive root since $s_\alpha(\alpha) = -\alpha$. Take $\alpha = \sum_{\gamma\in\Pi} c_\gamma \gamma \in \Phi^+ - \Pi$; as $0 < (\alpha, \alpha) = \sum_{\gamma\in\Pi} c_\gamma (\alpha, \gamma)$ there exists $\gamma \in \Pi$ such that $(\alpha, \gamma) > 0$. Then $\alpha' = s_\gamma(\alpha)$ is still positive by 2.15 and is obtained by removing a positive multiple of γ to α. Thus if we set $h(\alpha) = \sum_\gamma c_\gamma$ we have $h(\alpha') < h(\alpha)$. We can repeat this process as long as $\alpha' \notin \Pi$. As Φ^+ is finite this process must eventually stop, at a root in Π. $\square$

The proof of the corollary shows more, that every root is conjugate to an element of Π by a sequence of $s_\gamma, \gamma \in \Pi$. In particular every s_α is in the group generated by $\{s_\gamma\}_{\gamma\in\Pi}$, thus W itself is generated by $\{s_\gamma\}_{\gamma\in\Pi}$.

We show now that W is a Coxeter group using characterization 2.4.

Proposition 2.17. *(W, S) where $S = \{s_\alpha \mid \alpha \in \Pi\}$ is a Coxeter system.*

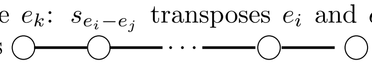
Proof. We apply 2.4 with $D_{s_\alpha} = \{w \in W \mid w^{-1}(\alpha) > 0\}$. That $D_{s_\alpha} \cap s_\alpha D_{s_\alpha} = \emptyset$ is clear. Now take $w \in D_{s_\alpha}$ such that $ws_{\alpha'} \notin D_{s_\alpha}$, that is $w^{-1}(\alpha) > 0$ and $s_{\alpha'} w^{-1}(\alpha) < 0$. As $s_\alpha{}'$ changes the sign of only α', we must have $w^{-1}(\alpha) = \alpha'$. As w preserves the scalar product, it conjugates $s_{\alpha'}$ to s_α, whence the result. □

Lemma 2.18. *(i) The set $N(w)$ of 2.2(ii) is $\{s_\alpha \mid \alpha \in \Phi^+, w(\alpha) < 0\}$.*

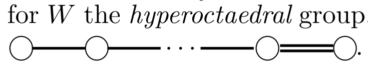
(ii) The element w_0 of 2.3 is such that $w_0(\Phi^+) = \Phi^-$.

Proof. If we set $N'(w) = \{s_\alpha \mid \alpha \in \Phi^+, w(\alpha) < 0\}$ we will show by induction on $l(w)$ that $N(w) = N'(w)$: let $w = vs$ with $s \in S, l(w) > l(v)$; it follows from the definitions of N and N' that $N(w) = s \cup sN(v)s$ and $N'(w) = s \cup sN'(v)s$. This proves (i).

For (ii), if $w \in W$ and $N(w) \neq \Phi^+$ then there exists $\alpha \in \Pi$ such that $w(\alpha) > 0$. Then by 2.17 and 2.4 we have $l(s_\alpha w) > l(w)$. If we iterate this we have to stop at w_0 and we must have $N(w_0) = \Phi^+$. □

Example 2.19. Root system of type A_{n-1}. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$. Then $\Phi = \{e_i - e_j\}_{i,j \in [1,\ldots,n], i \neq j}$ is a root system of cardinality $n(n-1)$ in the subspace V of dimension $n-1$ it generates. The vectors where $i > j$ are a positive subsystem relative to the linear form $x \mapsto (x, ne_1 + (n-1)e_2 + \cdots + e_n)$. We have $\Pi = \{e_i - e_{i+1}\}_{i=1,\ldots,n-1}$. If we set $\alpha_i = e_i - e_{i+1}$, we have $e_i - e_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ for $i < j$. The group W is the symmetric group, permuting the e_k: $s_{e_i - e_j}$ transposes e_i and e_j and fixes the other e_k. The Coxeter graph is  .

Example 2.20. Root system of type C_n.

It is formed of the $2n^2$ roots in $\mathbb{R}^n$ given by $\pm 2e_i$ and $\pm e_i \pm e_j$. For the same linear form as above we have $\Phi^+ = \{2e_i\}_i \cup \{e_i \pm e_j\}_{i<j}$ and $\Pi = \{e_1 - e_2, \ldots, e_{n-1} - e_n, 2e_n\}$. Here $s_{e_i - e_j}$ transposes e_i and e_j, $s_{e_i + e_j}$ transposes e_i and $-e_j$ and s_{2e_i} transposes e_i and $-e_i$; we get for W the *hyperoctaedral* group, which permutes the $\pm e_i$. The Coxeter graph is .

Example 2.21. we get type B_n replacing $2e_i$ by e_i.

3 Structure of reductive groups

Properties 3.1. Let $\mathbf{G}$ be a connected reductive group over k, and let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$. Then

(i) The minimal closed unipotent subgroups of $\mathbf{G}$ normalized by $\mathbf{T}$ are isomorphic to $\mathbb{G}_a$. Choosing such an isomorphism $x \mapsto \mathbf{u}(x) : \mathbb{G}_a \xrightarrow{\sim} \mathbf{U}$, for $t \in \mathbf{T}$ define $\alpha(t) \in k^\times$ by $t\mathbf{u}(x)t^{-1} = \mathbf{u}(\alpha(t)x)$; then $\alpha \in X(\mathbf{T})$.

The collection Φ of α thus obtained has no repetition, thus is a set and $\alpha \in \Phi$ determines a subgroup $\mathbf{U}_\alpha$ isomorphic to $\mathbb{G}_a$.

(ii) $\Phi = -\Phi$, and for any $\alpha \in \Phi$, there exists a homomorphism $\phi_\alpha : \mathrm{SL}_2 \to \mathbf{G}$ whose image is $\langle \mathbf{U}_\alpha, \mathbf{U}_{-\alpha}\rangle$, and which is injective or has kernel $\pm\,\mathrm{Id} = Z(\mathrm{SL}_2)$, and is such that

$$\phi_\alpha \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} = \mathbf{U}_\alpha, \quad \phi_\alpha \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} = \mathbf{U}_{-\alpha}, \quad \check{\alpha}(x) := \phi_\alpha \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in \mathbf{T}.$$

(iii) Φ is a reduced root system in $X(\mathbf{T}) \otimes \mathbb{R}$. We have $C_{\mathbf{G}}(\mathbf{T}) = \mathbf{T}$ and the natural map $W := N_{\mathbf{G}}(\mathbf{T})/\mathbf{T} \to \mathrm{GL}(X(\mathbf{T}) \otimes \mathbb{R})$ identifies W to the reflection group defined by Φ; s_α is the image of $\dot{s_\alpha} := \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(iv) Any closed connected subgroup of $\mathbf{G}$ containing $\mathbf{T}$ is generated by $\mathbf{T}$ and the $\mathbf{U}_\alpha$ it contains.

(v) A unipotent subgroup $\mathbf{H}$ of $\mathbf{G}$ normalized by $\mathbf{T}$ is equal to $\prod_{\mathbf{U}_\alpha \subset \mathbf{H}} \mathbf{U}_\alpha$ in any order.

(vi) Borel subgroups containing $\mathbf{T}$ are in bijection with positive subsystems of Φ: if $\mathbf{B}$ corresponds to Φ^+ then $\mathrm{R_u}(\mathbf{B}) = \prod_{\alpha \in \Phi^+} \mathbf{U}_\alpha$.

(vii) If $\alpha \neq -\beta$ then $[\mathbf{U}_\alpha, \mathbf{U}_\beta] \subset \prod_{\{\lambda,\mu \in \mathbb{N}^\times \mid \lambda\alpha+\mu\beta \in \Phi\}} \mathbf{U}_{\lambda\alpha+\mu\beta}$.

Note that (i) implies that for $w \in W$ and $\alpha \in \Phi$, we have ${}^w\mathbf{U}_\alpha = \mathbf{U}_{w(\alpha)}$.

In a reductive group we have $C_{\mathbf{G}}(\mathbf{T}) = \mathbf{T}$. Indeed $C_{\mathbf{G}}(\mathbf{T})$ is connected by 1.15, thus by (iv) is generated by $\mathbf{T}$ and some $\mathbf{U}_\alpha$. But no non-trivial element of an $\mathbf{U}_\alpha$ is in $C_{\mathbf{G}}(\mathbf{T})$ since by (i) $\mathbf{T}$ acts non-trivially on $\mathbf{U}_\alpha$.

(iv) can be applied to $\mathbf{G}$ itself. This can be used to describe $Z\mathbf{G}$: by the previous paragraph, $Z\mathbf{G} \subset \mathbf{T}$, and is thus the intersection of the kernels in $\mathbf{T}$ of all the roots.

Example 3.2. Let $\mathbf{G} = \mathrm{GL}_n$ and choose for $\mathbf{T}$ the diagonal matrices; then $N_{\mathbf{G}}(\mathbf{T})$ is the set of monomial matrices. The permutation matrices are a section (representing W) of the quotient $N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$. We have $X(\mathbf{T}) \simeq \mathbb{Z}^n$. The set $\Phi = e_i - e_j$ is a root system (for the natural scalar product) in the subspace it generates (of vectors in $X(\mathbf{T})$ with 0 sum). An isomorphism $k^+ \xrightarrow{\sim} \mathbf{U}_{e_i - e_j}$ is given by $x \mapsto \mathrm{Id} + xE_{i,j}$. The positive subsystem of 2.19 defines the Borel subgroup of upper triangular matrices. The image of $\phi_{e_i - e_j}$ is an SL_2 in position the intersections of the lines and columns i, j.

Example 3.3. In SL_n the elements of $\mathbf{T}$ satisfy $t_1 \cdots t_n = 1$. The coroots generate $Y(\mathbf{T})$; the kernel of the roots is $\mathrm{diag}(\zeta, \ldots, \zeta)$ where $\zeta^n = 1$. The Weyl group has no section in $N_{\mathbb{G}}(\mathbf{T})$ since $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = -\,\mathrm{Id}$.

Example 3.4. In PGL_n the roots generate $X(\mathbf{T})$; the images of the ϕ_α are isomorphic to PGL_2.

Example 3.5. In Sp_{2n}, with our basis $e_1, \ldots, e_n, e_{n'}, \ldots, e_{1'}$, there are 3 kinds of $\mathbf{U}_\alpha$, associated respectively to the morphisms $\mathbb{G}_a \to \mathbf{G}$ given by:

- $\lambda \mapsto \mathrm{Id} + \lambda E_{i,j} - \lambda E_{j',i'}$ for $\alpha = e_i - e_j$
- $\lambda \mapsto \mathrm{Id} + \lambda E_{i,j'} + \lambda E_{j,i'}$ for $\alpha = e_i + e_j$
- $\lambda \mapsto \mathrm{Id} + \lambda E_{i,i'}$ for $\alpha = 2e_i$

4 (B,N)-pairs

Definition 4.1. *We say that two subgroups B and N of a group G form a (B,N)-pair (called also a Tits system) for G if*

(i) B and N generate G and $T := B \cap N$ is normal in N.

(ii) The group $W := N/T$ is generated by a set S of involutions such that:

(iii) For $s \in S$, $w \in W$ we have $BsB.BwB \subset BwB \cup BswB$.

(iv) For $s \in S$, we have $sBs \neq B$.

We will see that under the assumptions 4.1 we have $S = \{w \in W \mid B \cup BwB \text{ is a group}\}$ thus S is determined by (B,N).

Proposition 4.2. *If $\mathbf{G}$ is a connected reductive group and $\mathbf{T} \subset \mathbf{B}$ is a pair of a maximal torus and a Borel subgroup, then $(\mathbf{B}, N_{\mathbf{G}}(\mathbf{T}))$ is a (B,N)-pair for $\mathbf{G}$.*

Proof. We show first that $\mathbf{B} \cap N_{\mathbf{G}}(\mathbf{T}) = \mathbf{T}$. By 1.7 we have $N_{\mathbf{B}}(\mathbf{T}) = C_{\mathbf{B}}(\mathbf{T}) \subset C_{\mathbf{G}}(\mathbf{T}) = \mathbf{T}$ by 3.1(iii). By definition $\mathbf{T}$ is normal in $N_{\mathbf{G}}(\mathbf{T})$. To show (i) it remains to show that $\mathbf{B}$ and $N_{\mathbf{G}}(\mathbf{T})$ generate $\mathbf{G}$. Since s_α conjugates $\mathbf{U}_\alpha$ to $\mathbf{U}_{s_\alpha(\alpha)} = \mathbf{U}_{-\alpha}$, the group generated by $\mathbf{B}$ and $N_{\mathbf{G}}(\mathbf{T})$ contains $\mathbf{T}$ and all the $\mathbf{U}_\alpha (\alpha \in \Phi^+)$ by 3.1 (vi), thus by 3.1 (iv) it is equal to $\mathbf{G}$.

$\mathbf{B}$ defines an ordering Φ^+ and a basis Π and (ii) is obtained by taking for S the $\{s_\alpha \mid \alpha \in \Pi\}$.

(iv) reflects that ${}^s\mathbf{U}_\alpha = \mathbf{U}_{-\alpha}$ is not in $\mathbf{B}$.

It remains to show (iii). Let $s = s_\alpha$, and write $\mathbf{B} = \mathbf{T}\prod_{\beta \in \Phi^+} \mathbf{U}_\beta$. As s normalizes $\mathbf{T}$, as ${}^s\mathbf{U}_\beta = \mathbf{U}_{s_\alpha(\beta)}$ and as $s_\alpha(\beta) \in \Phi^+$ if $\beta \in \Phi^+ - \{\alpha\}$, we get $\mathbf{B}s\mathbf{B}w\mathbf{B} = \mathbf{B}s\mathbf{U}_\alpha w\mathbf{B}$. If $w^{-1}(\alpha) \in \Phi^+$ the rhs is equal to $\mathbf{B}sw\mathbf{B}$. Otherwise we write the rhs as $\mathbf{B}s\mathbf{U}_\alpha ssw\mathbf{B}$ where this time $(sw)^{-1}(\alpha) \in \Phi^+$. Let $\mathbf{B}_\alpha$ be the image by ϕ_α of the Borel of SL_2 of upper triangular matrices. If $c \neq 0$ we have in SL_2:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1/c & -a \\ 0 & -c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix}$$

which taking images shows that $s\mathbf{U}_\alpha s \subset \operatorname{Im} \phi_\alpha = \mathbf{B}_\alpha \cup \mathbf{B}_\alpha s \mathbf{U}_\alpha$, whence $\mathbf{B}s\mathbf{U}_\alpha ssw\mathbf{B} \subset \mathbf{B}s\mathbf{U}_\alpha sw\mathbf{B} \cup \mathbf{B}sw\mathbf{B}$ where the first term is $\mathbf{B}w\mathbf{B}$ since $(sw)^{-1}(\alpha) \in \Phi^+$. □

Theorem 4.3. *If G has a (B,N)-pair, then*

(i) $G = \coprod_{w \in W} BwB$ ("Bruhat decomposition").

(ii) (W,S) is a Coxeter group.

(iii) The condition (iii) of 4.1 can be refined to

$$BsB.BwB = \begin{cases} BswB & \text{if } l(sw) = l(w) + 1 \\ BswB \cup BwB & \text{otherwise} \end{cases}.$$

(iv) For any $t \in N(w)$ (see 2.2(ii)), we have $BtB \subset Bw^{-1}BwB$.

(v) $S = \{w \in W \mid B \cup BwB$ is a group$\}$.

(vi) We have $N_G(B) = B$.

Proof. Let us show (i). As B and N generate B, we have $G = \cup_i (BNB)^i$. Since $BNB = BWB$ we will get $G = BWB$ if we show that $BWBWB = BWB$. For this it is enough to show that $BwBWB \subset BWB$; writing $w = s_1 \cdots s_n$ since $BwB \subset Bs_1B \cdots Bs_nB$ it is enough to show $BsBWB \subset BWB$; but this results from 4.1(iii). It remains to show that $BwB \neq Bw'B$ if $w \neq w'$. We show this by induction on $\inf(l(w), l(w'))$; assume for instance that $l(w) \leq l(w')$. The start of the induction is $l(w) = 0$ and the result comes from $w' \notin B$. Otherwise, taking $s \in S$ such that $l(sw) < l(w)$, by induction $BswB$ is equal neither to $Bw'B$ nor to $Bsw'B$ thus $BswB \cap BsB.Bw'B = \emptyset$; as $BswB \subset BsB.BwB$ it follows that $BwB \neq Bw'B$.

For (ii), we use 2.4 with $D_s = \{w \in W \mid BsBwB = BswB\}$ (we note that if this does not hold then $BsBwB = BswB \coprod BwB$). Clearly $D_s \ni 1$.

If $w, sw \in D_s$, then from $BsBwB = BswB$ and $BsBswB = BwB$ we get $BsBsBwB = BwB$, a contradiction since $BsBsB = BsB \coprod B$ (since $sBs \neq B$ by 4.1(iv)).

It remains to see that $w \in D_s, ws' \notin D_s \Rightarrow ws' = sw$. The assumption $ws' \notin D_s$ implies $BsBws'B = Bsws'B \coprod Bws'B$; in particular $BsBws'$ meets $Bws'B$; multiplying on the right by $s'B$ it follows that $BsBwB$ meets $Bws'Bs'B \subset (BwB \coprod Bws'B)$ (this last inclusion follows from 4.1 (iii) reversed, which is obtained by taking inverses). Thus $BswB = BsBwB$ (since $w \in D_s$) is equal to $Bws'B$, or to BwB. This last cannot happen since $w \neq sw$, thus $sw = ws'$ as was to be shown.

We have also shown (iii) by the property of D_s.

Let us show (iv). If $w = s_1 \cdots s_k$ is a reduced expression, for all i we can write by (iii) $BwB = Bs_1 \cdots s_{i-1}Bs_iBs_{i+1} \cdots s_kB$ and similarly for $Bw^{-1}B$ whence

$$\begin{aligned} Bw^{-1}BwB &= Bs_k \cdots s_{i+1}Bs_iBs_{i-1} \cdots s_1Bs_1 \cdots s_{i-1}Bs_iBs_{i+1} \cdots s_kB \\ &\supset Bs_k \cdots s_{i+1}Bs_iBs_iBs_{i+1} \cdots s_kB \\ &\supset Bs_k \cdots s_{i+1}Bs_iBs_{i+1} \cdots s_kB \\ &\supset Bs_k \cdots s_{i+1}s_is_{i+1} \cdots s_kB \end{aligned}$$

whence the result.

(v) follows immediately from (iv) which implies that $B \cup BwB$ can be a group only if $|N(w)| = 1$.

(vi) also follows from (iv). For $g \in BwB$ we have ${}^gB = B \Leftrightarrow {}^wB = B \Leftrightarrow BwBw^{-1}B = B$ which by (iv) happens only for $w = 1$. □

Remark 4.4. In a group with a (B, N)-pair, we call *Borel subgroups* the conjugates of B. A statement equivalent to the Bruhat decomposition is that every pair of Borel subgroups is conjugate to a pair $(B, {}^wB)$ for $w \in W$. We say that the pair is *in relative position* w.

We call *maximal tori* of G the conjugates of T. It follows that the intersection of two Borel subgroups always contains a torus (since B and ${}^wB$ contain T).

Example 4.5. In GL_n a matrix m is in BwB if and only if it all bottom left minors have same ranks as for the permutation matrix w, that is the ranks of the submatrices $m_{i,j}$ on lines $i, \ldots, n$ and columns $1, \ldots, j$ coincide. Indeed:

- The ranks of $m_{i,j}$ are invariant by left or right multiplication of m by an upper triangular matrix.
- A permutation matrix w for the permutation σ is characterized by the ranks of $w_{i,j}$, given by $|\{k <= j \mid \sigma(k) \geq i\}|$.

If $\{F_i\}$ and $\{F'_i\}$ are two complete flags then the permutation matrix which measures their relative position is given by $\operatorname{rank} w_{i,j} = \dim \frac{F_i \cap F'_j}{(F_{i-1} \cap F'_j) + (F_i \cap F'_{j-1})}$

Example 4.6. An "exotic" (B, N)-pair: $G = \mathrm{GL}_n(\mathbb{Q}_p)$; N=monomial matrices, B=matrices whose coefficients in the upper triangular part lie in $\mathbb{Z}_p$ and under the diagonal lie in $p\mathbb{Z}_p$ (B is an *Iwahori subgroup*, that is a subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$ whose reduction in $\mathrm{GL}_n(\mathbb{F}_p)$ falls in a Borel subgroup). Then W is of type $\tilde{A}_n$ ("affine" A_n). For $n = 2$, W is the infinite dihedral group with Coxeter diagram $\underset{s}{\bigcirc} \overset{\infty}{\text{---}} \underset{t}{\bigcirc}$, generated by $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $t = \begin{pmatrix} 0 & p \\ -p^{-1} & 0 \end{pmatrix}$.

In reductive groups, we can refine the Bruhat decomposition to the "unique Bruhat decomposition" which is as follows:

Lemma 4.7. *Let* $\mathbf{G}$ *be a connected reductive group and* $\mathbf{B} = \mathbf{T} \ltimes \mathbf{U}$ *be a decomposition of* $\mathbf{B}$ *as in 1.6, where* $\mathbf{U} = \mathrm{R_u}(\mathbf{B})$. *Then* $\mathbf{B}w\mathbf{B}$ *is the direct product* $\mathbf{U}\mathbf{T}w\mathbf{U}_w$ *where* $\mathbf{U}_w := \prod_{\{\alpha \in \Phi^+ \mid w(\alpha) < 0\}} \mathbf{U}_\alpha$.

Proof. Notice first that $\mathbf{U}_w$ is a group since if in 3.1(vii) α and β are sent to negative roots by w the same holds for $\lambda\alpha + \mu\beta$. We have $\mathbf{U} = \mathbf{U}'\mathbf{U}_w$ where $\mathbf{U}' = \prod_{\{\alpha \in \Phi^+ \mid w(\alpha) > 0\}} \mathbf{U}_\alpha$ thus ${}^w\mathbf{U}' \subset \mathbf{U}$; thus $\mathbf{B}w\mathbf{B} = \mathbf{U}\mathbf{T}w\mathbf{U}'\mathbf{U}_w = \mathbf{U}\mathbf{T}w\mathbf{U}_w$. It remains to see the decomposition is unique, that is if $u\mathbf{T}wu' = \mathbf{T}w$ with $u \in \mathbf{U}, u' \in \mathbf{U}_w$ then $u = u' = 1$. The condition implies $u \cdot {}^wu' \in \mathbf{T}$; in particular ${}^wu' \in \mathbf{B}$. But ${}^w\mathbf{U}_w \cap \mathbf{B} = 1$ since all $\mathbf{U}_\alpha$ in ${}^w\mathbf{U}_w$ are for negative α. Thus $u' = 1$, whence $u = 1$. □

In a connected linear algebraic group any semisimple element lies in some maximal torus. Since every element lies in some Borel subgroup this results from 1.6.

Proposition 4.8. *Let* $\mathbf{G}$ *be as in 4.7 and let* $t \in \mathbf{T}$. *Then*

(i) $C_{\mathbf{G}}(t)^\circ$ *is generated by* $\mathbf{T}$ *and the* $\mathbf{U}_\alpha$ *such that* $\alpha(t) = 1$.

(ii) $C_{\mathbf{G}}(t)$ is generated by $C_{\mathbf{G}}(t)^\circ$ and the $n \in N_{\mathbf{G}}(\mathbf{T})$ such that ${}^n t = t$.

Proof. (i) is an immediate consequence of 3.1(iv).

Let us prove (ii). Conjugation by t permutes the cells $\mathbf{B}w\mathbf{B}$; for this cell to be t-stable it must contain a representative n of w such that ${}^n t = t$. Then $\mathbf{T}w \subset C_{\mathbf{G}}(t)$ and by the unique decomposition 4.7 an element $utwu'$ with $u \in \mathbf{U}$, $u' \in \mathbf{U}_w$ is in $C_{\mathbf{G}}(t)$ if and only if both u and u' are in $C_{\mathbf{G}}(t)$, thus in $C_{\mathbf{G}}(t)^\circ$. □

4.1 Parabolic subgroups

In a group G with a (B, N)-pair, we call parabolic subgroups the groups containing a Borel subgroup.

In a Coxeter system (W, S), for $I \subset S$, we denote W_I the subgroup of W generated by I (see 2.5).

Proposition 4.9. *In a group G with a (B, N)-pair:*

- *(i) the parabolic subgroups containing B are $P_I := BW_IB$ for $I \subset S$.*
- *(ii) if $g \in G$ satisfies ${}^g B \subset P_I$ then $g \in P_I$.*

(ii) reproves 1.14 by reproving that Borel subgroups of P_I are P_I-conjugate.

Proof. Let P be a subgroup containing B and let $w \in W$ be such that $BwB \subset P$. Then $Bw^{-1}BwB \subset P$ thus by 4.3 (iv), we have $BtB \subset P$ for all $t \in N(w)$. If $s_1 \cdots s_k$ is a reduced expression for w, we have $s_k \in P$, $s_k s_{k-1} s_k \in P, \ldots$ which inductively implies $s_i \in P$ for all i; whence $P \supset BW_IB$ where $I = \{s_1, \ldots, s_k\}$; conversely BW_IB is a subgroup by the argument of 4.3(i). whence (i).

Let us show (ii). Assume ${}^g B \subset P_I$ and let $w \in W$ be such that $g \in Bw^{-1}B$. Then $P_I \supset BgBg^{-1}B = BwBw^{-1}B$ whence by the same argument as (i) $w \in W_I$ thus $g \in P_I$. □

5 Isogenies

If $\mathbf{G}$ is a connected reductive group and $\mathbf{T}$ is a maximal torus, we call *root datum* of $\mathbf{G}$ the quadruple $(X, Y, \Phi, \Phi^\vee)$ where $X = X(\mathbf{T})$, $Y = Y(\mathbf{T})$ and Φ (resp. $\Phi^\vee$) are the roots (resp. coroots) relative to $\mathbf{T}$. We will see that the root datum determines $\mathbf{G}$ up to isomorphism.

Isogenies

An *isogeny* is a surjective morphism of algebraic groups with finite kernel.

Any finite normal subgroup of $\mathbf{G}$ is central: since conjugacy is continuous it is trivial on a finite, thus discrete, group. A central group of a reductive group is in every maximal torus.

Let $p = \operatorname{char} k$; a *p-morphism* of root data $(X, Y, \Phi, \Phi^\vee) \xrightarrow{f} (X_1, Y_1, \Phi_1, \Phi_1^\vee)$ is a morphism $X_1 \xrightarrow{f} X$ with finite cokernel inducing a bijection $\Phi \xrightarrow{\tau} \Phi_1$ such that $f(\tau(\alpha)) = q_\alpha \alpha$ and $f^\vee(\alpha^\vee) = q_\alpha \tau(\alpha)^\vee$ where q_α is a power of p ($q_\alpha = 1$ if $\operatorname{char} k = 0$) — and where $f^\vee : Y \to Y_1$ denotes the transpose of f.

Theorem 5.1. *Let* $\mathbf{G} \xrightarrow{\phi} \mathbf{G}_1$ *be an isogeny and let* $\mathbf{T}_1 = \phi(\mathbf{T})$. *Then* ϕ *induces a* p*-morphism* $(X(\mathbf{T}), Y(\mathbf{T}), \Phi, \Phi^\vee) \to (X_1(\mathbf{T}_1), Y_1(\mathbf{T}_1), \Phi_1, \Phi_1^\vee)$ *where* τ *and the* q_α *are determined by the formula* $\phi(\mathbf{u}_\alpha(x)) = \mathbf{u}_{\tau(\alpha)}(\lambda_\alpha x^{q_\alpha})$ *for some scalar* λ_α. *Conversely, every* p*-morphism is induced by an isogeny, unique up to conjugacy by an element of* $\mathbf{T}$.

Proof of the first part. The isogeny ϕ induces $X(\mathbf{T}_1) \xrightarrow{f} X(\mathbf{T})$ given by $\alpha \mapsto \alpha \circ \phi$ and $Y(\mathbf{T}) \xrightarrow{f^\vee} Y(\mathbf{T}_1)$ given by $\alpha^\vee \mapsto \phi \circ \alpha^\vee$. If $\mathbf{u}_\alpha$ is a root subgroup, then $\phi(\mathbf{u}_\alpha)$ is another one $\mathbf{u}_{\tau(\alpha)}$, which defines a bijection τ. We define a polynomial P by $\phi(\mathbf{u}_\alpha(x)) = \mathbf{u}_{\tau(\alpha)}(P(x))$; the compatibility with the action of $\mathbf{T}$ gives $\phi({}^t\mathbf{u}_\alpha(x)) = \phi(\mathbf{u}_\alpha(\alpha(t)x)) = \mathbf{u}_{\tau(\alpha)}(P(\alpha(t)x))$ and $\phi({}^t\mathbf{u}_\alpha(x)) = {}^{\phi(t)}\mathbf{u}_{\tau(\alpha)}(P(x)) = \mathbf{u}_{\tau(\alpha)}(\tau(\alpha)(\phi(t))P(x))$ whence $P(\alpha(t)x) = \tau(\alpha)(\phi(t))P(x)$ which implies that P is a monomial; the compatibility to the group law of $\mathbb{G}_a$ gives $P(x+y) = P(x) + P(y)$. This forces $P = \lambda x^{q_\alpha}$ where q_α is a power of $p = \operatorname{char} k$ and λ a constant ($q_\alpha = 1$ if $\operatorname{char} k = 0$). The constants λ can be changed by composing ϕ with an element of $\operatorname{ad} \mathbf{T}$. □

We give now some examples of isogenies, defined by the corresponding p-morphism.

Example 5.2. The opposition automorphism: $q_\alpha = 1$ and $\tau(\alpha) = -\alpha$ for all α. It is $\operatorname{ad} w_0$ if w_0 is central in W, and transpose $\circ$ inverse $\circ \operatorname{ad} w_0$ in type A.

Example 5.3. An automorphism of the root system: τ is defined by the chosen automorphism and $q_\alpha = 1$.

Example 5.4. A split Frobenius: we assume $k = \overline{\mathbb{F}}_p$, we let $\tau(\alpha) = \alpha$ for all α, and set all $q_\alpha = q$ a given power of $p = \operatorname{char} k$. The corresponding isogeny $\mathbf{G} \xrightarrow{F} \mathbf{G}$ is called a *split Frobenius* and $\mathbf{G}^F = \mathbf{G}(\mathbb{F}_q)$; we will see later how to build it.

Example 5.5. We assume $\mathbf{G}$ has a root system Φ of type C_2, and $\Pi = \{\alpha = e_1 - e_2, \beta = 2e_2\}$. If $\operatorname{char} k = 2$ the formulae $\phi(\mathbf{u}_\alpha(x)) = \mathbf{u}_\beta(x^2), \phi(\mathbf{u}_{\alpha+\beta}(x)) = \mathbf{u}_{2\alpha+\beta}(x^2)), \phi(\mathbf{u}_\beta(x)) = \mathbf{u}_\alpha(x), \phi(\mathbf{u}_{2\alpha+\beta(x)}) = \mathbf{u}_{\alpha+\beta}(x)$ define an isogeny. If $t = \operatorname{diag}(t_1, t_2, t_2^{-1}, t_1^{-1}) \in \mathbf{T}$ we have $\phi(t) = \operatorname{diag}(t_1t_2, t_1t_2^{-1}, t_1^{-1}t_2, t_1^{-1}t_2^{-1})$. One checks that ϕ^2 raises all coordinates to the square; it is the split Frobenius F over $\mathbb{F}_2$. Then for any r the isogeny $\phi \circ F^r$ has $2^{2r+1} - 1$ fixed points on the torus. The group of fixed points $\mathbf{G}^{\phi F^r}$ is the *Suzuki group* $\operatorname{Sz}(2^{2r+1})$.

Theorem 5.1 shows that groups with isomorphic root data are isomorphic; this implies the classification if we can show the existence of the corresponding groups.

6 Rationality questions

Let k_0 be a subfield of k.

A k_0-*structure* on a vector space V is a sub-k_0-space $V(k_0)$ such that $V = V(k_0) \otimes_{k_0} k$.

A k_0-structure on a k-algebra A is a k_0-algebra of finite type $A(k_0)$ such that $A = A(k_0) \otimes_{k_0} k$.

A k_0-structure on an affine or projective variety $\mathbf{V}$ is the k_0-variety defined by $A(k_0)$ where the algebra A of $\mathbf{V}$ has a k_0-structure $A(k_0)$.

In general a k_0-structure on a variety is given by a finite open affine covering where each open affine has a k_0-structure.

In our lectures, all the varieties we will need to consider will be quasi-projective varieties, that is open subvarieties of projective varieties. We assume all varieties quasi-projective from now on.

Definition 6.1. *An algebraic variety* $\mathbf{V}$ *over* k *is said to be* defined over k_0*, if it has a* k_0*-structure* $\mathbf{V}(k_0)$*. In this case we write* $\mathbf{V} = \mathbf{V}(k_0) \otimes_{k_0} k$*.*

If the variety $\mathbf{V}$ has a k_0-structure, an element $\sigma \in \mathrm{Gal}(k/k_0)$ acts on $\mathbf{V}$ by $x \otimes \lambda \mapsto x \otimes \sigma(\lambda)$. If k/k_0 is a Galois extension, for instance if k is the separable closure of k_0, one can find $\mathbf{V}(k_0)$ as the fixed points of the action of $\mathrm{Gal}(k/k_0)$. This results from

Proposition 6.2. *If* V *is a* k*-vector space (resp. a* k*-algebra) with a continuous action of* $\mathrm{Gal}(k/k_0)$ *(as a profinite group, thus continuous means that* $V = \cup_G V^G$ *where* G *runs over subgroups of finite index) (resp. compatible to the algebra structure), the fixed points of the action define a* k_0*-structure.*

Proof. See [Springer, 11.1.6]. □

Example 6.3. When k is an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_p$, we have $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) = \hat{\mathbb{Z}}$; an element of $\hat{\mathbb{Z}}$ is defined by a sequence $k_n \in \mathbb{Z}$ subject to the only condition $k_n \equiv k_m \pmod{m}$ if m divides n; this element acts on $\mathbb{F}_{q^n}$ by $x \mapsto x^{q^{k_n}}$. We have $\hat{\mathbb{Z}} \simeq \prod_p \mathbb{Z}_p$.

Proposition 6.4. *A subvariety (resp. subalgebra, sub-vector space) is defined over* k_0 *(equivalently has a* k_0*-structure which is a subvariety (resp. subalgebra, subspace)) if and only if it is stable under the action of* $\mathrm{Gal}(k/k_0)$*.*

Proof. See [Springer, 11.1.4]. □

Example 6.5. The affine line is $\mathbb{A}^1 = \operatorname{Spec} k[T]$. The affine line on k_0, defined by the k_0-algebra $k_0[T]$, is a k_0-structure since $k[T] = k_0[T] \otimes_{k_0} k$. An element $\sigma \in \mathrm{Gal}(k/k_0)$ acts as $\sum_i a_i T^i \mapsto \sum_i \sigma(a_i) T^i$. A k-point of $\mathbb{A}^1$ is given by $a \in k$ (or by the ideal which is the kernel of the morphism $P \mapsto P(a) : k[T] \to k$); this point is defined over k_0 if $a \in k_0$.

6.1 Frobenius endomorphism

Definition 6.6. *Let* $\mathbf{V}$ *be an* $\overline{\mathbb{F}}_q$*-variety with an* $\mathbb{F}_q$*-structure* $\mathbf{V}(\mathbb{F}_q)$*. The associated* geometric Frobenius endomorphism $F : \mathbf{V} \to \mathbf{V}$ *is* $F_0 \otimes \mathrm{Id}$ *where* F_0 *is the endomorphism of* $\mathbf{V}(\mathbb{F}_q)$ *which raises the functions on* $\mathbf{V}(\mathbb{F}_q)$ *to the* q*-th power.*

The endomorphism Φ of $\mathbf{V}$ induced by $(\lambda \mapsto \lambda^q) \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is called the *arithmetic Frobenius endomorphism*.

On an affine variety $\operatorname{Spec} A$ the $\mathbb{F}_q$-structure is of the form $A = A(\mathbb{F}_q) \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ and the geometric Frobenius endomorphism corresponds to a morphism $F^* : a\otimes\lambda \mapsto a^q\otimes\lambda$ — in a coordinate system for the variety, the geometric Frobenius raises each coordinate to the q-th power). The arithmetic Frobenius is given by $\Phi : a \otimes \lambda \mapsto a \otimes \lambda^q$. The composition $F^* \circ \Phi$ raises each element of A to the q-th power, which acts trivially on the $\overline{\mathbb{F}}_q$-points of $\operatorname{Spec} A$.

Example 6.7. On $\mathbb{A}^1$, the geometric Frobenius is given by $F^* : P(T) \mapsto P(T^q)$; thus $F^* \circ \Phi$ maps $P(T)$ to $P(T)^q$. If $a \in \overline{\mathbb{F}}_q$ is an $\overline{\mathbb{F}}_q$-point of $\mathbb{A}^1$, the image of a by $F^* \circ \Phi$ is defined by the kernel of $P \mapsto P(a)^q$, which is the same as that of $P \mapsto P(a)$.

Note that the geometric Frobenius endomorphism is a morphism of $\overline{\mathbb{F}}_q$-varieties, while the arithmetic Frobenius endomorphism is only a morphism of $\mathbb{F}_q$-varieties. In the sequel we will only consider the geometric Frobenius endomorphism and just call it "the Frobenius endomorphism".

Proposition 6.8. *Let $\mathbf{V}$ be an affine or projective $\overline{\mathbb{F}}_q$-variety with algebra A. A surjective morphism $A \xrightarrow{F^*} A^q$ is the Frobenius endomorphism attached to an $\mathbb{F}_q$-structure on $\mathbf{V}$ if and only if for any $x \in A$ there exists n such that $F^{*n}(x) = x^{q^n}$. The corresponding $\mathbb{F}_q$-structure is $A(\mathbb{F}_q) = \{x \in A \mid x^q = F^*(x)\}$.*

Proof. If A has an $\mathbb{F}_q$-structure $A = A(\mathbb{F}_q) \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q \ni x = \sum_i x_i \otimes \lambda_i$ then $x^{q^n} = \sum_i x_i^{q^n} \otimes \lambda_i^{q^n}$ thus $x^{q^n} = F^{*n}(x)$ when n is such that all λ_i are in $\mathbb{F}_{q^n}$.

Conversely, if F^* is a surjective morphism as in the statement, since $x \mapsto x^{q^n}$ is injective then F^* must also be injective, thus bijective and we can define ϕ by $\phi(x) = F^{*-1}(x^q)$; then if we make the topological generator of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ act by ϕ, the assumptions of 6.2 are satisfied.

The fixed points of ϕ form the $\mathbb{F}_q$-structure by 6.2 and are as described in the statement. □

Proposition 6.9. *Let $\mathbf{V}$ be an $\overline{\mathbb{F}}_q$-variety and F be the Frobenius endomorphism corresponding to an $\mathbb{F}_q$-structure on $\mathbf{V}$.*

(i) If $\mathbf{V} = \operatorname{Spec} A$ then $A(\mathbb{F}_q) = \{x \in A \mid x^q = F^*(x)\}$.

(ii) A subvariety of $\mathbf{V}$ is defined over $\mathbb{F}_q$ if and only if it is F-stable; the corresponding Frobenius endomorphism is the restriction of F.

(iii) Let φ be an automorphism of $\mathbf{V}$ such that $(\varphi F)^n = F^n$ for some positive integer n; then φF is the Frobenius endomorphism attached to another $\mathbb{F}_q$-structure on $\mathbf{V}$.

(iv) If F' is a Frobenius endomorphism attached to another $\mathbb{F}_q$-structure on $\mathbf{V}$, there exists an integer $n > 0$ such that $F^n = F'^n$.

(v) F^n is the Frobenius endomorphism attached to an $\mathbb{F}_{q^n}$-structure on $\mathbf{V}$.

(vi) Every closed subvariety of a variety defined over $\mathbb{F}_q$ is defined over a finite extension of $\mathbb{F}_q$. Every morphism between varieties defined over $\mathbb{F}_q$ is defined over a finite extension of $\mathbb{F}_q$.

(vii) The orbits of F on the set of points of $\mathbf{V}$ are finite, as well as the set $\mathbf{V}^F$, also denoted $\mathbf{V}(\mathbb{F}_q)$, which consists of the points of $\mathbf{V}$ defined over $\mathbb{F}_q$.

Proof. (i) is clear by the proof of 6.8 and 6.2.

(ii) reflects 6.4.

(iii) results from the fact that φF still satisfies 6.8.

(iv): by considering an affine open covering it is sufficient to deal with the case $\mathbf{V} = \operatorname{Spec} A$. Then we use that A is of finite type, thus there exists n such that $F^{*n}(x) = F'^{*n}(x) = x^{q^n}$ for every generator x of A.

(v) results from 6.8.

(vi) has a proof similar to that of (iv): there exists n such that for any element a in a finite set of generators of the ideal I defining the subvariety (resp. any coefficient a of an equation of the morphism) we have $F^{*n}a = a^{q^n}$, thus $I \subset \sqrt{F^{*n}(I)}$.

Let us show (vii). As in (iv) we may assume $\mathbf{V} = \operatorname{Spec} A$. Let $\{a_1, \ldots, a_n\}$ be generators of $A(\mathbb{F}_q)$. A point $x \in \mathbf{V}$ is given by a morphism $x : A \to \overline{\mathbb{F}}_q$. It is F^{*n}-fixed if for any i we have $x(a_i) \in \mathbb{F}_{q^n}$, which happens for a sufficiently large n. It is F^*-fixed if $x(a_i) \in \mathbb{F}_q$, or equivalently if we are given a morphism $A(\mathbb{F}_q) \to \mathbb{F}_q$; there is a finite number of such morphisms. □

Proposition 6.10. *Let $\mathbf{V} \simeq \mathbb{A}^n$ as an $\overline{\mathbb{F}}_q$-variety. Then $|\mathbf{V}^F| = q^n$ for any $\mathbb{F}_q$-structure on $\mathbf{V}$.*

Proof. See [Geck, 4.2.4] for a (complicated) elementary proof in the case of unipotent groups. It is an immediate consequence of the Lefschetz theorem in ℓ-adic cohomology. □

7 The Lang-Steinberg theorem

We say an algebraic group over $\overline{\mathbb{F}}_q$ is defined over $\mathbb{F}_q$ if the corresponding Frobenius endomorphism is a group morphism.

Let $\mathbf{G}$ be a reductive group over $\overline{\mathbb{F}}_q$, let F be the Frobenius endomorphism attached to an $\mathbb{F}_q$-structure and let $\mathbf{T}$ be an F-stable maximal torus (we will see later that there always exists an F-stable maximal torus). There is a natural $\mathbb{F}_q$-structure on $X(\mathbf{T}) = \operatorname{Hom}(\mathbf{T}, \mathbb{G}_m)$ defined by the $\mathbb{F}_q$-structure $F(x) = x^q$ on $\mathbb{G}_m$ (the only $\mathbb{F}_q$-structure compatible with the group structure): for this $\mathbb{F}_q$-structure F acts on $X(\mathbf{T})$ by $(F(\alpha))({}^F t) = F(\alpha(t)) = \alpha(t)^q$. On the other hand F sends a root subgroup $\mathbf{u}_\alpha$ to another root subgroup $\mathbf{u}_{\tau(\alpha)}$ for some permutation τ so in the end we get $F(\mathbf{u}_\alpha(x)) = u_{\tau(\alpha)}(\lambda x^q)$ thus F is an isogeny associated to τ and such that $q_\alpha = q$ for any α; the p-morphism is $q\tau$.

Note that a Frobenius endomorphism, having trivial kernel and being bijective on points, is an isogeny; but it is not an isomorphism, since it is not invertible as a morphism of varieties.

Definition 7.1. *Let* $\mathbf{G}$ *be a reductive group over* $\overline{\mathbb{F}}_q$ *and let* $F : \mathbf{G} \to \mathbf{G}$ *be an isogeny such that some power is a Frobenius endomorphism. Then the group of fixed points* $\mathbf{G}^F$ *is called a* finite group of Lie type.

7.1 The Lang-Steinberg theorem

Lemma 7.2. *Let* $\mathbf{G}$ *be an affine algebraic group over* $\overline{\mathbb{F}}_q$ *and* F *be an isogeny such that some power is a Frobenius endomorphism. Then for* $g \in G$ *the map* $\operatorname{ad} gF$ *is still an isogeny such that some power is a Frobenius endomorphism.*

Proof. That $\operatorname{ad} gF$ is still an isogeny is obvious. It thus is enough to check that some power of $\operatorname{ad} gF$ is equal to the same power of F. We have $(gF)^n = yF^n$ where $y = g\,{}^F g \ldots {}^{F^{n-1}}g$; if n is such that g is F^n-stable then y is also F^n-stable and if y is of order e then $(xF)^{ne} = F^{ne}$. □

The fundamental theorem on connected algebraic groups over $\overline{\mathbb{F}}_q$ is

Theorem 7.3. *(Lang-Steinberg) Let* $\mathbf{G}$ *be a* **connected** *affine algebraic group over* $\overline{\mathbb{F}}_q$*, and* F *an isogeny such that some power is a Frobenius endomorphism. Then the* Lang map $\mathcal{L} : g \mapsto g^{-1}.{}^F g$ *is a surjective endomorphism of* $\mathbf{G}$.

Proof. The morphism $\mathcal{L}$ has fibers isomorphic to $\mathbf{G}^F$, thus finite, thus $\dim \operatorname{Im} \mathcal{L} = \dim \mathbf{G}$; as $\mathbf{G}$ is irreducible $\mathcal{L}$ is dominant (which means $\mathbf{G}$ is the closure of $\operatorname{Im} \mathcal{L}$), thus $\operatorname{Im} \mathcal{L}$ contains a nonempty open subset of $\mathbf{G}$.

For a given x, the morphism $g \mapsto g^{-1}.x.{}^F g$ has also finite fibers: indeed, a fiber has cardinality the number of solutions of $g^{-1}x{}^F g = x$, that is $g = {}^{xF}g$ and xF still has finitely many fixed points by Lemma 7.2. Thus the image of $g \mapsto g^{-1}.x.{}^F g$ contains also a nonempty open subset of $\mathbf{G}$, thus meets that of $\mathcal{L}$. Thus there exists g and h such that $g^{-1}.{}^F g = h^{-1}.x.{}^F h$, thus $x = \mathcal{L}(gh^{-1})$. □

[Steinberg68] has shown 7.3 under the only assumption that F is a surjective morphism such that $\mathbf{G}^F$ is finite.

A consequence of Lang's theorem is that for $g \in \mathbf{G}$ the group $\mathbf{G}^{gF}$ is isomorphic to $\mathbf{G}^F$. Indeed, write $g = h^{-1}F(h)$ then $\mathbf{G}^{gF} = h^{-1}\mathbf{G}^F h$.

7.2 Galois cohomology exact sequence

Here we follow [Serre, §5]. If G is a profinite group acting continuously on a set E we set $H^0(G, E) = E^G$ and if E is a group (on which G acts as a group endomorphism) Serre defines a set $H^1(G, E)$ (we do not give the definition in general, we will give it below when $G = \hat{\mathbb{Z}}$). If $A \subset B$ is a group inclusion we have the "Galois cohomology exact sequence"

$$1 \to H^0(G, A) \to H^0(G, B) \to H^0(G, B/A) \xrightarrow{p} H^1(G, A) \xrightarrow{i} H^1(G, B) \qquad (*)$$

When F is a topological generator of $G = \hat{\mathbb{Z}}$, we will denote $H^i(F, E)$ for $H^i(\hat{\mathbb{Z}}, E)$; in this case $H^1(F, E)$ is the set of F-classes of E, equal to the E-conjugacy orbits in $E.F$, or the classes of E under the "twisted conjugacy" $e \mapsto e'eF(e'^{-1})$.

The maps in (*) are the obvious ones excepted perhaps p which maps an F-stable coset bA to the F-class of $b^{-1}F(b)$ (an element of A since bA is F-stable). The "exactness of the sequence at $H^0(F,B/A)$" is that $H^0(F,B)=B^F$ acts naturally on $H^0(F,B/A)=(B/A)^F$ and that the elements of a given orbit have the same image in $H^1(F,A)$. For the next step, the image of p is the preimage by i of the F-class of 1: this is an "exact sequence of pointed sets".

The Lang theorem can be rephrased as:

Proposition 7.4. *If* $\mathbf{G}$*,* F *are as in 7.3, then* $H^1(F,\mathbf{G})=1$.

Proposition 7.5. *Let* $\mathbf{G},F$ *be as in 7.3 and let* $\mathbf{V}$ *be a variety with an action of* F *on which* $\mathbf{G}$ *acts transitively and compatibly with* F*. Then* $\mathbf{V}^F\neq\emptyset$.

Proof. Since the action is transitive, given $v\in\mathbf{V}$, there exists $g\in\mathbf{G}$ such that ${}^Fv=gv$. Write $g^{-1}=h^{-1}\,{}^Fh$, then ${}^F(hv)={}^Fhgv=hg^{-1}gv=hv$. □

Lemma 7.6. *Let* $A\subset B$ *be two closed and* F*-stable subgroups of* $\mathbf{G}$*, where* A *is connected. then*

(i) We have $(B/A)^F=B^F/A^F$.

(ii) If in addition A *is normal in* B*, the quotient map induces a bijection* $H^1(F,B)\to H^1(F,B/A)$.

Proof. (i) is (*) since $H^1(F,A)=1$ but let us give a naive proof. By 7.5, any F-stable coset bA contains an F-stable element, thus the natural map $B^F/A^F\to(B/A)^F$ is surjective. It is injective since if $x,y\in B^F$ are in the same A-coset, then $x^{-1}y\in A^F$.

Let us show (ii). Surjectivity is clear. Conversely, if $b,b'\in B$ are F-conjugate modulo A, we have $ab=xb'\,{}^Fx^{-1}$, with $x\in B$ and $a\in A$. We must see that ab is F-conjugate to b, that is there exists $y\in B$ such that $yab\,{}^Fy^{-1}=b$ or equivalently $a=y^{-1}\,{}^{bF}y$. This comes from 7.2 which shows that we may still apply Lang's theorem to $\operatorname{ad} bF$. □

Proposition 7.7. *Let* $\mathbf{G},F,\mathbf{V}$ *be as in 7.5, and let* $x\in\mathbf{V}^F$ *and* $g\in\mathbf{G}$*. Then*

(i) We have $gx\in\mathbf{V}^F$ *if and only if* $g^{-1}\,{}^Fg\in C_{\mathbf{G}}(x)$.

(ii) The map which sends the $\mathbf{G}^F$*-orbit of* $gx\in\mathbf{V}^F$ *to the* F*-conjugacy class of the image of* $g^{-1}\,{}^Fg$ *in* $C_{\mathbf{G}}(x)/C_{\mathbf{G}}(x)^\circ$ *is well-defined and bijective.*

Proof. The proposition translates (*) applied to the inclusion $C_{\mathbf{G}}(x)\subset\mathbf{G}$, which gives $1\to C_{\mathbf{G}}(x)^F\to\mathbf{G}^F\to\mathbf{V}^F\to H^1(F,C_{\mathbf{G}}(x))\to 1$ since $H^1(F,C_{\mathbf{G}}(x))=H^1(F,C_{\mathbf{G}}(x)/C_{\mathbf{G}}(x)^\circ)$ by 7.6(ii). Again we will give a naive proof.

(i) is an immediate computation. Let us show (ii). Let $x\in\mathbf{V}^F,h,g\in\mathbf{G}$ be such that $hx,gx\in\mathbf{V}^F$. Note that $hx=gx$ if and only if h and g differ by an element of $C_{\mathbf{G}}(x)$, and then $h^{-1}\,{}^Fh$ and $g^{-1}\,{}^Fg$ are F-conjugate in $C_{\mathbf{G}}(x)$. We have thus a well-defined map from $\mathbf{V}^F$ to the F-classes of $C_{\mathbf{G}}(x)$. On the other hand, if $h\in\mathbf{G}^F$, then gx and hgx have the same image $g^{-1}\,{}^Fg=(hg)^{-1}\,{}^F(hg)$. Thus the map goes from the $\mathbf{G}^F$-orbits in $\mathbf{V}^F$ to the F-classes of $C_{\mathbf{G}}(x)$. If

$g^{-1}{}^F g$ and $h^{-1}{}^F h$ are F-conjugate by $n \in C_{\mathbf{G}}(x)$, then $gnh^{-1} \in \mathbf{G}^F$ and sends hx to gx. The map is thus injective. By Lang's theorem any element of $C_{\mathbf{G}}(x)$ is of the form $g^{-1}{}^F g$ with $g \in \mathbf{G}$, which shows the surjectivity of the map. We finish the proof using 7.6(ii). $\square$

Corollary 7.8. *Let* $\mathbf{G}$ *as in 7.3.*

(i) F-stable Borel subgroups exist and are all $\mathbf{G}^F$-conjugate.

(ii) Let us define a geometric conjugacy class *as the intersection with $\mathbf{G}^F$ of an F-stable conjugacy class of $\mathbf{G}$. Then a geometric conjugacy class is non-empty, and if x is an element of such a class, the class splits under $\mathbf{G}^F$-conjugacy into classes parameterized by $H^1(F, C_{\mathbf{G}}(x)/C_{\mathbf{G}}(x)^\circ)$.*

Proof. (i) comes from 7.7 applied with $\mathbf{V}$ the variety of Borel subgroups, using that for $\mathbf{B}$ a Borel subgroup $N_{\mathbf{G}}(\mathbf{B}) = \mathbf{B}$ is connected.

For (ii) we apply 7.7 with $\mathbf{V}$ the geometric class (and the action of $\mathbf{G}$ by conjugacy). $\square$

All centralizers in GL_n are connected, thus geometric conjugacy classes do not split. Indeed, the centralizer in M_n of a matrix is an affine space, thus its intersection with GL_n is an open subspace of an affine space, which is always connected.

Example 7.9.

Let $\mathbf{G} = \mathrm{PGL}_2(\overline{\mathbb{F}}_q)$ where $q \not\equiv 0 \pmod 2$ and let F define an $\mathbb{F}_q$-structure. If $s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ then $C_{\mathbf{G}}(s) = \{\mathrm{Id}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$ is disconnected. For $\lambda \in \mathbb{F}_{q^2}, \lambda^{q-1} = -1$ we have $m = \begin{pmatrix} 0 & \lambda^{-1} \\ \lambda & 0 \end{pmatrix} \in \mathbf{G}^F$ (since ${}^F m = -m$ in GL_2) and if $x = \begin{pmatrix} 1 & 1 \\ \lambda & -\lambda \end{pmatrix}$ then $xsx^{-1} = m$ and $x^{-1}{}^F x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ thus m is geometrically conjugate but not $\mathbf{G}^F$-conjugate to s.

Similarly to (i) in 7.8 the F-stable maximal tori in an F-stable Borel subgroup $\mathbf{B}$ exist and are $\mathbf{B}^F$-conjugate, since for a torus $\mathbf{T}$ the group $N_{\mathbf{B}}(\mathbf{T}) = C_{\mathbf{B}}(\mathbf{T})$ (by 1.7) is connected by 1.15. Thus we may find an F-stable pair $\mathbf{T} \subset \mathbf{B}$ of a maximal torus and a Borel subgroup containing it.

Proposition 7.10. *Let $\mathbf{G}$ be as in 7.1 and let $\mathbf{T}$ be an F-stable maximal torus. Then the $\mathbf{G}^F$-conjugacy classes of F-stable maximal tori are parameterized by $H^1(F, W_{\mathbf{G}}(\mathbf{T}))$; given another F-stable maximal torus ${}^g\mathbf{T}$ with $g \in \mathbf{G}$ we call* **type** *of ${}^g\mathbf{T}$ with respect to $\mathbf{T}$ the F-class of w, the image in $W_{\mathbf{G}}(\mathbf{T})$ of $g^{-1}{}^F g \in N_{\mathbf{G}}(\mathbf{T})$.*

Proof. We apply 7.7 with $\mathbf{V}$ the variety of maximal tori of $\mathbf{G}$, on which $\mathbf{G}$ acts by conjugacy. $\square$

Note that the pair $({}^g\mathbf{T}, F)$ is sent by g^{-1}-conjugacy to the pair $(\mathbf{T}, wF)$.

Proposition 7.11. *Let* $\mathbf{G}$ *as in 7.3. Then every* F*-stable semisimple element lies in some* F*-stable maximal torus of* $\mathbf{G}$.

Proof. Let $s \in \mathbf{G}$ semisimple; then $s \in C_{\mathbf{G}}(s)^\circ$ by 4.8, and s being central in this group is in all maximal tori, thus in particular in the F-stable maximal tori of $C_{\mathbf{G}}(s)^\circ$ which are also maximal in $\mathbf{G}$. □

7.3 The relative (B,N)-pair.

Proposition 7.12. *Let* $\mathbf{G}$ *be as in 7.1.*

(i) Let $\mathbf{T}$ *be an* F*-stable maximal torus of* $\mathbf{G}$*. Then* $W^F = N_{\mathbf{G}}(\mathbf{T})^F/\mathbf{T}^F$.

Let $\mathbf{T} \subset \mathbf{B}$ *be an* F*-stable pair of a maximal torus and a Borel subgroup. Then*

(ii) $\mathbf{G}^F = \coprod_{w\in W^F} \mathbf{B}^F w \mathbf{B}^F$*, which we will recognize as the Bruhat decomposition attached to a relative* (B,N)*-pair (see 7.13 below).*

(iii) $|\mathbf{G}^F| = q^{l(w_0)}|\mathbf{T}^F|(\sum_{w\in W^F} q^{l(w)})$ *where* $q \in \mathbb{R}_{>0}$ *is defined by some power* F^a *being a split Frobenius attached to an* $\mathbb{F}_{q^a}$*-structure.*

(iv) $\mathrm{R_u}(\mathbf{B})^F$ *is a Sylow* p*-subgroup of* $\mathbf{G}^F$.

Proof. (i) comes from 7.6(i).

For (ii) we use the "unique Bruhat decomposition" 4.7 which implies that an F-stable element of $\mathbf{B}w\mathbf{B}$ is in $\mathbf{B}^F n \mathbf{B}^F = \mathbf{B}^F n \mathbf{U}_w^F$ where $n \in N_{\mathbf{G}}(\mathbf{T})^F$ is a representative of w.

Let us show (iii). By the proof of (ii) $|\mathbf{G}^F/\mathbf{B}^F| = \sum_{w\in W^F} |\mathbf{U}_w^F|$, and using $|\mathbf{B}^F| = |\mathbf{T}^F||\mathbf{U}^F|$ we get the stated formula if we show $|\mathbf{U}_w| = q^{l(w)}$ (since $\mathbf{U} = \mathbf{U}_{w_0}$). If F is a Frobenius attached to an $\mathbb{F}_q$-structure, this results from 6.10 and the fact that $\mathbf{U}_w$ is an affine space of dimension $l(w)$. We will admit the formula in other cases — one may use 2.6 and an explicit description of $\mathbf{U}_w^F$.

As $|\mathbf{G}^F/\mathbf{U}^F| = |\mathbf{T}^F|(\sum_{w\in W^F} q^{l(w)})$ is prime to p (since $\mathbf{T}$ is a p'-group, and $\sum_{w\in W^F} q^{l(w)} \equiv 1 \bmod.\ q$) we see that $\mathbf{U}^F$ is a Sylow p-subgroup of $\mathbf{G}^F$ (we have $N_{\mathbf{G}^F}(\mathbf{U}^F) = \mathbf{B}^F$, thus $\sum_{w\in W^F} q^{l(w)}$ is the number of Sylow p-subgroups of $\mathbf{G}^F$). □

Note that the fixed points of the unipotent radical of a Borel being a Sylow p-subgroup extends to non-reductive groups, since $\mathrm{R_u}(\mathbf{G})$ is a p-group, it is in all unipotent radicals of Borel subgroups, and being connected we have $|\mathbf{G}^F| = |(\mathbf{G}/\mathrm{R_u}(\mathbf{G}))^F||\mathrm{R_u}(\mathbf{G})^F|$.

Corollary 7.13. *Let* $\mathbf{G}$ *be as in 7.12. Then* $(\mathbf{B}^F, N_{\mathbf{G}}(\mathbf{T})^F)$ *is a* (B,N)*-pair for* $\mathbf{G}^F$ *with Weyl group* W^F*. Recall that* $(W^F, \{w_I\}_{I\in S/F})$ *is a Coxeter system where* I *runs over the* F*-orbits in* S *and where* w_I *is the longest element in* W_I.

Proof. The corollary follows immediately from the definition of (B,N)-pairs, from 7.12(ii) and from 2.6. We must check that for $I \in S/F$ and $w \in W^F$, then $\mathbf{B}^F w \mathbf{B}^F w_I \mathbf{B}^F \subset \mathbf{B}^F w \mathbf{B}^F \cup \mathbf{B}^F w w_I \mathbf{B}^F$. We use that either $l(w)+l(w_I) = l(ww_I)$, in which case $\mathbf{B}^F w \mathbf{B}^F w_I \mathbf{B}^F = \mathbf{B}^F w w_I \mathbf{B}^F$, or $w = w'w_I$ where $l(w')+$

$l(w_I) = l(w'w_I)$ in which case $\mathbf{B}^F w \mathbf{B}^F w_I \mathbf{B}^F \subset \mathbf{B}^F w' \mathbf{B}^F w_I \mathbf{B}^F w_I \mathbf{B}^F$, and $\mathbf{B}^F w_I \mathbf{B}^F w_I \mathbf{B}^F \subset \mathbf{B}^F \cup \mathbf{B}^F w_I \mathbf{B}^F$ since 1 and w_I are the only F-stable elements of W_I. $\square$

We complete the order formula for $\mathbf{G}^F$ by

Proposition 7.14. *Let $\mathbf{T}_w$ be a torus of type w with respect to $\mathbf{T}$. Then $|\mathbf{T}_w^F| = \det(wF - 1 \mid X(\mathbf{T}))$.*

Proof. It is enough to prove this formula for the pair $(\mathbf{T}, F)$. Applying $\mathrm{Hom}(-, \mathbb{G}_m)$ to the exact sequence $1 \to \mathbf{T}^F \to \mathbf{T} \xrightarrow{F-1} \mathbf{T} \to 1$ (where the surjectivity on the right is Lang's theorem) we get $1 \to X(\mathbf{T}) \xrightarrow{F-1} X(\mathbf{T}) \xrightarrow{p} \mathrm{Hom}(\mathbf{T}^F, \mathbb{G}_m)$; since the formula in the statement is the cokernel of $F-1$, we have to see the surjectivity of p. This comes from the fact that the dual map $\mathrm{Hom}(\mathrm{Hom}(\mathbf{T}^F, \mathbb{G}_m), \mathbb{G}_m) \to \mathrm{Hom}(X(\mathbf{T}), \mathbb{G}_m)$ is the inclusion $\mathbf{T}^F \hookrightarrow \mathbf{T}$; indeed the left-hand side is the dual of the dual of $\mathbf{T}^F$, isomorphic to $\mathbf{T}^F$. For the right-hand side, the algebra of $\mathbf{T}$, equal to $A = \overline{\mathbb{F}}_q[T_1, \ldots, T_n]$ identifies to $\overline{\mathbb{F}}_q[X(\mathbf{T})]$, and $\mathrm{Hom}(X(\mathbf{T}), \mathbb{G}_m)$ identifies to $\mathrm{Hom}(A, \overline{\mathbb{F}}_q)$ which is the set of points of $\mathbf{T}$. $\square$

7.4 Classification of finite groups of Lie type

Let us start with $\mathbf{G} = \mathrm{GL}_n(\overline{\mathbb{F}}_q)$. It has a natural $\mathbb{F}_q$-structure since its algebra is $\overline{\mathbb{F}}_q[T_{i,j}, \det(T_{i,j})^{-1}] = \mathbb{F}_q[T_{i,j}, \det(T_{i,j})^{-1}] \otimes \overline{\mathbb{F}}_q$. If F is the corresponding Frobenius endomorphism, we have $\mathbf{G}^F = \mathrm{GL}_n(\mathbb{F}_q)$; F raises all entries of a matrix to the q-th power.

The same kind of construction can be done with SL_n, the orthogonal, symplectic, ... groups. This gives the *split* $\mathbb{F}_q$-structure, giving an isogeny F such that $\tau = 1$.

An example of non-split $\mathbb{F}_q$-structure is the unitary group $\mathrm{GL}_n^{F'}$ where F' is defined by $F'(x) = F({}^t x^{-1})$, where F is split. Here $\tau(\alpha) = -\alpha$.

We will classify now the finite groups of Lie type which correspond to simple algebraic groups, that is adjoint groups with an irreducible Weyl group. We may start with an F-stable pair $\mathbf{T} \subset \mathbf{B}$, and, using the isogeny theorem, look at the corresponding root datum $(X(\mathbf{T}), Y(\mathbf{T}), \Phi, \check{\Phi})$. Since the group is adjoint $X(\mathbf{T})$ is determined by Φ; in addition giving q and τ determine the pair $(\mathbf{G}, F)$, thus $\mathbf{G}^F$, up to isomorphism.

We have a connected (since $\mathbf{G}$ is simple) Dynkin diagram and the possibilities for τ correspond to automorphisms of the Dynkin diagram.

The possibilities for a non-trivial τ on an irreducible root system are ${}^2A_n (n \geq 2)$, ${}^2D_n$, ${}^3D_4$ and ${}^2E_6$; here the exponent on the left is the order of τ. We have thus the following possibilities for $(\mathbf{G}, F)$, where $\mathbf{G}^F$ is simple unless noted otherwise.

- $A_n (n \geq 1)$ — the simple algebraic group is $\mathbf{G} = \mathrm{PGL}_n \simeq \mathrm{PSL}_n$.

  Remark 7.15. However, $\mathbf{G}^F$ is not in general simple. The simple finite group is $\mathrm{SL}_n^F / Z(\mathrm{SL}_n^F)$, which in general is not $\mathbf{G}^F$ but its derived subgroup. Indeed, in general $\mathrm{PSL}_n^F \neq \mathrm{SL}_n^F / Z(\mathrm{SL}_n^F)$, (this phenomenon $(A/B)^F \neq A^F/B^F$ comes from the fact that B is not connected). The center $Z\,\mathrm{SL}_n$ identifies to the group $\mu_{n_{p'}}$ of n-th roots of unity in $\overline{\mathbb{F}}_q$. The exact sequence (*) applied to the inclusion $\mu_{n_{p'}} \subset \mathrm{SL}_n$ gives $1 \to \mu_{n_{p'}}^F \to \mathrm{SL}_n^F \to \mathrm{PSL}_n^F \to H^1(F, \mu_{n_{p'}}) \to 1$ where $H^1(F, \mu_{n_{p'}}) = \mu_{n_{p'}}/(\mu_{n_{p'}})^{q-1}$ so the cokernel is non trivial if $q-1$ is not prime to $n_{p'}$.

We also have the small value $n = 2$ and $q = 2$ (resp. 3) where $\mathrm{SL}_2^F / Z(\mathrm{SL}_2^F) = \mathfrak{S}_3$ (resp. $\mathfrak{A}_4$) is solvable.

- ${}^2A_n (n \geq 2)$ — Special projective unitary group $\mathrm{PSU}_n \simeq \mathrm{PU}_n$ (the same remark on $\mathbf{G}^F$ applies as in the split case). Further for $q = 2$ and $n \in \{2, 3\}$ or $q = 3$ and $n = 2$ we get a non-simple group.

- $C_n (n \geq 2)$ — We get the projective symplectic group $\mathrm{Sp}_{2n}(\mathbb{F}_q)/Z(\mathrm{Sp}(\mathbb{F}_q))$ which is simple excepted $\mathrm{PSp}_4(\mathbb{F}_2) \simeq \mathfrak{S}_6$.

- $B_n (n \geq 2)$ — We get the orthogonal group SO_{2n+1} (B_2 and C_2 give isomorphic groups, non-simple for $q = 2$, see above).

- D_n (resp. ${}^2D_n$)$(n \geq 4)$ — Projective orthogonal group PSO_{2n}^+ (resp. PSO_{2n}^-).

- G_2 (for $q = 2$ the group $\mathbf{G}^F$ is not simple; its derived subgroup, of index 2, is).

- ${}^3D_4$ — The triality group.

- F_4, E_6, ${}^2E_6$, E_7, E_8.

There are in addition "exceptional" isogenies which correspond to automorphisms of the root system up to a scalar. In each case we have an automorphism of the Coxeter system. Such automorphisms which did not appear in the above list are ${}^2B_2$, ${}^2F_4$ (resp. ${}^2G_2$). To make them automorphisms of the root system we have to scale by $\sqrt{2}$ (resp. $\sqrt{3}$). With p the square of the scaling factor, we get a p-morphism defining an isogeny whose square is a Frobenius on a field of characteristic p. The corresponding groups $\mathbf{G}^F$ are the Suzuki and Ree groups, which are simple excepted ${}^2B_2$ for $q = 2$ (which is solvable), ${}^2G_2$ for $q = 3$ (whose derived subgroup is simple, isomorphic to $\mathrm{SL}_2(\mathbb{F}_8)$), and ${}^2F_4$ for $q = 2$ whose derived subgroup, of index 2, is simple. Adding to the above list the alternating groups, we have all the non-sporadic finite simple groups.

8 Parabolic subgroups and Levi subgroups

8.1 Levi decompositions

Let $\mathbf{G}$ be reductive group. Recall that a parabolic subgroup of $\mathbf{G}$ is a subgroup of $\mathbf{G}$ containing a Borel subgroup. Fix a pair $\mathbf{T} \subset \mathbf{B}$ consisting of a maximal torus and a Borel subgroup of $\mathbf{G}$. Then a subgroup containing $\mathbf{B}$ is called a *standard* parabolic subgroup. For such a group, there exists a subset I of simple reflections such that $\mathbf{B}$ and W_I generate the group. More precisely, the Bruhat decomposition yields

$$\mathbf{P}_I = \bigsqcup_{w \in W_I} \mathbf{B} w \mathbf{B}.$$

To the set I of simple reflections corresponds a set of simple roots $\Pi_I \subset \Pi$. We will denote by $\Phi_I \subset \Phi$ the set of roots which are linear combination of these simple roots. Then from Section 2.2 one deduces that Φ_I is a root system with basis Π_I and Weyl group W_I.

We will assume without proof that standard parabolic subgroups have a *Levi decomposition* (see Definition 8.6).

Proposition 8.1. *Let* $\mathbf{L}_I = \langle \mathbf{T}, \mathbf{U}_\alpha \,|\, \alpha \in \Phi_I \rangle$ *and* $\mathbf{U}_I = \prod_{\alpha \in \Phi^+ \setminus \Phi_I} \mathbf{U}_\alpha$. *Then* $\mathbf{L}_I$ *is a reductive group,* $\mathbf{U}_I = \mathrm{R_u}(\mathbf{P}_I)$ *is the unipotent radical of* $\mathbf{P}_I$ *and*

$$\mathbf{P}_I = \mathbf{L}_I \ltimes \mathrm{R_u}(\mathbf{P}_I) = \mathbf{L}_I \ltimes \mathbf{U}_I.$$

Remark 8.2. The fact that both $\mathbf{U}_I$ and $\mathbf{L}_I$ are groups, and that $\mathbf{L}_I$ normalizes $\mathbf{U}_I$ is a direct consequence from Chevalley's commutator formula (see 3.1.(vii)).

Remark 8.3. As a consequence, if $\mathbf{U}_\alpha \subset \mathbf{P}_I$ then $\alpha \in \Phi_I \cup \Phi^+$. Indeed, the image of $\mathbf{U}_\alpha$ by the quotient map $\mathbf{P}_I \twoheadrightarrow \mathbf{L}_I$ is either trivial (in which case $\alpha \in \Phi^+ \setminus \Phi_I$) or non-trivial (in which case $\alpha \in \Phi_I$).

Example 8.4. (a) For the Borel subgroup of $\mathbf{GL}_n$ consisting of upper-triangular matrices, the standard parabolic subgroups of $\mathbf{GL}_n$ are upper block triangular matrices. More precisely, given a composition $n_1 + n_2 + \cdots + n_r = n$ of n, the standard parabolic subgroup corresponding to $I = \{1, \ldots, n_1 - 1\} \sqcup \{n_1 + 1, \ldots, n_1 + n_2 - 1\} \sqcup \cdots \sqcup \{n_1 + \cdots + n_{r-1} + 1, \ldots, n-1\}$ is

$$\mathbf{P}_I = \begin{pmatrix} \mathbf{GL}_{n_1} & * & * \\ & \mathbf{GL}_{n_2} & * \\ (0) & & \ddots \end{pmatrix}$$

and its standard Levi complement is $\mathbf{L}_I \simeq \mathbf{GL}_{n_1} \times \mathbf{GL}_{n_2} \times \cdots \times \mathbf{GL}_{n_r}$.

(b) For the Borel subgroup of $\mathbf{Sp}_4$ consisting of upper-triangular matrices in $\mathbf{Sp}_4$, the standard parabolic subgroups are $\mathbf{B}$, $\mathbf{Sp}_4$ and two parabolic subgroups corresponding respectively to the short simple root and the long simple root. Their standard Levi complement is $\mathbf{L}_{\mathrm{short}} \simeq \mathbf{GL}_2$ and $\mathbf{L}_{\mathrm{long}} \simeq \mathbf{SL}_2 \times \mathbf{G}_m$

Proposition 8.5. *Let I be a subset of S. Then* $N_{\mathbf{G}}(\mathbf{L}_I)/\mathbf{L}_I = N_{\mathbf{G}}(\mathbf{L}_I)/N_{\mathbf{G}}(\mathbf{L}_I)^\circ \simeq N_W(W_I)/W_I$.

Proof. Recall that $\mathbf{L}_I$ is generated by $\mathbf{T}$ and the one-parameter subgroups $\mathbf{U}_\alpha$ for $\alpha \in \Phi_I$. Therefore if $w \in W$, then ${}^w\mathbf{L}_I = \mathbf{L}_I$ if and only if ${}^w\Phi_I = \Phi_I$. We claim that this is equivalent to $w \in N_W(W_I)$; indeed, $\alpha \in \Phi_I$ if and only if $w_\alpha \in W_I$, therefore normalizing Φ_I amounts to normalizing the set of reflections in W_I. In particular, there is a well-defined map from $N_W(W_I)$ to $N_{\mathbf{G}}(\mathbf{L}_I)/\mathbf{L}_I$, and from the Bruhat decomposition of $\mathbf{P}_I$ we see that its kernel is exactly W_I. To prove that this map is surjective, let $g \in N_{\mathbf{G}}(\mathbf{L}_I)$. The maximal tori $\mathbf{T}$ and ${}^g\mathbf{T}$ are contained in $\mathbf{L}_I$, and as such they are conjugate under an element of $\mathbf{L}_I$. This means that there exists $l \in \mathbf{L}_I$ such that $gl \in N_{\mathbf{G}}(\mathbf{L}_I) \cap N_{\mathbf{G}}(\mathbf{T})$. From the previous argument we deduce that the image of gl in W normalizes W_I, which proves that $N_W(W_I)/W_I \simeq N_{\mathbf{G}}(\mathbf{L}_I)/\mathbf{L}_I$.

Since $\mathbf{L}_I$ is connected, then $\mathbf{L}_I = N_{\mathbf{G}}(\mathbf{L}_I)^\circ$ then follows from the fact that $N_{\mathbf{G}}(\mathbf{L}_I)/\mathbf{L}_I$ is finite. □

Definition 8.6. *A* Levi decomposition *of a parabolic subgroup* $\mathbf{P}$ *of* $\mathbf{G}$ *is a decomposition* $\mathbf{P} = \mathbf{L} \ltimes \mathrm{R_u}(\mathbf{P})$*. The group* $\mathbf{L} \simeq \mathbf{P}/\mathrm{R_u}(\mathbf{P})$ *is a reductive group called a* Levi subgroup *of* $\mathbf{G}$ *and a* Levi complement *of* $\mathbf{P}$*.*

Proposition 8.7. *Let* $\mathbf{P}$ *be a parabolic subgroups of* $\mathbf{G}$ *containing* $\mathbf{T}$*.*

(i) There exists a unique Levi complement of $\mathbf{P}$ *containing* $\mathbf{T}$*.*

(ii) Two Levi complements of $\mathbf{P}$ *are conjugate under a unique element of* $\mathrm{R_u}(\mathbf{P})$*.*

Proof. One can choose a Borel subgroup $\mathbf{B}$ of $\mathbf{P}$ so that $\mathbf{P}$ is standard for the system of positive roots corresponding to $\mathbf{B}$. The existence follows from Proposition 8.1. In addition, if $\mathbf{L}$ is any Levi complement of $\mathbf{P}$ containing $\mathbf{T}$, then it is generated by $\mathbf{T}$ and the one-parameter subgroups $\mathbf{U}_\alpha$ that it contains. These are exactly the one-parameter subgroups whose image under the map $\mathbf{P} \twoheadrightarrow \mathbf{P}/\mathrm{R_u}(\mathbf{P})$ is non-trivial (see Remark 8.3), which proves (i).

The maximal tori of $\mathbf{P}$ are conjugate under $\mathbf{P}$. Therefore (i) shows that two Levi complements are conjugate under $\mathbf{P}$, hence under $\mathrm{R_u}(\mathbf{P})$. Furthermore, if $u \in \mathrm{R_u}(\mathbf{P})$ normalizes $\mathbf{L}$, then for any $l \in \mathbf{L}$, $[v, l] \in \mathrm{R_u}(\mathbf{P}) \cap \mathbf{L} = 1$ hence $v \in C_{\mathbf{G}}(\mathbf{L}) \subset C_{\mathbf{G}}(\mathbf{T}) = \mathbf{T}$ so it must be trivial. This shows that the action of $\mathrm{R_u}(\mathbf{P})$ on the set of Levi complements is regular (*i.e.* free and transitive). □

We will need another characterization of parabolic subgroups which will be useful for computing intersections of parabolic subgroups and their decomposition.

Proposition 8.8. *Let* $\mathbf{P}$ *be a subgroup of* $\mathbf{G}$ *containing* $\mathbf{T}$*. Let* Ψ *be the set of roots such that* $\mathbf{U}_\alpha \subset \mathbf{P}$*. Then*

(i) $\mathbf{P}$ *is a parabolic subgroup of* $\mathbf{G}$ *if and only if* $\Phi = \Psi \cup -\Psi$*.*

(ii) If $\mathbf{P}$ *is parabolic, then its unique Levi complement containing* $\mathbf{T}$ *is*

$$\mathbf{L} = \langle \mathbf{T}, \mathbf{U}_\alpha \,|\, \alpha \in \Psi \cap (-\Psi) \rangle.$$

Proof. It is clear that any parabolic subgroup satisfies (i) (see Remark 8.3). To prove the converse, let $w \in W$ be such that $|\Psi \cap {}^w\Phi^+|$ is maximal. If ${}^w\Phi^+ \not\subseteq \Psi$, then there exists a simple root α such that $w(\alpha) \notin \Psi$. Therefore

$-w(\alpha) = w(-\alpha) \in \Psi$ and $\Psi \cap {}^w\Phi^+$ is a proper subset of $\Psi \cap ({}^w\Phi^+ \cup \{w(-\alpha)\}) = \Psi \cap {}^{ws_\alpha}\Phi^+$ which contradicts the maximality of $|\Psi \cap {}^w\Phi^+|$. This proves that ${}^w\mathbf{B} \subset \mathbf{P}$, hence $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$. In particular, ${}^{w^{-1}}\mathbf{P}$ is a standard parabolic subgroup $\mathbf{P}_I$, and (ii) follows from the fact that in that case $\Psi \cap (-\Psi) = {}^w\Phi_I$ and $\mathbf{L} = {}^w\mathbf{L}_I$. □

Proposition 8.9. *Let $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$ and $\mathbf{Q} = \mathbf{M} \ltimes \mathbf{V}$ be two parabolic subgroups with Levi complements $\mathbf{L}$ and $\mathbf{M}$ containing $\mathbf{T}$. Then*

(i) $(\mathbf{P} \cap \mathbf{Q}) \cdot \mathbf{U}$ *is a parabolic subgroup of $\mathbf{G}$ with Levi complement $\mathbf{L} \cap \mathbf{M}$.*

(ii) $\mathbf{P} \cap \mathbf{Q}$ *is connected and it has the following factorization:*

$$\mathbf{P} \cap \mathbf{Q} = (\mathbf{L} \cap \mathbf{M}) \ltimes \big((\mathbf{L} \cap \mathbf{V}) \cdot (\mathbf{M} \cap \mathbf{U}) \cdot (\mathbf{U} \cap \mathbf{V})\big).$$

Furthermore, the expression of an element of $\mathbf{P} \cap \mathbf{Q}$ with respect to this factorization is unique.

Proof. Without loss of generality one can assume that $\mathbf{P} = \mathbf{P}_I$, $\mathbf{L} = \mathbf{L}_I$, $\mathbf{Q} = {}^w\mathbf{P}_J$ and $\mathbf{M} = \mathbf{L}_J$ with w an I-reduced element of W. Then $\mathbf{P}_I \cap {}^w\mathbf{P}_J \supset \mathbf{L}_I \cap {}^w\mathbf{B} = \mathbf{L}_I \cap \mathbf{B}$. Therefore $(\mathbf{P}_I \cap {}^w\mathbf{P}_J) \cdot \mathbf{U}_I$ contains $\mathbf{B}$, hence it is a parabolic subgroup. This forces $\mathbf{P}_I \cap {}^w\mathbf{P}_J$ to be connected. In addition, if $\mathbf{U}_\alpha \subset \mathbf{P}_I \cap {}^w\mathbf{P}_J$ and $\mathbf{U}_{-\alpha} \subset \mathbf{P}_I \cap {}^w\mathbf{P}_J$ then $\alpha \in \Phi_I$ and $w^{-1}(\alpha) \in \Phi_J$. Therefore by Proposition 8.8 the Levi complement of $(\mathbf{P}_I \cap {}^w\mathbf{P}_J) \cdot \mathbf{U}_I = (\mathbf{P} \cap \mathbf{Q}) \cdot \mathbf{U}$ containing $\mathbf{T}$ is $\mathbf{L}_I \cap {}^w\mathbf{L}_J = \mathbf{L} \cap \mathbf{M}$.

Now $\mathbf{U} \cap \mathbf{Q}$ is a unipotent subgroup of $\mathbf{P} \cap \mathbf{Q}$ normalized by $\mathbf{T}$, therefore it is the product of the $\mathbf{U}_\alpha$'s that it contains. In particular, one can write $\mathbf{U} \cap \mathbf{Q} = (\mathbf{U} \cap \mathbf{M}) \cdot (\mathbf{U} \cap \mathbf{Q})$. Moreover, it is normalized by $\mathbf{L} \cap \mathbf{V}$. Since $\mathbf{P} \cap \mathbf{Q}$ is generated by $\mathbf{T}$ and the one-parameter subgroups that it contains, it is generated by $\mathbf{L} \cap \mathbf{M}$ and the unipotent group $\mathbf{H} = (\mathbf{L} \cap \mathbf{V}) \ltimes \mathbf{U} \cap \mathbf{Q}$. This corresponds indeed to the decomposition

$$(\Phi^+ \cup \Phi_I) \cap {}^w(\Phi^+ \cup \Phi_J) = (\Phi_I \cap {}^w\Phi_J) \bigsqcup \big((\Phi_I \cap {}^w(\Phi^+ \setminus \Phi_J)) \sqcup ((\Phi^+ \setminus \Phi_I) \cap {}^w(\Phi^+ \cup \Phi_J))\big).$$

Finally, since $\mathbf{L} \cap \mathbf{M}$ normalizes $\mathbf{H}$, we deduce the factorization of $\mathbf{P} \cap \mathbf{Q}$ given in (ii) and the uniqueness property for decompositions of elements in $\mathbf{P} \cap \mathbf{Q}$. □

Proposition 8.10. *(i) Let $\mathbf{P}$ and $\mathbf{Q}$ be two parabolic subgroups of $\mathbf{G}$ with $\mathbf{Q} \subset \mathbf{P}$. Then $\mathrm{R_u}(\mathbf{Q}) \supset \mathrm{R_u}(\mathbf{P})$ and given any Levi complement $\mathbf{M}$ of $\mathbf{Q}$, there is a unique Levi complement $\mathbf{L}$ of $\mathbf{P}$ such that $\mathbf{M} \subset \mathbf{L}$.*

(ii) Given a Levi complement $\mathbf{L}$ of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, the following are equivalent:

(a) $\mathbf{M}$ *is a Levi complement of a parabolic subgroup of $\mathbf{L}$;*

(b) $\mathbf{M}$ *is a Levi complement of a parabolic subgroup of $\mathbf{G}$, and $\mathbf{M} \subset \mathbf{L}$.*

Proof. For (i), let $\mathbf{T}$ be a maximal torus of $\mathbf{M}$ and let $\mathbf{L}$ be the unique Levi complement of $\mathbf{P}$ containing $\mathbf{T}$ (see Proposition 8.7). Then by Proposition 8.9, $\mathbf{L}\cap\mathbf{M}$ is a Levi complement of $(\mathbf{P}\cap\mathbf{Q})\cdot \mathrm{R_u}(\mathbf{P}) = \mathbf{P}$, therefore it must be equal to $\mathbf{M}$.

Let $\mathbf{P_L}$ be a parabolic subgroup of $\mathbf{L}$, and let $\mathbf{T}$ be a maximal torus of $\mathbf{P}_L$. Then using Proposition 8.8 one checks easily that $\mathbf{P_L} \ltimes \mathrm{R_u}(\mathbf{P})$ is a parabolic subgroup of $\mathbf{G}$, and that the Levi complements of $\mathbf{P}$ and $\mathbf{P}_L$ are equal, which proves (a) $\Rightarrow$ (b). Conversely, let $\mathbf{M} \subset \mathbf{L}$ be a Levi complement of a parabolic subgroup $\mathbf{Q}$ of $\mathbf{G}$. By Proposition 8.8, $\mathbf{L}\cap\mathbf{Q}$ is a parabolic subgroup of $\mathbf{L}$ whose Levi complement is $\mathbf{L}\cap\mathbf{M} = \mathbf{M}$. □

Remark 8.11. It follows from assertion (ii) that we can refer to $\mathbf{M}$ as a *Levi subgroup* with no mention of the underlying reductive group ($\mathbf{G}$ or $\mathbf{M}$).

8.2 Rational Levi subgroups

Maximal tori are particular Levi subgroups, and we have seen in 7.10 that the G-conjugacy classes of F-stable maximal tori are parameterized by F-conjugacy classes of W. We will state below a similar statement for Levi subgroups.

¿From now on we will assume that both $\mathbf{T}$ and $\mathbf{B}$ are F-stable, so that for standard Levi subgroups we have $F(\mathbf{L}_I) = \mathbf{L}_{F(I)}$. In particular $\mathbf{L}_I$ is F-stable if and only if $F(I) = I$. Now, if $\mathbf{L}$ is any F-stable Levi subgroup of $\mathbf{G}$, then it contains a maximal F-stable torus and there exists $g \in \mathbf{G}$ such that $(\mathbf{L}, \mathbf{T}') = ({}^g\mathbf{L}_I, {}^g\mathbf{T})$ for some $I \subset S$ which might not be F-stable. Consequently, if $w = g^{-1}F(g) \in N_G(\mathbf{T})$ then F acts on the pair $(\mathbf{L}, \mathbf{T}')$ as wF acts on the pair $(\mathbf{L}_I, \mathbf{T})$. In particular, ${}^wF(I) = I$, therefore wF normalizes W_I. Up to multiplication by W_I on the left (or $W_{F(I)}$ on the right), the class of the element wF in WF is uniquely determined and (I, wF) is called the *type* of $\mathbf{L}$. More precisely, 7.7 yields:

Proposition 8.12. *Let $w \in W$ and I be a wF-stable subset of S. Then there is a bijection between*

(a) $\mathbf{G}^F$*-conjugacy classes of F-stable Levi subgroups which are (geometrically) conjugate to $\mathbf{L}_I$ and*

(b) wF-conjugacy classes of $N_W(W_I)/W_I$.

Remark 8.13. It is important to note that an F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ might not be contained in any F-stable parabolic subgroup, for the same reason that not every F-stable maximal torus is in an F-stable Borel subgroup. For example ${}^wF(\mathbf{P}_I) \neq \mathbf{P}_I$ in general. This is a major obstruction to constructing every representation using parabolic induction (see the next two sections). We will say that an F-stable Levi subgroup is $\mathbf{G}$*-split* if it is a Levi subgroup of an F-stable parabolic subgroup of $\mathbf{G}$.

Example 8.14. We have seen in Example 8.4 that the standard Levi subgroups are of the form $\mathbf{L}_I \simeq \mathbf{GL}_{n_1} \times \cdots \times \mathbf{GL}_{n_r}$. We give here some examples of the finite Levi that can occur:

(a) With $w = 1$, we obtain the $\mathbf{G}$-split Levi subgroups

$$\mathbf{L}_I^F \simeq \mathrm{GL}_{n_1}(q) \times \cdots \times \mathrm{GL}_{n_r}(q).$$

(b) Assume $n = dr$ and take $(n_i) = (m, \ldots, m)$. Then $w = (1, m+1, 2m+1, \ldots, (d-1)m+1) \cdots (m, d+m, \ldots, dm)$ permutes cyclically the summands $\mathbf{GL}_m$ in $\mathbf{L}_I = (\mathbf{GL}_m)^d$ and we get

$$\mathbf{L}_I^{wF} \simeq \mathrm{GL}_m(q^d).$$

The case $m = 1$ corresponds to the *Coxeter torus* $\mathbf{T}^{wF} \simeq \mathrm{GL}_1(q^n) = \mathbb{F}_{q^n}^\times$.

9 Parabolic induction and restriction

Let Λ be a commutative ring with unit. Given a finite group H, we denote by ΛH-mod the category of finite dimensional ΛH-modules. By a *representation of H over* Λ we mean an object of the category ΛH-mod.

9.1 Invariants and coinvariants

Recall that given a representation M of H over Λ, and a subgroup K we can form the following Λ-modules:

- The *invariants* M^K of M under K:

$$M^K = \mathrm{Hom}_{\Lambda K}(\Lambda, M) \simeq \mathrm{Hom}_{\Lambda H}(\Lambda H/K, M).$$

  More concretely: $M^K \simeq \{m \in M \,|\, \forall\, k \in K,\ k \cdot m = m\}$, which is the largest ΛK-submodule of M on which K acts trivially.

- The *coinvariants* M_K of M under K:

$$M_K = \Lambda \otimes_{\Lambda K} M \simeq \Lambda H/K \otimes_{\Lambda H} M.$$

  More concretely: $M_K \simeq M/\langle m - k \cdot m \,|\, k \in K, m \in M\rangle$, which is the largest quotient of M (as a ΛK-module) on which K acts trivially.

The tensor-hom adjunction shows that "duality" $M^\vee = \mathrm{Hom}_\Lambda(M, \Lambda)$ exchanges the two notions. More precisely, $(M^\vee)^K \simeq (M_K)^\vee$. In addition, if L is another subgroup of H with $K \trianglelefteq L$, then both M^K and M_K have a structure of L/K-module, compatible with the previous isomorphism. In general, invariants and coinvariants do not coincide. However, if the order of K is invertible in Λ, then $\Lambda H/K \simeq e_K \Lambda H$ with $e_K = |K|^{-1} \sum_{k \in K} k$ so that $M^K \simeq M_K \simeq e_K M$. In that case, invariants and coinvariants are exact functors.

Given a right ΛH-module N, we define $N \otimes_{\Lambda K} M := (N \otimes_\Lambda M)_K$ where the action of K on $N \otimes_\Lambda M$ is diagonal, given by $k \cdot (m \otimes n) = (m \cdot k^{-1} \otimes k \cdot n)$.

Example 9.1. (a) Let X be a finite set with a left action of H, and let ΛX be the corresponding permutation module. Let $\pi : X \longrightarrow X/H$ be the canonical quotient map. It induces linear maps $\pi_* : \Lambda X \longrightarrow \Lambda X/H$ and $\pi^* : \Lambda X/H \longrightarrow \Lambda X$ given on the basis by $\pi_*(x) = xH$ and $\pi^*(yH) = \sum_{x \in yH} x$. As an exercise, one checks that they induce isomorphisms $(\Lambda X)_K \simeq \Lambda X/H$ and $\Lambda X/H \simeq (\Lambda X)_K$. In particular, invariants and coinvariants of the permutation module ΛX are isomorphic. However, the composition $\pi_* \circ \pi^* = |H| \operatorname{Id}_{\Lambda X/H}$ is not invertible in general.

(b) If Y is another set, now with a right action of H, then we can form the amalgamated product $Y \times_H X$ as the quotient of $Y \times X$ by the diagonal action of H. Then from (a) we obtain

$$\Lambda[Y \times_K X] \simeq \Lambda Y \otimes_{\Lambda K} \Lambda X.$$

9.2 Parabolic induction and restriction

Recall that $\mathbf{G}$ is a connected reductive group over $\overline{\mathbb{F}}_p$. In these notes we will focus on representations in *non-defining* characteristic, which means that we will study representations over fields of characteristic different from p. Following this assumption, we will assume from now on that

$$\boxed{p \text{ is invertible in } \Lambda}$$

Under this assumption one can define a good notion of induction and restriction for finite reductive groups.

An efficient method for constructing representations of a finite group is to induce representations from smaller subgroups. Here, since we are interested in finite reductive groups, we will consider induction from particular reductive subgroups which will correspond to split Levi subgroups. Furthermore, the usual induction from Levi subgroups is in a sense "too big" and hard to decompose into indecomposable summands (their number depend on q). To solve this problem, we will proceed in two steps, by first inflating the representation from the Levi subgroup to a parabolic subgroup, and then inducing. It turns out that this induction process, call Harish-Chandra induction or parabolic induction has particularly nice properties.

Definition 9.2. *Let $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$ be a F-stable parabolic subgroup of $\mathbf{G}$ with F-stable Levi complement $\mathbf{L}$. The* Harish-Chandra *or* parabolic *induction and restriction functors are*

$$\begin{aligned} R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : \Lambda\mathbf{L}^F\text{-}\mathsf{mod} &\longrightarrow \Lambda\mathbf{G}^F\text{-}\mathsf{mod} \\ M &\longmapsto \Lambda\mathbf{G}^F/\mathbf{U}^F \otimes_{\Lambda\mathbf{L}^F} M \\ {}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : \Lambda\mathbf{G}^F\text{-}\mathsf{mod} &\longrightarrow \Lambda\mathbf{L}^F\text{-}\mathsf{mod} \\ N &\longmapsto \operatorname{Hom}_{\Lambda\mathbf{G}^F}(\Lambda\mathbf{G}^F/\mathbf{U}^F, N) \end{aligned}$$

Remark 9.3. A more concrete description is

$$ {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(N) \simeq N^{\mathbf{U}^F} \quad \text{and} \quad R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(M) \simeq \mathrm{Ind}_{\mathbf{P}^F}^{\mathbf{G}^F} \circ \mathrm{Inf}_{\mathbf{L}^F}^{\mathbf{P}^F}(M). $$

with $\mathbf{L}^F$ acting on $N^{\mathbf{U}^F}$ via the isomorphism $\mathbf{P}^F/\mathbf{U}^F \simeq \mathbf{L}^F$. To see the second equality, we write $\Lambda\mathbf{G}^F/\mathbf{U}^F \otimes_{\Lambda\mathbf{L}^F} M \simeq \Lambda\mathbf{G}^F \otimes_{\Lambda\mathbf{P}^F} (\Lambda\mathbf{P}^F/\mathbf{U}^F \otimes_{\Lambda\mathbf{L}^F} M)$, which shows that $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ is isomorphic to the composition of the inflation functor $\mathrm{Inf}_{\mathbf{L}^F}^{\mathbf{P}^F} = \Lambda\mathbf{P}^F/\mathbf{U}^F \otimes_{\Lambda\mathbf{L}^F} -$ and the induction functor $\mathrm{Ind}_{\mathbf{P}^F}^{\mathbf{G}^F} = \Lambda\mathbf{G}^F \otimes_{\Lambda\mathbf{P}^F} -$.

Proposition 9.4. *Let $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$ be a F-stable parabolic subgroup of $\mathbf{G}$ with F-stable Levi complement $\mathbf{L}$. Then*

(i) ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \simeq \Lambda\mathbf{U}^F\backslash\mathbf{G}^F \otimes_{\Lambda\mathbf{G}^F} -$ *and* $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \simeq \mathrm{Hom}_{\Lambda\mathbf{L}^F}(\Lambda\mathbf{U}^F\backslash\mathbf{G}^F, -)$.

(ii) $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ *and* ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ *are exact functors.*

(iii) $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ *and* ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ *are biadjoint. In particular,*

$$ \begin{aligned} &\mathrm{Hom}_{\Lambda\mathbf{G}^F}(R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(M), N) \simeq \mathrm{Hom}_{\Lambda\mathbf{L}^F}(M, {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(N)) \\ \text{and} \qquad &\mathrm{Hom}_{\Lambda\mathbf{G}^F}(N, R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(M)) \simeq \mathrm{Hom}_{\Lambda\mathbf{L}^F}({}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(N), M) \end{aligned} $$

(iv) $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ *and* ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ *preserve injectivity and projectivity.*

(v) If Λ is a principal domain, then $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ *and* ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ *map Λ-free modules to Λ-free modules.*

Proof. Let $e_{\mathbf{U}^F} = |\mathbf{U}^F|^{-1}\sum_{u\in\mathbf{U}^F} u$. Since $\mathbf{U}^F$ is a p-group, then the order of $\mathbf{U}^F$ is invertible in Λ, therefore $e_{\mathbf{U}^F}$ is a well-defined element of $\Lambda\mathbf{U}^F \subset \Lambda\mathbf{G}^F$. From Example 9.1 we have

$$ \Lambda\mathbf{G}^F/\mathbf{U}^F \simeq \Lambda\mathbf{G}^F e_{\mathbf{U}^F} \quad \text{and} \quad \Lambda\mathbf{U}^F\backslash\mathbf{G}^F \simeq e_{\mathbf{U}^F}\Lambda\mathbf{G}^F. $$

Therefore we can write $\Lambda\mathbf{G}^F = \Lambda\mathbf{G}^F e_{\mathbf{U}^F} \oplus \Lambda\mathbf{G}^F(1-e_{\mathbf{U}^F}) \simeq \Lambda\mathbf{G}^F/\mathbf{U}^F \oplus \Lambda\mathbf{G}^F(1-e_{\mathbf{U}^F})$, from which we deduce that $\Lambda\mathbf{G}^F/\mathbf{U}^F$ is a projective left $\Lambda\mathbf{G}^F$-module and a projective (hence flat) right $\Lambda\mathbf{L}^F$-module. This proves (ii), and from it we deduce (v). In addition, with the property of the group algebra $\Lambda\mathbf{G}^F$ to be symmetric, we have

$$ (\Lambda\mathbf{G}^F/\mathbf{U}^F)^\vee \simeq (\Lambda\mathbf{G}^F e_{\mathbf{U}^F})^\vee \simeq e_{\mathbf{U}^F}(\Lambda\mathbf{G}^F)^\vee \simeq e_{\mathbf{U}^F}\Lambda\mathbf{G}^F \simeq \Lambda\mathbf{U}^F\backslash\mathbf{G}^F $$

and we get (i) by using these explicit descriptions and tensor-hom adjunction.

Property (iii) comes from the usual tensor-hom adjunction. Together with (ii), it proves (iv). Indeed, if M is projective then $\mathrm{Hom}_{\Lambda\mathbf{G}^F}(R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(M), -)$ is exact as the composition of the exact functors ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ and $\mathrm{Hom}_{\Lambda\mathbf{L}^F}(M, -)$. $\square$

Proposition 9.5. *Let $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$ and $\mathbf{Q} = \mathbf{M} \ltimes \mathbf{V}$ be two F-stable parabolic subgroups of $\mathbf{G}$ with F-stable Levi complements $\mathbf{L}$ and $\mathbf{M}$. Assume that $\mathbf{Q} \subset \mathbf{P}$ and $\mathbf{M} \subset \mathbf{L}$. Then*

$$ R^{\mathbf{G}}_{\mathbf{M}\subset\mathbf{Q}} \simeq R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \circ R^{\mathbf{L}}_{\mathbf{M}\subset\mathbf{L}\cap\mathbf{Q}} \quad \textit{and} \quad {}^*R^{\mathbf{G}}_{\mathbf{M}\subset\mathbf{Q}} \simeq {}^*R^{\mathbf{L}}_{\mathbf{M}\subset\mathbf{L}\cap\mathbf{Q}} \circ {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}. $$

Proof. Recall from Propositions 8.9 and 8.10 that $\mathbf{L}\cap\mathbf{Q}$ is a parabolic subgroup of $\mathbf{L}$ with Levi decomposition $\mathbf{L}\cap\mathbf{Q} = \mathbf{M} \ltimes \mathbf{L}\cap\mathbf{V}$. Given a $\Lambda\mathbf{M}^F$-module M, The composition $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} \circ R_{\mathbf{M}\subset\mathbf{L}\cap\mathbf{Q}}^{\mathbf{L}}(M)$ is given by

$$\Lambda\mathbf{G}^F/\mathbf{U}^F \otimes_{\Lambda\mathbf{L}^F} \Lambda\mathbf{L}^F/\mathbf{L}^F\cap\mathbf{V}^F \otimes_{\mathbf{M}^F} M.$$

Now, we have seen in Example 9.1 that

$$\Lambda\mathbf{G}^F/\mathbf{U}^F \otimes_{\Lambda\mathbf{L}^F} \Lambda\mathbf{L}^F/\mathbf{L}^F\cap\mathbf{V}^F \simeq \Lambda[\mathbf{G}^F/\mathbf{U}^F \times_{\mathbf{L}^F} \mathbf{L}^F/\mathbf{L}^F\cap\mathbf{V}^F]$$

so it amounts to produce a bijection between the sets $\mathbf{G}^F/\mathbf{U}^F \times_{\mathbf{L}^F} \mathbf{L}^F/\mathbf{L}^F\cap\mathbf{V}^F$ and $\mathbf{G}^F/\mathbf{V}^F$ which is equivariant for the left action of $\mathbf{G}^F$ and the right action of $\mathbf{M}^F$. In addition, since the stabilizers of the various actions are connected, it is enough to prove that the map

$$\phi : (g\mathbf{U}, l\mathbf{L}\cap\mathbf{V}) \in \mathbf{G}/\mathbf{U} \times_{\mathbf{L}} \mathbf{L}/\mathbf{L}\cap\mathbf{V} \longmapsto gl\mathbf{V} \in \mathbf{G}/\mathbf{V}.$$

is a bijective morphism of algebraic varieties (see 7.7). It is well-defined since $\mathbf{L}$ normalizes $\mathbf{U}$ and $\mathbf{U}\subset\mathbf{V}$. It is also clearly $\mathbf{G}\times\mathbf{M}^{\mathrm{op}}$-equivariant and surjective. Assume that $gl\mathbf{V} = g'l'\mathbf{V}$. Then $l^{-1}g^{-1}g'l' \in \mathbf{V} = \mathbf{U} \rtimes (\mathbf{L}\cap\mathbf{V})$. Up to multiplying g on the right by an element of $\mathbf{U}$ and l' by an element of $\mathbf{L}\cap\mathbf{V}$, we can assume that $l^{-1}g^{-1}g'l' = 1$, that is $gl = g'l'$. This proves that ϕ induces a bijection between $\mathbf{G}/\mathbf{U} \times_{\mathbf{L}} \mathbf{L}/\mathbf{L}\cap\mathbf{V}$ and $\mathbf{G}/\mathbf{V}$. □

9.3 Mackey formula

We shall now prove a fundamental property for parabolic induction and restriction. It is the analogue of the classical Mackey formula for usual induction and restriction.

Theorem 9.6. *Let $\mathbf{P} = \mathbf{L}\ltimes\mathbf{U}$ and $\mathbf{Q} = \mathbf{M}\ltimes\mathbf{V}$ be two F-stable parabolic subgroups of $\mathbf{G}$ with F-stable Levi complements $\mathbf{L}$ and $\mathbf{M}$. Then*

$${}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} \circ R_{\mathbf{M}\subset\mathbf{Q}}^{\mathbf{G}} \simeq \sum_{x\in\mathbf{L}^F\backslash\mathcal{S}(\mathbf{L},\mathbf{M})^F/\mathbf{M}^F} R_{\mathbf{L}\cap{}^x\mathbf{M}\subset\mathbf{L}\cap{}^x\mathbf{Q}}^{\mathbf{L}} \circ {}^*R_{\mathbf{L}\cap{}^x\mathbf{M}\subset\mathbf{P}\cap{}^x\mathbf{M}}^{{}^x\mathbf{M}} \circ \mathrm{ad}\, x$$

where $\mathrm{ad}\, x : \Lambda M\text{-}\mathsf{mod} \longrightarrow \Lambda^x M\text{-}\mathsf{mod}$ *denotes the action of x by conjugation on the representations and* $\mathcal{S}(\mathbf{L},\mathbf{M}) = \{x\in\mathbf{G} \mid \mathbf{L}\cap{}^x\mathbf{M}$ *contains a maximal torus of* $\mathbf{G}\}$.

Proof. The proof of the Mackey formula will be in several steps. We first use Proposition 9.4.(i) to get

$$\begin{aligned} {}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} \circ R_{\mathbf{M}\subset\mathbf{Q}}^{\mathbf{G}} &\simeq \Lambda\mathbf{U}^F\backslash\mathbf{G}^F \otimes_{\mathbf{G}^F} \Lambda\mathbf{G}^F/\mathbf{V}^F \otimes_{\Lambda\mathbf{M}^F} - \\ &\simeq \Lambda[\mathbf{U}^F\backslash\mathbf{G}^F \times_{\mathbf{G}^F} \mathbf{G}^F/\mathbf{V}^F] \otimes_{\Lambda\mathbf{M}^F} - \\ &\simeq \Lambda[\mathbf{U}^F\backslash\mathbf{G}^F/\mathbf{V}^F] \otimes_{\Lambda\mathbf{M}^F} -. \end{aligned}$$

¿From Lemma 9.11 we deduce the following decomposition for the set $\mathbf{U}^F\backslash\mathbf{G}^F/\mathbf{V}^F$:

$$\mathbf{U}^F\backslash\mathbf{G}^F/\mathbf{V}^F = \bigsqcup_{x\in\mathbf{L}^F\backslash\mathcal{S}(\mathbf{L},\mathbf{M})^F/\mathbf{M}^F} \mathbf{U}^F\backslash\mathbf{P}^F x\,\mathbf{Q}^F/\mathbf{V}^F. \tag{9.7}$$

Then each piece of this decomposition can be expressed in terms of smaller subgroups using Lemma 9.10. Indeed, given $x \in \mathcal{S}(\mathbf{L},\mathbf{M})^F/\mathbf{M}^F$ we have an $\mathbf{L}^F \times (\mathbf{M}^F)^{\mathrm{op}}$-equivariant bijection

$$\mathbf{L}^F/(\mathbf{L} \cap {}^x\mathbf{V})^F \times_{(\mathbf{L}\cap{}^x\mathbf{M})^F} ({}^x\mathbf{M} \cap \mathbf{U})^F \backslash {}^x\mathbf{M}^F \overset{\sim}{\longrightarrow} \mathbf{U}^F\backslash \mathbf{P}^F x\, \mathbf{Q}^F/\mathbf{V}^F. \quad (9.8)$$

where on the left-hand side, $\mathbf{L}^F$ acts by multiplication on the left and $m \in \mathbf{M}^F$ by multiplication by ${}^xm$ on the right. Combining Equations (9.7) and (9.8) we get

$$\Lambda[\mathbf{U}^F\backslash\mathbf{G}^F/\mathbf{V}^F] \simeq \bigoplus_{x\in \mathbf{L}^F\backslash \mathcal{S}(\mathbf{L},\mathbf{M})^F/\mathbf{M}^F} \Lambda[\mathbf{L}^F/(\mathbf{L}\cap{}^x\mathbf{V})^F] \otimes_{\Lambda(\mathbf{L}\cap{}^x\mathbf{M})^F} \Lambda[({}^x\mathbf{M}\cap\mathbf{U})^F\backslash{}^x\mathbf{M}^F].$$

Finally, we use Proposition 9.4.(i) to see that the functor

$$\Lambda[\mathbf{L}^F/(\mathbf{L} \cap {}^x\mathbf{V})^F] \otimes_{\Lambda(\mathbf{L}\cap{}^x\mathbf{M})^F} \Lambda[({}^x\mathbf{M} \cap \mathbf{U})^F\backslash {}^x\mathbf{M}^F] \otimes_{\Lambda\mathbf{M}^F} -$$

is isomorphic to

$$R^{\mathbf{L}}_{\mathbf{L}\cap{}^x\mathbf{M}\subset\mathbf{L}\cap{}^x\mathbf{Q}} \circ {}^*R^{{}^x\mathbf{M}}_{\mathbf{L}\cap{}^x\mathbf{M}\subset\mathbf{P}\cap{}^x\mathbf{M}} \circ \mathrm{ad}\, x$$

for all $x \in \mathcal{S}(\mathbf{L},\mathbf{M})^F/\mathbf{M}^F$, which yields the Mackey formula. $\square$

We now prove the results which we used for the proof of the Mackey formula. The first one is a generalization to parabolic subgroups of the Bruhat decomposition $\mathbf{B}\backslash\mathbf{G}/\mathbf{B} \simeq W$.

Lemma 9.9. *Let $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$ and $\mathbf{Q} = \mathbf{M} \ltimes \mathbf{V}$ be two parabolic subgroups with Levi complements $\mathbf{L}$ and $\mathbf{M}$ containing $\mathbf{T}$. Then $\mathbf{P}\backslash\mathbf{G}/\mathbf{Q} \simeq W_{\mathbf{L}}\backslash W/W_{\mathbf{M}}$.*

Proof. As shown in the proof of Proposition 8.8, there exist $x, y \in N_{\mathbf{G}}(\mathbf{T})$ such that $\mathbf{P} = {}^x\mathbf{P}_I$ and $\mathbf{Q} = {}^y\mathbf{P}_J$. Then the map $g \longmapsto x^{-1}gy$ induces bijections $\mathbf{P}\backslash\mathbf{G}/\mathbf{Q} \overset{\sim}{\longrightarrow} \mathbf{P}_I\backslash\mathbf{G}/\mathbf{P}_J$ and $W_{\mathbf{L}}\backslash W/W_{\mathbf{M}} \overset{\sim}{\longrightarrow} W_I\backslash W/W_J$, therefore one can assume that all the parabolic groups and Levi complements are standard.

By the usual Bruhat decomposition, the inclusion of $N_{\mathbf{G}}(\mathbf{T})$ in $\mathbf{G}$ induces a surjective map $W \twoheadrightarrow \mathbf{P}_I\backslash\mathbf{G}/\mathbf{P}_J$. We claim that it gives the expected bijection. This amounts to showing that $\mathbf{P}_I w\mathbf{P}_J = \mathbf{B}W_I wW_J\mathbf{B}$. Taking for w a I-reduced representative, we have $\mathbf{P}_I w\mathbf{P}_J = \mathbf{B}W_I w\mathbf{P}_J$ by 2.5. Now, for each $vw \in W_I w$, one can choose the corresponding reduced-J element $z_x \in vwW_J \subset W_I wW_J$ and $\mathbf{B}vw\mathbf{P}_J = \mathbf{B}z_x W_J\mathbf{B} \subset \mathbf{B}W_I wW_J\mathbf{B}$. This proves that $\mathbf{P}_I w\mathbf{P}_J \subset \mathbf{B}W_I wW_J\mathbf{B}$. The other inclusion is straightforward. $\square$

Lemma 9.10. *Let $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$ and $\mathbf{Q} = \mathbf{M} \ltimes \mathbf{V}$ be two F-stable parabolic subgroups of $\mathbf{G}$ with F-stable Levi complements $\mathbf{L}$ and $\mathbf{M}$. Then the inclusion $\mathcal{S}(\mathbf{L},\mathbf{M}) \hookrightarrow \mathbf{G}$ induces bijections*

$$\mathbf{L}\backslash\mathcal{S}(\mathbf{L},\mathbf{M})/\mathbf{M} \overset{\sim}{\longrightarrow} \mathbf{P}\backslash\mathbf{G}/\mathbf{Q} \quad \textit{and} \quad \mathbf{L}^F\backslash\mathcal{S}(\mathbf{L},\mathbf{M})^F/\mathbf{M}^F \overset{\sim}{\longrightarrow} \mathbf{P}^F\backslash\mathbf{G}^F/\mathbf{Q}^F.$$

Proof. The second bijection can be deduced from the first one using 7.7. Indeed, the stabilizer of $x \in \mathbf{G}$ under the action of $\mathbf{P} \times \mathbf{Q}^{\mathrm{op}}$ is $\mathbf{P} \cap {}^x\mathbf{Q}$, which is connected by Proposition 8.9. Similarly, the stabilizer under the action of $\mathbf{L} \times \mathbf{M}^{\mathrm{op}}$ is $\mathbf{L} \cap {}^x\mathbf{M}$, which is a Levi subgroup (hence connected) whenever $x \in \mathcal{S}(\mathbf{L}, \mathbf{M})$ (see Proposition 8.9).

We first observe that any double coset $\mathbf{P}x\mathbf{Q}$ contains an element of $\mathcal{S}(\mathbf{L}, \mathbf{M})$. Indeed, any pair of Borel subgroups have a common maximal torus (see 4.4), therefore by Proposition 8.7 there exist Levi complements of $\mathbf{P}$ and ${}^x\mathbf{Q}$ which contain a common maximal torus. These complement are of the form ${}^u\mathbf{L}$ for $u \in \mathbf{U}$ and ${}^{xv}\mathbf{M}$ for $v \in \mathbf{V}$, therefore $u^{-1}xv \in \mathcal{S}(\mathbf{L}, \mathbf{M})$. Consequently, we have a natural surjective map $\mathbf{L} \backslash \mathcal{S}(\mathbf{L}, \mathbf{M})/\mathbf{M} \twoheadrightarrow \mathbf{P} \backslash \mathbf{G}/\mathbf{Q}$. To prove that it is injective we now show that $|\mathbf{L} \backslash \mathcal{S}(\mathbf{L}, \mathbf{M})/\mathbf{M}| \leq |\mathbf{P} \backslash \mathbf{G}/\mathbf{Q}|$. We proceed as follows: fix x_0 such that $\mathbf{L} \cap {}^{x_0}\mathbf{M}$ contains a maximal torus $\mathbf{T}$. Given $x \in \mathcal{S}(\mathbf{L}, \mathbf{M})$, a maximal torus of $\mathbf{L} \cap {}^x\mathbf{M}$ is of the form ${}^l\mathbf{T} = {}^{xmx_0^{-1}}\mathbf{T}$ for some $l \in \mathbf{L}$ and $m \in \mathbf{M}$. This shows that $l^{-1}xmx_0^{-1} \in N_{\mathbf{G}}(\mathbf{T})$. Multiplying this element by $N_{\mathbf{L}}(\mathbf{T})$ on the left and $N_{{}^{x_0}\mathbf{M}}(\mathbf{T})$ on the right does not change the class of x in $\mathbf{L} \backslash \mathcal{S}(\mathbf{L}, \mathbf{M})/\mathbf{M}$. In particular, there are at most $N_{\mathbf{L}}(\mathbf{T}) \backslash N_{\mathbf{G}}(\mathbf{T})/N_{{}^{x_0}\mathbf{M}}(\mathbf{T}) \simeq W_{\mathbf{L}} \backslash W/W_{{}^{x_0}\mathbf{M}}$ elements in $\mathbf{L} \backslash \mathcal{S}(\mathbf{L}, \mathbf{M})/\mathbf{M}$. By Lemma 9.9, this number is exactly the number of double cosets $\mathbf{P} \backslash \mathbf{G}/\mathbf{Q}$ and we get the injectivity. $\square$

Lemma 9.11. *Assume $\mathbf{L} \cap {}^x\mathbf{M}$ contains a maximal torus. Then the map*

$$\begin{array}{ccc} \mathbf{L}^F/(\mathbf{L} \cap {}^x\mathbf{V})^F \times_{(\mathbf{L} \cap {}^x\mathbf{M})^F} ({}^x\mathbf{M} \cap \mathbf{U})^F \backslash {}^x\mathbf{M}^F & \xrightarrow{\sim} & \mathbf{U}^F \backslash \mathbf{P}^F x\, \mathbf{Q}^F/\mathbf{V}^F \\ (l(\mathbf{L} \cap {}^x\mathbf{V})^F, ({}^x\mathbf{M} \cap \mathbf{U})^F m) & \longmapsto & \mathbf{U}^F lmx\mathbf{V}^F \end{array}$$

is a $\mathbf{L}^F \times (\mathbf{M}^F)^{op}$-equivariant bijection.

Proof. The stabilizer of $x \in \mathbf{G}$ under the action of $\mathbf{U} \times \mathbf{V}^{\mathrm{op}}$ is the unipotent group $\mathbf{U} \cap {}^x\mathbf{V}$, hence it is connected. The stabilizer of any element of $\mathbf{L}^F/(\mathbf{L} \cap {}^x\mathbf{V})^F \times ({}^x\mathbf{M} \cap \mathbf{U})^F \backslash {}^x\mathbf{M}^F$ under the diagonal action of $\mathbf{L} \cap {}^x\mathbf{M}$ is trivial, hence connected. Therefore by 7.7 it is enough to prove the following isomorphism:

$$\begin{array}{ccc} \mathbf{L}/(\mathbf{L} \cap {}^x\mathbf{V}) \times_{\mathbf{L} \cap {}^x\mathbf{M}} ({}^x\mathbf{M} \cap \mathbf{U}) \backslash {}^x\mathbf{M} & \xrightarrow{\sim} & \mathbf{U} \backslash \mathbf{P}x\, \mathbf{Q}/\mathbf{V} \\ (l(\mathbf{L} \cap {}^x\mathbf{V}), ({}^x\mathbf{M} \cap \mathbf{U})m) & \longmapsto & \mathbf{U}lmx\mathbf{V} \end{array}$$

The map is well-defined (since $\mathbf{L}$ normalizes $\mathbf{U}$ and $\mathbf{M}$ normalizes $\mathbf{V}$) and it is clearly surjective. To prove the injectivity, assume that $\mathbf{U}lmx\mathbf{V} = \mathbf{U}l'm'x\mathbf{V}$. Then there exists $u \in \mathbf{U}$ and $v \in {}^x\mathbf{V}$ such that $lm = l'uvm'$. Consequently, $u^{-1}l'^{-1}l = vm'm^{-1} \in \mathbf{P} \cap {}^x\mathbf{Q}$ which decomposes as $\mathbf{P} \cap {}^x\mathbf{Q} = (\mathbf{U} \cap {}^x\mathbf{Q}) \rtimes (\mathbf{L} \cap {}^x\mathbf{Q})$ by Proposition 8.9. Therefore by unicity of the decomposition $\mathbf{P} = \mathbf{U} \rtimes \mathbf{L}$, we have $l'^{-1}l \in \mathbf{L} \cap {}^x\mathbf{Q}$. Similarly, $m'm^{-1} \in \mathbf{P} \cap {}^x\mathbf{M}$, and they have the same projection on $\mathbf{L} \cap {}^x\mathbf{M}$. This means that there exists $y \in \mathbf{L} \cap {}^x\mathbf{M}$ such that $yl'^{-1}l \in \mathbf{L} \cap {}^x\mathbf{V}$ and $ym'm^{-1} \in \mathbf{U} \cap {}^x\mathbf{M}$. Up to multiplying l on the right by an element of $\mathbf{L} \cap {}^x\mathbf{V}$ and m on the left by an element of $\mathbf{U} \cap {}^x\mathbf{M}$ we can therefore assume that $yl'^{-1}l = ym'm^{-1} = 1$ that is $l'm' = lm$, which proves the injectivity. $\square$

In the case where $\mathbf{P}$ and $\mathbf{Q}$ are standard parabolic subgroups, we can also use Lemmas 9.9 and 9.10 to obtain the following particular case of the Mackey formula.

Corollary 9.12. *Let I and J be two F-stable subsets of S. Then*

$$^*R^{\mathbf{G}}_{\mathbf{L}_I\subset\mathbf{P}_I}\circ R^{\mathbf{G}}_{\mathbf{L}_J\subset\mathbf{P}_J} \simeq \sum_{w\in W_I^F\backslash W^F/W_J^F} R^{\mathbf{L}_I}_{\mathbf{L}_I\cap{}^w\mathbf{L}_J\subset\mathbf{L}_I\cap{}^w\mathbf{P}_J}\circ{}^*R^{^w\mathbf{L}_J}_{\mathbf{L}_I\cap{}^w\mathbf{L}_J\subset\mathbf{P}_I\cap{}^w\mathbf{L}_J}\circ\mathrm{ad}\, w.$$

Example 9.13. For $\mathbf{G} = \mathbf{GL}_n$ with standard Frobenius F, and $I = J = \{1,\ldots,n-2\}$, we have $\mathbf{L}_I \simeq \mathbf{GL}_{n-1}\times\mathbf{GL}_1$. There are only two cosets in $W_I^F\backslash W^F/W_J^F \simeq \mathfrak{S}_{n-1}\backslash\mathfrak{S}_n/\mathfrak{S}_{n-1}$; they correspond to 1 and $(1,n)$. Furthermore, $\mathbf{L}_I\cap{}^{(1,n)}\mathbf{L}_J$ is the standard Levi subgroup $\mathbf{L}_{\{2,\ldots,n-3\}}\simeq\mathbf{GL}_1\times\mathbf{GL}_{n-2}\times\mathbf{GL}_1$ of $\mathbf{L}_I$. We deduce the following formula

$$^*R^{\mathbf{GL}_n}_{\mathbf{GL}_{n-1}}\circ R^{\mathbf{GL}_n}_{\mathbf{GL}_{n-1}}\simeq \mathrm{Id} + R^{\mathbf{GL}_{n-1}}_{\mathbf{GL}_{n-2}}\circ{}^*R^{\mathbf{GL}_{n-1}}_{\mathbf{GL}_{n-2}}$$

where we have omitted the parabolic subgroups involved and the copies of $\mathbf{GL}_1$ in the various Levi subgroups.

Exercise 9.14. Do the same computation for twisted type A, with $I = J = \{2,\ldots,n-2\}$ a maximal F-stable subset of $\{1,\ldots,n-1\}$. What is the difference with the previous case?

10 Harish-Chandra theory

In this section we use the previous induction and restriction functor to decompose the set of irreducible representations into series. Unlike the usual induction and restriction for finite groups, parabolic induction from proper Levi subgroups does not reach all the representations. The missing ones are the so-called *cuspidal* representations. Therefore the first steps towards the classification of the irreducible representations are:

(a) Find the irreducible cuspidal representations; these representations are usually constructed using geometric methods (Deligne-Lusztig theory) and have been classified by Lusztig in the case where $\Lambda = \mathbb{C}$.

(b) Determine which representation can be reach from the parabolic induction of a cuspidal one. This amounts to studying the representation theory of the endomorphism algebra of the induced representation, which will be done in the next section.

Note that this program if far from being achieved for representations in positive characteristic, and is still a very active area of research.

10.1 Independence of the parabolic subgroup

The parabolic induction and restriction functors are defined in terms of F-stable parabolic subgroups and their F-stable Levi complement. It turns out that they depend only on the choice of the Levi, and not on the parabolic subgroup.

Theorem 10.1. *Let* $\mathbf{L}$ *be an common* F*-stable Levi complement of the* F*-stable parabolic subgroups* $\mathbf{P}$ *and* $\mathbf{Q}$*. Then*

$$ {}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} \simeq {}^*R_{\mathbf{L}\subset\mathbf{Q}}^{\mathbf{G}} \quad \textit{and} \quad R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} \simeq R_{\mathbf{L}\subset\mathbf{Q}}^{\mathbf{G}}. $$

Proof. The result over Λ is due to Howlett-Lehrer [Howlett-Lehrer] and Dipper-Du [Dipper-Du]. It is a consequence of the existence of an isomorphism of $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{op}}$-modules between $\Lambda\mathbf{G}^F e_{\mathrm{R_u}(\mathbf{P})^F}$ and $\Lambda\mathbf{G}^F e_{\mathrm{R_u}(\mathbf{Q})^F}$. Here, we shall give a proof in the case $\Lambda = \mathbb{C}$ only.

We proceed by induction on the semi-simple rank of $\mathbf{G}$. If $\mathbf{G}$ is a torus, then $\mathbf{L} = \mathbf{P} = \mathbf{Q} = \mathbf{G}$ and the result is obvious. Otherwise one can assume that $\mathbf{L} \neq \mathbf{G}$. Let λ be an irreducible character of $\mathbf{L}^F$. By the Mackey formula and the adjunction, we have

$$ \langle R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}\lambda\,; R_{\mathbf{L}\subset\mathbf{Q}}^{\mathbf{G}}\lambda\rangle_{\mathbf{G}^F} = \sum \langle {}^*R_{\mathbf{L}\cap{}^x\mathbf{L}\subset\mathbf{L}\cap{}^x\mathbf{Q}}^{\mathbf{L}}\lambda\,; R_{\mathbf{L}\cap{}^x\mathbf{L}\subset\mathbf{P}\cap{}^x\mathbf{L}}^{{}^x\mathbf{L}}\,{}^x\lambda\rangle_{\mathbf{L}^F\cap{}^x\mathbf{L}^F}. $$

If we assume by induction that the right-hand side does not depend on $\mathbf{P}$ and $\mathbf{Q}$, then we deduce that

$$ \langle R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}\lambda\,; R_{\mathbf{L}\subset\mathbf{Q}}^{\mathbf{G}}\lambda\rangle_{\mathbf{G}^F} = \langle R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}\lambda\,; R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}\lambda\rangle_{\mathbf{G}^F} = \langle R_{\mathbf{L}\subset\mathbf{Q}}^{\mathbf{G}}\lambda\,; R_{\mathbf{L}\subset\mathbf{Q}}^{\mathbf{G}}\lambda\rangle_{\mathbf{G}^F} $$

from which we get

$$ \langle R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}\lambda - R_{\mathbf{L}\subset\mathbf{Q}}^{\mathbf{G}}\lambda\,; R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}\lambda - R_{\mathbf{L}\subset\mathbf{Q}}^{\mathbf{G}}\lambda\rangle_{\mathbf{G}^F} = 0. $$

This proves that $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}\lambda = R_{\mathbf{L}\subset\mathbf{Q}}^{\mathbf{G}}\lambda$ so that $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ does not depend on $\mathbf{P}$. In addition, we can use adjunction to show that the restriction ${}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ does not depend on $\mathbf{P}$ either. $\square$

We will not use the parabolic subgroups anymore but we still need to assume that all the Levi subgroups involved are F-stable complements of F-stable Levi subgroups, that is are $\mathbf{G}$-split.

10.2 Cuspidality and Harish-Chandra series

For usual induction and restriction, every representation can be reached by induction from a proper subgroup. This is no longer true for parabolic induction, the main difference being than the parabolic restriction can kill some representations. This motivates the following definition.

Definition 10.2. *A* $\Lambda\mathbf{G}^F$*-module* M *is said to be* cuspidal *if* ${}^*R_{\mathbf{L}}^{\mathbf{G}}(M) = 0$ *for all proper* $\mathbf{G}$*-split Levi subgroups* $\mathbf{L}$*.*

Recall from Proposition 9.4 that the induction and restriction functors are adjoint. In particular, given a $\Lambda\mathbf{L}^F$-module N we have

$$\mathrm{Hom}_{\mathbf{L}^F}(N, {}^*R_{\mathbf{L}}^{\mathbf{G}}(M)) \simeq \mathrm{Hom}_{\mathbf{G}^F}(R_{\mathbf{L}}^{\mathbf{G}}(N), M).$$

Consequently, in the case where M is simple, the property of being non-cuspidal is equivalent to the existence of a pair $(\mathbf{L}, N)$ an a surjective map $R_{\mathbf{L}}^{\mathbf{G}}(N) \twoheadrightarrow M$. We can even assume that N is cuspidal:

Lemma 10.3. *Let M be a simple $\Lambda\mathbf{G}^F$-module. Then there exist a $\mathbf{G}$-split Levi subgroup $\mathbf{L}$ and a (simple) cuspidal $\mathbf{L}^F$-module N such that M is in the head of $R_{\mathbf{L}}^{\mathbf{G}}(N)$, i.e. such that there exists a $\mathbf{G}^F$-equivariant surjective map $R_{\mathbf{L}}^{\mathbf{G}}(N) \twoheadrightarrow M$.*

Proof. Let $\mathbf{L}$ to be minimal Levi subgroup for the property that ${}^*R_{\mathbf{L}}^{\mathbf{G}}(M) \neq 0$. Then by transitivity of the parabolic restriction, ${}^*R_{\mathbf{L}}^{\mathbf{G}}(M)$ is a cuspidal $\Lambda\mathbf{L}^F$-module. Since the parabolic restriction is exact (see Proposition 9.4), we can take for N any simple submodule of ${}^*R_{\mathbf{L}}^{\mathbf{G}}(M)$. $\square$

Remark 10.4. Using the other adjunction one can prove that there exists a cuspidal module $\mathbf{L}^F$-module N' such that M is in the socle of $R_{\mathbf{L}}^{\mathbf{G}}(N')$, i.e. such that there exists a $\mathbf{G}^F$-equivariant injective map $R_{\mathbf{L}}^{\mathbf{G}}(N') \hookrightarrow M$. Note however that if Λ is not a field, it is unclear whether in that case N can be assumed to be simple.

Definition 10.5. *A* cuspidal pair *is a pair $(\mathbf{L}, N)$ where $\mathbf{L}$ is an $\mathbf{G}$-split Levi subgroup and N is a cuspidal simple $\Lambda\mathbf{L}^F$-module. The* Harish-Chandra series *corresponding to such a pair is*

$$\mathrm{Irr}\left(\mathbf{G}^F|(\mathbf{L}, N)\right) = \{M \in \mathrm{Irr}\ \mathbf{G}^F \mid R_{\mathbf{L}}^{\mathbf{G}}(N) \twoheadrightarrow M\}.$$

The previous lemma ensures that any simple representation lies in at least one Harish-Chandra series. However, a representation can lie in different Harish-Chandra series, but in that case they will be conjugate under $\mathbf{G}^F$, as shown in the following proposition.

Proposition 10.6. *Assume that Λ is a field. Let $(\mathbf{L}, N)$ and $(\mathbf{L}', N')$ be two cuspidal pairs. Then $\mathrm{Irr}\left(\mathbf{G}^F|(\mathbf{L}, N)\right) \cap \mathrm{Irr}\left(\mathbf{G}^F|(\mathbf{L}', N')\right) \neq \emptyset$ if and only if there exists $g \in \mathbf{G}^F$ such that $(\mathbf{L}', N') = {}^g(\mathbf{L}, N)$.*

Proof. We first note that for $g \in \mathbf{G}^F$, we have $R_{{}^g\mathbf{L}}^{\mathbf{G}}({}^gN) \simeq {}^g(R_{\mathbf{L}}^{\mathbf{G}}(N))$ therefore the series corresponding to conjugate cuspidal pairs have isomorphic constituents.

To prove the converse, let M be a $\Lambda\mathbf{G}^F$-module such that $R_{\mathbf{L}}^{\mathbf{G}}(N) \twoheadrightarrow M$ and $R_{\mathbf{L}'}^{\mathbf{G}}(N') \twoheadrightarrow M$. Since Λ is a field, there exists a projective cover P_N (resp. P_M) of N (resp. M). Now, by Proposition 9.4, the map $R_{\mathbf{L}}^{\mathbf{G}}(P_N) \twoheadrightarrow R_{\mathbf{L}}^{\mathbf{G}}(N)$ is surjective and by composition we get a surjective map $R_{\mathbf{L}}^{\mathbf{G}}(P_N) \twoheadrightarrow M$. Hence P_M must be a direct summand of $R_{\mathbf{L}}^{\mathbf{G}}(P_N)$. Now, since M is in the head of $R_{\mathbf{L}'}^{\mathbf{G}}(N')$, it is a composition factor and we have $\mathrm{Hom}_{\mathbf{G}^F}(P_M, R_{\mathbf{L}'}^{\mathbf{G}}(N')) \neq 0$. We deduce that $\mathrm{Hom}_{\mathbf{G}^F}(R_{\mathbf{L}}^{\mathbf{G}}(P_N), R_{\mathbf{L}'}^{\mathbf{G}}(N')) \neq 0$. By the Mackey formula, the latter is isomorphic to

$$\bigoplus_{x\in\mathbf{L}^F\backslash\mathcal{S}(\mathbf{L},\mathbf{L}')^F/\mathbf{L}'^F} \mathrm{Hom}_{(\mathbf{L}\cap{}^x\mathbf{L}')^F}\left({}^*R^{\mathbf{L}}_{\mathbf{L}\cap{}^x\mathbf{L}'}(P_N), {}^*R^{{}^x\mathbf{L}'}_{\mathbf{L}\cap{}^x\mathbf{L}'}({}^xN')\right)$$

Now, since N is cuspidal, the restriction ${}^*R^{{}^x\mathbf{L}'}_{\mathbf{L}\cap{}^x\mathbf{L}'}({}^xN') \simeq {}^x\big({}^*R^{\mathbf{L}'}_{\mathbf{L}^x\cap\mathbf{L}'}(N')\big)$ is zero whenever $\mathbf{L}^x\cap\mathbf{L}'$ is a proper Levi subgroup of $\mathbf{L}'$. This proves that $\mathbf{L}'\subset\mathbf{L}^x$ whenever $x\in\mathcal{S}(\mathbf{L},\mathbf{L}')^F$. Exchanging the role of $(\mathbf{L},N)$ and $(\mathbf{L}',N')$ we see that we must have an equality. Consequently, the Mackey formula takes the following very simple form

$$0\neq \mathrm{Hom}_{\mathbf{G}^F}(R_{\mathbf{L}}^{\mathbf{G}}(P_N), R_{\mathbf{L}'}^{\mathbf{G}}(N')) \simeq \bigoplus_{x\in N_{\mathbf{G}}(\mathbf{L}',\mathbf{L})^F/\mathbf{L}^F} \mathrm{Hom}_{\mathbf{L}^F}(P_N, {}^xN')$$

where $N_{\mathbf{G}}(\mathbf{L}',\mathbf{L})^F = \{g\in\mathbf{G}\mid {}^x\mathbf{L}'=\mathbf{L}\}$. Finally, since N' (hence ${}^xN'$) is simple there is a non-zero map between P_N and ${}^xN'$ if and only if $N\simeq{}^xN'$. $\square$

Let $\mathcal{C}$ be the set of cuspidal classes of $\mathbf{G}$. When Λ is a field, the combination of Lemma 10.3 and Proposition 10.6 yields a partition of the set of isomorphism classes of irreducible representations of $\mathbf{G}^F$ into Harish-Chandra series:

$$\mathrm{Irr}\ \mathbf{G}^F = \bigsqcup_{(\mathbf{L},N)\in\mathcal{C}/\mathbf{G}^F} \mathrm{Irr}\left(\mathbf{G}^F|(\mathbf{L},N)\right).$$

In addition, if $\mathbf{L}$ is a Levi subgroup of a Levi subgroup $\mathbf{L}'$ of $\mathbf{G}$, then by the transitivity of the induction $R_{\mathbf{L}}^{\mathbf{G}}N = R_{\mathbf{L}'}^{\mathbf{G}}\circ R_{\mathbf{L}}^{\mathbf{L}'}N$. Therefore if $M\in\mathrm{Irr}\left(\mathbf{L}'^F|(\mathbf{L},N)\right)$ then any constituent in the head of $R_{\mathbf{L}'}^{\mathbf{G}}M$ lies in $\mathrm{Irr}\left(\mathbf{G}^F|(\mathbf{L},N)\right)$. This means Harish-Chandra series are compatible with Harish-Chandra induction.

Remark 10.7. The property that Λ is a field was needed to make sure that the category $\Lambda\mathbf{G}^F$ had enough projective objects. Note that this holds for a more general class of rings, such as complete discrete valuation ring (*e.g.* $\Lambda=\mathbb{Z}_\ell$).

10.3 Endomorphism algebras

To finish this series of general results, we shall now describe the set $\mathrm{Irr}\left(\mathbf{G}^F|(\mathbf{L},N)\right)$ for a given cuspidal pair $(\mathbf{L},N)$. This will be related to the representation theory of the endomorphism algebra $\mathcal{H}(\mathbf{L},N)=\mathrm{End}_{\mathbf{G}^F}(R_{\mathbf{L}}^{\mathbf{G}}(N))$. To this end, let us consider the Hom-functor

$$\begin{array}{rcl}\Theta:\Lambda\mathbf{G}^F\text{-}\mathsf{mod} & \longrightarrow & \mathcal{H}(\mathbf{L},N)^{\mathrm{op}}\text{-}\mathsf{mod}\\ M & \longmapsto & \mathrm{Hom}_{\mathbf{G}^F}(R_{\mathbf{L}}^{\mathbf{G}}(N),M)\end{array}$$

Proposition 10.8. *If Λ is a field, the functor Θ induces a bijection*

$$\mathrm{Irr}\left(\mathbf{G}^F|(\mathbf{L},N)\right)\stackrel{1:1}{\longleftrightarrow}\mathrm{Irr}\ \mathcal{H}(\mathbf{L},N).$$

Proof. We will write $\mathcal{H}$ for $\mathcal{H}(\mathbf{L}, N)$ and $X = R_{\mathbf{L}}^{\mathbf{G}}(N)$ to simplify the notation. Let I be the annihilator of X in $\Lambda\mathbf{G}$. We proceed in three steps, showing that

(i) X is a projective $\Lambda\mathbf{G}/I$-module;

(ii) if M is such that $IM = 0$ then $\Theta(M) \simeq \mathrm{Hom}_{\Lambda\mathbf{G}^F/I}(X, M) =: \overline{\Theta}(M)$;

(iii) $\overline{\Theta}$ induces a bijection between the isomorphism classes of the simple constituents in the head of the projective $\Lambda\mathbf{G}^F/I$-module X and $\mathrm{Irr}\ \mathcal{H}$.

Choose a projective cover P_N of N and let $P = R_{\mathbf{L}}^{\mathbf{G}}(P_N)$. We have a surjective map $P \twoheadrightarrow X$ which yields an injective linear map $\mathrm{Hom}_{\mathbf{G}^F}(X, X) \hookrightarrow \mathrm{Hom}_{\mathbf{G}^F}(P, X)$. Now, we have seen in the course of the proof of Proposition 10.6 that when N is cuspidal these two spaces have the same dimension. Therefore $P \twoheadrightarrow X$ induces an isomorphism

$$\mathrm{Hom}_{\mathbf{G}^F}(X, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{G}^F}(P, X). \tag{10.9}$$

Since P is projective, we have $\mathrm{Hom}_{\mathbf{G}^F}(P, \Lambda\mathbf{G}^F) \otimes_{\Lambda\mathbf{G}^F} X \simeq \mathrm{Hom}_{\mathbf{G}^F}(P, X)$ via the map $f \otimes x \longmapsto (\phi_{f,x} : p \longmapsto f(p)x)$. By the isomorphism (10.9), each $\phi_{f,x} : P \longrightarrow X$ factors through $P \twoheadrightarrow X$. Consequently

$$\mathrm{Ker}(P \twoheadrightarrow X) \ \subset\ \bigcap \mathrm{Ker}\ \phi_{f,x} \ \subset\ IP$$

which shows that $P/IP \twoheadrightarrow X$ is injective, hence bijective and (i) follows.

The inflation functor is a fully-faithful functor from $\Lambda G^F/I$-mod to ΛG^F-mod. It induces an equivalence between $\Lambda G^F/I$-mod and the full subcategory of ΛG^F-mod on which I acts by zero. Now, if $X \twoheadrightarrow M$ is a surjective morphism of $\Lambda\mathbf{G}^F$-modules, then by equivariance $IM = 0$ so that M can be viewed as an object of that category, which yields (ii).

To conclude it is enough to prove that $\overline{\Theta}$ induces a bijection between the isomorphism classes of the simple constituents in the head of the projective $\Lambda\mathbf{G}^F/I$-module X and $\mathrm{Irr}\ \overline{\Theta}(X) = \mathrm{Irr}\ \mathcal{H}$. Let M be a simple module. Since X is projective, any non-zero map $f : X \twoheadrightarrow M$ is sent to a surjective map $\overline{\Theta}(X) = \mathcal{H} \twoheadrightarrow \overline{\Theta}(M)$ which means that $f \in \overline{\Theta}(M)$ generates $\overline{\Theta}(M)$, and proves that it is simple. Therefore $\overline{\Theta}$ induces a well-defined map

$$\{\text{isoclasses of simple modules in the head of } X\} \longrightarrow \mathrm{Irr}\ \mathcal{H}.$$

This map is injective: if M and M' are simple modules in the head of X, then P_M is isomorphic to a direct summand of X and one can take $\pi_M \in \mathcal{H}$ to be the projection to this direct summand. If $\overline{\Theta}(M) \simeq \overline{\Theta}(M')$ as $\mathcal{H}$-modules, then $\pi_M \cdot f = f \circ \pi_M$ is non-zero for all non-zero $f \in \overline{\Theta}(M')$ and in particular there will be a non-zero map from P_M to M'. This forces $M' \simeq M$. For the surjectivity, let S be a simple $\mathcal{H}$-module and consider $M = X \otimes_{\mathcal{H}} S$. Then using tensor-hom adjunction and the fact that X is projective we obtain

$$\begin{aligned}
\overline{\Theta}(V) &= \mathrm{Hom}_{\Lambda\mathbf{G}^F/I}(X, X \otimes_{\mathcal{H}} S) \\
&\simeq \mathrm{Hom}_{\Lambda\mathbf{G}^F/I}(X, \Lambda\mathbf{G}^F/I) \otimes_{\Lambda\mathbf{G}^F/I} X \otimes_{\mathcal{H}} S \\
&\simeq \mathcal{H} \otimes_{\mathcal{H}} S \\
\overline{\Theta}(V) &\simeq S
\end{aligned}$$

In particular $\overline{\Theta}(V) \neq 0$. Since $\overline{\Theta}$ is exact, there exists a composition factor M of V such that $\overline{\Theta}(M) \neq 0$, and by simplicity of S it will satisfy $\overline{\Theta}(M) \simeq S$. $\square$

Remark 10.10. Note that the proof of (iii) only uses the projectivity of X, and nothing from the theory of Harish-Chandra induction or restriction. The functors $\mathrm{Hom}_A(X, -)$ for X a projective A-module are sometimes called *Schur functors*.

We summarize the results of this section in the following main theorem.

Theorem 10.11. *Assume that Λ is a field. Let $\mathcal{C}$ be the set of cuspidal pairs of $\mathbf{G}^F$. Then the irreducible representations of $\mathbf{G}^F$ fall into Harish-Chandra series*

$$\mathrm{Irr}\ \mathbf{G}^F = \bigsqcup_{(\mathbf{L},N)\in\mathcal{C}/\mathbf{G}^F} \mathrm{Irr}\left(\mathbf{G}^F|(\mathbf{L},N)\right).$$

Furthermore, there is a natural parametrization of each series

$$\mathrm{Irr}\left(\mathbf{G}^F|(\mathbf{L},N)\right) \xleftrightarrow{1:1} \mathrm{Irr}\left(\mathrm{End}_{\mathbf{G}^F}(R_{\mathbf{L}}^{\mathbf{G}}(N))\right).$$

11 Endomorphism algebras as Hecke algebras

We have seen in the previous section how to classify the irreducible representations of $\mathbf{G}^F$ in terms of Harish-Chandra series. Each series can be parametrized by means of the irreducible representations of some endomorphism algebra $\mathcal{H}(\mathbf{L},N) = \mathrm{End}_{\mathbf{G}^F}(R_{\mathbf{L}}^{\mathbf{G}}(N))$. The purpose of this section is to study explicity the algebra structure of $\mathcal{H}(\mathbf{L},N)$, starting from the example of $(\mathbf{L},N) = (\mathbf{T},\Lambda)$.

11.1 First example

Since a maximal torus $\mathbf{T}$ has a unique Levi subgroup (itself), every representation of $\mathbf{T}$ is cuspidal. Therefore from the previous section it makes sense to consider the algebra

$$\mathcal{H} := \mathcal{H}(\mathbf{T},\Lambda) = \mathrm{End}_{\mathbf{G}^F}(R_{\mathbf{T}}^{\mathbf{G}}(\Lambda)).$$

By assumption $\mathbf{T}$ is $\mathbf{G}$-split, so it is contained in an F-stable Borel subgroup $\mathbf{B}$, with unipotent radical $\mathbf{U}$. By definition $R_{\mathbf{T}}^{\mathbf{G}}(\Lambda) = \Lambda\mathbf{G}^F/\mathbf{U}^F \otimes_{\Lambda\mathbf{T}^F} \Lambda \simeq \Lambda\mathbf{G}^F/\mathbf{B}^F$. In order to simplify the argument in this case we will now assume that $(\mathbf{G},F)$ is *split* (which means that F acts trivially on W) and $q(q-1) \in$

$\Lambda^\times$. This last assumption ensures that the order of $\mathbf{B}^F$ is invertible in Λ. As a consequence $\Lambda\mathbf{G}^F/\mathbf{B}^F \simeq \Lambda\mathbf{G}^F e_{\mathbf{B}^F}$ where $e_{\mathbf{B}^F} = |\mathbf{B}^F|^{-1}\sum_{b\in\mathbf{B}^F} b$ is the idempotent corresponding to the trivial representation of $\mathbf{B}^F$. We deduce that $\mathcal{H}$ and $e_{\mathbf{B}^F}\Lambda\mathbf{G}^F e_{\mathbf{B}^F}$ are isomorphic algebras. For $w \in W$, let us define

$$T_w = q^{\ell(w)}\, e_{\mathbf{B}^F} w e_{\mathbf{B}^F}.$$

Note that this element is well-defined since $\mathbf{T} \subset \mathbf{B}$. It follows from the Bruhat decomposition that the family $(T_w)_{w\in W}$ form a Λ-basis of $\mathcal{H}$.

Let $w, w' \in W$ such that $\ell(ww') = \ell(w) + \ell(w')$. By the Bruhat decomposition every element of $w\mathbf{B}^F w'$ lies in $\mathbf{B}^F ww'\mathbf{B}^F$. Consequently, for all $b \in \mathbf{B}^F$ we have $e_{\mathbf{B}^F} wbw' e_{\mathbf{B}^F} = e_{\mathbf{B}^F} ww' e_{\mathbf{B}^F}$ and we deduce that

$$T_{ww'} = T_w T_{w'}. \tag{11.1}$$

This proves that $T_w = T_{s_1}T_{s_2}\cdots T_{s_r}$ whenever $w = s_1 s_2 \cdots s_r$ is a reduced expression of w.

Let $s \in S$ be a simple reflection. Then an element of $s\mathbf{B}^F s$ lies either in $\mathbf{B}^F s\mathbf{B}^F$ or in $\mathbf{B}^F$. Now the group $s\mathbf{B}s \cap \mathbf{B} = \mathbf{B} \cap {}^s\mathbf{B}$ is generated by $\mathbf{T}$ and the $\mathbf{U}_\alpha$'s for $\alpha \in \Phi^+ \smallsetminus \{\alpha_s\}$, therefore $|\mathbf{B}^F|/|(\mathbf{B}\cap {}^s\mathbf{B})^F| = |\mathbf{U}_\alpha^F| = q$. We deduce that

$$(e_{\mathbf{B}^F} s e_{\mathbf{B}^F})^2 = e_{\mathbf{B}^F} s e_{\mathbf{B}^F} s e_{\mathbf{B}^F} = \frac{q-1}{q}\, e_{\mathbf{B}^F} s e_{\mathbf{B}^F} + \frac{1}{q}\, e_{\mathbf{B}^F}$$

which we can write as the quadratic relation

$$T_s^2 = (q-1)T_s + q. \tag{11.2}$$

In fact, relations (11.1) and (11.2) generate all relations in $\mathcal{H}$, as we will see from the general theory for these algebras.

11.2 Hecke algebras

Let (W, S) be a Coxeter system. Let $\mathbf{q} = (q_s)_{s\in S}$ be set of indeterminates satisfying $q_s = q_t$ whenever s and t are conjugate in W.

Definition 11.3. *The* generic Hecke algebra $\mathcal{H}_{\mathbf{q}}(W,S)$ *is the* $\mathbb{Z}[\mathbf{q},\mathbf{q}^{-1}]$*-algebra generated by the elements* $(T_s)_{s\in S}$ *subject to the following relations*

- $(T_s - q_s)(T_s + 1) = 0$ *for all* $s \in S$ *(order relation)*
- $T_sT_tT_s\cdots = T_tT_sT_t\cdots$ *whenever* $sts\cdots = tst\cdots$ *in* W *(braid relations)*

Note that the first relation can be written $T_s^2 = (q_s - 1)T_s + q_s$, as in the previous example. Each T_s is invertible with inverse

$$T_s^{-1} = \frac{1}{q_s}\, T + \frac{1-q}{q}.$$

The braid relations ensure that we can define elements T_w by setting

$$T_w = T_{s_1} T_{s_2} \cdots T_{s_r}$$

for any reduced expression $w = s_1 s_2 \cdots s_r$ of $w \in W$. Indeed, by Matsumoto Lemma this does not depend on the reduced expression.

Theorem 11.4. *The family $(T_w)_{w\in W}$ is a basis of $\mathcal{H}_q(W,S)$ is over $\mathbb{Z}[q_s, q_s^{-1}]$.*

Proof. The fact that (T_w) generates $\mathcal{H}_q(W,S)$ is obvious. To show that they are linearly independent, we construct two representations of $\mathcal{H}_q(W,S)$ on $\mathbb{Z}[q_s, q_s^{-1}]W$ as follows:

$$\lambda(T_s)(w) \quad = \begin{cases} (q_s-1)sw + q_s w & \text{if } sw < w \\ sw & \text{otherwise} \end{cases}$$

$$\rho(T_s)(w) \quad = \begin{cases} (q_s-1)ws + q_s w & \text{if } \ ws < w \\ ws & \text{otherwise} \end{cases}$$

We claim that $\lambda(T_s)$ and $\rho(T_t)$ commute for any $s,t \in S$. There are many cases to look at. For example when $l(sw) > l(w)$ and $l(swt) < l(wt)$, we must have $sw = wt$ by the exchange lemma and therefore

$$\begin{aligned} &\lambda(T_s)\rho(T_t)(w) = \lambda(T_s)(wt) = (q_s-1)swt + q_s wt \\ &\rho(T_t)\lambda(T_s)(w) = \rho(T_t)(sw) = (q_t-1)swt + q_t sw = (q_t-1)swt + q_t wt \end{aligned}$$

and we conclude using $q_s = q_t$ since $s = wtw^{-1}$.

Next we show that ρ and λ extend to algebra homomorphisms. The order relation are clearly satisfied. For the braid relation, we consider two reduced expressions $w = s_1 \cdots s_r = t_1 \cdots t_r$ of a given $w \in W$, and we set $\Delta = \lambda(T_{s_1}) \cdots \lambda(T_{s_r}) - \lambda(T_{t_1}) \cdots \lambda(T_{t_r})$. We show by induction on $\ell(v)$ that $\Delta(v)$ is zero: it is clear if $v = 1$ since $\Delta(1) = w - w = 0$. Otherwise take $s \in S$ such that $vs < v$ and assume that $\Delta(vs) = 0$. Then $vs < vss = v$ therefore $\Delta(v) = \Delta(\rho(T_s)vs) = \rho(T_s)(\Delta(vs)) = 0$ since $\rho(T_s)$ commutes with each $\lambda(T_t)$, hence with Δ.

Finally, the evaluation of the family $(\lambda(T_w))_{w\in W}$ at 1 is the basis $(w)_{w\in W}$ of $\mathbb{Z}[q_s, q_s^{-1}]W$, which proves that the elements T_w's are linearly independent. $\square$

Order relations for the elements T_s are deformation of the relation $s^2 = 1$ in the group algebra $\mathbb{Z}[W]$ obtained by specializing each q_s to 1. The previous result ensures that the Hecke algebras are flat deformations of the group algebra $\mathbb{Z}W$, and we can invoke Tits' deformation theorem to get a bijection between $\operatorname{Irr}_K \mathcal{H}$ and $\operatorname{Irr}_L W$ for suitable fields K and L. We

Proposition 11.5. *There exists a finite extension K of $\mathbb{C}(\mathbf{q})$ such that the specialization map $q_s \longmapsto 1$ induces a bijection*

$$\begin{array}{ccc} \operatorname{Irr}_K \mathcal{H} & \stackrel{1:1}{\longleftrightarrow} & \operatorname{Irr}_{\mathbb{C}} W \\ \chi_{\mathbf{q}} & \longmapsto & \chi \end{array}$$

Remark 11.6. Using the fact that $K\mathcal{H}$ is a symmetric algebra, one can actually prove that this result remains true for any specialization of q_s outside the roots of unity.

11.3 The Hecke algebra $\mathcal{H}(\mathbf{L}, N)$

For $(\mathbf{L}, N) = (\mathbf{T}, \Lambda)$ we constructed for each $w \in W$ an endomorphism T_w satifying Hecke relations. We are now going to extend this construction to any cuspidal pair.

Assume that Λ is a field, which is large enough for $\mathbf{L}^F$. Let $W(\mathbf{L}, N) = N_{\mathbf{G}}(\mathbf{L}, N)/\mathbf{L}$ be the automizer of the pair $(\mathbf{L}, N)$. For $n \in N(\mathbf{L}, N)$, the $\mathbf{L}^F$-modules N and ${}^nN$ are isomorphic, therefore there exists a bijective linear map $\gamma_n : N \xrightarrow{\sim} N$ such that $\gamma_n(l \cdot x) = l^n \cdot \gamma_n(x)$ for all $l \in \mathbf{L}^F$ and $x \in N$. Given two elements $n, n' \in N(\mathbf{L}, N)$, the composition $\gamma_{nn'}^{-1} \circ \gamma_{n'} \circ \gamma_n$ is an endomorphism of the $\mathbf{L}^F$-module N. Since N is simple and Λ is big enough, $\mathrm{End}_{\mathbf{L}^F}(N) \simeq \Lambda$, therefore there exists a scalar $\lambda(n, n')$ such that

$$\gamma_{n'} \circ \gamma_n = \lambda(n, n')\gamma_{ww'}.$$

Furthermore, $\lambda : N(\mathbf{L}, N) \times N(\mathbf{L}, N) \longrightarrow \Lambda$ is a 2-cocycle.

Since N is cuspidal, any restriction of N to a proper $\mathbf{G}$-split Levi subgroup is zero. This simplifies the expression of the Mackey formula. More precisely, Theorem 9.6 yields the following isomorphism of Λ-modules

$$\mathcal{H}(\mathbf{L}, N) = \mathrm{End}_{\mathbf{G}^F}(R_{\mathbf{L}}^{\mathbf{G}} N) \simeq \bigoplus_{w \in W(\mathbf{L}, N)^F} \mathrm{Hom}_{\mathbf{L}^F}(N, {}^{n_w}N) \tag{11.7}$$

where n_w is any representative of $w \in W(\mathbf{L}, N)^F$ in $N(\mathbf{L}, N)^F$. Note that it is unclear how to describe the algebra structure on the right-hand side, and this is exactly the problem we would like to address. Since N is simple and Λ large enough, each $\mathrm{Hom}_{\mathbf{L}^F}(N, {}^nN)$ is one-dimensional and generated by γ_n. In particular, $\mathcal{H}(\mathbf{L}, N)$ is free of rank $|W(\mathbf{L}, N)^F|$ over Λ.

Recall that the isomorphism (11.7) comes from a decomposition of $\mathbf{U}^F \backslash \mathbf{G}^F / \mathbf{U}^F$ into double cosets $\mathbf{U}^F \backslash \mathbf{P}^F n \mathbf{P}^F / \mathbf{U}^F$. Following the proof of the Mackey formula we see that the linear maps γ_n correspond via (11.7) to the operators

$$B_n \,:\, g e_{\mathbf{U}^F} \otimes x \longmapsto g e_{\mathbf{U}^F} n e_{\mathbf{U}^F} \otimes \gamma_n(x).$$

Note that this linear map is well-defined since $n e_{\mathbf{U}^F} \otimes \gamma_n(lx) = n e_{\mathbf{U}^F} \otimes l^n \gamma_n(x) = n e_{\mathbf{U}^F} l^n \otimes \gamma_n(x) = l n e_{\mathbf{U}^F} \otimes \gamma_n(x)$. Moreover it is clearly $\mathbf{G}^F$-equivariant, and one can check as an exercise that it does correspond to γ_n via (11.7). This yields the following result:

Lemma 11.8. *For any set of representatives n_w of $W(\mathbf{L}, N)^F$ in $N(\mathbf{L}, N)^F$, the family $(B_{n_w})_{w \in W(\mathbf{L}, N)^F}$ is a basis of $\mathcal{H}(\mathbf{L}, N)$ over Λ.*

We now fix a specific set of representatives n_w of $W(\mathbf{L}, N)^F$ in $N(\mathbf{L}, N)^F$ such that $n_{ww'} = n_w n_{w'}$ whenever $\ell(ww') = \ell(w) + \ell(w')$. To simplify the notation we will denote by $\dot{w} = n_w$ such representative and by $B_w := B_{\dot{w}}$ the corresponding element of $\mathcal{H}(\mathbf{L}, N)$.

Lemma 11.9. *Given $w, w' \in W(\mathbf{L}, N)^F$ such that $\ell(ww') = \ell(w) + \ell(w')$ we have*

$$B_{ww'} = \lambda(\dot{w}, \dot{w}')B_w \circ B_{w'}$$

in the algebra $\mathcal{H}(\mathbf{L}, N)$.

Proof. Let $\alpha \in \Phi^+$ be a positive root. Assume that ${w'}^{-1}(\alpha) < 0$ and let $\beta = -{w'}^{-1}(\alpha) > 0$. We have $w'(\beta) < 0$ and therefore $-w(\alpha) = ww'(\beta) < 0$ since $\ell(w) + \ell(w') = \ell(ww')$. This proves that ${w'}^{-1}(\alpha) > 0$ or $w(\alpha) > 0$. Consequently, with $\mathbf{U}$ being generated by the $\mathbf{U}_\alpha$ for $\alpha > 0$, we have $\dot{w}u\dot{w}' \in \mathbf{U}^F \dot{w}\dot{w}' \mathbf{U}^F$ for all $u \in \mathbf{U}^F$. This proves that

$$e_{\mathbf{U}^F} \dot{w} e_{\mathbf{U}^F} \dot{w}' e_{\mathbf{U}^F} = e_{\mathbf{U}^F} \dot{w}\dot{w}' e_{\mathbf{U}^F}$$

and the lemma follows. □

If $\mathbf{M}$ is a $\mathbf{G}$-split Levi subgroup containing $\mathbf{L}$, we can consider a smaller Hecke algebra $\mathrm{End}_{\mathbf{M}^F}(R_{\mathbf{L}}^{\mathbf{M}}(N))$. Let $w \in W_{\mathbf{M}}(\mathbf{L}, N)^F$. Using the equation (11.7) for $\mathbf{M}$ instead of $\mathbf{G}$ one can define an element $B_{\mathbf{M},w} \in \mathrm{End}_{\mathbf{M}^F}(R_{\mathbf{L}}^{\mathbf{M}}(N))$ corresponding to $\gamma_{\dot{w}}$. Then by construction B_w is the image of $B_{\mathbf{M},w}$ under the parabolic induction functor $R_{\mathbf{M}}^{\mathbf{G}}$. We deduce the following result:

Lemma 11.10. *Let $\mathbf{M}$ be a $\mathbf{G}$-split Levi subgroup containing $\mathbf{L}$. Assume that $W_{\mathbf{M}}(\mathbf{L}, N)^F$ has order 2. Then for $w \in N_{\mathbf{M}}(\mathbf{L}, N)^F$ the operator B_w of $\mathcal{H}$ satisfies a quadratic relation.*

Proof. If $w = 1$ it is obvious since in that case B_w is the identity. Otherwise, $\mathrm{End}_{\mathbf{M}^F}(R_{\mathbf{L}}^{\mathbf{M}}(N))$ has a basis given by the identity $B_{\mathbf{M},1}$ and $B_{\mathbf{M},w}$. Writing $(B_{\mathbf{M},w})^2$ in this basis gives a quadratic relation for $B_{\mathbf{M},w}$ and hence a quadratic relation for $B_w = R_{\mathbf{M}}^{\mathbf{G}}(B_{\mathbf{M},w})$. □

To finish with the description of $\mathcal{H}$, and endow it with a structure of a Hecke algebra, one would need to solve the following problems:

(P1) Describe the (normal) subgroup of $W_{\mathbf{G}}(\mathbf{L}, N)^F$ generated by the non-trivial involution in the various $W_{\mathbf{M}}(\mathbf{L}, N)^F$ when $\mathbf{M}$ is such that $W_{\mathbf{M}}(\mathbf{L}, N)^F$ has order 2.

(P2) Give the explicit quadratic equations satisfied by $B_{\mathbf{M},w}$ when $W_{\mathbf{M}}(\mathbf{L}, N)^F$ has order 2; this is related to computing the parameters of the Hecke algebra structure describing $\mathcal{H}$.

(P3) Compute the 2-cocycle λ.

A general solution to these problems has yet to be found. However, it has been completely solved in the case where $\Lambda = \mathbb{C}$, by a combination of work by Lusztig, Howlett-Lehrer and Geck. In that case $W_{\mathbf{G}}(\mathbf{L}, N)^F$ is a Coxeter group and is generated by the various subgroups $W_{\mathbf{M}}(\mathbf{L}, N)^F$ of order 2, the quadratic relations are explicit in terms of the dimensions of the two irreducible summands of $R_{\mathbf{L}}^{\mathbf{M}}(N)$ and the 2-cocycle λ is trivial. The following theorem summarizes their results:

Theorem 11.11. *Assume $\Lambda = \mathbb{C}$ and let $(\mathbf{L}, N)$ be a cuspidal pair. Then there exists*

- *a set of involution $S(\mathbf{L}, N) \subset W(\mathbf{L}, N)^F$ making $(W(\mathbf{L}, N)^F, S(\mathbf{L}, N))$ a Coxeter system;*
- *for each $s \in S(\mathbf{L}, N)$, a power of q denoted by q_s such that $q_s = q_t$ whenever s and t are conjugate in $W(\mathbf{L}, N)^F$;*
- *for each $w \in W(\mathbf{L}, N)^F$ a scalar λ_w;*

such that the map

$$\begin{array}{rcl} \mathcal{H}_{\mathbf{q}}(W(\mathbf{L}, N)^F, S(\mathbf{L}, N)) & \xrightarrow{\sim} & \mathcal{H}(\mathbf{L}, N) \\ T_w & \longmapsto & \lambda_w B_w \end{array}$$

is an isomorphism of algebras.

Recall from Proposition 10.8 that the irreducible representations lying in the Harish-Chandra series above the cuspidal pair $(\mathbf{L}, N)$ are parameterized by irreducible representations of $\mathcal{H}(\mathbf{L}, N)$. The previous theorem ensures that the latter are parametrized by representations of the Coxeter group $W(\mathbf{L}, N)^F$ (see Proposition 11.5 and Remark 11.6).

Corollary 11.12. *Let $(\mathbf{L}, N)$ be a cuspidal pair over $\mathbb{C}$. Then*

$$\operatorname{Irr}\left(\mathbf{G}^F | (\mathbf{L}, N)\right) \stackrel{1:1}{\longleftrightarrow} \operatorname{Irr}_{\mathbb{C}} W(\mathbf{L}, N)^F.$$

Example 11.13. Let $\mathbf{G} = \mathbf{GL}_n$, and F be the Frobenius endomorphism on $\mathbf{G}$ giving the general unitary group $\mathbf{G}^F = \mathbf{GU}_n(q)$. If there exists a *unipotent* cuspidal complex character ρ of $\mathbf{GU}_n(q)$ if and only if $n = t(t-1)/2$ for some $t \in \mathbb{Z}_{\geq 0}$ (see 15.1 for the definition of unipotent characters). Furthermore, if such a character exists it is unique, and we will denote it by ρ_t. If $t = 0$ or $t = 1$, ρ_t is just the trivial character.

Fix $t \geq 0$, $n = t(t-1)/2$ and consider the group $\mathbf{GU}_{n+2m}(q)$. A $\mathbf{GU}_{n+2m}$-split Levi subgroup of $\mathbf{GL}_{n+2m}$ of semisimple rank $n-1$ is given by the standard Levi subgroup corresponding to the subset $I = \{\alpha_{m+1}, \alpha_{m+2}, \ldots, \alpha_{m+n-1}\}$ of the set of simple roots $\{\alpha_1, \ldots, \alpha_{n+2m-1}\}$ of $\mathbf{GL}_{n+2m}$. Since $F(\alpha_i) = \alpha_{n+2m-i}$, it is stable by F (see Figure 1). Therefore $\mathbf{L}_I$ is F-stable and we have $\mathbf{L}_I^F \simeq \mathbf{GU}_n(q) \times (\mathbb{F}_{q^2}^\times)^m$. The character ρ_t can be inflated to an irreducible character of $\mathbf{L}_I^F$. It is still the unique unipotent cuspidal character of $\mathbf{L}_I^F$ and therefore

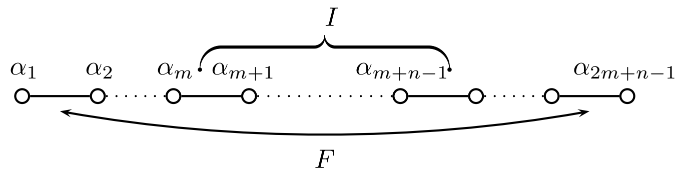


Figure 1: Dynkin diagram of $\mathbf{GU}_{n+2m}(q)$

any element of the normalizer of $\mathbf{L}_I$ normalizes the pair $(\mathbf{L}_I, \rho_t)$. In particular $W(\mathbf{L}_I, \rho_t)^F = N_{\mathbf{G}}(\mathbf{L}_I)^F/\mathbf{L}_I^F \simeq N_W(W_I)^F/W_I^F$.

The general procedure for constructing generators of $N_W(W_I)^F/W_I^F$ is as follows: take $\alpha \notin I$, consider the orbit $\omega_\alpha = \{\alpha, F(\alpha), \ldots\}$ of simple roots under F, and set $w_\alpha = w_{I\cup\omega_\alpha} w_I$. Here, if $j \notin \{m, \ldots, n+m\}$ then α_j is orthogonal to every element in I so that $w_{\alpha_j} = s_j s_{n+2m-j}$, which corresponds to the permutation $(j, j+1)(n+2m-j, n+2m-j+1)$. Otherwise if $j \in \{m, n+m\}$ then w_{α_j} corresponds to the permutation $(m, n+m+1)$. One can check that these involutions endow $N_W(W_I)^F/W_I^F$ of a structure of Coxeter group of type B_m. By the previous theorem, $\mathrm{End}_{\mathbf{G}^F}(R_{\mathbf{L}_I}^{\mathbf{G}} \rho_t)$ is a Hecke algebra of type B_m. The parameters of this Hecke algebra are given in Figure 2. Note that w_{α_i} and w_{α_j} are conjugate if and only if $i, j < m$.

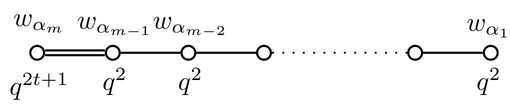


Figure 2: Parameters of the Hecke algebra

To finish this section, we state a compatibility theorem due to Howlett-Lehrer between parabolic induction and usual induction in the Coxeter groups. We still work in the particular case where $\Lambda = \mathbb{C}$.

Theorem 11.14. *Assume that $\Lambda = \mathbb{C}$. Let $\mathbf{L} \subset \mathbf{M}$ be two quasi-simple Levi subgroups of $\mathbf{G}$ and N be a simple cuspidal $\Lambda\mathbf{L}^F$-module. Then we have the* *following commutative diagrams:*

$$\begin{tikzcd}
\mathbb{Z}\operatorname{Irr}\big(\mathbf{G}^F,(\mathbf{L},N)\big) \arrow[r, leftrightarrow] & \mathbb{Z}\operatorname{Irr} W_{\mathbf{G}}(\mathbf{L},N)^F \\
\mathbb{Z}\operatorname{Irr}\big(\mathbf{M}^F,(\mathbf{L},N)\big) \arrow[r, leftrightarrow] \arrow[u, "R_{\mathbf{M}}^{\mathbf{G}}"'] & \mathbb{Z}\operatorname{Irr} W_{\mathbf{M}}(\mathbf{L},N)^F \arrow[u, "\operatorname{Ind}_{W_{\mathbf{M}}(\mathbf{L},N)^F}^{W_{\mathbf{G}}(\mathbf{L},N)^F}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{Z}\operatorname{Irr}\big(\mathbf{G}^F,(\mathbf{L},N)\big) \arrow[r, leftrightarrow] \arrow[d, "{}^*R_{\mathbf{M}}^{\mathbf{G}}"] & \mathbb{Z}\operatorname{Irr} W_{\mathbf{G}}(\mathbf{L},N)^F \arrow[d, "\operatorname{Res}_{W_{\mathbf{M}}(\mathbf{L},N)^F}^{W_{\mathbf{G}}(\mathbf{L},N)^F}"'] \\
\mathbb{Z}\operatorname{Irr}\big(\mathbf{M}^F,(\mathbf{L},N)\big) \arrow[r, leftrightarrow] & \mathbb{Z}\operatorname{Irr} W_{\mathbf{M}}(\mathbf{L},N)^F
\end{tikzcd}$$

12 Alvis-Curtis-Kawanaka duality

Throughout this section we shall assume that $\Lambda = \mathbb{C}$ is the field of complex numbers. In that case the category $\mathbb{C}\mathbf{G}^F$-mod is semi-simple, and we shall focus on its Grothendieck group $K_0(\mathbb{C}\mathbf{G}^F\text{-mod})$. A $\mathbb{Z}$-basis of this group is given by the complex irreducible characters, and as such any element of $K_0(\mathbb{C}\mathbf{G}^F\text{-mod})$ can be thought of as a *virtual character*. The Harish-Chandra induction and restriction functors induce linear maps on virtual characters that we will still denote by $R_{\mathbf{L}}^{\mathbf{G}}$ and ${}^*R_{\mathbf{L}}^{\mathbf{G}}$.

To avoid technical difficulties, we will always assume in this section that F is a Frobenius endomorphism. Under this assumption we will define and study the F-rank of an algebraic group, which we will use to define the duality operator.

12.1 $\mathbb{F}_q$-rank

Let $\mathbf{T}$ be a torus, and F be an endomorphism of $\mathbf{T}$ corresponding to an $\mathbb{F}_q$-structure. Then the map $\tau = q^{-1}F$ is an linear endomorphism of $X(\mathbf{T}) \otimes \mathbb{R}$ which has finite order (the order is 1 if and only if $\mathbf{T}$ is split).

Definition 12.1. *The* $\mathbb{F}_q$-rank *of a torus* $\mathbf{T}$ *is*

$$\mathbb{F}_q\text{-}\mathit{rank}(\mathbf{T}) = \dim\big(X(\mathbf{T}) \otimes \mathbb{R}\big)^{\tau}.$$

It is also the rank of the largest $\mathbb{F}_q$*-split torus in* $\mathbf{T}$*.*

Let $\mathbf{H}$ be an algebraic group, and F be a Frobenius endomorphism of $\mathbf{H}$ defining an $\mathbb{F}_q$-structure on $\mathbf{H}$. Since two $\mathbf{H}$-split tori of $\mathbf{H}$ are $\mathbf{H}^F$-conjugate, the following definition makes sense:

Definition 12.2. *Let* $\mathbf{H}$ *be an algebraic group endowed with an* $\mathbb{F}_q$*-structure via a Frobenius endomorphism* F*.*

(i) The $\mathbb{F}_q$-rank *of* $\mathbf{H}$ *is the* $\mathbb{F}_q$*-rank of any* $\mathbf{H}$*-split maximal torus of* $\mathbf{H}$*.*

(ii) The $\mathbb{F}_q$-semisimple rank $r(\mathbf{H})$ *of* $\mathbf{H}$ *is the* $\mathbb{F}_q$*-rank of* $\mathbf{H}/R(\mathbf{H})$*.*

Remark 12.3. Let $\mathbf{T}$ be a $\mathbf{H}$-split maximal torus of $\mathbf{H}$. Using the exact sequence $0 \longrightarrow X(\mathbf{T}/R(\mathbf{H})) \longrightarrow X(\mathbf{T}) \longrightarrow X(R(\mathbf{H})) \longrightarrow 0$ and the fact that τ has finite order, we get

$$r(\mathbf{H}) = \mathbb{F}_q\text{-rank}(\mathbf{H}) - \mathbb{F}_q\text{-rank}(R(\mathbf{H})). \tag{12.4}$$

In the case where $\mathbf{H}$ is a reductive group, then the semisimple rank can be computed in terms of the root system defined from a $\mathbf{H}$-split torus.

Lemma 12.5. *Let* $\mathbf{G}$ *be a connected reductive group,* $\mathbf{T}$ *be a maximal torus, and* $\Phi = \Phi(\mathbf{G}, \mathbf{T})$ *be the corresponding root system. Set* $V = X(\mathbf{T}) \otimes \mathbb{R}$ *and* $\tau = q^{-1}F \in \operatorname{End}(V)$*. Then*

(i) $\mathbb{F}_q$*-rank*$(R(\mathbf{G})) = \dim\big(\langle \Phi^\vee \rangle^\perp \cap V^\tau\big)$*.*

(ii) If $\mathbf{T}$ *is* $\mathbf{G}$*-split,* $r(\mathbf{G}) = \dim\big(\langle \Phi \rangle \cap V^\tau\big)$*.*

Proof. By Proposition 7.10, there exists $g \in \mathbf{G}$ such that ${}^g\mathbf{T}$ is $\mathbf{G}$-split and $w := g^{-1}F(g) \in N_{\mathbf{G}}(\mathbf{T})$. If $\mathbf{B}$ is an F-stable Borel containing ${}^g\mathbf{T}$, then $\mathbf{B}^g$ is an wF-stable Borel containing $\mathbf{T}$. Since w acts trivially on $\langle \Phi^\vee \rangle^\perp$, we deduce that

$$\dim\big(\langle \Phi^\vee \rangle^\perp \cap V^\tau\big) = \dim\big(\langle \Phi^\vee \rangle^\perp \cap V^{w\tau}\big)$$

and therefore we can assume that $\mathbf{T}$ is $\mathbf{G}$-split without loss of generality.

The torus $\mathbf{T}/R(\mathbf{G})$ is a $\mathbf{G}/R(\mathbf{G})$-split maximal, and Φ is the image of $\Phi(\mathbf{G}/R(\mathbf{G}), \mathbf{T}/R(\mathbf{G}))$ via the F-equivariant embedding $X(\mathbf{T}/R(\mathbf{G})) \hookrightarrow X(\mathbf{T})$. Since $\mathbf{G}/R(\mathbf{G})$ is semisimple, its root system spans the real vector space $X(\mathbf{T}/R(\mathbf{G})) \otimes \mathbb{R}$ and we deduce (ii). Assertion (i) follows from Equation (12.4). □

Remark 12.6. If $\mathbf{B}$ is an F-stable Borel subgroup containing $\mathbf{T}$, then from Lemma 12.5 we deduce that for any F-stable subset I of S the $\mathbb{F}_q$-semisimple rank of the corresponding parabolic subgroup equals the number of orbits of F on I:

$$r(\mathbf{P}_I) = r(\mathbf{L}_I) = |I/F|.$$

12.2 Duality

Let $\mathbf{L}$ be a $\mathbf{G}$-split Levi subgroup. Then for $g \in \mathbf{G}$, we have $R_{{}^g\mathbf{L}}^{\mathbf{G}} \circ \operatorname{ad} g = \operatorname{ad} g \circ R_{\mathbf{L}}^{\mathbf{G}}$ and ${}^*R_{{}^g\mathbf{L}}^{\mathbf{G}} \circ \operatorname{ad} g = \operatorname{ad} g \circ {}^*R_{\mathbf{L}}^{\mathbf{G}}$. In particular, the composition $R_{\mathbf{L}}^{\mathbf{G}} \circ {}^*R_{\mathbf{L}}^{\mathbf{G}}$ depends only on the $\mathbf{G}^F$-conjugacy class of $\mathbf{L}$.

Definition 12.7. *Let* $\mathbf{B}$ *be an* F*-stable Borel subgroup of* $\mathbf{G}$*. The* Alvis-Curtis-Kawanaka duality *is the linear map on* $K_0(\mathbb{C}\mathbf{G}^F\text{-}\mathsf{mod})$ *defined by*

$$D_{\mathbf{G}} = \sum_{\mathbf{P} \subset \mathbf{B}} (-1)^{r(\mathbf{P})} R_{\mathbf{L}}^{\mathbf{G}} \circ {}^*R_{\mathbf{L}}^{\mathbf{G}}$$

where P *runs over the* F*-stable parabolic subgroups containing* $\mathbf{B}$ *and where* $\mathbf{L}$ *is any* F*-stable Levi complement of* $\mathbf{P}$*.*

Since any two F-stable Borel subgroups are conjugate under $\mathbf{G}^F$, we deduce from the above discussion that definition of $D_{\mathbf{G}}$ does not depend on $\mathbf{B}$. If we fix an F-stable maximal torus $\mathbf{T}$ of $\mathbf{B}$, then all the parabolic subgroups and their Levi complements can be assumed to be standard. In that case we can use Remark 12.6 to write

$$D_{\mathbf{G}} = \sum_{I \subset S,\, F(I)=I} (-1)^{|I/F|} R_{\mathbf{L}_I}^{\mathbf{G}} \circ {}^*R_{\mathbf{L}_I}^{\mathbf{G}}.$$

Since $R_{\mathbf{L}}^{\mathbf{G}}$ and ${}^*R_{\mathbf{L}}^{\mathbf{G}}$ are adjoint maps, then $D_{\mathbf{G}}$ is clearly self-adjoint. In terms of characters, this means that given two complex characters ρ and χ of $\mathbf{G}^F$ we have

$$\langle D_{\mathbf{G}}(\rho)\,;\chi\rangle_{\mathbf{G}^F} = \langle \rho\,; D_{\mathbf{G}}(\chi)\rangle_{\mathbf{G}^F}.$$

The behaviour of $D_{\mathbf{G}}$ with respect to Harish-Chandra induction and restriction is much deeper, and the proof of the following result, due to Curtis, will be at the heart of this section.

Theorem 12.8. *Let $\mathbf{M}$ be a $\mathbf{G}$-split Levi subgroup. Then*

$$D_{\mathbf{G}} \circ R_{\mathbf{M}}^{\mathbf{G}} = R_{\mathbf{M}}^{\mathbf{G}} \circ D_{\mathbf{M}} \quad \textit{and} \quad D_{\mathbf{M}} \circ {}^*R_{\mathbf{M}}^{\mathbf{G}} = {}^*R_{\mathbf{M}}^{\mathbf{G}} \circ D_{\mathbf{G}}.$$

Proof. Computing $D_{\mathbf{G}} \circ R_{\mathbf{M}}^{\mathbf{G}}$ involves computing $R_{\mathbf{L}}^{\mathbf{G}} \circ {}^*R_{\mathbf{L}}^{\mathbf{G}} \circ R_{\mathbf{M}}^{\mathbf{G}}$ for $\mathbf{G}$-split Levi subgroups $\mathbf{L}$ and $\mathbf{M}$. To this end, we shall use the Mackey formula. For each $\mathbf{L}$, we fix a system of representatives $\mathcal{R}_{\mathbf{L}}$ for $\mathbf{L}^F\backslash\mathcal{S}(\mathbf{L},\mathbf{M})^F/\mathbf{M}^F$. Then

$$\begin{aligned}
D_{\mathbf{G}} \circ R_{\mathbf{M}}^{\mathbf{G}} &= \sum_{\mathbf{P}\supset\mathbf{B}} (-1)^{r(\mathbf{P})} R_{\mathbf{L}}^{\mathbf{G}} \circ {}^*R_{\mathbf{L}}^{\mathbf{G}} \circ R_{\mathbf{M}}^{\mathbf{G}} \\
&= \sum_{\mathbf{P}\supset\mathbf{B}} (-1)^{r(\mathbf{P})} R_{\mathbf{L}}^{\mathbf{G}} \circ \Big(\sum_{x\in\mathcal{R}_{\mathbf{L}}} R_{\mathbf{L}\cap{}^x\mathbf{M}}^{\mathbf{L}} \circ {}^*R_{\mathbf{L}\cap{}^x\mathbf{M}}^{{}^x\mathbf{M}} \circ \mathrm{ad}\, x\Big) \\
&= \sum_{\mathbf{P}\supset\mathbf{B}} (-1)^{r(\mathbf{P})} \Big(\sum_{x\in\mathcal{R}_{\mathbf{L}}} R_{\mathbf{L}\cap{}^x\mathbf{M}}^{\mathbf{G}} \circ {}^*R_{\mathbf{L}\cap{}^x\mathbf{M}}^{{}^x\mathbf{M}} \circ \mathrm{ad}\, x\Big) \\
&= \sum_{\mathbf{P}\supset\mathbf{B}} (-1)^{r(\mathbf{P})} \Big(\sum_{x\in\mathcal{R}_{\mathbf{L}}} R_{\mathbf{L}^x\cap\mathbf{M}}^{\mathbf{G}} \circ {}^*R_{\mathbf{L}^x\cap\mathbf{M}}^{\mathbf{M}}\Big) \\
D_{\mathbf{G}} \circ R_{\mathbf{M}}^{\mathbf{G}} &= R_{\mathbf{M}}^{\mathbf{G}} \circ \Big(\sum_{\mathbf{P}\supset\mathbf{B}} (-1)^{r(\mathbf{P})} \Big(\sum_{x\in\mathcal{R}_{\mathbf{L}}} R_{\mathbf{L}^x\cap\mathbf{M}}^{\mathbf{M}} \circ {}^*R_{\mathbf{L}^x\cap\mathbf{M}}^{\mathbf{M}}\Big)\Big)
\end{aligned}$$

where we used $R_{\mathbf{L}\cap{}^x\mathbf{M}}^{\mathbf{G}} \circ \mathrm{ad}\, x = R_{\mathbf{L}^x\cap\mathbf{M}}^{\mathbf{G}}$ for the before to last equality (note that $x \in \mathbf{G}^F$).

We now want to relate the right-hand side to the duality $D_{\mathbf{M}}$. Let $\mathbf{B}_{\mathbf{M}}$ be an F-stable Borel subgroup of $\mathbf{M}$. For each $x \in \mathcal{S}(\mathbf{L},\mathbf{M})^F$, the group $\mathbf{M}\cap\mathbf{P}^x$ is an F-stable parabolic subgroup of $\mathbf{M}$, containing the F-stable Borel subgroup $\mathbf{M}\cap\mathbf{B}^x$. Up to multipliying x by an element of $\mathbf{M}^F$ on the right, we can assume that $\mathbf{M}\cap\mathbf{B}^x = \mathbf{B}_M$. This yields a map

$$\varphi_{\mathbf{P}} \,:\, \mathbf{L}^F\backslash\mathcal{S}(\mathbf{L},\mathbf{M})^F/\mathbf{M}^F \;\longrightarrow\; \{\mathbf{P}^g \,|\, g\in\mathbf{G}^F \;\text{ and }\; \mathbf{B}_{\mathbf{M}} \subset \mathbf{P}^g\}.$$

First note that this map is well-defined: if $x \in \mathcal{S}(\mathbf{L}, \mathbf{M})^F$ is such that $\mathbf{P}^x$ and $\mathbf{P}^{xm}$ contain $\mathbf{B}_M$, then $\mathbf{P}^x \cap \mathbf{M}$ and $\mathbf{P}^{xm} \cap \mathbf{M}$ are two conjugate parabolic subgroup which contain $\mathbf{B_M}$, so they must be equal. Since parabolic subgroups are self-normalizing, then $m \in \mathbf{P}^x \cap \mathbf{M}$ and therefore $\mathbf{P}^{xm} = \mathbf{P}^x$. The map $\varphi_{\mathbf{P}}$ is also clearly surjective since F-stable Levi subgroups of $\mathbf{P}$ are conjugate under $\mathbf{P}^F$. Finally, we show that it is injective: if $\mathbf{P}^g = \mathbf{P}^h$ then $hg^{-1} \in \mathbf{P}^F$. We deduce from Proposition 8.9 that $\mathbf{L}^g \cap \mathbf{M}$ and $\mathbf{L}^h \cap \mathbf{M}$ are both rational Levi subgroups of $\mathbf{M} \cap \mathbf{P}^g$, therefore they must be conjugate under $(\mathbf{M} \cap \mathbf{U}^g)^F$. Therefore up to multipliying h on the right by an element of $(\mathbf{M} \cap \mathbf{U}^g)^F$, we can assume that $\mathbf{L}^g \cap \mathbf{M} = \mathbf{L}^h \cap \mathbf{M}$. But then $\mathbf{L}^g$ and $\mathbf{L}^h$ contain a common maximal torus of $\mathbf{P}^g$ so that they must be equal by Proposition 8.7. Consequently $hg^{-1} \in N_{\mathbf{P}^F}(\mathbf{L}) = \mathbf{L}^F$, which proves that g and h are equal in $\mathbf{L}^F \backslash \mathcal{S}(\mathbf{L}, \mathbf{M})^F / \mathbf{M}^F$.

The map $(\mathbf{P}, x) \longmapsto \varphi_{\mathbf{P}}(x)$ induces a bijection between the pairs $(\mathbf{P}, x)$ where $\mathbf{P} \supset \mathbf{B}$ and $x \in \mathbf{L}^F \backslash \mathcal{S}(\mathbf{L}, \mathbf{M})^F / \mathbf{M}^F$ and the parabolic subgroups $\mathbf{Q}$ of $\mathbf{M}$ containing $\mathbf{B_M}$. Note that this bijection preserves the $\mathbb{F}_q$-semisimple rank of the parabolic subgroups. As a consequence, we can transform the expression of $D_{\mathbf{G}} \circ R_{\mathbf{M}}^{\mathbf{G}}$ obtained previously into

$$D_{\mathbf{G}} \circ R_{\mathbf{M}}^{\mathbf{G}} = R_{\mathbf{M}}^{\mathbf{G}} \circ \Big(\sum_{\mathbf{Q} \supset \mathbf{B_M}} (-1)^{r(\mathbf{Q})} R_{\mathbf{L}' \cap \mathbf{M}}^{\mathbf{M}} \circ {}^*R_{\mathbf{L}' \cap \mathbf{M}}^{\mathbf{M}} \Big).$$

where the sum runs over the F-stable parabolic subgroups $\mathbf{Q}$ containing $\mathbf{B_M}$, and where $\mathbf{L}'$ is any F-stable Levi complement of $\mathbf{Q}$ containing a maximal torus of $\mathbf{M}$. For such a group $\mathbf{Q}$, we can consider the parabolic subgroup $\mathbf{P}_M = \mathbf{Q} \cap \mathbf{M}$ of $\mathbf{M}$ containing the Borel subgroup $\mathbf{B_M}$ of $\mathbf{M}$. The group $\mathbf{L_M} = \mathbf{L}' \cap \mathbf{M}$ is an F-stable Levi complement of $\mathbf{Q} \cap \mathbf{M}$ and we can write the previous sum as

$$D_{\mathbf{G}} \circ R_{\mathbf{M}}^{\mathbf{G}} = R_{\mathbf{M}}^{\mathbf{G}} \circ \Big(\sum_{\mathbf{P_M} \supset \mathbf{B_M}} \Big(\sum_{\mathbf{Q} \cap \mathbf{M} = \mathbf{P}_M} (-1)^{r(\mathbf{Q})} \Big) R_{\mathbf{L_M}}^{\mathbf{M}} \circ {}^*R_{\mathbf{L_M}}^{\mathbf{M}} \Big).$$

where $\mathbf{P}_M$ runs over the F-stable parabolic subgroups of $\mathbf{M}$ containing $\mathbf{B_M}$ (with Levi $\mathbf{L_M}$) and $\mathbf{Q}$ over the F-stable parabolic subgroups of $\mathbf{G}$ such that $\mathbf{Q} \cap \mathbf{M} = \mathbf{P_M}$. The formula stated in the theorem now follows from the following Lemma.

Lemma 12.9. *Let $\mathbf{H}$ be an F-stable Levi subgroup of $\mathbf{G}$ and $\mathbf{R}$ be an F-stable parabolic subgroup of $\mathbf{H}$. Then*

$$\sum_{\mathbf{Q} \cap \mathbf{H} = \mathbf{R}} (-1)^{r(\mathbf{Q})} = (-1)^{r(\mathbf{R}) + \mathbb{F}_q\text{-}rank(\mathbf{G}) + \mathbb{F}_q\text{-}rank(\mathbf{H})}$$

where $\mathbf{Q}$ runs over the F-stable parabolic subgroups of $\mathbf{G}$.

Proof of the Lemma. Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{H}$. We work in the real vector space $V = X(\mathbf{T}) \otimes \mathbb{R}$. Recall that $\tau = q^{-1}F$ is an isomorphism of V of finite order. Given $\mathbf{K}$ a subgroup of $\mathbf{G}$ containing $\mathbf{T}$, we denote by $\Phi_{\mathbf{K}}$ the set of roots α such that $\mathbf{U}_\alpha \subset \mathbf{K}$. Recall from Proposition 8.8 that if $\mathbf{L}$ is the unique Levi complement of $\mathbf{Q}$ containing $\mathbf{T}$ then $\Phi_{\mathbf{L}} = \Phi_{\mathbf{Q}} \cap (-\Phi_{\mathbf{Q}})$. Since

$\mathbf{Q}$ is F-stable, we have $\mathbb{F}_q$-rank$(\mathbf{Q}) = \mathbb{F}_q$-rank$(\mathbf{G})$ and therefore using Lemma 12.5 we obtain

$$\begin{aligned} r(\mathbf{Q}) &= \mathbb{F}_q\text{-rank}(\mathbf{Q}) - \dim\langle \Phi_{\mathbf{Q}}^\vee \cap (-\Phi_{\mathbf{Q}})^\vee \rangle^\perp \cap V^\tau \\ &= \mathbb{F}_q\text{-rank}(\mathbf{G}) - \dim\langle \Phi_{\mathbf{Q}}^\vee \cap (-\Phi_{\mathbf{Q}})^\vee \rangle^\perp \cap V^\tau. \end{aligned}$$

To a subgroup $\mathbf{K}$ of $\mathbf{G}$ containing $\mathbf{T}$ we associate the following set:

$$\mathcal{F}_{\mathbf{K}} = \left\{ x \in V \;\middle|\; \begin{array}{l} \langle \alpha\,; x\rangle = 0 \text{ for } \alpha \in \Phi_{\mathbf{K}}^\vee \cap (-\Phi_{\mathbf{K}}^\vee) \\ \langle \alpha\,; x\rangle > 0 \text{ for } \alpha \in \Phi_{\mathbf{K}}^\vee \setminus (-\Phi_{\mathbf{K}}^\vee) \end{array} \right\}.$$

Given $x \in V$, we can consider $\Psi_x = \{\alpha \in \Phi \mid \langle \alpha\,; x\rangle \geq 0\}$. Then $\Phi = \Psi_x \cup (-\Psi_x)$ and we can use Proposition 8.8 to see that the subgroup generated by $\mathbf{T}$ and $\mathbf{U}_\alpha$ for $\alpha \in \Psi_x$ is a parabolic subgroup containing $\mathbf{T}$. This has the following consequences:

(i) Each element $x \in V$ lies in a set $\mathcal{F}_{\mathbf{Q}}$ for a unique parabolic subgroup $\mathbf{Q}$ containing $\mathbf{T}$ (with the previous notation $\Psi_x = \Phi_{\mathbf{Q}}$).

(ii) $\mathcal{F}_{\mathbf{Q}} \cap \tau(\mathcal{F}_{\mathbf{Q}}) \neq \emptyset$ if and only if $\mathcal{F}_{\mathbf{Q}} = \tau(\mathcal{F}_{\mathbf{Q}})$, in which case $\mathbf{Q}$ is F-stable.

(iii) $\mathcal{F}_{\mathbf{Q}} \cap \mathcal{F}_{\mathbf{R}} \neq \emptyset$ if and only if $\mathcal{F}_{\mathbf{Q}} \subset \mathcal{F}_{\mathbf{R}}$, which in turn is equivalent to $\Phi_{\mathbf{R}} = \Phi_{\mathbf{Q}} \cap \Phi_{\mathbf{H}}$ (or equivalently $\mathbf{R} = \mathbf{Q} \cap \mathbf{H}$).

Each set $\mathcal{F}_{\mathbf{Q}}$ is open in its support $\langle \Phi_{\mathbf{Q}}^\vee \cap (-\Phi_{\mathbf{Q}}^\vee)\rangle^\perp$, therefore its dimension as a subvariety of V equals the dimension of this vector space. Moreover, if $\tau(\mathcal{F}_{\mathbf{Q}}) = \mathcal{F}_{\mathbf{Q}}$ (*i.e.* if $\mathbf{Q}$ is F-stable) then $\mathcal{F}_{\mathbf{Q}} \cap V^\tau$ is a convex set. In addition it is non-empty since it contains the average of any orbit under τ of elements of $\mathcal{F}_{\mathbf{Q}}$. We deduce that

$$\dim(\mathcal{F}_{\mathbf{Q}} \cap V^\tau) = \dim\langle \Phi_{\mathbf{Q}}^\vee \cap (-\Phi_{\mathbf{Q}}^\vee)\rangle^\perp \cap V^\tau \;=\; \mathbb{F}_q\text{-rank}(\mathbf{G}) - r(\mathbf{Q}).$$

The same argument using $\mathbf{R}$ instead of $\mathbf{Q}$ shows that the formula we want to prove is equivalent to

$$\sum_{\substack{\tau(\mathcal{F}_{\mathbf{Q}})=\mathcal{F}_{\mathbf{Q}} \\ \mathcal{F}_{\mathbf{Q}}\subset\mathcal{F}_{\mathbf{R}}}} (-1)^{\dim \mathcal{F}_{\mathbf{Q}} \cap V^\tau} = (-1)^{\dim \mathcal{F}_{\mathbf{R}} \cap V^\tau}.$$

By restriction to V^τ, the sets $\mathcal{F}_{\mathbf{Q}} \cap V^\tau$ for $\mathbf{Q}$ an F-stable parabolic subgroup containing $\mathbf{T}$ form a partition of V^τ. By (iii), the condition $\mathcal{F}_{\mathbf{Q}} \subset \mathcal{F}_{\mathbf{R}}$ is equivalent to the fact that the convex sets $\mathcal{F}_{\mathbf{Q}} \cap V^\tau$ and $\mathcal{F}_{\mathbf{R}} \cap V^\tau$ have a non-trivial intersection. Now the sets $\mathcal{F}_{\mathbf{Q}} \cap V^\tau$ are exactly the facets of the set of hyperplanes $V^\tau \cap \operatorname{Ker}\langle \alpha\,; -\rangle$ of the vector space V^τ. Therefore it is enough to prove that given a vector space E and a convex union of facets C of an hyperplane arrangement, we have

$$\sum_{\mathcal{F}\subset C} (-1)^{\dim \mathcal{F}} = (-1)^{\dim C}$$

where $\mathcal{F}$ runs over the facets of the hyperplane arrangement such that $\mathcal{F} \subset C$. This is the expression of the Euler characteristic of C. $\square$

Proof of the theorem. We apply the Lemma to $\mathbf{H} = \mathbf{M}$ and $\mathbf{R} = \mathbf{P_M}$. Note that the $\mathbb{F}_q$-ranks of $\mathbf{G}$ and $\mathbf{Q}$ are equal since $\mathbf{M}$ is $\mathbf{G}$-split. This gives the equality $D_{\mathbf{G}} \circ R_{\mathbf{M}}^{\mathbf{G}} = R_{\mathbf{M}}^{\mathbf{G}} \circ D_{\mathbf{M}}$. The corresponding relation for the parabolic restriction follows by adjunction. $\square$

¿From Theorem 12.8 we deduce two important results. First, that $D_{\mathbf{G}}$ is an involution on the space of virtual characters, and second that it sends an irreducible character to a signed irreducible character.

Corollary 12.10. *$D_{\mathbf{G}} \circ D_{\mathbf{G}}$ is the identity map on $K_0(\mathbb{C}\mathbf{G}^F\text{-}\mathsf{mod})$.*

Proof. We use the expression of $D_{\mathbf{G}}$ in terms of standard Levi subgroups. Using in addition Theorem 12.8 we get

$$\begin{aligned} D_{\mathbf{G}} \circ D_{\mathbf{G}} &= \sum_{I \subset S/F} (-1)^{|I|} R_{\mathbf{L}_I}^{\mathbf{G}} \circ {}^*R_{\mathbf{L}_I}^{\mathbf{G}} \circ D_{\mathbf{G}} \\ &= \sum_{I \subset S/F} (-1)^{|I|} R_{\mathbf{L}_I}^{\mathbf{G}} \circ D_{\mathbf{L}_I} \circ {}^*R_{\mathbf{L}_I}^{\mathbf{G}}. \end{aligned}$$

The group $\mathbf{B}_I = \mathbf{B} \cap \mathbf{L}_I$ is a Borel subgroup of $\mathbf{L}_I$ containing $\mathbf{T}$. By Proposition 8.10, the F-stable standard parabolic subgroups of $\mathbf{L}_I$ are of the form $\mathbf{P}_J \cap \mathbf{L}_I$ for $J \subset I$, with a Levi complement $\mathbf{L}_J$. Therefore

$$\begin{aligned} D_{\mathbf{G}} \circ D_{\mathbf{G}} &= \sum_{J \subset I \subset S/F} (-1)^{|I|+|J|} R_{\mathbf{L}_I}^{\mathbf{G}} \circ R_{\mathbf{L}_J}^{\mathbf{L}_I} \circ {}^*R_{\mathbf{L}_J}^{\mathbf{L}_I} \circ {}^*R_{\mathbf{L}_I}^{\mathbf{G}} \\ &= \sum_{J \subset I \subset S/F} (-1)^{|I|+|J|} R_{\mathbf{L}_J}^{\mathbf{G}} \circ {}^*R_{\mathbf{L}_J}^{\mathbf{G}} \\ &= \sum_{J \subset S/F} (-1)^{|J|} \Big(\sum_{I \supset J} (-1)^{|I|} \Big) R_{\mathbf{L}_J}^{\mathbf{G}} \circ {}^*R_{\mathbf{L}_J}^{\mathbf{G}} \end{aligned}$$

which equals Id since $\sum_{J \subset I \subset S/F} (-1)^{|I|} = 0$ whenever $J \neq S/F$ (it is the expansion of $(1-1)^n$ with n being the number of elements in the complement of J in S/F). $\square$

Corollary 12.11. *Let χ be an irreducible character of $\mathbf{G}^F$, and $(\mathbf{L}, \rho)$ be the cuspidal pair such that $\chi \in \mathrm{Irr}\big(\mathbf{G}^F, (\mathbf{L}, \rho)\big)$. Then $(-1)^{r(\mathbf{L})} D_{\mathbf{G}}(\chi)$ is an irreducible character of $\mathbf{G}^F$.*

Proof. If χ is irreducible, the virtual character $D_{\mathbf{G}}(\chi)$ satifies

$$\langle D_{\mathbf{G}}(\chi)\,; D_{\mathbf{G}}(\chi) \rangle_{\mathbf{G}^F} = \langle \chi\,; D_{\mathbf{G}} \circ D_{\mathbf{G}}(\chi) \rangle_{\mathbf{G}^F} = \langle \chi\,; \chi \rangle_{\mathbf{G}^F} = 1$$

since $D_{\mathbf{G}} \circ D_{\mathbf{G}}$ is the identity by Corollary 12.10. Therefore one of $D_{\mathbf{G}}(\chi)$ or $-D_{\mathbf{G}}(\chi)$ is an irreducible character. Now we use Theorem 12.8 and the previous corollary to get

$$0 < \langle \chi\,; R_{\mathbf{L}}^{\mathbf{G}}(\rho) \rangle_{\mathbf{G}^F} = \langle D_{\mathbf{G}}(\chi)\,; D_{\mathbf{G}}(R_{\mathbf{L}}^{\mathbf{G}}(\rho)) \rangle_{\mathbf{G}^F} = \langle D_{\mathbf{G}}(\chi)\,; R_{\mathbf{L}}^{\mathbf{G}}(D_{\mathbf{L}}(\rho)) \rangle_{\mathbf{G}^F}.$$

But with ρ being cuspidal, we find $D_{\mathbf{L}}(\rho) = (-1)^{r(\mathbf{L})}\rho$ using the definition of $D_{\mathbf{L}}$, and the result follows from the previous inequality. $\square$

13 l-adic cohomology

Given $\mathbf{X}$ an $\overline{\mathbb{F}}_q$-variety and ℓ a prime number not dividing q, Grothendieck has constructed "l-adic cohomology groups with compact support" $H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell)$ which are finite dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces. Here compact support could have been called "proper support"; *proper* morphisms are the algebraic geometry equivalent of compact morphisms: technically they are separated, of finite type, closed and they remain closed by base change. The important points for us is that finite morphisms are proper, and if $f \circ g$ is proper and f separated, then g is proper. In particular, a finite order automorphisms, or an endomorphism which has a power equal to a Frobenius endomorphism, is proper.

Proposition 13.1.

(i) $H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell) = 0$ *if* $i \notin [0, \ldots, 2\dim \mathbf{X}]$.

(ii) Every proper morphism $f : \mathbf{X} \to \mathbf{X}$ *induces a linear map* f^* *on* $H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell)$ *and* $f \mapsto f^*$ *is functorial.*

(iii) "Trace formula": If F *is the Frobenius endomorphism attached to some* $\mathbb{F}_q$*-structure on* $\mathbf{X}$*, then* F^* *is invertible and* $|\mathbf{X}^F| = \mathrm{Trace}(F^* \mid H^*_c(\mathbf{X}))$*, where* H^*_c *denote the virtual vector space* $\sum_i (-1)^i H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell)$.

The trace formula is an analogue of the Lefschetz theorem in algebraic topology; if $g \in \mathrm{Aut}(\mathbf{X})$ is of finite order, we define the *Lefschetz number* of g on $\mathbf{X}$ as $\mathcal{L}(g, \mathbf{X}) = \mathrm{Trace}(g^* \mid H^*_c(\mathbf{X}))$.

Corollary 13.2. *For* $g \in \mathrm{Aut}(\mathbf{X})$ *of finite order and* F*-stable, let* $R(t) = -\sum_{n>0} |\mathbf{X}^{gF^n}| t^n$*. Then* $\mathcal{L}(g, \mathbf{X}) = R(t)|_{t=\infty}$*, and is an integer independent of* ℓ*.*

Proof. By 6.9 (iii) for any $n > 0$ the morphism gF^n is a Frobenius endomorphism so verifies (iii) thus $R(t) = -\sum_{n>0} \mathrm{Trace}(gF^n \mid H^*_c(\mathbf{X}))t^n$.

Since F and g commute we may choose a basis of $H^*_c(\mathbf{X})$ where they are both triangular. If $\lambda_1, \ldots, \lambda_k$ are the eigenvalues of F and $x_1, \ldots, x_k$ those of g we have

$$R(t) = -\sum_{n>0} \sum_{i=1}^{k} \lambda_i^n x_i t^n = \sum_{i=1}^{k} x_i \frac{-\lambda_i t}{1 - \lambda_i t}.$$

It follows that $R(t)|_{t=\infty} = \sum_{i=1}^k x_i = \mathcal{L}(g, \mathbf{X})$. The independence of ℓ follows since ℓ does not appear in the definition of $R(t)$. Further, the formula above shows that $R(t)$ is a rational fraction. As a formal series with integer coefficients, it is a rational fraction with integer coefficients, thus $\mathcal{L}(g, \mathbf{X})$ is a rational number. It is also an algebraic integer as the trace of g on a representation, thus it is an integer. $\square$

We give now the main properties of Lefschetz number, which reflect properties of ℓ-adic cohomology we will mention. Sometimes the properties result directly from 13.2 and we will give the proof.

Proposition 13.3.

(i) Let $\mathbf{X}_1 \subset \mathbf{X}$ be a closed subvariety and let $\mathbf{X}_2$ be the complement open subvariety. We then have a long exact sequence

$$\ldots \to H^i_c(\mathbf{X}_1, \overline{\mathbb{Q}}_\ell) \to H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell) \to H^i_c(\mathbf{X}_2, \overline{\mathbb{Q}}_\ell) \to H^{i+1}_c(\mathbf{X}_1, \overline{\mathbb{Q}}_\ell) \to \ldots;$$

in particular $H^*_c(\mathbf{X}) = H^*_c(\mathbf{X}_1) + H^*_c(\mathbf{X}_2)$. The boundary maps vanish if $\mathbf{X}_1$ and $\mathbf{X}_2$ are connected components.

(ii) Let $\mathbf{X} = \coprod_j \mathbf{X}_j$ be a finite partition into locally closed subvarieties; if $g \in \operatorname{Aut}\mathbf{X}$ of finite order stabilizes the partition then $\mathcal{L}(g, \mathbf{X}) = \sum_{\{j \mid {}^g\mathbf{X}_j = \mathbf{X}_j\}} \mathcal{L}(g, \mathbf{X}_j)$.

Proof. (ii) follows from (i). It results also from 13.2 taking a Frobenius endomorphism F commuting to g and stabilizing each $\mathbf{X}_j$, in which case it is clear that $|\mathbf{X}^{gF^n}| = \sum_j |\mathbf{X}_j^{gF^n}|$. $\square$

Corollary 13.4. *Let $\mathbf{X}$ be an $\overline{\mathbb{F}}_q$-variety of dimension 0. Then*

(i) $H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell) = 0$ if $i \neq 0$ and $H^0_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell) \simeq \overline{\mathbb{Q}}_\ell[\mathbf{X}]$.

(ii) $\operatorname{Aut}(\mathbf{X})$ identifies to the symmetric group on $\mathbf{X}$, and $H^*_c(\mathbf{X}) \simeq \overline{\mathbb{Q}}_\ell[\mathbf{X}]$ is the corresponding permutation module; for $g \in \operatorname{Aut}(\mathbf{X})$ we have $\mathcal{L}(g, \mathbf{X}) = |\mathbf{X}^g|$.

Proof. These facts follow immediately from 13.1 and 13.3. $\square$

Proposition 13.5. *Let $\mathbf{X}$ and $\mathbf{X}'$ be two $\overline{\mathbb{F}}_q$-varieties. Then*

(i) $H^k_c(\mathbf{X} \times \mathbf{X}', \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{i+j=k} H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell) \otimes_{\overline{\mathbb{Q}}_\ell} H^j_c(\mathbf{X}', \overline{\mathbb{Q}}_\ell)$ ("Kunneth theorem").

(ii) Let $g \in \operatorname{Aut}\mathbf{X}$, $g' \in \operatorname{Aut}\mathbf{X}'$ be of finite order. Then $\mathcal{L}(g \times g', \mathbf{X} \times \mathbf{X}') = \mathcal{L}(g, \mathbf{X})\mathcal{L}(g', \mathbf{X}')$.

Proof. (ii) follows from (i) but again can be deduced from 13.2. Let $r * r'$ denote the Hadamard product $\sum_{i\geq 0} a_i b_i t^i$ of two series $r = \sum_{i\geq 0} a_i t^i$ and $r' = \sum_{i\geq 0} b_i t^i$. We need to show that when $r = \sum_{n\geq 1} |\mathbf{X}^{gF^n}| t^n$ and $r' = \sum_{n\geq 1} |\mathbf{X}'^{g'F^n}| t^n$ then $-(r * r')|_{t=\infty} = -r|_{t=\infty} \times -r'|_{t=\infty}$. This follows from the proof of 13.2 which showed that these series are linear combination of series $\frac{t}{1-\lambda t}$ which have this property. $\square$

Proposition 13.6. *Let $H \subset \operatorname{Aut}\mathbf{X}$ be a finite subgroup, and let $g \in C_{\operatorname{Aut}\mathbf{X}}(H)$ of finite order. Then*

(i) We have an isomorphism of $\overline{\mathbb{Q}}_\ell[g]$-modules: $H^i_c(\mathbf{X})^H \simeq H^i_c(\mathbf{X}/H)$.

(ii) $\mathcal{L}(g, \mathbf{X}/H) = |H|^{-1} \sum_{h \in H} \mathcal{L}(gh, \mathbf{X})$.

Proof. Note that the quotient $\mathbf{X}/H$ always exists since we assumed our varieties quasi-projective. Again, (ii) results from (i) or from choosing a Frobenius endomorphism F commuting to g and all elements of H; then

$$|(\mathbf{X}/H)^{gF^n}| = |H|^{-1} \sum_{h \in H} |\mathbf{X}^{ghF^n}|$$

□

Proposition 13.7. *Let $\mathbf{X} \simeq \mathbb{A}^n$ be an affine space. Then*

(i) $\dim H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell) = \begin{cases} 1, & \text{if } i=2n \\ 0, & \text{otherwise} \end{cases}$.

(ii) For any Frobenius endomorphism F on $\mathbf{X}$ defining an $\mathbb{F}_q$-structure, we have $|\mathbf{X}^F| = q^n$.

(iii) For any $g \in \operatorname{Aut}(\mathbf{X})$ of finite order we have $\mathcal{L}(g, \mathbf{X}) = 1$.

Proof. (ii) and (iii) follows from (i): the endomorphism F acts on the 1-dimensional space $H^{2n}_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell)$ by some scalar λ; for any $m > 0$ we have $|\mathbf{X}^{F^m}| = \lambda^m$ thus λ is a non-zero integer. Let A_0 be the $\mathbb{F}_q$-structure on $A = \overline{\mathbb{F}}_q[T_1, \ldots, T_n]$ attached to F as in (ii); then for some extension $\mathbb{F}_{q^{n_0}}$ we have $A_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^{n_0}} \simeq \mathbb{F}_{q^{n_0}}[T_1, \ldots, T_n]$; indeed, take n_0 such that the generators of A_0 lie in $\mathbb{F}_{q^{n_0}}[T_1, \ldots, T_n]$; then $A_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^{n_0}} \subset \mathbb{F}_{q^{n_0}}[T_1, \ldots, T_n]$ which implies equality since these two $\mathbb{F}_{q^{n_0}}$-vector spaces become equal after tensoring by $\overline{\mathbb{F}}_q$. Thus for m multiple of n_0 we have $|\mathbf{X}^{F^m}| = q^{mn}$, which proves $\lambda = q^n$ and gives (ii); (iii) follows. □

Proposition 13.8. *Let $\mathbf{X} \xrightarrow{\pi} \mathbf{X}'$ be a surjective morphism with fibers isomorphic to $\mathbb{A}^n$. Then:*

(i) $H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell) \simeq H^{i-2n}_c(\mathbf{X}', \overline{\mathbb{Q}}_\ell)(-n)$, a "Tate twist", which means an isomorphism as vector spaces and that for any $\mathbb{F}_q$-structure on $\mathbf{X}$ the action of F on $H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell)$ is q^n times that of F on $H^{i-2n}_c(\mathbf{X}', \overline{\mathbb{Q}}_\ell)$.

(ii) If $g \in \operatorname{Aut} \mathbf{X}$ (resp. $g' \in \operatorname{Aut} \mathbf{X}'$) are of finite order and such that $g'\pi = \pi g$ then the isomorphism of (i) sends g^* to g'^*.

(iii) In the situation of (ii) we have $\mathcal{L}(g, \mathbf{X}) = \mathcal{L}(g', \mathbf{X}')$.

Proof. (iii) follows from (i) and (ii) but can also be deduced by choosing $\mathbb{F}_q$-structures on $\mathbf{X}$ and $\mathbf{X}'$ compatible with π, g and g'. For the corresponding Frobenius F we have by 13.7 that $|\mathbf{X}^{gF^m}| = \sum_{y \in \mathbf{X}'^{g'F^m}} |\pi^{-1}(y)^{gF^m}| = |\mathbf{X}'^{g'F^m}| q^{mn}$, whence the result. □

Proposition 13.9. *Let $\mathbf{G}$ be a connected linear algebraic group acting on $\mathbf{X}$. Then*

(i) $\mathbf{G}$ acts trivially on $H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell)$ for all i.

(ii) For all $g \in \mathbf{G}$ we have $\mathcal{L}(g, \mathbf{X}) = \mathcal{L}(1, \mathbf{X})$.

Proof. Again, (ii) follows from (i) but also from choosing $\mathbb{F}_q$-structures on $\mathbf{G}$ and $\mathbf{X}$ which make the action compatible with the corresponding Frobenius, that is $F(gx) = F(g)F(x)$ for all $(g, x) \in \mathbf{G} \times \mathbf{X}$. Then for every n, by Lang's theorem there exists $h \in \mathbf{G}$ such that $h.^{F^n}h^{-1} = g$ and $x \mapsto h^{-1}x$ gives a bijection $\mathbf{X}^{gF^n} \simeq \mathbf{X}^{F^n}$, thus $|\mathbf{X}^{gF^n}| = |\mathbf{X}^{F^n}|$ which gives the result. $\square$

Proposition 13.10. *Let $g = su$ be the decomposition in a p'-part and a p-part of $g \in \operatorname{Aut} \mathbf{X}$ of finite order. Then $\mathcal{L}(g, \mathbf{X}) = \mathcal{L}(u, \mathbf{X}^s)$.*

Proof. This proposition cannot be deduced from the definition of the Lefschetz number directly; it reflects deeper properties of ℓ-adic cohomology. $\square$

14 Deligne-Lusztig induction

Here again $\mathbf{G}$ is a connected reductive group over $\overline{\mathbb{F}}_q$, with an isogeny F such that some power is a Frobenius.

We would like to construct an induction $R_{\mathbf{L}}^{\mathbf{G}}$ when $\mathbf{L}$ is an F-stable Levi which is not the Levi of any F-stable parabolic subgroup.

Example 14.1. In the case of the unitary group, if $n \neq m$, the F-stable Levi $\begin{pmatrix} \mathbf{U}_n & 0 \\ 0 & \mathbf{U}_m \end{pmatrix}$ is not the Levi of any F-stable parabolic subgroup; F exchanges the upper and lower triangular matrices.

The idea of Deligne and Lusztig is to construct an $\overline{\mathbb{F}}_q$-variety $\mathbf{X}$ attached to the unipotent radical $\mathbf{U}$ of a parabolic subgroup of Levi $\mathbf{L}$, with commuting actions of $\mathbf{G}^F$ and $\mathbf{L}^F$ on $\mathbf{X}$, and use $H_c^*(\mathbf{X})$ as a module to define induction. Further, we want this construction to generalize Harish-Chandra induction, that is when ${}^F\mathbf{U} = \mathbf{U}$ we should have $H_c^*(\mathbf{X}) \simeq \overline{\mathbb{Q}}_\ell[\mathbf{G}^F/\mathbf{U}^F]$.

Definition 14.2. *Let $\mathbf{X}_{\mathbf{U}}$ be the variety $\{g\mathbf{U} \in \mathbf{G}/\mathbf{U} \mid g\mathbf{U} \cap F(g\mathbf{U}) \neq \emptyset\}$. It has an obvious left $\mathbf{G}^F$-action, and a right $\mathbf{L}^F$-action since $\mathbf{L}$ normalizes $\mathbf{U}$. We define*

(i) Lustig induction of a $\overline{\mathbb{Q}}_\ell[\mathbf{L}^F]$-module by $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(M) = H_c^*(\mathbf{X}_{\mathbf{U}}) \otimes_{\overline{\mathbb{Q}}_\ell[\mathbf{L}^F]} M$. If λ is the character of M, then

$$(R_{\mathbf{L}}^{\mathbf{G}}\lambda)(g) = |\mathbf{L}^F|^{-1} \sum_{l \in \mathbf{L}^F} \mathcal{L}((g, l), \mathbf{X}_{\mathbf{U}})\lambda(l^{-1}).$$

(ii) Lustig restriction of a $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]$-module by ${}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(M) = M \otimes_{\overline{\mathbb{Q}}_\ell[\mathbf{G}^F]} H_c^*(\mathbf{X}_{\mathbf{U}})$. If γ is the character of M, then

$$({}^*R_{\mathbf{L}}^{\mathbf{G}}\gamma)(l) = |\mathbf{G}^F|^{-1} \sum_{g \in \mathbf{G}^F} \mathcal{L}((g, l), \mathbf{X}_{\mathbf{U}})\gamma(g^{-1}).$$

If $\mathbf{P}$ is F-stable, then $\mathbf{U}$ too, and $g\mathbf{U} \cap F(g\mathbf{U}) \neq \emptyset$ if and only if $g\mathbf{U} = F(g\mathbf{U})$. In this case $\mathbf{X}_{\mathbf{U}} \simeq \mathbf{G}^F/\mathbf{U}^F$, a discrete variety, and $H_c^*(\mathbf{X}_{\mathbf{U}})$ reduces to H_c^0, which equals $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F/\mathbf{U}^F]$.

We will denote the variety $\mathbf{X}_{\mathbf{U}}^{\mathbf{G}}$ if we need to specify the ambient group.

Note the following alternative models for $\mathbf{X}_{\mathbf{U}}$:

$$\begin{aligned}\mathbf{X}_{\mathbf{U}} &= \{g\mathbf{U} \in \mathbf{G}/\mathbf{U} \mid g^{-1}F(g) \in \mathbf{U}\cdot F(\mathbf{U})\}\\ &= \{g(\mathbf{U}\cap F(\mathbf{U})) \in \mathbf{G}/(\mathbf{U}\cap F(\mathbf{U})) \mid g^{-1}F(g) \in F(\mathbf{U})\}\end{aligned}$$

where we have still a left action of $\mathbf{G}^F$ and a right action of $\mathbf{L}^F$. Note that the last model shows that $\dim \mathbf{X}_{\mathbf{U}} = \dim \mathbf{U} - \dim(\mathbf{U}\cap F(\mathbf{U}))$.

Lusztig induction is transitive, as is Harish-Chandra induction:

Proposition 14.3. *(Transitivity) Let $\mathbf{P} \subset \mathbf{Q}$ be two parabolic subgroups of $\mathbf{G}$, such that there are F-stable Levi subgroups $\mathbf{L}$ of $\mathbf{P}$ and $\mathbf{M}$ of $\mathbf{Q}$ with $\mathbf{M} \subset \mathbf{L}$. Then $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \circ R^{\mathbf{L}}_{\mathbf{M}\subset\mathbf{L}\cap\mathbf{Q}} = R^{\mathbf{G}}_{\mathbf{M}\subset\mathbf{Q}}$.*

Proof. We have to show (see 9.5) that

$$H^*_c(\mathbf{X}_{\mathbf{U}}) \otimes_{\overline{\mathbb{Q}}_\ell[\mathbf{L}^F]} H^*_c(\mathbf{X}_{\mathbf{V}\cap\mathbf{L}}) \simeq H^*_c(\mathbf{X}_{\mathbf{V}})$$

where $\mathbf{P} = \mathbf{L}\mathbf{U}$ and $\mathbf{Q} = \mathbf{M}\mathbf{V}$ are the Levi decompositions. If G is a finite group with a right action of the variety $\mathbf{X}$ and a left action on the variety $\mathbf{Y}$, we denote $\mathbf{X} \times_G \mathbf{Y}$ the quotient of $\mathbf{X} \times \mathbf{Y}$ by the "diagonal" action where $g \in G$ acts by (g, g^{-1}). From the properties 13.5 and 13.6 of the cohomology, we see that the statement would come from $\mathbf{X}_{\mathbf{U}} \times_{\mathbf{L}^F} \mathbf{X}_{\mathbf{V}\cap\mathbf{L}} \simeq \mathbf{X}_{\mathbf{V}}$. This has a proof similar to 9.5 since if we set $\mathbf{V}' = \mathbf{V}\cap\mathbf{L}$ and $g\mathbf{U}\cap F(g\mathbf{U}) \neq \emptyset$ and $l\mathbf{V}'\cap F(l\mathbf{V}') \neq \emptyset$ then $gl\mathbf{V}\cap F(gl\mathbf{V}) \neq \emptyset$. Conversely, if $gl\mathbf{V}\cap F(gl\mathbf{V}) \neq \emptyset$ then $g \in F(g) \in \mathbf{P}F(\mathbf{P}) = \mathbf{L}\mathbf{U}F(\mathbf{U})$, so by modifying g by some element of $\mathbf{L}$ we may assume that $g^{-1}F(g) \in \mathbf{U}F(\mathbf{U})$. Then $g\mathbf{U}\cap F(g\mathbf{U}) \neq \emptyset$ and $gl\mathbf{V}\cap F(gl\mathbf{V}) \neq \emptyset$ can be written $\mathbf{U}g^{-1}F(g)F(\mathbf{U}) \cap F(l\mathbf{V}')\mathbf{V}'l^{-1} \neq \emptyset$. As the left term is in $\mathbf{U}F(\mathbf{U})$ and the right one in $\mathbf{L}$ the intersection must be 1 (since $\mathbf{P}\cap F(\mathbf{P}) = \mathbf{L}\cdot(\mathbf{U}\cap F(\mathbf{U}))$ by 8.9) so $l\mathbf{V}'\cap F(l\mathbf{V}') \neq \emptyset$. □

It is conjectured that Lusztig induction satisfies "Mackey formula":

Conjecture-Theorem 14.4. *Let $\mathbf{P}$ and $\mathbf{Q}$ be two parabolic subgroups of $\mathbf{G}$ and $\mathbf{L}$ (resp. $\mathbf{M}$) and F-stable Levi of $\mathbf{P}$ (resp. $\mathbf{Q}$). Then, conjecturally:*

$${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \circ R^{\mathbf{G}}_{\mathbf{M}\subset\mathbf{Q}} = \sum_x R^{\mathbf{L}}_{\mathbf{L}\cap{}^x\mathbf{M}\subset\mathbf{L}\cap{}^x\mathbf{Q}} \circ {}^*R^{{}^x\mathbf{M}}_{\mathbf{L}\cap{}^x\mathbf{M}\subset\mathbf{P}\cap{}^x\mathbf{M}} \circ \operatorname{ad} x,$$

where x runs over representatives of $\mathbf{L}^F\backslash\mathcal{S}(\mathbf{L},\mathbf{M})^F/\mathbf{M}^F$, where $\mathcal{S}(\mathbf{L},\mathbf{M}) = \{x \in \mathbf{G} \mid \mathbf{L}\cap{}^x\mathbf{M}$ contain a maximal torus of $\mathbf{G}\ \}$.

This is known in the following cases (the proofs are rather complicated):

- $\mathbf{L}$ or $\mathbf{M}$ is a torus (Deligne-Lusztig, 1983; see [Digne-Michel, 11.13]).
- When $q \neq 2$ or $\mathbf{G}$ has no component of type ${}^2E_6$, E_7 or E_8 (see [Bonnafé-Michel]).

As in 10.1 the Mackey formula shows the independence from $\mathbf{P}$ of $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ in the above cases.

It also allows to follow the proof of 12.8. But since $\varepsilon_{\mathbf{G}} := (-1)^{\mathbb{F}_q\text{-rank}(\mathbf{G})}$ is not necessarily equal in our case to $\varepsilon_{\mathbf{L}} := (-1)^{\mathbb{F}_q\text{-rank}(\mathbf{L})}$ the application of Lemma 12.9 gives

Theorem 14.5. *We have* $D_{\mathbf{G}} \circ R^{\mathbf{G}}_{\mathbf{L}} = \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{L}} R^{\mathbf{G}}_{\mathbf{L}}$.

When $\mathbf{L}$ is a torus $\mathbf{T}$ and $\theta \in \operatorname{Irr}(\mathbf{T}^F)$ the characters $R^{\mathbf{G}}_{\mathbf{T}}(\theta)$ where introduced in [Deligne-Lusztig] (1976) and are called Deligne-Lusztig characters. We will now concentrate on Deligne-Lusztig characters to simplify the exposition, though most theorems we give have (more complicated) analogues for general $R^{\mathbf{G}}_{\mathbf{L}}$.

The $R^{\mathbf{G}}_{\mathbf{T}}(\theta)$ generate "almost all" class functions on $\mathbf{G}^F$. A class fonction is said *uniform* if it is a linear combination of Deligne-Lusztig characters. Lusztig has shown that the characteristic function of a geometric conjugacy class is uniform; thus for instance in $\mathrm{GL}_n(\mathbb{F}_q)$ and $U_n(\mathbb{F}_q)$ all class functions are uniform.

For Deligne-Lusztig characters, the Mackey formula reduces to a scalar product formula:

Corollary 14.6. *Let* $\mathbf{T}$ *and* $\mathbf{T}'$ *be to* F*-stable maximal tori, and* $\theta \in \operatorname{Irr}(\mathbf{T}^F)$, $\theta' \in \operatorname{Irr}(\mathbf{T}'^F)$. *Then*

$$\langle R^{\mathbf{G}}_{\mathbf{T}}(\theta), R^{\mathbf{G}}_{\mathbf{T}'}(\theta')\rangle = \begin{cases} |N_{\mathbf{G}^F}(\mathbf{T},\theta)/\mathbf{T}^F|, & \textit{if } (\mathbf{T},\theta) \textit{ and } (\mathbf{T}',\theta') \textit{ are } \mathbf{G}^F\textit{-conjugate} \\ 0 & \textit{otherwise.} \end{cases}$$

We note in particular that $R^{\mathbf{G}}_{\mathbf{T}}(\theta)$ is an irreducible character up to sign if θ is in *general position*, that is is not invariant by a non-trivial element of $W_{\mathbf{G}}(\mathbf{T})$; this is the case for almost all θ (a Zariski open set).

14.1 The character formula

We denote by $\mathbf{G}_u$ the set of unipotent elements of an algebraic group $\mathbf{G}$.

Definition 14.7. *Given* $\mathbf{T}$, *an* F*-stable maximal torus of* $\mathbf{G}$ *and a Levi decomposition* $\mathbf{B} = \mathbf{T}\mathbf{U}$ *of a (possibly non* F*-stable) Borel subgroup containing* $\mathbf{T}$, *the* Green function $Q^{\mathbf{G}}_{\mathbf{T}} : \mathbf{G}^F_u \to \mathbb{Z}$ *is defined by* $u \mapsto R^{\mathbf{G}}_{\mathbf{T}}(\mathrm{Id})(u)$.

Proposition 14.8. *We have* $Q^{\mathbf{G}}_{\mathbf{T}}(u) = |\mathbf{T}^F|^{-1}\mathcal{L}(u, \mathbf{X}_{\mathbf{U}})$.

Proof. By definition and 13.10 we have $Q^{\mathbf{G}}_{\mathbf{T}}(u) = |\mathbf{T}^F|^{-1}\sum_{t\in\mathbf{T}^F}\mathcal{L}((u,t), \mathbf{X}_{\mathbf{U}}) = |\mathbf{T}^F|^{-1}\sum_{t\in\mathbf{T}^F}\mathcal{L}(u, \mathbf{X}^t_{\mathbf{U}})$. But $\mathbf{X}^t_{\mathbf{U}} = \emptyset$ unless $t = 1$. □

Proposition 14.9. *(character formula for* $R^{\mathbf{G}}_{\mathbf{T}}$ *and* ${}^*R^{\mathbf{G}}_{\mathbf{T}}$*) Let* $\mathbf{T}$ *be a* F*-stable maximal torus of* $\mathbf{G}$ *and let* $\theta \in \operatorname{Irr}(\mathbf{T}^F)$ *and* su *be the Jordan decomposition of an element of* $\mathbf{G}^F$, *then*

$$(R^{\mathbf{G}}_{\mathbf{T}}\theta)(su) = |C_{\mathbf{G}}(s)^{\circ F}|^{-1} \sum_{\{h\in\mathbf{G}^F \mid s\in{}^h\mathbf{T}\}} Q^{C_{\mathbf{G}}(s)^\circ}_{{}^h\mathbf{T}}(u)\,{}^h\theta(s) \qquad (i)$$

and let $\gamma \in \operatorname{Irr}(\mathbf{G}^F)$ *and* $t \in \mathbf{T}^F$, *then*

$$({}^*R_{\mathbf{T}}^{\mathbf{G}}\gamma)(t) = |\mathbf{T}^F||C_{\mathbf{G}}(t)^{\circ F}|^{-1} \sum_{u \in C_{\mathbf{G}}(t)^{\circ F}_u} Q_{\mathbf{T}}^{C_{\mathbf{G}}(t)^\circ}(u)\gamma(tu) \qquad (ii)$$

Proof. The main step is the following lemma.

Lemma 14.10. *With the above notation, we have*

$$\mathcal{L}((su,t), \mathbf{X_U}) = |\mathbf{T}^F||C_{\mathbf{G}}(t)^{\circ F}|^{-1} \sum_{\{h \in \mathbf{G}^F \mid {}^h t = s^{-1}\}} Q_{\mathbf{T}}^{C_{\mathbf{G}}(t)^\circ}({}^{h^{-1}}u)$$

Proof. From 13.10 we get $\mathcal{L}((su,t), \mathbf{X_U}) = \operatorname{Trace}(u \mid H_c^*(\mathbf{X}_{\mathbf{U}}^{s,t}))$. Let $\mathbf{V} := \mathbf{U} \cap F(\mathbf{U})$, $\mathbf{U}_t := \mathbf{U} \cap C_{\mathbf{G}}(t)^\circ$ and $\mathbf{V}_t := \mathbf{V} \cap C_{\mathbf{G}}(t)^\circ$. We first show that

$$\mathbf{X}_{\mathbf{U}}^{s,t} \simeq \coprod_{\{h \in [\mathbf{G}^F/C_{\mathbf{G}}(t)^{\circ F}] \mid s^h = t^{-1}\}} \mathbf{X}_{\mathbf{U}_t}^{C_{\mathbf{G}}(t)^\circ}$$

where $[\mathbf{G}^F/C_{\mathbf{G}}(t)^{\circ F}]$ denotes a set of representatives of $\mathbf{G}^F/C_{\mathbf{G}}(t)^{\circ F}$. Choosing the model $\mathbf{X}_{\mathbf{U}_t}^{C_{\mathbf{G}}(t)^\circ} = \{z\mathbf{V}_t \mid z^{-1}F(z) \in F(\mathbf{U}_t)\}$ and the model $\mathbf{X}_{\mathbf{U}}^{\mathbf{G}} = \{g\mathbf{V} \mid g^{-1}F(g) \in F(\mathbf{U})\}$, the isomorphism is given by mapping $z\mathbf{V}_t$ in the piece indexed by h to $hz\mathbf{V}$. This last element is in $\mathbf{X}_{\mathbf{U}}^{s,t}$ since $shz\mathbf{V}t = shzt\mathbf{V} = shtz\mathbf{V} = hz\mathbf{V}$ since $sht = h$. To show that each element of $\mathbf{X}_{\mathbf{U}}^{s,t}$ is in the image of a piece, we use

Lemma 14.11. *Any semisimple element of* $t\mathbf{V}^F$ *is* $\mathbf{V}^F$*-conjugate to* t.

Proof. Let t' be a semisimple element of $t\mathbf{V}^F$. Since $\mathbf{V}^F$ is normal in the group $K = \langle t, \mathbf{V}^F \rangle$, t' and $\mathbf{V}^F$ also generate this group, since t and t' have same image in $K/\mathbf{V}^F$. By the Schur-Zassenhaus the two p'-complements $\langle t \rangle$ and $\langle t' \rangle$ of the p-Sylow subgroup $\mathbf{V}^F$ are conjugate, and this conjugation conjugates t to t' since they are determined in the cyclic subgroup they generate by their image in $K/\mathbf{V}^F$. □

Now if $g\mathbf{V} \in \mathbf{X}_{\mathbf{U}}^{s,t}$ then $sg\mathbf{V}t = g\mathbf{V}$ thus $s^g \in t^{-1}\mathbf{V}$. By the lemma we can change the representative in $g\mathbf{V}$ to k such that $s^k = t^{-1}$. Applying F we have $s^{F(k)} = t^{-1}$ thus $k^{-1}F(k) \in C_{\mathbf{G}}(t)$. Since $k^{-1}F(k) \in F(\mathbf{U})$ it is even in $C_{\mathbf{G}}(t)^\circ$ by 4.8, thus by Lang's theorem there exists $z \in C_{\mathbf{G}}(t)^\circ$ such that $k^{-1}F(k) = z^{-1}F(z)$. Then $h := kz^{-1}$ is in $\{h \in \mathbf{G}^F \mid {}^h t = s^{-1}\}$ and $z\mathbf{V}_t$ is in $\mathbf{X}_{\mathbf{U}_t}^{C_{\mathbf{G}}(t)^\circ}$. In our construction, k can be changed by $C_{\mathbf{G}}(t)^{\circ F}$ which is why we can choose for h a given representative modulo $C_{\mathbf{G}}(t)^{\circ F}$. Since $uhz\mathbf{V} = hu^h z\mathbf{V}$, the element u acts on the piece $\mathbf{X}_{\mathbf{U}_t}^{C_{\mathbf{G}}(t)^\circ}$ indexed by h by u^h. We thus get

$$\operatorname{Trace}(u \mid H_c^*(\mathbf{X}_{\mathbf{U}}^{s,t})) = |C_{\mathbf{G}}(t)^{\circ F}|^{-1} \sum_{\{h \in \mathbf{G}^F \mid {}^h t = s^{-1}\}} \operatorname{Trace}({}^{h^{-1}}u | H_c^*(\mathbf{X}_{\mathbf{U}_t}^{C_{\mathbf{G}}(t)^\circ})),$$

whence lemma 14.10. □

We now prove proposition 14.9. Expanding 14.2 (i) by lemma 14.10 we get

$$(R_{\mathbf{T}}^{\mathbf{G}}\theta)(su) = \sum_{\{h\in\mathbf{G}^F \mid {}^{h^{-1}}s\in\mathbf{T}\}} |C_{\mathbf{G}}({}^{h^{-1}}s)^{0F}|^{-1} Q_{\mathbf{T}}^{C_{\mathbf{G}}({}^{h^{-1}}s)^\circ}({}^{h^{-1}}u)\theta(h^{-1}s),$$

whence (i) of 14.9 by conjugating the inner terms by h. Expanding similarly (ii) of 14.2 we get by lemma 14.10, using that if $g = su$ is a Jordan decomposition then $u \in C_{\mathbf{G}}(s)^{0F}_u$

$$({}^*R_{\mathbf{T}}^{\mathbf{G}}\gamma)(t) = |\mathbf{G}^F|^{-1}|C_{\mathbf{G}}(t)^{\circ F}|^{-1}|\mathbf{T}^F| \sum_{h\in\mathbf{G}^F} \sum_{u\in C_{\mathbf{G}}({}^h t)^{0F}_u} Q_{\mathbf{T}}^{C_{\mathbf{G}}(t)^\circ}({}^{h^{-1}}u)\gamma({}^h t u^{-1}).$$

whence (ii) of 14.9 changing the variable on which we sum to ${}^{h^{-1}}u^{-1}$. ∎

The Green functions can be computed by the theory of *Springer representations.*

Proposition 14.12. *Let* $\mathbf{T}$ *be a* F*-stable maximal torus subgroup of* $\mathbf{G}$*, let* $\gamma \in \mathrm{Irr}(\mathbf{G}^F)$ *and let* $s \in \mathbf{T}^F$*. Then*

$$({}^*R_{\mathbf{T}}^{\mathbf{G}}\gamma)(s) = ({}^*R_{\mathbf{T}}^{C_{\mathbf{G}}(s)^\circ}(\mathrm{Res}^{\mathbf{G}^F}_{C_{\mathbf{G}}(s)^{\circ F}}\gamma))(s).$$

Proof. This results from the remark that, in the character formula 14.9 for ${}^*R_{\mathbf{T}}^{\mathbf{G}}\gamma$, the right-hand side does not change if we replace $\mathbf{G}$ by $C_{\mathbf{G}}(s)^\circ$. ∎

Proposition 14.13. *Let* f *be a class function on* $\mathbf{G}^F$ *which depends only on the semisimple part — that is* $f(su) = f(s)$ *if* su *is a Jordan decomposition — then, for any* F*-stable maximal torus* $\mathbf{T}$ *of* $\mathbf{G}$ *and any function* θ *on* $\mathbf{T}^F$ *(resp.* γ *on* $\mathbf{G}^F$*), we have*

(i) $R_{\mathbf{T}}^{\mathbf{G}}(\theta.\mathrm{Res}^{\mathbf{G}^F}_{\mathbf{T}^F} f) = (R_{\mathbf{T}}^{\mathbf{G}}\theta).f$

(ii) $({}^*R_{\mathbf{T}}^{\mathbf{G}}\gamma).\mathrm{Res}^{\mathbf{G}^F}_{\mathbf{T}^F} f = {}^*R_{\mathbf{T}}^{\mathbf{G}}(\gamma.f)$

(iii) ${}^*R_{\mathbf{T}}^{\mathbf{G}} f = \mathrm{Res}^{\mathbf{G}^F}_{\mathbf{T}^F} f$

Proof. The character formula 14.9(i) gives

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta.\mathrm{Res}^{\mathbf{G}^F}_{\mathbf{T}^F} f)(su) = |C_{\mathbf{G}}(s)^{\circ F}|^{-1} \sum_{\{h\in\mathbf{G}^F \mid s\in{}^h\mathbf{T}\}} Q_{{}^h\mathbf{T}}^{C_{\mathbf{G}}(s)^\circ}(u)\,{}^h\theta(s)\,{}^h f(s),$$

which gives (i) using ${}^h f(su) = f(s) = f(su)$; equality (ii) results from (i) by adjunction.

(iii) follows from the special case of (ii) where $\gamma = \mathrm{Id}_{\mathbf{G}}$, and the special case of (iii) where $f = \mathrm{Id}_{\mathbf{G}}$. Let us prove this last fact: by definition ${}^*R_{\mathbf{T}}^{\mathbf{G}}(\mathrm{Id}_{\mathbf{G}})(t) = |\mathbf{G}^F|^{-1}\sum_{g\in\mathbf{G}^F}\mathcal{L}((g,t),\mathbf{X}_{\mathbf{U}})$. Looking at the model $\mathbf{X}_{\mathbf{U}} = \{g\mathbf{V} \in \mathbf{G}/\mathbf{V} \mid g^{-1}F(g) \in F(\mathbf{U})\}$, we see that $\mathbf{X}_{\mathbf{U}} = \tilde{\mathbf{X}}_{\mathbf{U}}/\mathbf{V}$, where $\tilde{\mathbf{X}}_{\mathbf{U}} = \{g \in \mathbf{G} \mid g^{-1}F(g) \in F(\mathbf{U})\}$, so by 13.8(iii) we have $\mathcal{L}((g,t),\mathbf{X}_{\mathbf{U}}) = \mathcal{L}((g,t),\tilde{\mathbf{X}}_{\mathbf{U}})$.

Now by 13.6(ii) we have $|\mathbf{G}^F|^{-1}\sum_{g\in\mathbf{G}^F}\mathcal{L}((g,t),\tilde{\mathbf{X}}_{\mathbf{U}}) = \mathcal{L}(t,\tilde{\mathbf{X}}_{\mathbf{U}}^{\mathbf{G}^F})$ and the map $g\mapsto g^{-1}F(g)$ gives an isomorphism $\tilde{\mathbf{X}}_{\mathbf{U}}^{\mathbf{G}^F}\simeq F(\mathbf{U})$; and by 13.7 we have $\mathcal{L}(t,F(\mathbf{U}))=1$, whence the result. □

Propositions 14.12 and 14.13 are still valid for $R_{\mathbf{L}}^{\mathbf{G}}$, using the character formula for $R_{\mathbf{L}}^{\mathbf{G}}$ (see [Digne-Michel, 12.5 and 12.6], or [Digne-Michel, 7.4, 7.5 and 7.6] in the setting of Harish-Chandra induction).

Proposition 14.14. *Let s be the semi-simple part of an element $l\in\mathbf{L}^F$; then* $(\operatorname{Res}^{\mathbf{L}^F}_{C_{\mathbf{L}}(s)^{\circ F}}{}^*R_{\mathbf{L}}^{\mathbf{G}}\chi)(l) = ({}^*R^{C_{\mathbf{G}}(s)^\circ}_{C_{\mathbf{L}}(s)^\circ}\operatorname{Res}^{\mathbf{G}^F}_{C_{\mathbf{G}}(s)^{\circ F}}\chi)(l)$.

Proposition 14.15. *If f is a class function on $\mathbf{G}^F$ which depends only on the semisimple part and $\gamma\in\operatorname{Irr}(\mathbf{G}^F)$, $\lambda\in\operatorname{Irr}(\mathbf{L}^F)$ then*

(i) $R_{\mathbf{L}}^{\mathbf{G}}(\lambda\operatorname{Res}^{\mathbf{G}^F}_{\mathbf{L}^F}f) = (R_{\mathbf{L}}^{\mathbf{G}}\lambda)f$.

(ii) $({}^*R_{\mathbf{L}}^{\mathbf{G}}\gamma).\operatorname{Res}^{\mathbf{G}^F}_{\mathbf{L}^F}f = {}^*R_{\mathbf{L}}^{\mathbf{G}}(\gamma f)$.

(iii) ${}^*R_{\mathbf{L}}^{\mathbf{G}}f = \operatorname{Res}^{\mathbf{G}^F}_{\mathbf{L}^F}f$.

Corollary 14.16. *(of 12.9) If s is the semi-simple part of $x\in\mathbf{G}^F$ and $\chi\in\operatorname{Irr}(\mathbf{G}^F)$, then* $(D_{\mathbf{G}}\chi)(x) = \varepsilon_{\mathbf{G}}\varepsilon_{C_{\mathbf{G}}(s)^\circ}(D_{C_{\mathbf{G}}(s)^\circ}\circ\operatorname{Res}^{\mathbf{G}}_{C_{\mathbf{G}}(s)^\circ}\chi)(x)$.

Proof. If $\mathbf{L}$ is a F-stable Levi subgroup of a F-stable parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, we have (see 9.3) $R_{\mathbf{L}}^{\mathbf{G}}\circ{}^*R_{\mathbf{L}}^{\mathbf{G}}\chi = \operatorname{Ind}^{\mathbf{G}^F}_{\mathbf{P}^F}\operatorname{Inf}^{\mathbf{P}^F}_{\mathbf{L}^F}{}^*R_{\mathbf{L}}^{\mathbf{G}}(\chi)$, whence

$$\begin{aligned}(R_{\mathbf{L}}^{\mathbf{G}}\circ{}^*R_{\mathbf{L}}^{\mathbf{G}}\chi)(x) &= |\mathbf{P}^F|^{-1}\sum_{\{g\in\mathbf{G}^F\mid {}^g\mathbf{P}\ni x\}}(\operatorname{Inf}^{\mathbf{P}^F}_{\mathbf{L}^F}{}^*R_{\mathbf{L}}^{\mathbf{G}}\chi)({}^{g^{-1}}x)\\ &= \sum_{\{\mathbf{P}'\sim_{\mathbf{G}^F}\mathbf{P}\mid\mathbf{P}'\ni x\}}(\operatorname{Inf}^{\mathbf{P}'^F}_{\mathbf{L}'^F}{}^*R_{\mathbf{L}'}^{\mathbf{G}}\chi)(x)\end{aligned}$$

where in the last summation $\mathbf{L}'$ is a F-stable Levi subgroup of $\mathbf{P}'$. As any F-stable parabolic subgroup of $\mathbf{G}$ is $\mathbf{G}^F$-conjugate to some $\mathbf{P}\supset\mathbf{B}$, we have

$$\begin{aligned}(D_{\mathbf{G}}\chi)(x) &= \sum_{\mathbf{P}\supset\mathbf{B}}(-1)^{r(\mathbf{P})}\sum_{\{\mathbf{P}'\sim_{\mathbf{G}^F}\mathbf{P}\mid\mathbf{P}'\ni x\}}(\operatorname{Inf}^{\mathbf{P}'^F}_{\mathbf{L}'^F}{}^*R_{\mathbf{L}'}^{\mathbf{G}}\chi)(x)\\ &= \sum_{\mathbf{P}'\ni x}(-1)^{r(\mathbf{P}')}(\operatorname{Inf}^{\mathbf{P}'^F}_{\mathbf{L}'^F}{}^*R_{\mathbf{L}'}^{\mathbf{G}}\chi)(x).\end{aligned}\tag{1}$$

By 14.14 we then get

$$\begin{aligned}(D_{\mathbf{G}}\chi)(x) &= \sum_{\mathbf{P}\ni x}(-1)^{r(\mathbf{P})}(\operatorname{Inf}^{\mathbf{P}^F\cap C_{\mathbf{G}}(s)^\circ}_{\mathbf{L}^F\cap C_{\mathbf{G}}(s)^\circ}{}^*R^{C_{\mathbf{G}}(s)^\circ}_{\mathbf{L}\cap C_{\mathbf{G}}(s)^\circ}\chi)(x)\\ &= \sum_{\mathbf{P}'}\Big(\sum_{\{\mathbf{P}\mid\mathbf{P}\cap C_{\mathbf{G}}(s)^\circ=\mathbf{P}'\}}(-1)^{r(\mathbf{P})}\Big)(\operatorname{Inf}^{\mathbf{P}'^F}_{\mathbf{L}'^F}{}^*R^{C_{\mathbf{G}}(s)^\circ}_{\mathbf{L}'}\chi)(x)\end{aligned}$$

where $\mathbf{P}'$ runs over the set of F-stable parabolic subgroup $C_{\mathbf{G}}(s)^\circ$ and where $\mathbf{L}'$ is any F-stable Levi subgroup of $\mathbf{P}'$. We use now that it is possible to apply 12.9 with $\mathbf{H}=C_{\mathbf{G}}(s)^\circ$ (see for instance [Digne-Michel, 8.12]) and compare with the equality (1) applied in $C_{\mathbf{G}}(s)^\circ$, which gives the result. □

15 The Steinberg character and applications

In this chapter we use the duality introduced in Section 12 to define and study the famous "Steinberg character" which was originally defined by Steinberg in 1956.

Definition 15.1. *The irreducible character* $\mathrm{St}_{\mathbf{G}} = D_{\mathbf{G}}(\mathrm{Id}_{\mathbf{G}})$ *where* $\mathrm{Id}_{\mathbf{G}}$ *is the trivial character of* $\mathbf{G}^F$ *is called* the Steinberg character *of* $\mathbf{G}^F$.

We get using 14.5 and 14.15(iii)

$$^*R^{\mathbf{G}}_{\mathbf{L}}\mathrm{St}_{\mathbf{G}} = {}^*R^{\mathbf{G}}_{\mathbf{L}}D_{\mathbf{G}}(\mathrm{Id}_{\mathbf{G}}) = \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{L}}D_{\mathbf{L}}{}^*R^{\mathbf{G}}_{\mathbf{L}}(\mathrm{Id}_{\mathbf{G}}) = \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{L}}D_{\mathbf{L}}(\mathrm{Id}_{\mathbf{L}}) = \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{L}}\,\mathrm{St}_{\mathbf{L}}\,.$$

For $\mathbf{L}$ a $\mathbf{G}$-split torus $\mathbf{T}$, since $\mathrm{St}_{\mathbf{T}} = \mathrm{Id}_{\mathbf{T}}$, we get ${}^*R^{\mathbf{G}}_{\mathbf{T}}\mathrm{St}_{\mathbf{G}} = \mathrm{Id}_{\mathbf{T}}$, thus $\langle \mathrm{St}_{\mathbf{G}}, R^{\mathbf{G}}_{\mathbf{T}}(\mathrm{Id}_{\mathbf{T}})\rangle_{\mathbf{G}^F} = 1$ thus we get by 12.11 that $\mathrm{St}_{\mathbf{G}}$ is a true character (not the opposite of one). We have the following more precise result.

Lemma 15.2. *Let* $\mathbf{T}$ *be an* F*-stable maximal torus of an* F*-stable Borel subgroup* $\mathbf{B}$ *of* $\mathbf{G}$*; then* $\mathrm{Res}^{\mathbf{G}^F}_{\mathbf{B}^F}\mathrm{St}_{\mathbf{G}} = \mathrm{Ind}^{\mathbf{B}^F}_{\mathbf{T}^F}\mathrm{Id}_{\mathbf{T}}$.

Proof. Using the definitions of $\mathrm{St}_{\mathbf{G}}$ and of $D_{\mathbf{G}}$ we get

$$\mathrm{St}_{\mathbf{G}} = \sum_{I\subset S/F}(-1)^{|I|}\,\mathrm{Ind}^{\mathbf{G}^F}_{\mathbf{P}_I^F}\mathrm{Id}_{\mathbf{P}_I},$$

where the notation is the same as in the proof of 12.10. So we have

$$\begin{aligned}\mathrm{Res}^{\mathbf{G}^F}_{\mathbf{B}^F}(\mathrm{St}_{\mathbf{G}}) &= \sum_{I\subset S/F}(-1)^{|I|}\,\mathrm{Res}^{\mathbf{G}^F}_{\mathbf{B}^F}\mathrm{Ind}^{\mathbf{G}^F}_{\mathbf{P}_I^F}\mathrm{Id}_{\mathbf{P}_I}\\ &= \sum_{I\subset S/F}(-1)^{|I|}\sum_{w\in {}^IW^F}\mathrm{Ind}^{\mathbf{B}^F}_{\mathbf{B}^F\cap{}^w\mathbf{P}_I^F}\mathrm{Res}^{\mathbf{P}_I^F}_{\mathbf{B}^F\cap{}^w\mathbf{P}_I^F}\mathrm{Id}_{\mathbf{P}_I},\end{aligned}$$

the last equality following from the Mackey formula for induction and restriction, where we have denoted by ${}^IW^F$ the set of reduced-I elements of W^F, which is a set of representatives for the double cosets $\mathbf{B}^F\backslash\mathbf{G}^F/\mathbf{P}_I^F$ by the existence of the relative (B,N)-pair. But we have $\mathbf{B}\cap{}^w\mathbf{P}_I = \mathbf{B}\cap{}^w\mathbf{B}$ since $\mathbf{B}w\cap w\mathbf{P}_I = \coprod_{v\in W_I}(\mathbf{B}w\cap w\mathbf{B}v\mathbf{B})$ and as the lengths add we have $w\mathbf{B}v\mathbf{B}\subset\mathbf{B}wv\mathbf{B}$ thus meets $\mathbf{B}w\mathbf{B}$ only for $v=1$. Using this result in the formula for $\mathrm{Res}^{\mathbf{G}^F}_{\mathbf{B}^F}\mathrm{St}_{\mathbf{G}}$ and exchanging the summations gives

$$\mathrm{Res}^{\mathbf{G}^F}_{\mathbf{B}^F}\mathrm{St}_{\mathbf{G}} = \sum_{w\in W}\Big(\sum_{\{I\subset S/F\mid w\in{}^IW^F\}}(-1)^{|I|}\Big)\,\mathrm{Ind}^{\mathbf{B}^F}_{\mathbf{B}^F\cap{}^w\mathbf{B}^F}\mathrm{Id}_{\mathbf{B}\cap{}^w\mathbf{B}}\,.$$

By lemma 2.8 we have $w\in{}^IW^F$ if and only if $I\cap N(w)=\emptyset$, where N is computed in W^F, so the inner sum is $\sum_{I\cap N(w)=\emptyset}(-1)^{|I|}$, which is different from zero only if $N(w)=S/F$, thus w is the longest element of W^F and in that case we have $\mathbf{B}\cap{}^w\mathbf{B}=\mathbf{T}$, whence the result. $\square$

Corollary 15.3. *We have*

$$\mathrm{St}_{\mathbf{G}}(x) = \begin{cases} \varepsilon_{\mathbf{G}}\varepsilon_{(C_{\mathbf{G}}(x)^\circ)}|C_{\mathbf{G}}(x)^{\circ F}|_p & \text{if } x \text{ is semi-simple,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let s be the semi-simple part of x. We have

$$\mathrm{St}_{\mathbf{G}}(x) = D_{\mathbf{G}}(\mathrm{Id}_{\mathbf{G}})(x) = \varepsilon_{\mathbf{G}}\varepsilon_{C_{\mathbf{G}}(s)^\circ}(D_{C_{\mathbf{G}}(s)^\circ}(\mathrm{Id}_{C_{\mathbf{G}}(s)^\circ}))(x) = \varepsilon_{\mathbf{G}}\varepsilon_{C_{\mathbf{G}}(s)^\circ}\,\mathrm{St}_{C_{\mathbf{G}}(s)^\circ}(x)$$

by 14.16. So we may assume that s is central in $\mathbf{G}$. But then there exists an F-stable Borel subgroup $\mathbf{B}$ which contains x. Indeed the unipotent part of x is contained in an F-stable Borel subgroup by 7.12(iv), and s, being central, is contained in any Borel subgroup. So by lemma 15.2 we have $\mathrm{St}_{\mathbf{G}}(x) = (\mathrm{Res}^{\mathbf{G}^F}_{\mathbf{B}^F}\mathrm{St}_{\mathbf{G}})(x) = (\mathrm{Ind}^{\mathbf{B}^F}_{\mathbf{T}^F}\mathrm{Id}_{\mathbf{T}})(x)$. Thus $\mathrm{St}_{\mathbf{G}}(x) = 0$ unless x has a conjugate in $\mathbf{T}^F$, that is is semi-simple, thus $x = s$ and we get $\mathrm{St}_{\mathbf{G}}(x) = |\mathbf{B}^F|/|\mathbf{T}^F| = |\mathbf{G}^F|_p$, the last equality by 7.12, whence the result. □

Corollary 15.4. *The dual of the regular representation* $\mathrm{reg}_{\mathbf{G}}$ *of* $\mathbf{G}^F$ *is* $D_{\mathbf{G}}(\mathrm{reg}_{\mathbf{G}}) = \gamma_p$*, where* γ_p *is the function whose value is* $|\mathbf{G}^F|_{p'}$ *on unipotent elements and* 0 *elsewhere.*

Proof. By 14.15 we have $D_{\mathbf{G}}(\chi.f) = D_{\mathbf{G}}(\chi).f$ for any $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ and any class function f on $\mathbf{G}^F$ which depends only on the semisimple part. Applying this to $f = \gamma_p$ we have

$$D_{\mathbf{G}}(\gamma_p) = D_{\mathbf{G}}(\mathrm{Id}_{\mathbf{G}}.\gamma_p) = D_{\mathbf{G}}(\mathrm{Id}_{\mathbf{G}})\gamma_p = \mathrm{St}_{\mathbf{G}}.\gamma_p = \mathrm{reg}_{\mathbf{G}},$$

the second equality by the definition of $D_{\mathbf{G}}$ and 14.15 (i) and (ii), and the last equality by 15.3. □

Corollary 15.5. *The number of unipotent elements in* $\mathbf{G}^F$ *is equal to* $(|\mathbf{G}^F|_p)^2$.

Proof. This results from 15.4 by writing $\langle \mathrm{reg}_{\mathbf{G}}, \mathrm{reg}_{\mathbf{G}}\rangle = \langle \gamma_p, \gamma_p\rangle$. □

We can now give the dimension of the (virtual) characters $R^{\mathbf{G}}_{\mathbf{T}}(\theta)$.

Proposition 15.6. *For any F-stable maximal torus* $\mathbf{T}$ *and any* $\theta \in \mathrm{Irr}(\mathbf{T}^F)$*, we have* $\dim R^{\mathbf{G}}_{\mathbf{T}}(\theta) = \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}|\mathbf{G}^F|_{p'}|\mathbf{T}^F|^{-1}$.

Proof. By 14.9 the dimension we want to compute does not depend on θ. On the other hand, taking the scalar product of the equality ${}^*R^{\mathbf{G}}_{\mathbf{T}}\mathrm{St}_{\mathbf{G}} = \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}\,\mathrm{Id}_{\mathbf{T}}$ with θ, we get $\langle R^{\mathbf{G}}_{\mathbf{T}}(\theta), \mathrm{St}_{\mathbf{G}}\rangle_{\mathbf{G}^F} = \varepsilon_{\mathbf{T}}\varepsilon_{\mathbf{G}}\delta_{1,\theta}$, whence $\langle \sum_\theta R^{\mathbf{G}}_{\mathbf{T}}(\theta), \mathrm{St}_{\mathbf{G}}\rangle_{\mathbf{G}^F} = \varepsilon_{\mathbf{T}}\varepsilon_{\mathbf{G}}$. But by 14.9 $\sum_\theta R^{\mathbf{G}}_{\mathbf{T}}(\theta)$ vanishes on all elements with non-trivial semi-simple part. Since $\mathrm{St}_{\mathbf{G}}$ vanishes outside semi-simple elements, the scalar product above reduces to

$$|\mathbf{G}^F|^{-1}|\mathbf{T}^F|\mathrm{St}_{\mathbf{G}}(1)\dim(R^{\mathbf{G}}_{\mathbf{T}}(\theta)).$$

This gives the result after replacing $\mathrm{St}_{\mathbf{G}}(1)$ by its value. □

Remark 15.7. If $\mathbf{T}_w$ is a maximal F-stable torus of type $w \in W(\mathbf{T})$ with respect to some $\mathbf{G}$-split torus $\mathbf{T}$, we have $\varepsilon_{\mathbf{T}_w}\varepsilon_{\mathbf{G}} = (-1)^{l(w)}$, where $l(w)$ is the length of w in $W(\mathbf{T})$.

Proof. Let $V = X(\mathbf{T})\otimes\mathbb{R}$; by 12.2 and 12.5(i) we have $\varepsilon_{\mathbf{G}} = (-1)^{\dim(V^\tau)}$ and, as $(\mathbf{T}_w, F)$ is conjugate to $(\mathbf{T}, wF)$ (see 7.10), we have $\varepsilon_{\mathbf{T}_w} = (-1)^{\dim(V^{w\tau})}$. Since τ is an automorphism of finite order of the lattice $X(\mathbf{T})$, we have $(-1)^{\dim(V)-\dim(V^\tau)} = \det(\tau)$, and similarly $(-1)^{\dim(V)-\dim(V^{w\tau})} = \det(w\tau)$, which gives the result as $\det(w) = (-1)^{l(w)}$, since the determinant of a reflection is -1. $\square$

Proposition 15.8. *For any $\theta \in \operatorname{Irr}(\mathbf{T}^F)$ (resp. $\gamma \in \operatorname{Irr}(\mathbf{G}^F)$) we have*

- $\operatorname{St}_{\mathbf{G}} \otimes R_{\mathbf{T}}^{\mathbf{G}}\theta = \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}} \operatorname{Ind}_{\mathbf{T}^F}^{\mathbf{G}^F}(\theta)$
- ${}^*R_{\mathbf{T}}^{\mathbf{G}}(\gamma \otimes \operatorname{St}_{\mathbf{G}}) = \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}} \operatorname{Res}_{\mathbf{T}^F}^{\mathbf{G}^F} \gamma$.

Proof. As (i) is the adjoint of (ii) it is enough to prove (ii). By the character formula 14.9, and taking in account that $\operatorname{St}_{\mathbf{G}}$ vanishes outside semisimple elements, we have to check that for $t \in \mathbf{T}^F$ we have

$$|\mathbf{T}^F||C_{\mathbf{G}}(t)^{\circ F}|^{-1} Q_{\mathbf{T}}^{C_{\mathbf{G}}(t)^\circ}(1)\gamma(t)\operatorname{St}_{\mathbf{G}}(t) = \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}\gamma(t).$$

This results that from the previous proposition we have $Q_{\mathbf{T}}^{C_{\mathbf{G}}(t)^\circ}(1) = \dim R_{\mathbf{T}}^{C_{\mathbf{G}}(t)^\circ} = |C_{\mathbf{G}}(t)^{\circ F}|_{p'}|\mathbf{T}^F|^{-1}\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}}$, and from the value of $\operatorname{St}_{\mathbf{G}}(t)$. $\square$

We now prove that the identity and the regular representation are both linear combinations of Deligne-Lusztig characters.

Proposition 15.9. *The orthogonal projection of class functions onto the subspace of uniform functions is given by the operator:*

$$p = \sum_{w \in H^1(F,W)} |C_W(wF)|^{-1} R_{\mathbf{T}_w}^{\mathbf{G}} \circ {}^*R_{\mathbf{T}_w}^{\mathbf{G}}$$

where w runs over representatives of the F-classes of W and we take for model of $(\mathbf{T}_w, F)$ the pair $(\mathbf{T}_1, wF)$ where $\mathbf{T}_1$ is $\mathbf{G}$-split.

Proof. Since $p(\gamma)$ is clearly uniform for any $\gamma \in \operatorname{Irr}(\mathbf{G}^F)$, it is enough to check that for any $(\mathbf{T}, \theta)$ we have $\langle \gamma, R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle_{\mathbf{G}^F} = \langle p(\gamma), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle_{\mathbf{G}^F}$. We have

$$\begin{aligned}\langle p(\gamma), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle_{\mathbf{G}^F} &= \langle \sum_{w \in H^1(F,W)} |C_W(wF)|^{-1} R_{\mathbf{T}_w}^{\mathbf{G}} \circ {}^*R_{\mathbf{T}_w}^{\mathbf{G}}, R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle_{\mathbf{G}^F} \\ &= \langle \sum_{w \in H^1(F,W)} |C_W(wF)| {}^*R_{\mathbf{T}_w}^{\mathbf{G}}, {}^*R_{\mathbf{T}_w}^{\mathbf{G}} R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle_{\mathbf{G}^F}\end{aligned}$$

but, by 14.6 we have:

$${}^*R_{\mathbf{T}_w}^{\mathbf{G}} R_{\mathbf{T}}^{\mathbf{G}}(\theta) = \begin{cases} \sum_{v \in C_W(wF)} {}^v\theta & \text{if } \mathbf{T} = \mathbf{T}_w \\ 0 & \text{if } \mathbf{T} \text{ is not of type } w \end{cases}$$

so

$$\langle p(\gamma), R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle_{\mathbf{G}^F} = \langle {}^*R_{\mathbf{T}}^{\mathbf{G}}(\gamma), |C_W(wF)|^{-1} \sum_{v\in C_W(wF)} {}^v\theta\rangle_{\mathbf{G}^F} = \langle \gamma, R_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle_{\mathbf{G}^F}$$

the rightmost equality since $R_{\mathbf{T}_w}^{\mathbf{G}}({}^v\theta) = R_{\mathbf{T}_w}^{\mathbf{G}}(\theta)$. □

Proposition 15.10. $\mathrm{Id}_{\mathbf{G}}$ *and* $\mathrm{St}_{\mathbf{G}}$ *are uniform functions; we have*

(i) $\mathrm{Id}_{\mathbf{G}} = \sum_{w\in H^1(F,W)} |C_W(wF)|^{-1} R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id}_{\mathbf{T}_w})$.

(ii) $\mathrm{St}_{\mathbf{G}} = \sum_{w\in H^1(F,W)} |C_W(wF)|^{-1} \varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}_w} R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id}_{\mathbf{T}_w})$.

Proof. Since by 14.13(iii) we have ${}^*R_{\mathbf{T}}^{\mathbf{G}}(\mathrm{Id}_{\mathbf{G}}) = \mathrm{Id}_{\mathbf{T}}$, expression (i) represents $p(\mathrm{Id}_{\mathbf{G}})$. It is enough to check that $\mathrm{Id}_{\mathbf{G}}$ has same scalar product with this expression as with itself. But indeed we have

$$\begin{aligned} &\langle \mathrm{Id}_{\mathbf{G}}, \sum_{w\in H^1(F,W)} |C_W(wF)|^{-1} R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id}_{\mathbf{T}_w})\rangle_{\mathbf{G}^F} \\ &\quad = \sum_{w\in H^1(F,W)} |C_W(wF)|^{-1} \langle {}^*R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id}_{\mathbf{G}}), \mathrm{Id}_{\mathbf{T}_w}\rangle_{\mathbf{T}_w^F} \\ &\quad = \sum_{w\in H^1(F,W)} |C_W(wF)|^{-1} = 1. \end{aligned}$$

We get (ii) from (i) by applying duality. □

Corollary 15.11. *The character* $\mathrm{reg}_{\mathbf{G}}$ *of the regular representation of* $\mathbf{G}^F$ *is a uniform function; we have*

$$\mathrm{reg}_{\mathbf{G}} = \sum_{w\in H^1(F,W)} |C_W(wF)|^{-1} \dim(R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id}_{\mathbf{T}_w})) R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{reg}_{\mathbf{T}_w})$$

Proof. We saw in the proof of 15.4 that $\mathrm{reg}_{\mathbf{G}} = \mathrm{St}_{\mathbf{G}} \cdot \gamma_p$. Using expression (ii) above for $\mathrm{St}_{\mathbf{G}}$, it is enough to see that

$$\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{T}_w} R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id}_{\mathbf{T}_w})\gamma_p = \dim R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id}_{\mathbf{T}}) R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{reg}_{\mathbf{T}_w}).$$

This comes from the equality $R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id}_{\mathbf{T}_w})\gamma_p = R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Res}_{\mathbf{T}_w^F}^{\mathbf{G}^F}(\gamma_p))$ given by 14.13, from the fact that $\mathrm{Res}_{\mathbf{T}_w^F}^{\mathbf{G}^F}(\gamma_p)$ has value $|\mathbf{G}^F|_{p'}$ at 1 and 0 elsewhere, so is equal to $|\mathbf{G}^F|_{p'}|\mathbf{T}_w^F|^{-1}\,\mathrm{reg}_{\mathbf{T}_w}$, and from 15.6. □

15.1 Unipotent characters

We call *unipotent irreducible characters* the irreducible characters which occur in some $R_{\mathbf{T}}^{\mathbf{G}}(\mathrm{Id})$.

Lusztig's "Jordan decomposition of characters" states that the decomposition of $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ into irreducible characters is the same as the decomposition of

$R_{\mathbf{T}}^{\mathbf{H}}(\mathrm{Id})$ in unipotent irreducible characters for some other group $\mathbf{H}$. Let us explain this; we have seen in 5 that $\mathbf{G}$ is defined up to isomorphism by its root datum $(X, Y, \Phi, \Phi^\vee)$ where for some maximal torus $\mathbf{T}$ we have set $X = X(\mathbf{T})$, $Y = Y(\mathbf{T})$ and where Φ (resp. $\Phi^\vee$) are the roots (resp. coroots) with respect to $\mathbf{T}$. Now the quadruple $(Y, X, \Phi^\vee, \Phi)$ is the root datum of another group, called the *Langlands dual* of $\mathbf{G}$. If we denote $\mathbf{G}^*$ this dual group, and $\mathbf{T}^*$ the dual torus (such that $X(\mathbf{T}^*) = Y(\mathbf{T})$ and $Y(\mathbf{T}^*) = X(\mathbf{T})$), we still have an isogeny F^* on $\mathbf{G}^*$ by taking the dual of the p-morphism and by the proof of 7.14 we get natural isomorphisms $\mathbf{T}^F \simeq \mathrm{Irr}(\mathbf{T}^{*F^*})$ and $\mathrm{Irr}(\mathbf{T}^F) \simeq \mathbf{T}^{*F^*}$. Thus to any pair $(\mathbf{T}, \theta)$ where $\mathbf{T}$ is an F-stable maximal torus and $\theta \in \mathrm{Irr}(\mathbf{T}^F)$ corresponds a pair $(\mathbf{T}^{*F^*}, s)$ where $\mathbf{T}^*$ is an F^*-stable maximal torus and $s \in \mathbf{T}^{*F^*}$, and this bijection respects $\mathbf{G}^F$-conjugacy classes. This allows to use the notation $R_{\mathbf{T}}^{\mathbf{G}}(s)$ for $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ when $\mathbf{T}^* \ni s$.

Definition 15.12 (Lusztig series)**.** *Given $s \in \mathbf{G}^{*F^*}$, we define*

$$\mathcal{E}(\mathbf{G}^F, s) = \{\gamma \in \mathrm{Irr}(\mathbf{G}^F) \mid \exists \mathbf{T}^* \ni s, \langle R_{\mathbf{T}}^{\mathbf{G}}(s), \gamma \rangle_{\mathbf{G}^F} \neq 0\}.$$

With this definition the unipotent characters form $\mathcal{E}(\mathbf{G}^F, 1)$.

We can now state Lusztig's Jordan decomposition of characters:

Theorem 15.13. *Assume $\mathbf{G}$ has connected center. Then there is a bijection between $\mathcal{E}(\mathbf{G}^F, s)$ and $\mathcal{E}(C_{\mathbf{G}^{*F^*}}(s), 1)$, and this bijection extended by linearity maps $\varepsilon_{\mathbf{G}} R_{\mathbf{T}}^{\mathbf{G}}(s)$ to $\varepsilon_{C_{\mathbf{G}^*}(s)} R_{\mathbf{T}}^{C_{\mathbf{G}^*}(s)}(\mathrm{Id})$.*

The condition of connected center is to ensure that $C_{\mathbf{G}^*}(s)$ is a connected reductive group. There is a more general statement if the center is not connected, see [Digne-Michel, 13.23].

The decomposition of $R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id})$ into unipotent irreducible characters is thus an important topic. To explore this, given an extension $\tilde{\chi}$ to $W \rtimes \langle F \rangle$ of $\chi \in \mathrm{Irr}(W)^F$ we define

$$R_{\tilde{\chi}} := |W|^{-1} \sum_{w \in W} \tilde{\chi}(wF) R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id}) = \sum_{w \in H^1(F,W)} |C_W(wF)|^{-1} \tilde{\chi}(wF) R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id}).$$

Proposition 15.14.

$$\langle R_{\tilde{\chi}}, R_{\tilde{\psi}} \rangle_{\mathbf{G}^F} = \begin{cases} 0 & \textit{unless } \tilde{\chi} \textit{ and } \tilde{\psi} \textit{ are extensions of the same character} \\ 1 & \textit{if } \tilde{\chi} = \tilde{\psi}. \end{cases}$$

Proof.

$$\begin{aligned} \langle R_{\tilde{\chi}}, R_{\tilde{\psi}} \rangle_{\mathbf{G}^F} &= \sum_{v,w \in H^1(F,W)} |C_W(wF)|^{-2} \tilde{\chi}(wF) \overline{\tilde{\psi}}(vF) \langle R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id}), R_{\mathbf{T}_v}^{\mathbf{G}}(\mathrm{Id}) \rangle_{\mathbf{G}^F} \\ &= \sum_{w \in H^1(F,W)} |C_W(wF)|^{-1} \tilde{\chi}(wF) \overline{\tilde{\psi}}(wF). \end{aligned}$$

□

The $R_{\tilde\chi}$ form an orthonormal basis, but in general they are not characters, but $\mathbb{Q}$-linear combinations of characters. However

- We can reformulate 15.10 as: $R_{\tilde{\mathrm{Id}}} = \mathrm{Id}$, and $R_{\tilde{\mathrm{sgn}}} = \mathrm{St}_{\mathbf{G}}$ where $\tilde{\mathrm{Id}}$ is the trivial extension of the identity and $\tilde{\mathrm{sgn}}$ the trivial extension of the "sign" character $w \mapsto (-1)^{l(w)}$.
- we will see that in $\mathbf{GL}_n$ the R_χ are irreducible characters.

We assume now that F acts trivially on W to simplify the computations. Then we do not need to take extensions and can extend the notation R_χ to any class function on W by linearity (the following computation can be done without assuming that $\mathbf{L}$ is $\mathbf{G}$-split or that F acts trivially but then we would have to define induction for F-class functions).

Lemma 15.15. *Let $\mathbf{L}$ be a $\mathbf{G}$-split Levi subgroup of $\mathbf{G}$ and let $\lambda \in \mathrm{Irr}(W_{\mathbf{L}})$. Then $R_{\mathbf{L}}^{\mathbf{G}}(R_\lambda) = R_{\mathrm{Ind}_{W_{\mathbf{L}}}^{W}(\lambda)}$.*

Proof.

$$\begin{aligned}
R_{\mathrm{Ind}_{W_{\mathbf{L}}}^{W}(\lambda)} &= \sum_\chi \langle \chi, \mathrm{Ind}_{W_L}^W \lambda\rangle_W R_\chi \\
&= |W|^{-1} \sum_{\chi,w} \langle \chi, \mathrm{Ind}_{W_L}^W(\lambda)\rangle_W \chi(w) R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id}) \\
&= |W|^{-1} \sum_{w} \mathrm{Ind}_{W_L}^W(\lambda)(w) R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id}) \\
&= |W|^{-1}|W_L|^{-1}|\{v \in W \mid {}^v w \in W_{\mathbf{L}}\}|\lambda(w) R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id}) \\
&= |W_{\mathbf{L}}|^{-1} \sum_{w\in W_{\mathbf{L}}} \lambda(w) R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id}) \\
&= R_{\mathbf{L}}^{\mathbf{G}}(|W_{\mathbf{L}}|^{-1} \sum_{w\in W_{\mathbf{L}}} \lambda(w) R_{\mathbf{T}_w}^{\mathbf{L}}(\mathrm{Id})) = R_{\mathbf{L}}^{\mathbf{G}}(R_\lambda)
\end{aligned}$$

□

Applying this for $\lambda = \mathrm{Id}$, since $R_{\mathrm{Id}} = \mathrm{Id}$ by 15.10, we get that $R_{\mathrm{Ind}_{W_{\mathbf{L}}}^{W}(\mathrm{Id})}$ is an actual character. It follows that in $\mathbf{GL}_n$, the R_χ are irreducible characters. Indeed, in the symmetric group, any irreducible character is a linear combination of the identity induced from various Young subgroups; this is because in the induced $\mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}\mathrm{Id}$ from the Young subgroup indexed by the partition λ, the only characters occuring correspond to the partitions μ which dominate λ, and the character corresponding to λ occurs with multiplicity 1. Thus the matrix is unitriangular.

Inverting the formula for $R_{\tilde\chi}$, we have in general $R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id}) = \sum_\chi \tilde\chi(wF) R_{\tilde\chi}$. This is the decomposition of $R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id})$ in irreducibles in the case of $\mathbf{GL}_n$. In particular if $\mathbf{T}_1$ is $\mathbf{G}$-split we have $R_{\mathbf{T}_1}^{\mathbf{G}}\,\mathrm{Id} = \sum_\chi \chi(1) R_\chi$ and this is the decomposition attached to the Hecke algebra as explained in Section 11.

We finish with two results which hint to the rich relationship between Deligne-Lusztig representations and modular representation theory.

Theorem 15.16 (Broué-Michel, 1986)**.** *Let $\ell \neq p$ be a prime; let $s \in \mathbf{G}^{*F^*}$ be a semi-simple ℓ'-element. Then $\coprod_t \mathcal{E}(\mathbf{G}^F, st)$ is a union of ℓ-blocks, where t runs over $C_{\mathbf{G}^*}(s)_\ell^{F^*}$.*

Theorem 15.17 (Bonnafé-Rouquier, 2003)**.** *In the situation of the above theorem, let $\mathbf{L}^*$ be an F-stable Levi subgroup which contains $C_{\mathbf{G}^*}(s)$. Denote by $e_s^{\mathbf{G}}$ (resp. $e_s^{\mathbf{L}}$) the central idempotent of $\overline{\mathbb{Z}}_\ell \mathbf{G}^F$ (resp. $\overline{\mathbb{Z}}_\ell \mathbf{L}^F$) corresponding to the set of characters $\coprod_t \mathcal{E}(\mathbf{G}^F, st)$ (resp. $\coprod_t \mathcal{E}(\mathbf{L}^F, st)$). Then "$R_{\mathbf{L}}^{\mathbf{G}}$ induces a Morita equivalence between $\overline{\mathbb{Z}}_\ell \mathbf{L}^F e_s^{\mathbf{L}}$ and $\overline{\mathbb{Z}}_\ell \mathbf{G}^F e_s^{\mathbf{G}}$": more specifically there exists a parabolic subgroup $\mathbf{P} = \mathbf{L}\mathbf{U}$ with Levi complement $\mathbf{L}$ such that*

(i) $e_s^{\mathbf{G}} H_c^i(\mathbf{X}_{\mathbf{U}}, \overline{\mathbb{Z}}_\ell) e_s^{\mathbf{L}}$ is non-zero for $i = \dim \mathbf{X}_{\mathbf{U}}$ only;

(ii) if $M = e_s^{\mathbf{G}} H_c^{\dim \mathbf{X}_{\mathbf{U}}}(\mathbf{X}_{\mathbf{U}}, \overline{\mathbb{Z}}_\ell) e_s^{\mathbf{L}}$ then $M \otimes_{\mathbf{L}^F} -$ induces a Morita equivalence between $\overline{\mathbb{Z}}_\ell \mathbf{L}^F e_s^{\mathbf{L}}$ and $\overline{\mathbb{Z}}_\ell \mathbf{G}^F e_s^{\mathbf{G}}$.

15.2 Further reading

Good references on algebraic groups are the books [Borel] and [Springer].

References on the topics of these lectures are the books [Carter], [Digne-Michel], [Geck] and [Srinivasan].

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