Asymptotic expansion of solutions of Schroedinger equations, the WKB expansion seen from topological recursion

Abstract:

Consider a matrix 1st order linear differential system: $\hbar d/dx\Psi(x) = A(x)\Psi(x)$, where $\Psi(x)$ is a matrix (in fact $\Psi(x) \in G$ a Lie group) and $A(x)$ is a matrix (in fact $A(x) \in g$ its Lie algebra, and $\nabla = d - \hbar^{-1}A(x)dx$ is a connection on the Lie-group bundle). Can we find its asymptotic behavior as $\hbar \to 0$? A formal asymptotic series is of the form (WKB):

$$\Psi(x) \sim_{\hbar\to 0} V(x)(Id + \sum_{k=1}^{\infty} \hbar^k \Psi_k(x)) e^{\hbar^{-1}\int^x Y dx} C$$

where $Y(x)$ is the matrix of eigenvalues of $A(x)$ and $V(x)$ its eigenvectors, i.e. $A = VYV^{-1}$, and C is a constant matrix, and $\sum_{k=1}^{\infty} \hbar^k \Psi_k(x)$ means a formal series $\in C(x)[[\hbar]]$. The goal is to find the coefficients $\Psi_k(x)$ of that series. The topological recursion is a procedure that associates to a spectral curve (an immersed Riemann surface, here the surface of equation $\det(y - A(x)) = 0$), a sequence of differential forms called $\omega_{g,n}$. A conjecture proposed from 2007, claimed that

$$\left(Id + \sum_{k=1}^{\infty} \hbar^k \Psi_k(x)\right)_{1,1} = \exp \sum_{g,n} \frac{\hbar^{2g-2+n}}{n!} \int^x \cdots \int^x \omega_{g,n}$$

In fact the 2007 conjecture also said that if the spectral curve $\det(y - A(x)) = 0$ has genus > 0, then the conjecture should be amended to include some theta-functions terms (not written here for short). The conjecture was then proved by various authors for various examples, and the most recent proof for a very large set of differential systems. We shall talk about the conjecture, illustrate with the simplest case (Airy equation) and its proofs and consequences.