AUTOMORPHISMS OF CHARACTER VARIETIES

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Abstract. We show that the algebraic automorphism group of the $\mathrm{SL}_2(\mathbb{C})$ character variety of a closed orientable surface with negative Euler characteristic is a finite extension of its mapping class group. Along the way, we provide a simple characterization of the valuations on the character algebra coming from measured laminations.

0. Introduction

Let Σ be a closed oriented surface of genus $g \geq 2$ and denote $\mathrm{Mod}(\Sigma)$ its mapping class group, which by the Dehn-Nielsen-Baer theorem can be identified with $\mathrm{Out}(\pi_1(\Sigma)) = \mathrm{Aut}(\pi_1(\Sigma))/\mathrm{Inn}(\pi_1(\Sigma))$.

The meaningful spaces which carry an action of this group often manifest rigidity properties, as explained for instance in [AS16] : whenever the mapping class group acts preserving some structure, it is almost the full automorphism group ; celebrated examples are the Teichmuller space with its Kähler structure, the space of measured laminations with its PL structure, or the curve complex with its simplicial structure.

Here we show the same kind of result for the character variety, that is the space $X(\Sigma)$ defined as the algebraic quotient of $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}_2(\mathbb{C}))$ by the conjugation action of $\mathrm{SL}_2(\mathbb{C})$. Recall that this affine variety is constructed from its algebra of functions

$$\mathbb{C}[X(\Sigma)] = \mathbb{C}[\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}_2(\mathbb{C}))]^{\mathrm{SL}_2(\mathbb{C})}.$$

The automorphism group $\mathrm{Aut}(X(\Sigma))$ of the affine variety $X(\Sigma)$ is by definition the group of automorphisms of the $\mathbb{C}$-algebra $\mathbb{C}[X(\Sigma)]$.

The action $[\phi]\cdot[\rho] = [\rho\circ\phi^{-1}]$ of $\phi \in \mathrm{Aut}(\pi_1(\Sigma))$ on $\rho \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}_2(\mathbb{C}))$ descends to an algebraic action of the mapping class group on the character variety. Moreover, multiplying all representations $\rho\colon \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ by a central one $\lambda\colon \pi_1(\Sigma) \to \{\pm \mathrm{Id}\}$ yields an action of $H^1(\Sigma, \mathbb{Z}/2\mathbb{Z})$ on $X(\Sigma)$. The purpose of this note is to prove the:

Theorem. *When $g \geq 3$, the so defined map $H^1(\Sigma, \mathbb{Z}/2\mathbb{Z}) \rtimes \mathrm{Mod}(\Sigma) \to \mathrm{Aut}(X(\Sigma))$ is an isomorphism. When $g = 2$ it is surjective with kernel the hyperelliptic involution.*

Similar results were obtained in [EH74] for the 1-punctured torus and 4-punctured sphere. We thank Serge Cantat for pointing this out and explaining the proof. The computation of the automorphism group for the character variety had been asked on many occasions to the first author by Juan Souto ; however this work stemmed from different motivations, including a better understanding of measured laminations in terms of valuations.

Our strategy is to find an action of $\mathrm{Aut}(X(\Sigma))$ on the space of measured laminations, and show that it preserves the set of simple curves and their disjointness relation, hence the curve complex. For this consider the action of $\mathrm{Aut}(X(\Sigma))$ on a space $\mathcal{V}$ of valuations. Those are by definition the functions $v\colon \mathbb{C}[X(\Sigma)] \to \{-\infty\} \cup [0, +\infty)$ which are null on $\mathbb{C}^*$, take finite values except for $v(0) = -\infty$, and satisfy for all f, g the relations $v(fg) = v(f) + v(g)$ and $v(f + g) \leq \max(v(f), v(g))$. We endow $\mathcal{V}$ with the topology given by pointwize convergence, for which the action of $\mathrm{Aut}(X(\Sigma))$ is continuous.

Although it is established (for instance in [Ota12]) that the space of measured laminations $\mathrm{ML}(\Sigma)$ embeds continuously in $\mathcal{V}$, it is not clear why this subset should be preserved by $\mathrm{Aut}(X(\Sigma))$, and this is one of the main steps in the proof. Corollary 2.2 gives a simple and presumably new characterization of measured laminations as the set of "simple valuations" which is analogous to the set of monomial valuations for polynomial algebras. We hope that this description will help understand the topology of the set of all valuations over the character variety, for instance by showing that it retracts by deformation on the space of simple valuations.

Section 1 is quite elementary : we define simple valuations and show that they contain an embedded copy of the space measured laminations. Section 2 is the technical heart of the note: we show that any valuation is (sharply) dominated by one which is associated to a measured lamination, for this we apply the Morgan-Otal and Skora theorems to some Bass-Serre tree. Then we identify a property of valuations, called untameability, which is obviously preserved by algebraic automorphism and we show that it defines a dense subset in ML to conclude that $\mathrm{Aut}(X(\Sigma))$ acts on $\mathrm{ML}(\Sigma)$. In Section 3, after identifying multicurves with simple and discrete valuations, we explain how to read the number of components and their geometric intersection in terms of their associated valuation rings. We deduce that $\mathrm{Aut}(X(\Sigma))$ acts on the curve complex, and conclude using Ivanov's Theorem.

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1. Simple valuations and measured laminations

For $\alpha \in \pi_1(\Sigma)$, define $t_\alpha \in \mathbb{C}[X(\Sigma)]$ by the formula $t_\alpha([\rho]) = \mathrm{Tr}\,\rho(\alpha)$. Let us recall a folklore presentation of the character variety, mainly due to Procesi. We refer to [CP17] for a full account and to [Che13, Prop. 2.3] for a proof of this specific statement.

Theorem (Algebra presentation)**.** *The algebra $\mathbb{C}[X(\Sigma)]$ is generated by the t_α for $\alpha \in \pi_1(\Sigma)$ with ideal of relations generated by $t_1 - 2$ and $t_\alpha t_\beta - t_{\alpha\beta} - t_{\alpha\beta^{-1}}$ for $\alpha, \beta \in \pi_1(\Sigma)$.*

We call *multicurve* on Σ an embedded one dimensional submanifold $\Gamma \subset \Sigma$ which is a union of curves homotopic to $\gamma_i \in \pi_1(\Sigma) \setminus \{1\}$, and set $t_\Gamma = \prod_{i=1}^n t_{\gamma_i} \in \mathbb{C}[X(\Sigma)]$.

Components of Γ must be simple and disjoint ; we allow the empty multicurve for which $t_\emptyset = 1$. The previous theorem has the following important consequence [PS00]:

Theorem (Linear basis for the character algebra)**.** *The family (t_Γ) where Γ ranges over the isotopy classes of multicurves forms a linear basis of $\mathbb{C}[X(\Sigma)]$.*

This allows to treat the elements t_Γ as if they were monomials (despite the fact that they are not stable by multiplication), and motivates the next definition.

Definition 1.1 (Simple valuations)**.** *A valuation $v \in \mathcal{V}$ is called* simple *if for any $f \in \mathbb{C}[X(\Sigma)] \setminus \{0\}$ decomposed as $f = \sum m_\Gamma t_\Gamma$ in the basis of multicurves, one has*

$$v(f) = \max\{v(t_\Gamma) \mid m_\Gamma \neq 0\}. \tag{max}$$

Remark. *It is not clear at this point why the set of simple valuations should be preserved by $\mathrm{Aut}(X(\Sigma))$, since we have not uniquely characterized our linear basis.*

Proposition 1.2 (Measured laminations are simple valuations)**.** *Fix λ a measured lamination and denote by $i(\cdot,\cdot)$ the intersection pairing between measured laminations. The map $v_\lambda : t_\alpha \mapsto i(\lambda, \alpha)$ for $\alpha \in \pi_1(\Sigma)$ extends to a unique simple valuation $v_\lambda \in \mathcal{V}$.*

Proof. First treat the case of the measured lamination associated to a simple curve δ. We define v_δ on the basis of multicurves by the expression $v_\delta\left(\prod_{i=1}^n t_{\gamma_i}\right) = \sum i(\delta, \gamma_i)$ and extend it to $\mathbb{C}[X(\Sigma)]$ using equation (max). We must verify that both definitions coincide meaning that given $\alpha \in \pi_1(\Sigma)$ such that $t_\alpha = \sum m_\Gamma t_\Gamma$, we have $i(\delta, \alpha) = \max\{i(\delta, \Gamma) \mid m_\Gamma \neq 0\}$, but this is the content of D. Thurston's [Thu09, Lemma 12].

All that remains, is to check the formula $v_\delta(fg) = v_\delta(f) + v_\delta(g)$. For this, consider the increasing filtration of $\mathbb{C}[X(\Sigma)]$ defined by $F_n = \mathrm{Span}\{t_\gamma \mid \gamma \in \pi_1(\Sigma), i(\delta, \gamma) \leq n\}$. Let $k = v_\delta(f)$ and $l = v_\delta(g)$ be such that $f \in F_k \setminus F_{k-1}$ and $g \in F_l \setminus F_{l-1}$. It is equivalent to prove $v_\delta(fg) = v_\delta(f) + v_\delta(g)$ and that fg is non zero in F_{k+l}/F_{k+l-1}. Hence, we are reduced to showing that the graded algebra $\bigoplus_{n\in\mathbb{N}} F_n/F_{n-1}$ is an integral domain, which is [PS19, Theorem 12] (this can also be derived from the proof of [CM12, Theorem 5.3]).

For the general case, we extend the map v_λ to $\mathbb{C}[X(\Sigma)]$ in the same way. The fact that it is indeed an extension and defines a simple valuation will follow from the case of simple curves by a limiting procedure. We know from [FLP79, Theorème 1.3] that any measured lamination λ is a limit of weighted simple curves $m_j\delta_j$ in the sense that for all $\alpha \in \pi_1(\Sigma)$ one has $i(\lambda, \alpha) = \lim_{j\to\infty} m_j i(\delta_j, \alpha)$. Hence by equation (max) the $m_j v_{\delta_j}$ converge pointwizely to v_λ, which indeed defines a valuation satisfying $v_\lambda(t_\alpha) = i(\lambda, \alpha)$ for all $\alpha \in \pi_1(\Sigma)$. One could also observe that the subset of simple valuations is closed in $\mathcal{V}$ for the topology of pointwize convergence. □

As a consequence of this proposition, we can identify a measured lamination λ with the corresponding valuation v_λ. Moreover, the usual topology on the space of measured laminations ML is the topology of weak convergence of lengths: in other terms, it is the topology induced by the embedding of ML in $\mathcal{V}$.

Definition 1.3 (Strict valuations)**.** *A valuation $v \in \mathcal{V}$ is* strict *if for all multicurves $\Gamma \neq \Delta$, we have $v(t_\Gamma) \neq v(t_\Delta)$.*

Definition 1.4 (Positive valuations)**.** *A valuation $v \in \mathcal{V}$* positive *when $v(t_\gamma) > 0$ for every simple curve γ.*

Remark. *Let us comment on the relations between simple, strict and positive:*

(i) *For simple valuations it is equivalent to be positive on t_γ for all simple γ, on t_α for all non-trivial $\alpha \in \pi_1(\Sigma)$ or, on non-constant elements in $\mathbb{C}[X(\Sigma)]$.*
(ii) *(Strict implies positive). By definition $v(t_\emptyset) = v(1) = 0$, it follows that if v is strict, then for any simple curve γ we must have $v(t_\gamma) > 0$.*
(iii) *(Strict implies simple). For any valuation $v \in \mathcal{V}$, the relation $v(f+g) \leq \max\{v(f), v(g)\}$ is an equality as soon as $v(f) \neq v(g)$. Therefore strict valuations are simple since they automatically satisfy the (*max*) relation.*

Recall that Thurston introduced a natural measure on the space of measured laminations that we use freely in this section. Masur showed that up to scaling, it is the only $\mathrm{Mod}(\Sigma)$-invariant measure on ML in its Lebesgue class. Therefore it is proportional to the Borelian measure assigning to every open set U the limit as $r \to \infty$ of the number of multicurves Γ in the dilated set $r \cdot U$ divided by r^{6g-6}. This may serve as a definition.

Proposition 1.5 (Most measured laminations are strict)**.** *The set of measured laminations λ such that v_λ is strict has full measure in* ML*.*

Proof. Let us first show that the set of $\lambda \in \mathrm{ML}$ such that v_λ is positive has full measure. By [Pap86a, Section 3, Lemma 2], for every simple curve γ the set $N(\gamma) = \{\lambda \in \mathrm{ML} \mid i(\gamma, \lambda) \neq 0\}$ is the complement of a codimension-1 PL-submanifold, in particular it has measure 0 (see also [Pap86b, Section 4]). The set of positive laminations is the intersection of all $N(\gamma)$, so it has full measure.

Inside that set of positive laminations, the complement to the set of strict valuations is the union over the countable set of pairs of distinct non-empty multicurves, of the $I(\Gamma_1, \Gamma_2) = \{\lambda \in \mathrm{ML} \mid i(\lambda, \Gamma_1) = i(\lambda, \Gamma_2)\}$: the result follows from the next lemma. ☐

Lemma 1.6 (Intersection with distinct multicurves seldom coincides)**.** *For distinct multicurves $\Gamma_1 \neq \Gamma_2$, the set $I(\Gamma_1, \Gamma_2) = \{\lambda \in \mathrm{ML} \mid i(\lambda, \Gamma_1) = i(\lambda, \Gamma_2)\}$ has measure 0 in the set of positive measured laminations.*

Proof. The space ML has a piecewise linear (PL) structure (given by Dehn-Thurston coordinates or train tracks) for which the functions $f_j(\lambda) = i(\Gamma_j, \lambda)$ are piecewise linear. In particular there exists an atlas $(U_k, \phi_k)_{k \in \mathbb{N}}$ of ML such that all functions $f_j \circ \phi_k^{-1}$ are PL. The subset $W = \{\lambda \in \mathrm{ML} \mid \exists k \in \mathbb{N}, f_1 \circ \phi_k^{-1} \text{ or } f_2 \circ \phi_k^{-1} \text{ is not smooth at } \phi_k(\lambda)\}$ is a PL-subcomplex of positive codimension so its complement has full measure.

Let $x \in \mathrm{ML} \setminus W$, choose $k \in \mathbb{N}$ such that $x \in U_k$, and denote by V the connected component of $\phi_k(U_k \setminus W)$ containing $\phi_k(x)$. Then f_1 and f_2 are linear on V: it is sufficient to show that these two maps are distinct to conclude that the locus where they coincide has zero measure.

Take a collection of $9g-9$ curves $\gamma_1, \dots, \gamma_{9g-9}$ such that

$$\lambda = \mu \iff i(\lambda, \gamma_l) = i(\mu, \gamma_l) \text{ for } l = 1, \dots, 9g-9.$$

Applying a power of a pseudo-Anosov map whose attracting point belongs to $\phi_k^{-1}(V)$, we may suppose that the curves γ_l belong to $\phi_k^{-1}(V)$ (up to a scalar). As we have $f_1(\phi_k(\gamma_l)) \neq f_2(\phi_k(\gamma_l))$ for some l, we can conclude that f_1 and f_2 are distinct. □

2. Domination of valuations

Definition (Order structure). *We shall work with the partial order structure on $\mathcal{V}$ called* domination *and defined by $v \leq w$ if we have $v(f) \leq w(f)$ for all $f \in \mathbb{C}[X(\Sigma)]$. A valuation u is called* untameable *if $u \leq v$ implies $v = Cu$ for some $C \in \mathbb{R}$.*

The set $\mathcal{U}$ of *untameable* valuations is clearly preserved by the action of $\mathrm{Aut}(X(\Sigma))$. The main purpose of this subsection is to obtain the following theorem, whose proof consists in a compilation of known results.

Theorem 2.1 (Domination by measured laminations). *For all $v \in \mathcal{V}$, there exists a unique $\lambda \in \mathrm{ML}$ such that $v \leq v_\lambda$ and satisfying $v(t_\alpha) = v_\lambda(t_\alpha)$ for all $\alpha \in \pi_1(\Sigma)$.*

Recall that $\mathbb{C}[X(\Sigma)]$ is an integral domain (see [PS19, Theorem 12], again this can also be derived from the proof of [CM12, Theorem 5.3]), so that we can consider its field of fraction $K = \mathbb{C}(X(\Sigma))$. We extend v to K by setting $v(f/g) = v(f) - v(g)$. Notice that our sign convention differs from the standard one in valuation theory: for instance our definition of the valuation ring associated to v is $\mathcal{O}_v = \{f \in K, v(f) \leq 0\}$. We first recall the standard construction of the so-called tautological representation.

Lemma (Tautological representation). *There exists a finite extension $\hat{K}$ of K along with a representation $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\hat{K})$ such that $\mathrm{Tr}\,\rho(\alpha) = t_\alpha$ for all $\alpha \in \pi_1(\Sigma)$.*

Proof. This follows from classical arguments in geometric invariant theory. In a nutshell, consider an algebraic closure $\overline{K}$ of K. One may interpret the inclusion $\mathbb{C}[X(\Sigma)] \to \overline{K}$ as a $\overline{K}$-point of $X(\Sigma)$. Since the map $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}_2(\overline{K})) \to X(\Sigma)(\overline{K})$ is surjective, we have a representation $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\overline{K})$ satisfying $\mathrm{Tr}\,\rho(\gamma) = t_\gamma \in K$. As $\pi_1(\Sigma)$ is finitely generated, the coefficients of ρ may be taken in a finite extension of K. One may also consult [Mar16, Proposition 3.3] for a down-to-earth proof using K. Saito's theorem which even shows that a quadratic extension suffices. □

Proof of Theorem 2.1. There exists (see [Vaq07] for a proof) a valuation $\hat{v} : \hat{K}^* \to \mathbb{R}$ extending v, so we may consider the Bass-Serre real tree T associated to the pair $(\hat{K}, \hat{v})$. We know from [MO93, Corollary III.1.2] that there exists a measured lamination μ with associated real tree T_μ and an equivariant morphism of $\mathbb{R}$-trees $\Phi : T_\mu \to T$ which decreases the distance (by Step 1 of Lemma I.1.1). Thus for any $\alpha \in \pi_1(\Sigma)$, the translation length of the action of α on T_μ, which equals $i(\mu, \alpha)$, is greater than the translation length of α acting on T which is $\max\{0, 2\hat{v}(t_\alpha)\} = 2v(t_\alpha)$.

It follows in particular that $2v(t_\gamma) \leq v_\mu(t_\gamma)$ for every simple curve γ, therefore this holds also over multicurves. For any $f \in \mathbb{C}[X(\Sigma)]$ expanded as $f = \sum w_\Gamma t_\Gamma$, we get

$$v(f) \leq \max\{v(t_\Gamma) \mid w_\Gamma \neq 0\} \leq \frac{1}{2}\max\{v_\mu(t_\Gamma) \mid w_\Gamma \neq 0\} = \frac{1}{2}v_\lambda(f)$$

where the last equality follows by Proposition 1.2. Hence the first part of the theorem holds if one sets $\lambda = \frac{1}{2}\mu$.

If we prove that Φ is an isometry on its image then we are done. Indeed this implies that the translation lengths of the actions of $\alpha \in \pi_1(\Sigma)$ on T_μ and T coincide, in other words that $v(t_\alpha) = \frac{1}{2}v_\lambda(t_\alpha)$. But by Skora's theorem [Sko96], the morphism of $\mathbb{R}$-trees $\Phi : T_\mu \to T$ mentionned above is an isometry on its image if and only if there does not exist any free subgroup $F_2 \subset \pi_1(\Sigma)$ stabilizing a non-trivial edge in $\Phi(T_\mu)$.

Hence suppose by contradiction there exist $\alpha, \beta \in \pi_1(\Sigma)$ generating a free subgroup which fixes a non-trivial edge in T of length l. Following for instance [Ota12, section 4.2], this implies that up to conjugation, the tautological representation restricted to F_2 has values in

$$G_l = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, ad - bc = 1,\, v(a) \leq 0,\, v(b) \leq 0,\, v(c) \leq -l,\, v(d) \leq 0 \right\}.$$

If $\mathcal{M}_v^l$ denotes the ideal of $\mathcal{O}_v$ defined by the equation $v \leq -l$ then G_l consists precisely of those elements in $\mathrm{SL}_2(\mathcal{O}_v)$ projecting to triangular matrices in $\mathrm{SL}_2(\mathcal{O}_v/\mathcal{M}_v^l)$. The commutator $\rho([\alpha,\beta])$ is then unipotent in this quotient and we get $\mathrm{Tr}\,\rho([\alpha,\beta]) = 2 \mod \mathcal{M}_v^l$. This means that $v(t_{[\alpha,\beta]} - 2) \leq -l$, contradicting the fact that v is non-negative over $\mathbb{C}[X(\Sigma)]$. □

Corollary 2.2 (Simple valuations are measured laminations)**.** *A valuation is simple if and only if it has the form v_λ for some $\lambda \in$* ML*. In particular* ML *is closed.*

Corollary 2.3 (Untameable implies Simple)**.** *We have an inclusion $\mathcal{U} \subset$* ML.

Recall that a lamination is called *maximal* if it is filling (that is, the valuation v_λ is positive) and its complementary regions are triangles ; or equivalently, when there are no lamination with a strictly larger support. A lamination is *uniquely ergodic* if it supports a unique transverse measure up to a scalar.

Lemma 2.4. *Almost all measured laminations are strict, maximal and uniquely ergodic. Such laminations are untameable. In particular $\mathcal{U}$ is dense in* ML.

Proof. Almost all measured valuations are maximal by [LM08, Lemma 2.3], uniquely ergodic by a theorem of Masur [Mas82, Theorem 2], and strict by lemma 1.5.

If λ is maximal and uniquely ergodic and μ is another measured lamination such that $v_\lambda \leq v_\mu$, then the support of λ is included in the support of μ. By maximality, the support of μ is the same as the support of λ: only the transverse measures may differ. By unique ergodicity, μ and λ must be proportional.

Let λ be a strict, maximal and uniquely ergodic measured lamination. Now suppose there exists another valuation $w \in \mathcal{V}$ with $v \leq w$. Applying Theorem 2.1, there exists another measured lamination μ with $w \leq v_\mu$ and $w(t_\alpha) = v_\mu(t_\alpha)$ for all $\alpha \in \pi_1(\Sigma)$. From the previous discussion there exists $C \geq 1$ such that $\mu = C\lambda$. But w and $C\lambda$ are strict and coincide on simple curves, so that $w = C\lambda$ and we are done. □

Corollary 2.5. *The action of* $\mathrm{Aut}(X(\Sigma))$ *on* $\mathcal{V}$ *preserves* ML.

Proof. The action of $\mathrm{Aut}(X(\Sigma))$ on $\mathcal{V}$ is continuous and preserves the subset $\mathcal{U} \subset \mathrm{ML}$ of untameable valuations, so it also preserves its closure $\overline{\mathcal{U}} = \mathrm{ML}$. □

3. Discrete valuations and disjointness

We call *discrete* a valuation in $\mathcal{V}$ whose finite values belong to $\mathbb{N}$, and likewize a measured lamination whose associated valuation is discrete. One expects the underlying lamination to be a multicurve, this is almost true :

Lemma (Discrete measured laminations)**.** *Discrete measured lamination are precisely the weighted multicurves* $\frac{1}{2}\Gamma$ *such that the class of* Γ *in* $H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$ *vanishes.*

Proof. This lemma can be shown using Dehn's coordinates, see [FLP79, Exposé 6]. Let us give a proof more in the spirit of this note as it follows they lines of reasoning for Theorem 2.1. For this, observe that the Bass-Serre tree associated to a discrete valuation has a simplicial structure.

We extend the discrete valuation v on K to a discrete valuation $\hat{v}$ on $\hat{K}$ whose Bass-Serre tree T is thus simplicial. The Morgan-Otal-Skora theorem reduces to a theorem of Stallings (see [Sha02, Chapter 2] for a nice account) stating that the action of $\pi_1(\Sigma)$ on T is dual to a multicurve Γ. This implies $v(t_\alpha) = \frac{1}{2}i(\Gamma, \alpha)$, so v is associated to the measured lamination $\lambda = \frac{1}{2}\Gamma$.

Computing modulo $\mathbb{Z}$ the weighted intersection of $\frac{1}{2}\Gamma$ with any curve γ one should get 0: by Poincaré duality this shows that the class of Γ in $H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$ vanishes. □

Of course, an automorphism of $\mathbb{C}[X(\Sigma)]$ must preserve the set of discrete valuations, and hence also the set of discrete measured laminations by Corollary 2.5. The previous lemma says that such a valuation has the form $v = \frac{1}{2}v_\Gamma$ for a multicurve Γ which is null in homology modulo 2.

We wish to show that an algebraic automorphism actually preserves the set of discrete valuations coming from simple curves, as well as their property of being disjoint. Thus we need an algebraic characterization for the number of components in a multicurve, as well as the disjointness property, in terms of their associated valuations.

For this recall the notion of valuation ring associated to the valuation $v \in \mathcal{V}$, defined by $\mathcal{O}_v^+ = \{f \in \mathbb{C}[X(\Sigma)] | v(f) \leq 0\}$; its Krull dimension $\dim \mathcal{O}_v^+$ is preserved by algebraic automorphisms. When $\lambda = \frac{1}{2}\Gamma$ for a multicurve Γ, we have $\mathcal{O}_\lambda^+ = \mathrm{Span}\{t_\Delta | i(\Gamma, \Delta) = 0\}$, which corresponds geometrically to the coordinate ring for the image of the restriction map $X(\Sigma) \to X(\Sigma \setminus \Gamma)$.

Lemma 3.1 (Number of components)**.** *The number of distinct homotopy classes of simple curves in a multicurve* Γ *is equal to* $\operatorname{codim} \mathcal{O}_\Gamma^+ = \dim \mathbb{C}[X(\Sigma)] - \dim \mathcal{O}_\Gamma^+$.

Proof. Let $\gamma_1,\dots,\gamma_n$ be the simple curves composing Γ. We let $U\subset X(\Sigma)$ be the Zariski open subset of $X(\Sigma)$ containing irreducible representations ρ satisfying $\operatorname{Tr}\rho(\gamma_i)\neq\pm 2$ for $i=1,\dots,n$. The restriction map $X(\Sigma)\to X(\Sigma\setminus\{\gamma_1,\dots,\gamma_n\})$ restricted to U and corestricted to its image is a fibration whose fibres are covered by the orbits of the (commuting) Goldman twist flows along $\gamma_1,\dots,\gamma_n$ on the character variety. Since the fiber has dimension n, the image has codimension n and the result follows. □

Lemma 3.2 (Disjointness property)**.** *For any pair of non-parallel simple curves γ,δ one has $\dim\mathcal{O}^+_\gamma\cap\mathcal{O}^+_\delta\leq\dim\mathbb{C}[X(\Sigma)]-2$, with equality if and only if $i(\gamma,\delta)=0$.*

Proof. If γ and δ are disjoint, Lemma 3.1 already gives $\operatorname{codim}\mathcal{O}^+_{\gamma\cup\delta}=2$ and the result follows from the observation that $\mathcal{O}^+_{\gamma\cup\delta}=\mathcal{O}^+_\gamma\cap\mathcal{O}^+_\delta$.

Suppose now that γ and δ intersect minimally and set $\Sigma'=\Sigma\setminus(\gamma\cup\delta)$. The ring $\mathcal{O}^+_\gamma\cap\mathcal{O}^+_\delta$ is the image of the natural map $\mathbb{C}[X(\Sigma')]\to\mathbb{C}[\Sigma]$. Geometrically, it is the coordinate ring for the image of the restriction map $X(\Sigma)\to X(\Sigma')$, hence it is sufficient to show $\dim X(\Sigma')<\dim X(\Sigma)-2$. The surface Σ' is a disjoint union of connected surfaces with connected boundary: their fundamental group is either trivial or free with rank ≥ 2. Denoting by F the number of simply connected components, we have $\dim X(\Sigma')=-3\chi(\Sigma')+3F$.

But $\chi(\Sigma')=\chi(\Sigma)+i(\gamma,\delta)$ so $\dim X(\Sigma')=\dim X(\Sigma)-3i(\gamma,\delta)+3F$, and we must show that $i(\gamma,\delta)>F$. Since $\gamma\cup\delta$ is a taut union of simple curves, the polygonal components have at least 4 corners, so $4F$ is smaller or equal to the total number of corners which is $4i(\gamma,\delta)$. But $F=i(\gamma,\delta)$ implies that Σ' is a union of quadrilaterals, so $F=\chi(\Sigma')=\chi(\Sigma)+i(\gamma,\delta)$ and thus $\chi(\Sigma)=0$: a contradiction. □

Corollary 3.3 (Action on the curve complex)**.** *The group $\operatorname{Aut}(X(\Sigma))$ preserves the set*

$$\mathcal{C}=\{v_\gamma|\gamma \text{ simple and non-separating}\}\cup\{v_{\gamma/2}|\gamma \text{ simple and separating}\}$$

along with its orthogonality relation $v_\gamma\perp v_\delta\iff i(\gamma,\delta)=0$ for $v_\gamma,v_\delta\in\mathcal{C}$.

Set $\operatorname{Mod}'(\Sigma)=\operatorname{Mod}(\Sigma)$ if $g\geq 3$ and $\operatorname{Mod}'(\Sigma)=\operatorname{Mod}(\Sigma)/\langle\tau\rangle$ if $g=2$ where τ denotes the hyperelliptic involution.

Theorem 3.4. *There is a natural split extension*

$$0\to H^1(\Sigma,\mathbb{Z}/2\mathbb{Z})\to\operatorname{Aut}(X(\Sigma))\to\operatorname{Mod}'(\Sigma)\to 0.$$

Proof. Let ϕ be an automorphism of $X(\Sigma)$. By Corollary 3.3, it acts on the curve complex $(\mathcal{C},\perp)$ and Ivanov's Theorem (see [Iva97] or [Luo00]) implies the existence of a unique element $\varphi\in\operatorname{Mod}(\Sigma)$ which acts in the same way: this defines the map $\operatorname{Aut}(X(\Sigma))\to\operatorname{Mod}(\Sigma)$, which is a group morphism. Since $\operatorname{Mod}(\Sigma)$ naturally acts on $X(\Sigma)$ by sending t_γ to $t_{\varphi(\gamma)}$, there is a natural section of the previous map whose kernel consists of mapping classes which fix all non-oriented simple curves. This reduces to the identity or the hyperelliptic involution in genus 2 (see for instance [FM12]) hence the surjection $\operatorname{Aut}(X(\Sigma))\to\operatorname{Mod}'(\Sigma)$.

Suppose $\phi \in \mathrm{Aut}(X(\Sigma))$ acts trivially on $\mathcal{C}$. Fix a simple curve γ and consider the possible values of $\phi(t_\gamma)$. As ϕ preserves the valuations v_δ for any simple curve δ we get $v_\delta(\phi(t_\gamma)) = v_\delta(t_\gamma) = i(\delta, \gamma)$. Write $\phi(t_\gamma) = \sum m_\Gamma t_\Gamma$: for any simple curve δ which does not intersect γ one has $0 = v_\delta(\phi(t_\gamma)) = \max\{i(\delta, \Gamma) \mid m_\gamma \neq 0\}$, so every Γ such that $m_\Gamma \neq 0$ must be a family of parallel copies of γ. Hence $\phi(t_\gamma)$ is a polynomial in t_γ, so ϕ induces an automorphism of the subalgebra $\mathbb{C}[t_\gamma]$, thus $\phi(t_\gamma) = a_\gamma t_\gamma + b_\gamma$ with $a_\gamma \neq 0$.

Lemma 3.5 shows that $b_\gamma = 0$ for every non trivial simple curve γ, so $\phi(\gamma) = a_\gamma t_\gamma$ and for any multicurve Γ with components γ_j, we thus have $\phi(t_\Gamma) = a_\Gamma t_\Gamma$ where $a_\Gamma = \prod a_{\gamma_j}$. Lemma 3.6 says that a_Γ does not change when Γ undergoes a saddle-move (we give the definition after Lemma 3.5). But two multicurves are related by a sequence of saddle moves when they belong to the same homology class modulo 2, so a factors to a map $a\colon H_1(\Sigma, \mathbb{Z}/2\mathbb{Z}) \to \mathbb{C}^*$, which according to Lemma 3.7 is in fact a morphism. Thus $a \in H^1(\Sigma, \mathbb{Z}/2\mathbb{Z})$ and we are done. □

Lemma 3.5. *For every non-trivial simple curve γ, we have $b_\gamma = 0$ and $a_\gamma = \pm 1$.*

Proof. Suppose γ is a non-separating simple curve. Then there is another one δ which intersects it once and a tubular neighborhood of their union is a π_1-injectively embedded punctured torus. In that case apply ϕ to the relation $t_\gamma t_\delta = t_{\gamma\delta} + t_{\gamma\delta^{-1}}$ and decompose in the basis of multicurves to get $b_\gamma = 0$ and then $a_\gamma a_\delta = a_{\gamma\delta}$. But the simple curve $\gamma\delta$ also intersects γ once, so the same reasoning also gives $a_{\gamma\delta} a_\gamma = a_\delta$. Combined with the previous relation this implies $a_\gamma a_\delta a_\gamma = a_\delta$, hence $a_\gamma = \pm 1$.

From now on γ is a separating simple curve. There exists a non-separating simple curve δ which intersects it twice with opposite signs. A neighborhood for their union $\gamma \cup \delta$ is an embedded four holed sphere $F \subset \Sigma$ with boundary components $\eta_1, \eta_2, \eta_3, \eta_4$ such that the triple η_1, η_2, γ bounds a three holed sphere. (This fixes the configuration up to the action of the mapping class group.)

Since $i(\gamma, \delta) = 2$ the η_j are non-trivial and as they are disjoint from γ and δ they must be homotopically distinct from δ and γ. Also, as γ is separating, η_1 and η_2 are both homotopically distinct from each of η_3 and η_4.

Using the trace relation on $t_\gamma t_\delta$ to resolve the intersections of $\gamma \cup \delta$, we decompose $t_\gamma t_\delta = t_{\eta_1} t_{\eta_3} + t_{\eta_2} t_{\eta_4} - t_\theta - t_\zeta$ where $\zeta, \theta \subset F$ are two simple curves both intersecting twice γ and δ, such that each triple η_1, η_3, θ and η_2, η_4, ζ bounds a three-holed sphere.

Considering intersection numbers as before we see that θ, ζ are homotopically distinct from each other and γ, δ, η_j : among all simple curves into play here, only the boundary components $\eta_1 3$ and η_2, or η_3 and η_4, may belong the same class. So when we decompose into the basis of multicurves after applying ϕ to the previous relation, we find that $b_\gamma = 0$ and then $a_\gamma a_\delta = a_\zeta$. Since γ is separating and δ is not, ζ must also be non-separating, and the previous discussion implies $a_\delta = \pm 1$ and $a_\zeta = \pm 1$, so $a_\gamma = \pm 1$. □

In order to introduce the notion of a saddle move, we will think of multicurves as isotopy classes of 1-submanifolds in Σ up to adding or removing a trivial component. Two multicurves are related by a saddle move if they are represented by two 1-manifolds which differ in a disc as in Figure 1.

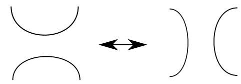

FIGURE 1. Saddle move

It is an easy exercise to show that two multicurves are related by a sequence of saddle moves if and only if they belong to the same homology class modulo 2. Let us sketch the argument. Let Γ_1, Γ_2 be two multicurves homologous modulo 2 in Σ and suppose that they intersect transversely. Any connected component of $\Sigma \backslash (\Gamma_1 \cup \Gamma_2)$ containing a corner should contain an even number of those, and two consecutive corners are cancelled by a saddle move. By induction, we are reduced to the case where $\Gamma_1 \cap \Gamma_2 = \emptyset$. As Γ_1 and Γ_2 are homologous modulo 2, there exists en embedded subsurface $S \subset \Sigma$ with $\partial S = \Gamma_1 \cup \Gamma_2$, and decomposing this cobordism into saddles shows how to transform Γ_1 into Γ_2.

Set $t_\epsilon = 2$ for the trace of a trivial component ϵ and extend the t_Γ by multiplicativity on all components γ_j to encompass the more general notion of multicurves. This remains consistent with the trace relation. As $\phi(t_\epsilon) = t_\epsilon$ we have $a_\epsilon = 1$, and according to the previous lemma we now have $a_\Gamma = \prod a_{\gamma_j}$ defined as a function on multicurves with values in ± 1.

Lemma 3.6. *The function a so defined on multicurves is invariant by saddle move.*

Proof. Before starting the proof, observe that a saddle move changes the number of components of a 1-submanifold by 0 (type 0) or 1 (type 1).

Consider first a saddle move of type 0: ignoring the components that are not affected, one can assume it transforms a simple curve η into another one θ. Basing the fundamental group at the singular point appearing half way during the saddle move, we see that there exists two simple curves γ, δ intersecting once such that $\eta = \gamma\delta$ and $\theta = \gamma\delta^{-1}$. As in the proof of Lemma 3.5, we deduce that $a_\eta = a_\gamma a_\delta = a_\gamma a_{\delta^{-1}} = a_\theta$, which concludes this case.

Now consider the case of a saddle move of type 1. Again, ignoring the components that are not affected, we observe that we must prove the formula $a_\gamma = a_{\eta_1} a_{\eta_2}$ where γ, η_1, η_2 are three boundary components of an embedded pair of pants P in Σ. Since $a_\gamma = \pm 1$ by Lemma 3.5, this can be written more symmetrically as $a_\gamma a_{\eta_1} a_{\eta_2} = 1$. This formula obviously holds if one of the three boundary components is homotopically trivial so that we can suppose that P is incompressible in Σ. Thus we may consider another embedded 3-holed sphere bounded by γ, η_3, η_4 such that the η_j bound an embedded 4-holed sphere F as in the proof of the previous lemma. This time, using the trace relation on $t_\theta t_\delta$ to resolve the intersections of $\theta \cup \delta$, we decompose $t_\theta t_\delta = t_{\eta_1} t_{\eta_2} + t_{\eta_3} t_{\eta_4} - t_\nu - t_\gamma$ where the simple curve ν corresponds to image of θ under the half Dehn twist along δ. Applying ϕ to it and decomposing into the basis of multicurves we find that $a_\gamma = a_{\eta_1} a_{\eta_2}$ and the lemma follows. □

Lemma 3.7. *The map induced on homology* $a\colon H_1(\Sigma; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{C}^*$ *is a morphism.*

Proof. Let α, β be two classes in $H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$. They can be represented by simple curves γ, δ which are either disjoint (if $\alpha \cdot \beta = 0$) and then $\alpha + \beta$ is represented by the disjoint union $\gamma \cup \delta$, or else intersect once (if $\alpha \cdot \beta = 1$) and then $\alpha + \beta$ is represented by the product $\gamma\delta$. In any case, we already proved that $a_{\alpha\beta} = a_\alpha a_\beta$. □

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