SL_2-character varieties of 2- and 3-manifolds through examples.

J. Marché and R. Santharoubane

1 Introduction

These notes deal with character varieties into $\mathrm{SL}_2(\mathbb{C})$ for fundamental groups of 2 and 3-manifolds. More generally, if Γ is a finitely generated group, the set $\mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is an algebraic variety. Considering conjugacy classes of representations, one faces the problem that $\mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$ is neither separated nor an algebraic variety. One has to take an algebraic quotient

$$X(\Gamma) = \mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))//\mathrm{SL}_2(\mathbb{C})$$

which is more complicated to handle!

Given $\gamma \in \Gamma$ the map :

$$f_\gamma : [\rho] \in X(\Gamma) \mapsto \mathrm{Tr}(\rho(\gamma)) \in \mathbb{C}$$

is a well-known algebraic function called the trace function. These functions generate the ring of algebraic functions on $X(\Gamma)$.

We can also define the real points of $X(\Gamma)$:

$$X(\Gamma)_\mathbb{R} = \{[\rho] \in X(\Gamma)/\mathrm{Tr}(\rho(\gamma)) \in \mathbb{R}\}.$$

One can show that $X(\Gamma)_\mathbb{R}$ is made of two components (not disjoint):

$$\mathrm{Hom}(\Gamma, \mathrm{SU}_2)/\mathrm{SU}_2 \quad \text{and} \quad \mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{R}))/\mathrm{SL}_2(\mathbb{R})$$

Let us motivate the study of character varieties by some examples.

- If Σ is a closed oriented surface, $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}_2(\mathbb{R}))/\mathrm{SL}_2(\mathbb{R})$ has finitely many connected components, one of which is the Teichmüller space of the surface Σ.
- $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SU}_2)/\mathrm{SU}_2$ is a symplectic manifold with singularities whose quantization of level k gives the so-called TQFT vector space $V_k(\Sigma)$ and finite dimensional representations of the mapping class group of Σ.
- For M a closed and oriented hyperbolic 3-manifold there exists a representation
$$\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C}) \simeq \mathrm{Isom}^+(\mathbb{H}^3)$$
which lifts to an isolated point $[\rho] \in X(\pi_1(M))$ (Weil-Mostow rigidity). If M has toric boundary, the variety $X(\pi_1(M))$ is also studied for deformation of hyperbolic structures related to the Dehn fillings of M.

- For K a knot in S^3 and $M = S^3 \setminus V(K)$ (the complement of an open tubular neighborhood of K), $X(\pi_1(M))$ is (usually) an affine algebraic curve whose ideal points correspond to surfaces in M. Moreover, $X(\pi_1(M))$ governs conjecturally the asymptotics of Topological Quantum Field Theory invariants related to M such as the colored Jones polynomials...

2 Formal definitions and examples

Let Γ be a finitely generated group, say by $\gamma_1, ..., \gamma_n$. The injective map

$$\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C})) \mapsto (\rho(\gamma_1), ..., \rho(\gamma_n)) \in \mathrm{SL}_2(\mathbb{C})^n$$

gives an induced topology independent on the choice of generators.

Problem : The quotient $\mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$ is not necessarily separated. For instance : if $\Gamma = \mathbb{Z}$, we have $\mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\mathrm{SL}_2(\mathbb{C})$. Now if we denote by

$$\mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

the limit :

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \underset{t \to 0}{\longrightarrow} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

says that

$$\overline{[\mathrm{Id}]} \cap \overline{[M]} \neq \emptyset$$

The well-known solution to this problem is to identify ρ_1 and ρ_2 if $\overline{[\rho_1]} \cap \overline{[\rho_2]} \neq \emptyset$.

Theorem 2.1. *(Geometric Invariant Theory): In each class, there is exactly one closed orbit corresponding to a semi-simple representation that is irreducible or diagonal.*

2.1 General definition using algebraic geometry

We define

$$A(\Gamma) = \mathbb{C}[M^{\gamma}_{i,j} \mid i, j \in \{1, 2\}, \gamma \in \Gamma] \Big/ \begin{pmatrix} \det(M^\gamma) = 1 & \\ M^{\gamma\delta} = M^\gamma M^\delta & \forall \gamma, \delta \in \Gamma \end{pmatrix}$$

In this formula, M^γ stands for the matrix with the indeterminate entries $(M^{\gamma}_{i,j})$. The algebra $A(\Gamma)$ is Noetherian and satisfies the following universal property: for any ring R, there is a natural isomorphism

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{alg}}(A(\Gamma), R) & \simeq & \mathrm{Hom}_{\mathrm{gr}}(\Gamma, \mathrm{SL}_2(R)) \\ \Phi & \longmapsto & (\rho : \gamma \mapsto \Phi(M_\gamma)) \end{array}$$

Now $\mathrm{SL}_2(R)$ acts on $A(\Gamma)$ by $g.P(M^\gamma) = P(g^{-1} M^\gamma g)$. We denote by $A(\Gamma)^{\mathrm{SL}_2}$ the algebra of invariants.

Let us illustrate the concept of algebraic quotient with a very elementary example. The algebra of regular functions on $\mathbb{C}$ is $\mathbb{C}[X]$. If $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{C}$ by $x \mapsto -x$, the ring of invariant functions is $\mathbb{C}[X^2]$. Setting $y = x^2$, the quotient is isomorphic to $\mathbb{C}$ with coordinate y and the quotient map is $x \mapsto x^2$.

Theorem 2.2. *We set :*

$$X(\Gamma) = \mathrm{Spec}(A(\Gamma)^{\mathrm{SL}_2}).$$

The inclusion $A(\Gamma)^{\mathrm{SL}_2} \hookrightarrow A(\Gamma)$ *gives* $\pi : \mathrm{Spec}(A(\Gamma)) \to X(\Gamma)$. *The fibers of* π *correspond to the equivalent classes defined above.*

For simplicity, if A is a Noetherian $\mathbb{C}$-algebra with spectrum X, we will denote by $X_{\mathbb{C}}$ its complex points, that is, the set of maximal ideals of A, or more conveniently:

$$X_{\mathbb{C}} = \mathrm{Hom}_{\mathrm{alg}}(A, \mathbb{C})$$

Hence, one can see $\mathrm{Hom}(\Gamma, \mathrm{SL}_2)$ as the spectrum of $A(\Gamma)$. In particular, the previous statement means that

- the map $\pi : \mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C})) \to X(\Gamma)_{\mathbb{C}}$ is surjective
- two representations satisfy $\pi(\rho_1) = \pi(\rho_2)$ if and only if $\overline{[\rho_1]} \cap \overline{[\rho_2]} \neq \emptyset$.

Example 2.3. For $\Gamma = \mathbb{Z}$, we have

$$A(\Gamma) \simeq \mathbb{C}[a,b,c,d] \Big/ \left(\ ad - bc = 1 \ \right)$$

and we can check that $A(\Gamma)^{\mathrm{SL}_2} \simeq \mathbb{C}[a+d]$ so that $X(\Gamma)_{\mathbb{C}} = \mathbb{C}$.

Example 2.4. For $\Gamma = \mathbb{Z}/n\mathbb{Z}$, we have

$$A(\Gamma) \simeq \mathbb{C}[a,b,c,d] \Big/ \left(\begin{array}{c} \det(M) = 1 \\ M^n = Id \end{array} \right)$$

We can check that $A(\Gamma)^{\mathrm{SL}_2} \simeq \mathbb{C}^{\lfloor n/2 \rfloor + 1}$. So $X(\Gamma)$ has only $\lfloor n/2 \rfloor + 1$ points.

Problem 2.5. Can we find generators and relations for $A(\Gamma)^{\mathrm{SL}_2}$?

2.2 The skein algebra

Definition 2.6. We set:

$$B(\Gamma) = \mathbb{C}[Y_\gamma \mid \gamma \in \Gamma] \Big/ \left(\ Y_e = 2, \quad Y_\gamma Y_\delta = Y_{\gamma\delta} + Y_{\gamma\delta^{-1}} \quad \forall \gamma, \delta \in \Gamma \ \right)$$

The algebra $B(\Gamma)$ is Noetherian. Moreover it is known (but not obvious at all) that the morphism

$$\begin{array}{ccc} B(\Gamma) & \xrightarrow{\Phi} & A(\Gamma)^{SL_2} \\ Y_\gamma & \longmapsto & \mathrm{Tr}(M^\gamma) \end{array}$$

is surjective and it kernel is in the nilradical of $B(\Gamma)$. Setting

$$X_{\mathrm{skein}}(\Gamma) = \mathrm{Spec}(B(\Gamma)) \quad \text{and} \quad X_{\mathrm{GIT}} = \mathrm{Spec}(A(\Gamma)^{\mathrm{SL}_2})$$

we have that the reduced varieties $X_{\mathrm{skein}}(\Gamma)^{\mathrm{red}}$ and $X_{\mathrm{GIT}}(\Gamma)^{\mathrm{red}}$ are the same (see [BH91, S93]).

Open question: Is Φ an isomorphism?

We notice that it is the case as soon as $B(\Gamma)$ is reduced. Let us concentrate on the case when Γ is the fundamental group of a surface. For a family $\gamma = (\gamma_1, \ldots, \gamma_n)$ of elements of Γ we set $Y_\gamma = \prod_i Y_i$.

Lemma 2.7. *If $\Gamma = \pi_1(\Sigma)$ where Σ is a surface with (possibly empty) boundary, then*

$$B(\Gamma) = \bigoplus_\gamma \mathbb{C} Y_\gamma$$

where γ runs over isotopy classes of multicurves (1-submanifolds of Σ without component bounding a disc).

Proof. (idea) Any element γ can be represented by an immersion with simple double points. Applying the defining equation of $B(\Gamma)$, one can reduce the number of crossings of γ. By recursion, it follows that the Y_γ where γ is a multicurve are generators. Using Reidemeister theorem on the classification of immersions, one shows that they are independent. □

Example 2.8. We denote by P the sphere with 3 holes. The fundamental group of P is the free group on two generators : $\mathbb{F}_2$. We denote by a and b the loops in P represented in Figure 1.

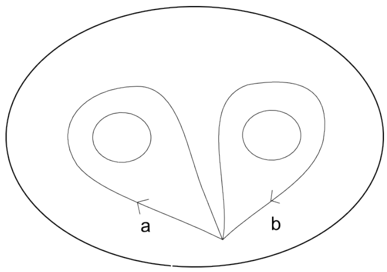


Figure 1: A pair of pants

Then we have $B(\pi_1(P)) = \mathbb{C}[x, y, z]$ where $x = Y_a, y = Y_b, z = Y_{ab}$ hence $B(\mathbb{F}_2)$ is reduced and isomorphic to $A(\mathbb{F}_2)^{\mathrm{SL}_2}$. So we have :

$$X(\mathbb{F}_2)_\mathbb{C} = \mathbb{C}^3$$

In other words, a class of representation $\rho : \mathbb{F}_2 \to \mathrm{SL}_2(\mathbb{C})$ is completely described by the triple $(\mathrm{Tr}(\rho(a)), \mathrm{Tr}(\rho(b)), \mathrm{Tr}(\rho(ab)))$.

Example 2.9. Let $\rho : \mathbb{Z}^n \to \mathrm{SL}_2(\mathbb{C})$ be upper triangular as one can suppose up to conjugation. We can write :

$$\rho(e_j) = \begin{pmatrix} \lambda_j & \star \\ 0 & \lambda_j^{-1} \end{pmatrix}$$

where $(e_1, ..., e_n)$ is the canonical basis of $\mathbb{Z}^n$ and $(\lambda_1, ..., \lambda_n) \in (\mathbb{C}^*)^n$. This representation is in the same class as the semi-simple representation

$$e_j \longmapsto \begin{pmatrix} \lambda_j & 0 \\ 0 & \lambda_j^{-1} \end{pmatrix}$$

hence $X(\mathbb{Z}^n) \simeq (\mathbb{C}^*)^n/\sigma$ where $\sigma(z_1, ..., z_n) = (z_1^{-1}, ..., z_n^{-1})$.

In the case of $\mathbb{Z}^2 = \pi_1(S^1 \times S^1)$, the isomorphism is given by

$$\gamma = (p, q) \in B(\pi_1(S^1 \times S^1)) \longmapsto X^p Y^q + X^{-p} Y^{-q} \in \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]^\sigma$$

Example 2.10. Let K be the figure eight knot in S^3 as shown in Figure 2.

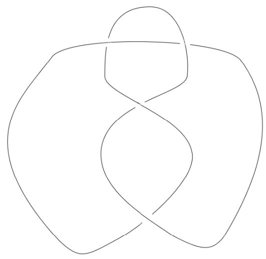


Figure 2: Figure eight knot

The fundamental group of its complement is

$$\pi_1(S^3 - K) = \langle u, v \mid wv = uw \rangle \quad \text{where} \quad w = v^{-1}uvu^{-1}$$

which can also be written as

$$\langle t, a, b \mid t^{-1}at = ab\,,\ t^{-1}bt = bab \rangle \quad \text{where} \quad t = u^{-1}\,,\ a = w\,,\ b = vu^{-1}$$

We use the following suggestive notation: for $w \in \Gamma$, we denote by Tr (w) the trace function $f_w \in A(\Gamma)^{\mathrm{SL}_2}$ and define

$$x = \mathrm{Tr}(u) = \mathrm{Tr}(v)\,, \quad y = \mathrm{Tr}(uv)\,, \quad x_1 = \mathrm{Tr}(a)\,, \quad x_2 = \mathrm{Tr}(b)$$

We have

$$x_2 = \mathrm{Tr}(uv^{-1}) = \mathrm{Tr}(u)\mathrm{Tr}(v^{-1}) - \mathrm{Tr}(uv) = x^2 - y \qquad (1)$$

in the same way we get

$$x_1 = \mathrm{Tr}(uvu^{-1}v^{-1}) = 2x^2 + y^2 - x^2y - 2 \qquad (2)$$

Moreover we have

$$x_2 = \mathrm{Tr}(t^{-1}bt) = \mathrm{Tr}(bab) = \mathrm{Tr}(abb) = \mathrm{Tr}(ab)\mathrm{Tr}(b) - \mathrm{Tr}(a) = x_1x_2 - x_1$$

from (1) and (2) we deduce that

$$(x_2 - 2)x_1 = x_2 - x_1 = x^2 - y - 2x^2 - y^2 + x^2y + 2 = (y - 1)(x^2 - y - 2)$$

which can be written as

$$(x^2 - y - 2)(2x^2 + y^2 - x^2y - y - 1) = 0$$

so that

$$X(\pi_1(S^3 - K))_{\mathbb{C}} = \{(x, y) \in \mathbb{C}^2 \mid (x^2 - y - 2)(2x^2 + y^2 - x^2y - y - 1) = 0\}$$

This can be represented as a nodal Riemann surface with punctures as in Figure 3. The sphere corresponds to the component of reducible representation. The ideal point it contains (denoted by +) corresponds to the Seifert surface (or fiber). The torus corresponds to the component of irreducible representations. The two ideal points correspond to the checkerboard surfaces of the figure eight knot. Finally, the two intersection points correspond to the zeroes of the Alexander polynomial $\Delta_K = -t + 3 - t^{-1}$.

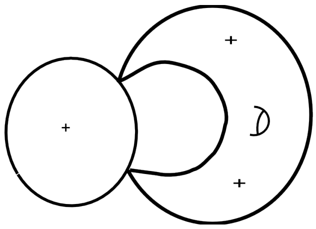


Figure 3: The character variety of the figure eight knot

3 Twisted Cohomology and tangent space

For a connected manifold M, we denote by $X(M)$ the character variety associated to its fundamental group $\pi_1(M)$. Many important properties of the character varieties come from the interpretation of $T_{[\rho]}X(M)$ in terms of the twisted cohomology of M.

- Weil rigidity : if M is an hyperbolic 3-manifold, the (lifted) holonomy representation $[\rho] \in X(M)$ is an isolated point i.e $T_{[\rho]}X(M) = 0$.

- Symplectic structure : if Σ is an oriented compact surface without boundary then
$$T_{[\rho]}X(\Sigma) \simeq H^1(\Sigma, \mathrm{Ad}_\rho)$$
has a cup-product composed with the Killing form:
$$H^1(\Sigma, \mathrm{Ad}_\rho) \times H^1(\Sigma, \mathrm{Ad}_\rho) \longrightarrow H^2(\Sigma, \mathbb{C}) \simeq \mathbb{C}$$
which gives a symplectic structure on $X(\Sigma)$. On $X(\Sigma)_{\mathbb{R}}$, this symplectic structure is either the Weil-Peterson form or the Goldman symplectic form.

- If M is a 3-manifold with boundary $r : X(M) \longrightarrow X(\partial M)$ is (generically) a Lagrangian immersion.

Let us start with the definition of twisted (co)homology. Let M be a connected finite CW-complex, $x_0 \in M$ a base point. Let V be a finite dimensional complex vector space with an action of $\pi = \pi_1(M, x_0)$ (i.e a $\mathbb{C}[\pi]$-module).

Then $C_*^{\text{cell}}(\tilde{M})$, the cellular complex of the universal covering is a representation of π. For V_1 and V_2 two representations of π, we define

$$V_1 \underset{\pi}{\otimes} V_2 = V_1 \otimes V_2 \big/ \left(\gamma v_1 \otimes v_2 = v_1 \otimes \gamma v_2 \quad \forall \gamma \in \pi \right)$$

$$\operatorname{Hom}_\pi(V_1, V_2) = \{ f : V_1 \longrightarrow V_2 \mid f(\gamma v) = \gamma f(v) \quad \forall \gamma \in \pi \quad \forall v \in V_1 \}$$

Now we set

$$H_*(M, V) = H_*(C_*^{\text{cell}}(\tilde{M}) \underset{\pi}{\otimes} V) \quad \text{and} \quad H^*(M, V) = H^*(\operatorname{Hom}_\pi(C_*^{\text{cell}}(\tilde{M}), V))$$

Example : For $M = S^1$, $\pi_1(S^1) = \mathbb{Z}$. Let V a finite dimensional complex vector space such that $\mathbb{Z}$ acts on V by powers of $\Phi \in \operatorname{Aut}(V)$. Let $\tilde{e_0}$ and $\tilde{e_1}$ a lift of the canonical cell decomposition of S^1. We have

$$C_0(\tilde{M}) = \mathbb{Z}[t^{\pm 1}]\tilde{e_0} \quad \text{and} \quad C_1(\tilde{M}) = \mathbb{Z}[t^{\pm 1}]\tilde{e_1}$$

Where t is a generator of $\pi_1(S^1)$. We can check that

$$\begin{aligned} C_1(\tilde{M}) \underset{\pi}{\otimes} V = V\tilde{e_1} &\xrightarrow{\partial} C_0(\tilde{M}) \underset{\pi}{\otimes} V = V\tilde{e_0} \\ \tilde{e_1} \otimes v &\longmapsto t\tilde{e_0} \otimes v - \tilde{e_0} \otimes v = \tilde{e_0} \otimes (\Phi(v) - v) \end{aligned}$$

So we get

$$H_0(M, V) = \operatorname{Coker}(\Phi - \operatorname{Id}) \quad \text{and} \quad H_1(M, V) = \operatorname{Ker}(\Phi - \operatorname{Id})$$

In the same way

$$H^1(M, V) = \operatorname{Coker}(\Phi - \operatorname{Id}) \quad \text{and} \quad H^1(M, V) = \operatorname{Ker}(\Phi - \operatorname{Id})$$

Group cohomology : Let Γ a group and V a Γ-module. For $n \geq 0$, we define the complex $C^n(\Gamma, V) = \{ f : \Gamma^n \to V \}$ and the coboundary homomorphisms $C^n(\Gamma, V) \xrightarrow{d^n} C^{n+1}(\Gamma, V)$ by the formula

$$\begin{aligned} (d^n f)(g_1, ..., g_{n+1}) = & g_1 f(g_2, ..., g_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(g_1, ..., g_{i-1}, g_i g_{i+1}, g_{i+2}, ..., g_{n+1}) \\ & + (-1)^{n+1} f(g_1, ..., g_n) \end{aligned}$$

We denote the cohomology of this complex by $H^*(\Gamma, V)$. The following facts are standard facts of algebraic topology transposed in the twisted case.

Theorem : If M is a $K(\pi, 1)$ (i.e $\tilde{M}$ is contractible) then

$$H^*(M, V) \simeq H^*(\pi_1(M), V)$$

One can show that the two cohomology spaces are always isomorphic in degree 0 and 1.

Poincaré duality : If M is a compact and oriented manifold of dimension n with (possibly empty) boundary then

$$\forall k \quad H^k(M,V) \simeq H_{n-k}(M,\partial M,V) \quad \text{and} \quad H^k(M,\partial M,V) \simeq H_{n-k}(M,V)$$

Universal coefficients :

$$H^k(X,V) \simeq H_k(X,V^*)^*$$

Observe that if there is a non-degenerate form $\langle \cdot\,,\cdot\rangle : V \times V \longrightarrow \mathbb{C}$ which is bi-invariant (i.e $\langle \gamma v, \gamma w\rangle = \langle v,w\rangle \quad \forall \gamma \in \pi_1(M)$) then V and V^* are isomorphic as π-modules and one has :

$$H^k(X,V) \simeq H_k(X,V)^*$$

Application to character variety : Define the following algebra morphism

$$\eta : \mathbb{C}[\epsilon]/(\epsilon^2) \longrightarrow \mathbb{C} \quad \text{by} \quad \eta(a+\epsilon b) = a.$$

By definition, the Zariski tangent space of $X = \operatorname{Spec} A$ at a complex point $\phi : A \to \mathbb{C}$ is the space of algebra morphisms $\Phi : A \to \mathbb{C}[\epsilon]/(\epsilon^2)$ such that $\phi = \epsilon \circ \Phi$. Using the universal property of the algebra $A(\Gamma)$ we get

$$TX(\Gamma)_{\mathbb{C}} = \operatorname{Hom}_{\text{alg}}(A(\Gamma), \mathbb{C}[\epsilon]/(\epsilon^2)) = \operatorname{Hom}_{\text{gr}}(\Gamma, \operatorname{SL}_2(\mathbb{C}[\epsilon]/(\epsilon^2))).$$

Let $\rho : \Gamma \longrightarrow \operatorname{SL}_2(\mathbb{C})$ be a representation. A tangent vector at ρ is then a representation $\rho_\epsilon : \Gamma \longrightarrow \operatorname{SL}_2(\mathbb{C}[\epsilon]/(\epsilon^2))$ such that $\eta \circ \rho_\epsilon = \rho$. Then define $z : \Gamma \longrightarrow sl_2(\mathbb{C})$ by

$$\forall \gamma \in \Gamma, \qquad z(\gamma) = \left.\frac{d}{dt}\right|_{\epsilon=0} \rho_\epsilon(\gamma)\rho(\gamma)^{-1}.$$

We can view $sl_2(\mathbb{C})$ as a Γ-module by the action

$$\forall \gamma \in \Gamma,\ \xi \in sl_2(\mathbb{C}) \qquad \gamma.\xi = \rho(\gamma)\xi\rho(\gamma)^{-1}$$

We denote by Ad_ρ this Γ-module. So from the point of view of group cohomology we can now see that $z \in C^1(\Gamma, sl_2(\mathbb{C})) = \{z : \Gamma \longrightarrow sl_2(\mathbb{C})\}$. Recall that $d^1 : C^1(\Gamma, sl_2(\mathbb{C})) \longrightarrow C^2(\Gamma, sl_2(\mathbb{C}))$ is the coboundary homomorphism defined previously. If we fix $(\gamma,\delta) \in \Gamma^2$ we have $\rho_\epsilon(\gamma\delta) = \rho_\epsilon(\gamma)\rho_\epsilon(\delta)$ so

$$\begin{aligned} z(\gamma\delta) &= \left.\frac{d}{dt}\right|_{\epsilon=0} \rho_\epsilon(\gamma)\rho_\epsilon(\delta)\rho(\delta)^{-1}\rho(\gamma)^{-1} \\ &= \left.\frac{d}{dt}\right|_{\epsilon=0} \rho_\epsilon(\gamma)\rho(\gamma)^{-1} + \rho(\gamma)\left.\frac{d}{dt}\right|_{\epsilon=0} \rho_\epsilon(\delta)\rho(\delta)^{-1}\rho(\gamma)^{-1} \\ &= z(\gamma) + \rho(\gamma)z(\gamma)\rho(\gamma)^{-1} \end{aligned}$$

we deduce that $d^1(z) = 0$, so $[z] \in H^1(M, \operatorname{Ad}_\rho)$. Intuitively if the quotient $R(\Gamma)/\operatorname{SL}_2(\mathbb{C})$ were the naive one, we would have $T_{[\rho]}X(\Gamma) \simeq H^1(M, \operatorname{Ad}_\rho)$. Luna's theorem states that the projection $R(\Gamma) \longrightarrow X(\Gamma)$ is an algebraic principal $\operatorname{SL}_2(\mathbb{C})$-bundle at the neighborhood of an irreducible representation. Thus there is a natural isomorphism $T_{[\rho]}X(\Gamma) \simeq H^1(M, \operatorname{Ad}_\rho)$ for any irreducible ρ.

Examples and counter examples :

- $M = S^1$ and $\Gamma = \mathbb{Z}$, we have $X(\mathbb{Z})_{\mathbb{C}} = \mathbb{C}$. Let $\rho : \mathbb{Z} \longrightarrow \mathrm{SL}_2(\mathbb{C})$ be a representation,

$$\begin{aligned} H^1(\mathbb{Z}, \mathrm{Ad}_\rho) &= H^1(S^1, \mathrm{Ad}_\rho) \quad \text{(using the fact that } S^1 \text{ is a } K(\pi,1)) \\ &= H_0(S^1, \mathrm{Ad}_\rho) \quad \text{(using Poincaré duality)} \\ &= H^0(S^1, \mathrm{Ad}_\rho^*)^* \quad \text{(using universal coefficients)} \\ &= H^0(S^1, \mathrm{Ad}_\rho)^* \qquad\qquad \text{(using the Killing form)} \\ &= \{\xi \in sl_2(\mathbb{C}) \mid \rho(\gamma)\xi\rho(\gamma)^{-1} = \xi \quad \forall \gamma \in \mathbb{Z}\}^* \end{aligned}$$

  Hence it follows that we have:

$$\begin{aligned} \rho(1) \neq \pm 1 \Rightarrow \dim(H^1(\mathbb{Z}, \mathrm{Ad}_\rho)) = 1 \\ \rho(1) = \pm 1 \Rightarrow \dim(H^1(\mathbb{Z}, \mathrm{Ad}_\rho)) = 3 \end{aligned}$$

  hence for non central representations we have the isomorphism $T_{[\rho]}X(\mathbb{Z}) = H^1(S^1, \mathrm{Ad}_\rho)$, but not for central ones although all points in the character variety are smooth.

- For $n \geq 1$, if we set $\Gamma = F_n$ (the free group on n generators) we see that Γ is the fundamental group of the wedge of n circles (denoted by B_n). Let $\rho : F_n \longrightarrow \mathrm{SL}_2(\mathbb{C})$ be an irreducible representation. We can check that if $k \geq 2$ then $H^k(B_n, \mathrm{Ad}_\rho) = 0$. Moreover $\chi(B_n, \mathrm{Ad}_\rho) = \dim(\mathrm{Ad}_\rho)\chi(B_n) = 3(1-n)$, therefore

$$\dim(H^0(B_n, \mathrm{Ad}_\rho)) - \dim(H^1(B_n, \mathrm{Ad}_\rho)) = 3 - 3n$$

  which implies that $X(F)$ is a smooth variety (at irreducible representations) of dimension $3n-3$. For instance if $n = 2$, we write $F_2 = \langle \alpha, \beta \rangle$ and let $\rho : F_2 \longrightarrow \mathrm{SL}_2(\mathbb{C})$ be a representation. We denote by

$$x = \mathrm{Tr}(\alpha)\,, \quad y = \mathrm{Tr}(\beta)\,, \quad z = \mathrm{Tr}(\alpha\beta)$$

  which gives a coordinate system for $X(F_2)_{\mathbb{C}} = \mathbb{C}^3$. Moreover ρ is irreducible if and only if $\mathrm{Tr}([\alpha,\beta]) \neq 2$, we deduce that the irreducible representations are

$$\{(x,y,z) \in \mathbb{C}^3 \mid x^2 + y^2 + z^2 - xyz - 2 \neq 2\}$$

- Let Σ_g be a surface of genus $g \geq 2$ and $\Gamma = \pi_1(\Sigma_g)$. Let $\rho : \Gamma \longrightarrow \mathrm{SL}_2(\mathbb{C})$ be an irreducible representation. We have $0 = H^0(\Gamma, \mathrm{Ad}_\rho) \simeq H^2(\Gamma, \mathrm{Ad}_\rho)$, therefore $\dim(H^1(\Gamma, \mathrm{Ad}_\rho)) = -3\chi(\Sigma_g) = 6g - 6$. Here irreducible conjugacy classes of representations correspond (bijectively) to smooth points of $X(\Gamma)$.

- For $K = \mathbb{R}$ or $\mathbb{Z}$ we define the Heisenberg group

$$H_K^3 = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(K) \mid (x,y,z) \in K \right\}$$

we then define the Heisenberg manifold $M = H^3_{\mathbb{R}}/H^3_{\mathbb{Z}}$. We have that $\pi_1(M) = H^3_{\mathbb{Z}}$ and $H_1(M,\mathbb{Z}) = \pi_1(M)^{\mathrm{ab}} = \mathbb{Z}^2$. More precisely, setting $\alpha = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, the group $H_{\mathbb{Z}}$ has the following presentation:

$$H_{\mathbb{Z}} = \langle \alpha, \beta, \gamma | \alpha\beta = \beta\alpha, \alpha\gamma = \gamma\alpha, \gamma\beta\gamma^{-1} = \alpha\beta \rangle$$

The variety $X(M)$ has two components

$$X(M) \simeq X(\mathbb{Z}^2) \sqcup \{[\rho_0]\}$$

where $[\rho_0]$ corresponds to the unique class of irreducible representation of $H^3_{\mathbb{Z}}$ in $\mathrm{SL}_2(\mathbb{C})$. This representation ρ_0 is given by

$$\rho_0(\alpha) = -\mathrm{Id}, \quad \rho_0(\beta) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \rho_0(\gamma) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and a computation shows that $H^1(M, \mathrm{Ad}_{\rho_0}) = 0$.

4 Symplectic structures on character varieties of surfaces

Let Σ_g be a surface of genus g and $\rho : \pi_1(\Sigma_g) \longrightarrow \mathrm{SL}_2(\mathbb{C})$ be an irreducible representation (non central if $g = 1$) then $T_{[\rho]}X(\Sigma_g) \simeq H^1(\Sigma_g, \mathrm{Ad}_\rho)$. Let ξ, $\eta \in H^1(\Sigma_g, \mathrm{Ad}_\rho)$, consider

$$\langle \xi \wedge \eta \rangle \in H^2(\Sigma_g, \mathbb{C})$$

which is the cup product followed by the Killing form on $sl_2(\mathbb{C})$. This cohomology class can be integrated, we define

$$\omega_{[\rho]}(\xi, \eta) = \int_{\Sigma_g} \langle \xi \wedge \eta \rangle \in \mathbb{C}.$$

This provides a non-degenerate antisymmetric 2-form on $X^{\mathrm{irr}}(\Sigma_g)$.

4.1 The Goldman bracket

We can give an explicit description of symplectic gradients of trace functions. Let $\rho : \pi_1(\Sigma_g) \longrightarrow \mathrm{SL}_2(\mathbb{C})$ be an irreducible representation. The linear form

$$\begin{aligned} df_\gamma([\rho]) : H^1(\Sigma_g, \mathrm{Ad}_\rho) &\longrightarrow \mathbb{C} \\ z &\longmapsto \left.\frac{d}{dt}\right|_{\epsilon=0} \mathrm{Tr}\,\rho_\epsilon(\gamma) = \mathrm{Tr}(z(\gamma)\rho(\gamma)) \end{aligned}$$

can be interpreted as an element of $H_1(\Sigma, \mathrm{Ad}_\rho)$.

Consider $\gamma \otimes \rho(\gamma)_0 \in H_1(\gamma, \mathrm{Ad}_\rho)$ where $\rho(\gamma)_0 = \rho(\gamma) - \frac{1}{2}\mathrm{Tr}(\rho(\gamma))\mathrm{Id} \in sl_2(\mathbb{C})$. We can take the image of $\gamma \otimes \rho(\gamma)_0$ by the natural map $H_1(\gamma, \mathrm{Ad}_\rho) \longrightarrow H_1(\Sigma_g, \mathrm{Ad}_\rho)$ which we denote by $X^\gamma_\rho \in H_1(\Sigma_g, \mathrm{Ad}_\rho)$.

For any $z \in H^1(\Sigma, \mathrm{Ad}_\rho)$, we compute :

$$\begin{aligned}\langle z, X_\rho^\gamma \rangle &= \mathrm{Tr}(z(\gamma)[\rho(\gamma) - \frac{1}{2}\mathrm{Tr}(\rho(\gamma))\mathrm{Id}]) = \mathrm{Tr}(z(\gamma)\rho(\gamma)) \\ &= df_\gamma([\rho]) \cdot z\end{aligned}$$

Hence $\gamma \otimes \rho(\gamma)_0 = df_\gamma([\rho])$ in $H_1(\Sigma, \mathrm{Ad}_\rho)$.

Let f, $g : X(\Sigma_g) \longrightarrow \mathbb{C}$ be two algebraic maps. We define for $\rho \in X^{\mathrm{irr}}(\Sigma)$

$$\{f, g\}(\rho) = \omega^{-1}(df(\rho), dg(\rho))$$

which gives a Poisson Bracket. Pick γ, δ two elements of $\pi_1(\Sigma_g)$, we suppose that theses two loops intersect transversally in a finite number of points. By definition, the pairing ω^{-1} on $H_1(\Sigma, \mathrm{Ad}_\rho)$ corresponds to the intersection product composed with the Killing form. Hence, we obtain the Goldman formula (see [G86])

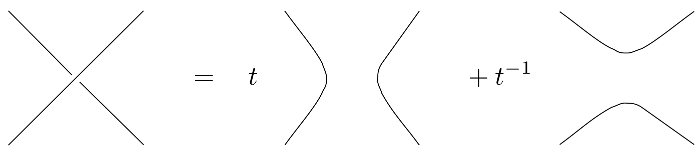
$$\{f_\gamma, f_\delta\}(\rho) = \langle \gamma \otimes \rho(\gamma)_0 \cdot \delta \otimes \rho(\delta)_0 \rangle = \sum_{p \in \gamma \cap \delta} \epsilon(p) \mathrm{Tr}(\rho(\gamma_p)_0 \, \rho(\delta_p)_0)$$

where $\epsilon(p) = \pm 1$ is the oriented intersection number of α and β at p and α_p denotes the loop α based at p. If we expand $\mathrm{Tr}(\rho(\gamma_p)_0 \, \rho(\delta_p)_0)$, we rewrite this formula in the following way:

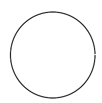
$$\{f_\gamma, f_\delta\}(\rho) = \sum_{p \in \gamma \cap \delta} \epsilon(p)[\mathrm{Tr}(\rho(\gamma_p)\, \rho(\delta_p)) - \mathrm{Tr}(\rho(\gamma_p)\, \rho(\delta_p)^{-1})].$$

4.2 The point of view of skein modules

Let R be a $\mathbb{C}[t, t^{-1}]$ algebra and M be a connected oriented compact 3-manifold. We define $S(M, R)$ as the R-module generated by isotopy classes of banded links factored by the following local relations :



(K1)


$= -t^2 - t^{-2}$ (K2)

We will make use of the following facts :

- $S(\Sigma \times [0, 1], \mathbb{C}[t, t^{-1}])$ is free with basis given by multicurves. Indeed if we consider a link in $\Sigma \times [0, 1]$, we can project it on Σ and apply (K1) to each crossing and remove trivial circles with (K2). We can then show that it is well defined using Reidemeister theorem (compare with Lemma 2.7).

- From the previous fact, we check that

$$S(\Sigma \times [0,1], \mathbb{C}[t,t^{-1}]) \simeq S(\Sigma \times [0,1], \mathbb{C}) \otimes \mathbb{C}[t,t^{-1}]$$

  where the isomorphism is given by sending a multicurve γ to $\gamma \otimes 1$.

- If Σ is a surface (maybe with boundary) then $S(\Sigma\times[0,1],R)$ has a natural product given by stacking : let $L \subset \Sigma \times [0,1] \simeq \Sigma \times [0,1/2]$ and $L' \subset \Sigma \times [0,1] \simeq \Sigma \times [1/2,1]$ so by gluing $\Sigma \times [0,1/2]$ and $\Sigma \times [1/2,1]$ along $\Sigma \times \{1/2\}$ we can define $L \cdot L' \in S(\Sigma \times [0,1], R)$. With this product, $S(\Sigma \times [0,1], R)$ becomes an associative algebra.

- We can check that $S(M,\mathbb{C}) \simeq B(\pi_1(M))$ where t acts by -1 on $\mathbb{C}$ and a banded knot γ in $S(M,\mathbb{C})$ corresponds to the generator $-Y_\gamma$ in $B(\pi_1(M))$.

- In the case of the torus, we can check that

$$\begin{array}{ccc} S(S^1\times S^1, \mathbb{C}[t^{\pm 1}]) & \xrightarrow{\Psi} & \mathbb{C}\langle X^{\pm1}, Y^{\pm1}\rangle/(YX = t^2XY) \\ (a,b) & \longmapsto & t^{ab}(X^aY^b + X^{-a}Y^{-b}) \end{array}$$

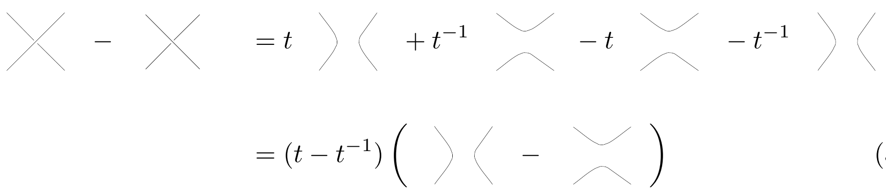
  is well defined (see [FG00]). Moreover it is injective and its image is the subalgebra of $\mathbb{C}\langle X^{\pm1}, Y^{\pm1}\rangle/(YX = t^2XY)$ invariant under $\sigma(X^aY^b) = X^{-a}Y^{-b}$.

4.3 Computation of the Poisson bracket

Theorem : The poisson bracket $\{\,,\,\}$ is obtained by

$$\{f\,,g\,\} = \frac{1}{2}\left.\frac{d}{dt}\right|_{t=-1} (fg - gf) \in S(\Sigma \times [0,1], \mathbb{C}[t,t^{-1}])$$

Proof : We show it for two simple closed curves γ and δ. We want to compute $f_\gamma f_\delta - f_\delta f_\gamma$. Remark that

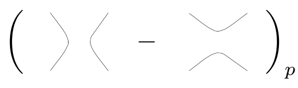


(3)

Number the crossings of $\gamma \cup \delta$ and denote by $(\pm, ..., \pm)$ the banded link $\gamma \cup \delta$ where γ is over δ at the n-th crossing if we have a $+$ at the n-th position and γ is under δ if it is a $-$.Then

$$\begin{aligned} f_\gamma f_\delta - f_\delta f_\gamma &= (+,\dots,+) - (-,\dots,-) \\ &= [(+,+,\dots,+) - (-,+,\dots,+)] + [(-,+,+,\dots,+) - (-,-,+,\dots,+)] \\ &\quad + \dots + [(-,\dots,-,+) - (-,\dots,-,-)] \end{aligned}$$

by using (3) we get that

$$\frac{1}{2}\left.\frac{d}{dt}\right|_{t=-1}(fg-gf) = \sum_p 2$$  $$= 2\sum_p \epsilon_p \mathrm{Tr}(\gamma_p\delta_p - \gamma_p\delta_p^{-1})$$

where we sum over intersection points. We conclude using the Goldman formula.

Hence, this symplectic structure is easily computable in terms of Poisson bracket. Moreover, the product in $S(\Sigma, \mathbb{C}[t^{\pm 1}])$ is obviously associative hence the commutator satisfies the Jacobi identity which implies that the Poisson bracket also satisfies the Jacobi identity. Finally we deduce that $d\omega = 0$ hence ω is indeed a symplectic form. The usual way to prove this fact uses the interpretation of the character variety as a symplectic reduction from the space of $sl_2(\mathbb{C})$-valued connections on Σ.

Example of the torus : We know that $X(S^1 \times S^1) \simeq \mathbb{C}^* \times \mathbb{C}^*/\sigma$ which is two dimensional. Pick $(x, y) \in \mathbb{C}^* \times \mathbb{C}^*/\sigma$ such that $x^2 \neq 1$ and $y^2 \neq 1$. At the point (x, y) the symplectic form is $\omega = \Omega(x, y)\, dx \wedge dy$, for $\Omega(x, y) \in \mathbb{C}^*$. We would like to compute Ω.

Let λ and μ the meridian and the longitude of the torus. We have $f_\lambda(x, y) = x + x^{-1}$ and $f_\mu(x, y) = y + y^{-1}$. So

$$df_\lambda = (1 - x^{-2})\, dx \quad \text{and} \quad df_\mu = (1 - y^{-2})\, dy$$

so the symplectic gradients have the form

$$X_\lambda = \begin{pmatrix} 0 \\ u \end{pmatrix} \quad \text{and} \quad X_\mu = \begin{pmatrix} v \\ 0 \end{pmatrix}$$

with

$$-\Omega u = 1 - x^{-2} \quad \text{and} \quad \Omega v = 1 - y^{-2} \tag{4}$$

moreover from the Goldman formula gives

$$\{f_\lambda, f_\mu\} = \Omega\, uv = (f_{\lambda\mu} - f_{\lambda\mu^{-1}}) = xy + x^{-1}y^{-1} - xy^{-1} - x^{-1}y \tag{5}$$

and from (4) and (5) we conclude that

$$(1 - x^{-2})(1 - y^{-2})(\Omega - x^{-1}y^{-1}) = 0$$

therefore we can explicit the symplectic form as follows:

$$\omega = \frac{dx}{x} \wedge \frac{dy}{y}.$$

5 Character varieties of 3-manifolds

Let M be a 3-manifold compact and oriented with boundary which may be disconnected. We set $X(\partial M) = \prod_i X(\partial_i M)$ where $\partial_i M$ denote the connected components of ∂M.

For each i, the inclusion $\partial_i M \to M$ induces a map $\pi_1(\partial_i M) \to \pi_1(M)$ which itself induces a map $r_i : X(M) \to X(\partial_i M)$ which we call restriction map. Finally, we set $r = (r_1, \ldots, r_n) : X(M) \to X(\partial M)$. The purpose of this section is to study this map.

5.1 Lagrangian property

Theorem 5.1. *Let $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ be a representation. The image of the natural map*

$$i^* : H^1(M, \mathrm{Ad}_\rho) \to H^1(\partial M, \mathrm{Ad}_\rho)$$

is a Lagrangian in $H^1(\partial M, \mathrm{Ad}_\rho)$.

Proof. We write a part of the long exact sequence of the pair $(M, \partial M)$ and Poincaré duality.

$$\begin{array}{ccccc} H^1(M,\mathrm{Ad}_\rho) & \xrightarrow{\alpha} & H^1(\partial M,\mathrm{Ad}_\rho) & \xrightarrow{\beta} & H^2(M,\partial M,\mathrm{Ad}_\rho) \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ H^2(M,\partial M,\mathrm{Ad}_\rho)^* & \xrightarrow{\beta^*} & H^1(\partial M,\mathrm{Ad}_\rho)^* & \xrightarrow{\alpha^*} & H^1(M,\mathrm{Ad}_\rho)^* \end{array}$$

We have $\mathrm{rk}(\alpha) = \dim\ker\beta = \mathrm{rk}\,\beta^* = \mathrm{rk}\,\beta$. From the equation $\dim\ker\beta + \mathrm{rk}\,\beta = \dim H^1(\partial M, \mathrm{Ad}\rho)$ we get the equality $\mathrm{rk}\, i^* = \mathrm{rk}\,\alpha = \frac{1}{2}\dim H^1(\partial M, \mathrm{Ad}\rho)$. The isotropy of the image of i^* comes directly from the diagram. ☐

This theorem has the following consequence: suppose that ρ and its restriction to the boundary is such that the first cohomology with twisted coefficient is the same as the Zariski tangent space of the character variety at $[\rho]$. Then, i^* is nothing more than the derivative of the restriction map i.e.

$$i^* = D_{[\rho]} r : T_{[\rho]} X(M) \to T_{r([\rho])} X(\partial M)$$

Hence, the image of the restriction map is locally a Lagrangian subspace of $X(\partial M)$.

5.2 Examples

Example 5.2. If $M = D^2 \times S^1$ then $\partial M = S^1 \times S^1$. The restriction map is the quotient of the map $\mathbb{C}^* \to \mathbb{C}^* \times \mathbb{C}^*$ defined by $z \mapsto (1, z)$.

Hence, the Lagrangian subspace $r(X(M)) \subset X(S^1 \times S^1)$ is the set of classes $(1, z) \in \mathbb{C}^* \times \mathbb{C}^*/\sigma$.

Example 5.3. If H is a handlebody of genus g and Σ be its boundary, let us describe the image $L = r(X(H)) \subset X(\Sigma)$.

Let $(\gamma_i)_{i=1,\ldots,3g-3}$ be a collection of disjoint and non parallel curves which bound a disc in H (we call this a pair of pants decomposition of Σ). Then, the corresponding trace functions Poisson commute meaning that $\{\mathrm{Tr}(\gamma_i), \mathrm{Tr}(\gamma_j)\} = 0$ and the Lagrangian L is given by

$$L = \{[\rho] |\, \mathrm{Tr}\rho(\gamma_i) = 2 \text{ for } i = 1, \ldots, 3g-3\}.$$

In that case, the character variety and this collection of trace functions form a completely integrable system.

Example 5.4. If M is the complement of the figure eight knot in S^3, that is $M = S^3 \setminus V(K)$, the image $L = r(X(M)) \subset X(\partial M)$ is an affine curve given in the following way

$$(x, y) \in \mathbb{C}^* \times \mathbb{C}^* \text{ belongs to } r(X(M)) \longleftrightarrow A(x, y) = 0.$$

where $A(x,y) = (x-1)(-x+xy^2+y^4+2xy^4+x^2y^4+xy^6-xy^8)$. Observe that the two factors of A correspond to the two components of $X(M)$ shown in Figure 2.

Example 5.5. Consider the gluing of two tetrahedra shown in Figure 4. The faces A and A', B and B' and C and C' are glued together in the unique way which preserves the arrows. We obtain a 3-manifold with a singular vertex whose link is a genus 2 surface. This means that by truncating a small tetrahedron in the neighborhood of every vertex, we find a 3-manifold M with $\partial M = \Sigma_2$. In Thurston's notes, it is explained that this manifold can be given a structure of hyperbolic manifold with totally geodesic boundary.

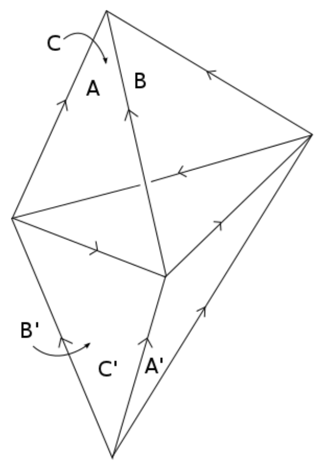


Figure 4: Gluing of two tetrahedra

The fundamental group of M is easily computed: we have

$$\pi_1(M) = \langle \alpha, \beta, \gamma | \alpha\beta\alpha^{-1}\beta\gamma\beta^{-1}\gamma\alpha\gamma^{-1} = 1 \rangle$$

One can show that the character variety of M is three dimensional, and Bezout theorem implies that the map $\chi : X(M) \to \mathbb{C}^3$ given by

$$\chi([\rho]) = (\mathrm{Tr}\rho(\alpha), \mathrm{Tr}\rho(\beta), \mathrm{Tr}\rho(\gamma))$$

is a ramified covering of degree 17.

5.3 The skein module point of view

If M is a manifold with boundary, there is an inclusion $i : \partial M \times [0,1] \to M$ given by a tubular neighborhood of the boundary. We denote by $i_* : S(\partial M \times [0,1], R) \to S(M,R)$ the map induced by the inclusion for any $\mathbb{C}[t,t^{-1}]$-algebra R.

For instance, take $R = \mathbb{C}$ with t acting by -1 and let $I = \ker i_*$. This is an ideal in $S(\partial M \times [0,1], \mathbb{C})$ which corresponds to a sub-variety of $X(\partial M)$. This is just the algebro-geometric way of defining the image $r(X(M)) \subset X(\partial M)$: an element $f \in I$ is a regular function on $X(\partial M)$ which vanishes on $r(X(M))$.

The fact that $r(X(M))$ is a Lagrangian sub-variety of $X(M)$ translates in the following way: if f, g belong to I then $\{f, g\} \in I$.

We propose a skein-theoretic explanation of this fact: let I^t be the kernel of the map $i^* : S(\partial M \times [0,1], \mathbb{C}[t^{\pm 1}]) \to S(M, \mathbb{C}[t^{\pm 1}])$. Putting $t = -1$ gives a map $I^t \to I$. Suppose that $f, g \in I$ lift to elements $f(t)$ and $g(t)$ in I^t. Then, their commutator still belong to I^t because I^t is a left ideal. Hence

$$\frac{f(t)g(t) - g(t)f(t)}{t+1} \in I^t$$

Taking the limit when t goes to -1, we get $\{f, g\} \in I$.

5.4 The transport equation

Let $S'(M)$ be the skein module of a 3-manifold M with coefficient $\mathbb{C}[\epsilon]/(\epsilon^2)$ where t acts by $-1 + \epsilon$. In other terms, we have

$$S'(M) = S(M, \mathbb{C}[t^{\pm 1}]/(t+1)^2).$$

If $M = \Sigma \times [0,1]$, we have the isomorphism $S'(M) = S(\Sigma, -1) \otimes \mathbb{C}[\epsilon]/(\epsilon^2)$. The stacking product has the following form through this isomorphism: if f, f', g, g' are complex combinations of multicurves in Σ, one has

$$(f + \epsilon f')(g + \epsilon g') = fg + \epsilon(fg' + f'g + \{f, g\})$$

Set $I' = \ker i_* : S'(\partial M \times [0,1]) \to S'(M)$. This is an left ideal in the non-commutative algebra $S'(\partial M \times [0,1])$ which fits in the following diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & I' & \longrightarrow & S'(\partial M \times [0,1]) & \xrightarrow{i*} & S'(M) \\ & \downarrow \eta & & \downarrow \eta & & \downarrow \eta \\ 0 \longrightarrow & I & \longrightarrow & S(\partial M \times [0,1], \mathbb{C}) & \xrightarrow{i^*} & S(M, \mathbb{C}) \end{array}$$

Conjecture 1. *There is a differential operator P acting on algebraic vector fields on $r(X(M))$ with values in algebraic functions on $r(X(M))$ with the property that*

$$f + \epsilon f' \in I' \Rightarrow f' = P(X_f) \in S(\partial M \times [0,1], \mathbb{C})/I$$

In this formula, X_f denotes the symplectic gradient of f. Moreover this operator satisfies $P(fX) = fP(X) + df(X)$.

This operator is related to the Reidemeister torsion T (a differential form on $r(X(M))$) in the following sense:

$$P(X) = \text{div}_T(X) = \frac{L_X T}{T}.$$

As I' is a left-ideal, if $f + \epsilon f' \in I'$, then $fg + \epsilon(gf' + g'f + \{f, g\}) \in I'$ whatever be g and g'. Hence, we should have $P(X_{fg}) = gf' + fg' + \{f, g\} = gf' + dg(X_f)$ mod I. But $X_{fg} = gX_f$ mod I, hence the formula makes sense.

This last equation is called the transport equation for the Reidemeister torsion and is a crucial ingredient in [CM11b].

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