Character varieties and skein modules

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Introduction

Given a finitely generated group Γ, its character variety will be informally the space

$$X(\Gamma) = \mathsf{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$$

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- 3-dimensional topology via "Culler-Shalen" theory: exceptional surgeries, Smith theorem, etc...

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- Geometrization in dimension 2: Teichmüller spaces of surfaces.
- Geometrization in dimension 3: hyperbolic manifold have a "geometric representation" into $\mathrm{SL}_2(\mathbb{C})$.
- 3-dimensional topology via "Culler-Shalen" theory: exceptional surgeries, Smith theorem, etc...
- Topological quantum field theory (Jones polynomials) is a "quantization" of character varieties.

Plan of the talk

First Part: Algebraic geometry of character varieties

- Construction as an algebraic quotient
- The skein module construction
- A theorem of K. Saito and its consequences
- Reidemeister torsion as a rational volume form

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Second part: Skein module at first order

- Symplectic structure of character varieties of surfaces
- Character varieties of 3-manifolds with boundary
- A conjecture on the skein module at first order
- Formal second derivative and self intersection

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- Study unreduced points: the tangent space of a representation has a topological interpretation.
- Character varieties are "defined over $\mathbb{Z}$". Its arithmetic properties should have relations with topology.
- The algebra defining the character variety has a topological interpretation (skein algebra).

The GIT quotient

Fix a ring k with characteristic 0 once for all and set

$$A(\Gamma) = k[X^{\gamma}_{i,j}, i, j \in \{1,2\}, \gamma \in \Gamma]/(\det(X^{\gamma}) - 1, X^{\gamma\delta} - X^{\gamma}X^{\delta})$$

This algebra defines the representation variety thanks to the following universal property for any k-algebra R:

$$\mathrm{Hom}_{k-\mathrm{alg}}(A(\Gamma), R) = \mathrm{Hom}(\Gamma, \mathrm{SL}_2(R))$$

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Definition

Let $\mathrm{SL}_2(k)$ act on the space $\mathrm{Hom}(\Gamma, \mathrm{SL}_2)$ by conjugation. We define the *character variety* of Γ and denote by $X(\Gamma)$ the spectrum of the algebra $A(\Gamma)^{\mathrm{SL}_2}$ of invariants.

Standard arguments of Geometric Invariant theory implies the following theorem:

Theorem

If k is algebraically closed, there is a bijection between the following sets:

- *The k-points of $X(\Gamma)$ or equivalently* $\mathrm{Hom}_{k-\mathrm{alg}}(A(\Gamma)^{\mathrm{SL}_2}, k)$
- *The closed orbits of* $\mathrm{SL}_2(k)$ *acting on* $\mathrm{Hom}(\Gamma, \mathrm{SL}_2(k))$
- *The conjugacy classes of semi-simple representations of* Γ *into* $\mathrm{SL}_2(k)$
- *The characters of representations in* $\mathrm{Hom}(\Gamma, \mathrm{SL}_2(k))$.

The scheme structure is encoded in the algebra $A(\Gamma)^{\mathrm{SL}_2}$. What are generators and relations for this algebra?

The skein algebra

Definition

We define the *skein character variety* $X_s(\Gamma)$ as the spectrum of

$$B(\Gamma) = k[Y_\gamma, \gamma \in \Gamma]/(Y_1 - 2, Y_{\alpha\beta} + Y_{\alpha\beta^{-1}} - Y_\alpha Y_\beta \text{ with } \alpha, \beta \in \Gamma)$$

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- $B(\Gamma)$ is a finitely generated k-algebra.
- Any representation $\rho : \Gamma \to \mathrm{SL}_2(k)$ gives rise to an algebra morphism $\chi_\rho : B(\Gamma) \to k$ by the formula $\chi_\rho(Y_\gamma) = \mathrm{Tr}\, \rho(\gamma)$.

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Remark

This is a consequence of the famous trace relation:

$$\mathrm{Tr}(AB) + \mathrm{Tr}(AB^{-1}) = \mathrm{Tr}(A)\,\mathrm{Tr}(B) \quad \forall A, B \in \mathrm{SL}_2(k)$$

The character of the tautological representation $\rho : \Gamma \to \mathrm{SL}_2(A(\Gamma))$ defined by $\rho(\gamma) = X^\gamma$ gives a map

$$\Phi : B(\Gamma) \to A(\Gamma)^{\mathrm{SL}_2}$$

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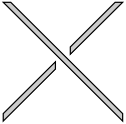
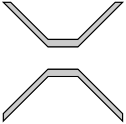
The Kauffman bracket

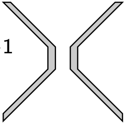
If M is an oriented compact 3-manifold (maybe with boundary), there is a topological interpretation of the algebra $B(\Gamma)$.
Let R be a ring and $t \in R^\times$ be an invertible element. We define $S(M,t)$ as the free R-module generated by banded links in M quotiented by the relations

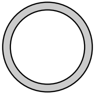


$= t$



$+t^{-1}$





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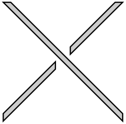

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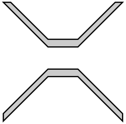

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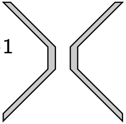

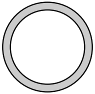

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Proposition

If $R = k$ and $t = -1$ then $S(M, -1) \simeq B(\Gamma)$ (disjoint union product) where $\Gamma = \pi_1(M)$.

Let Σ be a surface (maybe with boundary) and L be a banded link in $M = \Sigma \times [0,1]$. By resolving crossings of the projection on Σ and removing trivial circles one get the following

Theorem

The skein module $S(M,t)$ is a free R-module generated by multicurves (embedded curves in Σ without trivial components).

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Application

One has $X(F_2) = \mathbb{A}^3$.

Proof.

Set $F_2 = \langle a, b \rangle$ be the fundamental group of Σ, a disc with two holes. A multicurve is a disjoint union of copies of a, b and ab, hence $B(F_2) = k[Y_a, Y_b, Y_{ab}]$. □

K. Saito's theorem

Lemma (Culler-Shalen)

A representation $\rho : \Gamma \to \mathrm{SL}_2(k)$ *is absolutely irreducible if and only if there exists* $\alpha, \beta \in \Gamma$ *such that* $\mathrm{Tr}\, \rho[\alpha, \beta] \neq 2$.

Definition

Set $\Delta_{\alpha,\beta} = Y_{[\alpha,\beta]} - 2 = Y_\alpha^2 + Y_\beta^2 + Y_{\alpha\beta}^2 - Y_\alpha Y_\beta Y_{\alpha\beta} - 4$

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Theorem

Let R be a k-algebra, $\phi : B(\Gamma) \to R$ an algebra morphism, $\alpha, \beta \in \Gamma$, $A, B \in \mathrm{SL}_2(R)$ such that

- $\phi(\Delta_{\alpha,\beta}) \in R^\times$
- $\operatorname{Tr} A = \phi(Y_\alpha), \operatorname{Tr} B = \phi(Y_\beta)$ *and* $\operatorname{Tr}(AB) = \phi(Y_{\alpha\beta})$.

Then, there exists a unique representation $\rho : \Gamma \to \mathrm{SL}_2(R)$ such that $\rho(\alpha) = A, \rho(\beta) = B$ and $\operatorname{Tr}\rho(\gamma) = \phi(Y_\gamma)$ for all $\gamma \in \Gamma$.

Idea of the proof:

- Set $E_1 = Id, E_2 = A, E_3 = B, E_4 = AB$.
  Compute $\det(\mathrm{Tr}(E_i E_j)) = -\phi(\Delta_{\alpha,\beta})^2$ and deduce that these matrices form a basis of $\mathrm{M}_2(R)$.

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  Hence, one can write $\rho(\gamma)$ in the basis (E_i).

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If k is algebraically closed, a point of $X_s(\Gamma)$ is the character of a representation $\rho : \Gamma \to \mathrm{SL}_2(k)$.

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Proof.

Set $A = \begin{pmatrix} \phi(Y_\alpha) & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & u \\ -u^{-1} & \phi(Y_\beta) \end{pmatrix}$ where $u + u^{-1} = \phi(Y_{\alpha\beta})$. Then apply Saito's theorem. □

Brauer group

If k is arbitrary, one can solve the equation $u + u^{-1} = \phi(Y_{\alpha\beta})$ only in a quadratic extension $\hat{k}$ of k. The space

$$M(\rho) = \mathrm{Span}_k\{\rho(\gamma), \gamma \in \Gamma\}$$

is a central semi-simple k-algebra (indeed a quaternion algebra).

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Proposition

Given an irreducible character $\phi : B(\Gamma) \to k$, there is a representation $\rho : \Gamma \to \mathrm{SL}_2(k)$ with character ϕ iff $[M(\rho)] = 0$ in the Brauer group $Br(k)$.

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Example

There is a morphism $\phi : B(F_2) \to \mathbb{Q}$ given by $\phi(Y_\alpha) = \phi(Y_\beta) = \phi(Y_{\alpha\beta}) = 1$. Is it a character of a representation $\rho : F_2 \to \mathrm{SL}_2(\mathbb{Q})$?

Tangent space

Let $\rho : \Gamma \to \mathrm{SL}_2(k)$ be a representation and $\chi_\rho : B(\Gamma) \to k$ be its character. One has by definition the following $T_{\chi_\rho} X(\Gamma) = \{D : B(\Gamma) \to k, D(fg) = D(f)\chi_\rho(g) + \chi_\rho(f)D(g)\}$.

Theorem

If ρ is absolutely irreducible, the morphism $z \mapsto D$ where $D(Y_\gamma) = \mathrm{Tr}(\rho(\gamma)z(\gamma))$ from $H^1(\Gamma, \mathrm{Ad}_\rho)$ to $T_{\chi_\rho} X(\Gamma)$ is an isomorphism.

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Proof.

Construct the inverse map:

- from a derivation $D : B(\Gamma) \to k$ form the morphism $\phi_\epsilon = \chi_\rho + \epsilon D : B(\Gamma) \to k[\epsilon]/\epsilon^2$.

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- Invoke Saito's theorem to find a representation $\rho_\epsilon : \Gamma \to \mathrm{SL}_2(k[\epsilon]/\epsilon^2)$ with character ϕ_ϵ.
- Set $z(\gamma) = \frac{d}{d\epsilon}|_{\epsilon=0} \rho_\epsilon(\gamma)\rho^{-1}(\gamma)$.

Application

Let Γ be a finitely generated group and k be algebraically closed. The following properties are equivalent.

(i) $X^{\mathrm{irr}}(\Gamma)$ *is reduced of dimension 0*

(ii) *For all irreducible representations* $\rho : \Gamma \to \mathrm{SL}_2(k)$ *one has* $H^1(\Gamma, \mathrm{Ad}_\rho) = 0$.

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Example

If ρ is trivial, the previous result does not hold.
However the tangent space of $X(\Gamma)$ at the trivial representation is the space of maps $f : \Gamma \to k$ satisfying the parallelogram identity for any $\gamma, \delta \in \Gamma$.

$$f(\gamma\delta) + f(\gamma\delta^{-1}) = 2f(\gamma) + 2f(\delta)$$

Tautological representations

Let Y be an irreducible component of $X(\Gamma)$ containing the character of an irreducible representation.

Question

Can we find a tautological representation i.e. $\rho : \Gamma \to \mathrm{SL}_2(k(Y))$ *such that* $\mathrm{Tr}\,\rho(\gamma) = Y_\gamma$*?*

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Answer

Let $B(\Gamma) \to k[Y]$ *be the quotient map. There is an obstruction in* $Br(k(Y))$ *for the existence of a tautological representation. If* k *is alg. closed and* Y *has dimension 1, then* $Br(k(Y)) = 0$.

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The trefoil knot has fundamental group $\Gamma = \langle u, v | u^2 = v^3 \rangle$. The representation $\rho(u) = \begin{pmatrix} t & -1 \\ -1 - t^2 & -t \end{pmatrix}, \rho(v) = \begin{pmatrix} \omega & 0 \\ 0 & -\omega^2 \end{pmatrix}$ is tautological where $\omega^2 - \omega + 1 = 0$.

Reidemeister torsion

Let Γ be the fundamental group of a 3-manifold M with boundary. Let Y be a component of $X(\Gamma)$ and $\rho : \Gamma \to \mathrm{SL}_2(k(Y))$ be a tautological representation.
The Reidemeister torsion of M is an element of

$$\det H^0(\Gamma, \mathrm{Ad}_\rho) \otimes \det H^1(\Gamma, \mathrm{Ad}\,\rho)^* \otimes \det H^2(\Gamma, \mathrm{Ad}_\rho).$$

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Proposition (Some technical assumptions omitted)

- $H^1(M, \mathrm{Ad}_\rho) \simeq \Omega^1_{k(Y)/k}$ *that is rational differential forms on* Y.
- $H^2(M, \mathrm{Ad}_\rho) \simeq H^2(\partial M, \mathrm{Ad}_\rho) \simeq H^0(\partial M, \mathrm{Ad}_\rho) *$.

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Proposition

Choosing a natural basis for the latter space, one gets the following The Reidemeister torsion of M on Y, a d-dimensional component of $X(M)$ is a rational volume form on Y i.e. $\tau(M)\in\Omega^d_{k(Y)/k}$.

Example

Let M be a genus 2 handlebody.
Its fundamental group is $F_2 = \langle a, b \rangle$ and its character variety is $B(F_2) = k[x, y, z]$ where $x = Y_a, y = Y_b, z = Y_{ab}$. Then

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Let N be the complement of the figure eight knot.
Its fundamental group is $\Gamma = \langle t, a, b | t^{-1}at = ab, t^{-1}bt = bab \rangle$.
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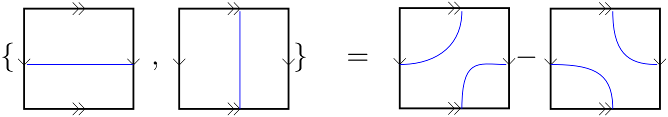
Question

- *Study poles and residues of the torsion (including at ideal points).*
- *Find differential equations satisfied by the torsion (to follow).*

Goldman Bracket

Let Σ be a closed surface. The Goldman bracket is a Poisson bracket $\{\cdot,\cdot\} : B(\Gamma)\otimes B(\Gamma)\to B(\Gamma)$ defined for simple curves γ,δ intersecting transversely by

$$\{Y_\gamma, Y_\delta\} = \sum_{p\in\gamma\cap\delta} \epsilon_p(Y_{\gamma_p\cup\delta_p} - Y_{\gamma_p\cup\delta_p^{-1}})$$



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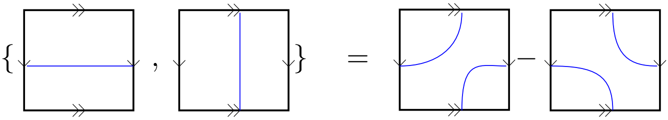

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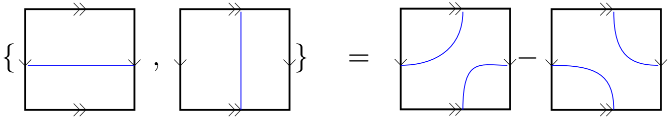

It comes from an (algebraic) symplectic structure ω on $X(\Sigma)$. There are two other ways for introducing it

- A cohomological one which will show that the form ω is non-degenerate.
- A skein module approach which will show that ω is closed.

Twisted cohomology perspective

Set $\Gamma = \pi_1(\Sigma)$ and pick $\rho : \Gamma \to \mathrm{SL}_2(k)$ irreducible. Then $T_{\chi_\rho} X(\Gamma) \simeq H^1(\Sigma, \mathrm{Ad}_\rho)$ and the cup product followed by the trace gives a non degenerate pairing:

$$\omega_\rho : H^1(\Sigma, \mathrm{Ad}_\rho) \otimes H^1(\Sigma, \mathrm{Ad}_\rho) \to H^2(\Sigma, k) \simeq k$$

This is related to the Goldman bracket by the formula

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If M is a 3-manifold with boundary Σ and $\rho : \Gamma \to \mathrm{SL}_2(k)$ is a representation, the natural map $H^1(M, \mathrm{Ad}_\rho) \to H^1(\Sigma, \mathrm{Ad}_\rho)$ is the derivative of the restriction map $r : X(M) \to X(\Sigma)$.

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Theorem (Consequence of Poincaré duality)

The image of $D_\rho r : H^1(M, \mathrm{Ad}_\rho) \to H^1(\Sigma, \mathrm{Ad}_\rho)$ is a Lagrangian subspace of $H^1(\Sigma, \mathrm{Ad}_\rho)$.

Skein module perspective

Let $M = \Sigma \times [0,1]$ and $R = k[t, t^{-1}]$. The skein module $S(M,t)$ has the structure of an associative algebra (stacking product): it becomes commutative when t goes to -1.

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Definition

Let M be a 3-manifold bounding Σ and denote by $\mathfrak{p}$ the kernel of the map $S(\Sigma, -1) \to S(M, -1)$.
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Proposition

The ideal $\mathfrak{p}$ is Lagrangian in the sense that

$$\forall f, g \in \mathfrak{p} \text{ one has } \{f, g\} \in \mathfrak{p}.$$

The derived skein module

Definition

For a 3-manifold M, we call derived skein module and denote by $S'(M,-1)$ the module $S(M,t)$ where we have set $R = k[\epsilon]/(\epsilon^2)$ and $t = -1+\epsilon$.

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Question

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Example

If $M = \Sigma \times [0,1]$, using the basis given by multicurves, we get an isomorphism $S'(M,-1) \simeq S(M,-1) \otimes k[\epsilon]/(\epsilon^2)$. The multiplication law reads
$(f+\epsilon f')\cdot(g+\epsilon g') = fg + \epsilon(fg' + f'g + \frac{1}{2}\{f,g\})$

A conjecture on the derived skein module

Let M be a 3-manifold with boundary Σ and let $B\Sigma \simeq \Sigma \times [0,1]$ be a tubular neighborhood of Σ in M.
Let $\mathfrak{p}'$ be the kernel of the map induced by the inclusion $S'(B\Sigma, -1) \to S'(M, -1)$. An element of $\mathfrak{p}'$ reads $f + \epsilon f'$ with $f \in \mathfrak{p}$.

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Definition

The quotient $R_Y = S(B\Sigma,-1)/\mathfrak{p}$ is the ring of functions on $Y = \overline{r(X(M))}$, a Lagrangian submanifold of $X(\Sigma)$. Given $f \in \mathfrak{p}$, the equation $\omega(X_f,\cdot) = df$ defines a vector field X_f on Y called Hamiltonian vector field of f.

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- *There is an algebraic operator P from vector fields on Y to functions on Y such that $f + \epsilon f' \in \mathfrak{p}' \iff f' = P(X_f)$.*

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- *There is an algebraic operator P from vector fields on Y to functions on Y such that $f + \epsilon f' \in \mathfrak{p}' \iff f' = P(X_f)$.*
- *This operator is determined by the Reidemeister torsion through the equation $P(X) = \mathrm{div}_\tau(X) = \frac{L_X\tau}{\tau}$.*

Evidences

Example (The handlebody)

If H is a handlebody with boundary Σ and γ is a curve on Σ bounding a disc in H then $Y_\gamma - 2 \in \mathfrak{p}'$ by the first Kauffman relation.
Hence writing $f = X_\gamma - 2$ we should have $P(X_f) = 0$. However any representation $\rho : \pi_1(H) \to \mathrm{SL}_2(k)$ satisfies $\rho(\gamma) = Id$ hence $df = \gamma \otimes \rho(\gamma)_0 = 0$. This implies that f vanishes identically on $\overline{X(H)}$ and $X_f = 0$.

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Remark

The operator P should satisfy $P(fX) = fP(X) + X \cdot f$.
If $f + \epsilon f' \in \mathfrak{p}'$, and $g + \epsilon g' \in S'(B\Sigma, -1)$ then $(f + \epsilon f') \cdot (g + \epsilon g') \in \mathfrak{p}'$. Hence one should verify
$P(X_{fg}) = fg' + f'g + \frac{1}{2}\{f, g\} = P(X_f)g + X_f \cdot g \mod \mathfrak{p}$.
But we check $X_{fg} = fX_g + gX_f = gX_f \mod \mathfrak{p}$.

Motivations

The question comes from asymptotics of quantum invariants. Let K be a knot in S^3 and (J_l^K) be the sequence of colored Jones polynomials. We let L and M act on such sequences by the formulas

$$(Lf)_l = f_{l+1}, \quad (Mf)_n = t^{2n} f_n.$$

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Conjecture (AJ-conjecture)

Let $\mathfrak{p}$ be the kernel of the inclusion $S(B\Sigma, -1) \to S(S^3 \setminus K, -1)$. The set of polynomials $\mathcal{A}(-1, L, M)$ for $\mathcal{A}$ annihilating the colored Jones polynomial generates $\mathfrak{p}$.

Proposition

Writing $t = -1 + \epsilon + o(\epsilon)$*, one has* $\mathcal{A} = f + \epsilon f' = o(\epsilon)$*. The quantum polynomial* $\mathcal{A}$ *annihilates* J^K *at first order iff* $f + \epsilon f' \in \mathfrak{p}'$*.*

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- *Such formulas are already conjectured by S. Garoufalidis, S. Gukov and T. Dimofte. We give here a more precise form.*
- *The relation between derived A-polynomial and torsion should shed some light on both invariants which are not fully understood.*

The 2-jet of the holonomy function

Let M be a 3-manifold and γ be a knot in M. For a 1-form $\alpha \in \Omega^1(M, \mathrm{SL}_2(\mathbb{C}))$, its holonomy may be computed through

$$\operatorname{Tr} \operatorname{Hol}_\gamma \alpha = \sum_{n \geq 0} \int_{0<t_1<\cdots<t_n<1} \operatorname{Tr}(\alpha(t_1) \cdots \alpha(t_n)).$$

From which we get
$D_\alpha \operatorname{Tr} \operatorname{Hol}_\gamma(\beta) = \int_\gamma \operatorname{Tr} \beta \operatorname{Hol}_\gamma(\alpha) = \langle \beta, \gamma \otimes \rho(\gamma)_0 \rangle$. Where $\gamma \otimes \rho(\gamma)_0 \in C_1(M, \mathrm{Ad}_\rho)$ is a twisted cycle.
In the same way we have

$$D^2_\alpha \operatorname{Tr} \operatorname{Hol}_\gamma(\beta_1, \beta_2) = \int_{\gamma \times \gamma} \operatorname{Tr}(\beta_1 \operatorname{Hol}_\gamma(\alpha)' \beta_2 \operatorname{Hol}_\gamma(\alpha)'')$$

We will interpret this formula with the help of a twisted 2-chain.

The formal second derivative

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The $z_2(\gamma)$ is the twisted-two chain supported by $C_2\gamma$ which associates to $\xi \in (\mathrm{Ad}_\rho)_x$ and $\eta \in (\mathrm{Ad}_\rho)_y$ the element $\mathrm{Tr}(\xi A \eta B)$ where A (resp. B) is the holonomy of ρ from x to y (resp. from y to x).

Proposition

- *One has* $\partial z_2\gamma = \gamma \otimes \phi$ *where* $\phi(\xi,\eta) = \mathrm{Tr}(\rho(\gamma)[\xi,\eta])$. *Hence* $z_2(\gamma) \in \Lambda = H_2(C_*(C_2M, \mathrm{Bil}_\rho)/C_*(SM, \mathrm{Alt}_\rho))$

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- *This construction proves half of the conjecture, i.e. the existence of the operator* P.
- *If* M *is closed and* $H^1(M,\mathrm{Ad}_\rho)=0$ *then* $\Lambda\simeq k$, *generated by the fiber of* $SM\to M$. *This gives an interpretation of the derived Kauffman bracket in that case.*