# Skein modules, AJ conjecture and knot state asymptotics

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## 1 Introduction

These lectures were given at the "Research school, CIMPA-GGTM-MAROC, symplectic geometry and topological geometry, Meknes, may-june 2102".

Skein modules were introduced independently by V. Turaev in 1988 and J. Przytycki in 1991, see [TU88, HP92]. They give a general framework for understanding the colored Jones polynomials and were a key ingredient for the construction of topological quantum field theory by C. Blanchet, N. Habegger, G. Masbaum and P. Vogel in 1995, see [BHMV2]. In 2002, C. Frohman, R. Gelca and W. LoFaro used it in [FGL02] to define a noncommutative generalization of the $A$-polynomial which is related to the $q$-differential relations satisfied by the colored Jones polynomials. Later, S. Garoufalidis and T. Q. T. Le showed by a different method that the colored Jones polynomials always satisfy non-trivial $q$-difference relations, see [GL05]. Recently, T.Q.T. Le and A. Tranh obtained interesting results towards the AJ-conjecture using skein modules, see [LT11]. To keep these notes elementary, we will not review their work here.

These lectures are motivated by the work of L. Charles and the author on the Witten asymptotic expansion conjecture, see [CM11a, CM11b]. Indeed, $q$-differential relations and their interpretation as Toeplitz operators are an essential ingredient for our approach. We will only reach the first step of this work during these lectures which may better be viewed as an introduction to skein modules, motivated by the AJ conjecture. Almost all the material of these notes is contained in the papers cited in the bibliography.

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# 2 Kauffman skein modules

## 2.1 Definition

Let $(R, t)$ be a pair where $R$ is a commutative ring and $t$ is an invertible element in $R$. We will mainly consider two cases:

- $\Lambda = (\mathbb{Z}[t, t^{-1}], t)$
- $\varepsilon = (\mathbb{Z}, -1)$

A morphism between two pairs $(R, t)$ and $(R', t')$ is a ring morphism from $R$ to $R'$ mapping $t$ to $t'$.

We will call manifold an oriented compact 3-manifold with (maybe empty) boundary. A banded link in $M$ is an oriented submanifold with boundary $L \subset \mathrm{Int}(M)$ which is homeomorphic to a finite collection of annuli. We will consider the empty set as a banded link.

Let $M$ be such a manifold and $\mathcal{R} = (R, t)$. We define

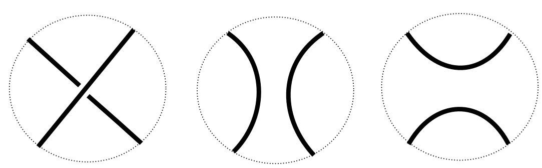
- $K_g(M, \mathcal{R})$ as the free $R$-module generated by isotopy classes of banded links in $M$. We will denote by $[L]$ the basis element corresponding to $L$.

- We denote by $K_r(M, \mathcal{R})$ the sub-module of $K_g(M, \mathcal{R})$ generated by the Kauffman relations that we describe now. We call Kauffman triple any triple of banded links $L_0, L_+, L_-$ which differ only in a ball as in Figure 1, and set $[L_0, L_+, L_-] = [L_0] - t[L_+] - t^{-1}[L_-]$. We call Kauffman pair a banded link $L_0$ whose intersection with a ball is a trivial banded unknot. Let $L$ be the banded link $L_0$ with the trivial unknot removed: we set $[L_0, L] = [L_0] + (t^2 + t^{-2})[L]$.

  Finally, $K_r(M, \mathcal{R})$ is the sub-$R$-module generated by $[L_0, L_+, L_-]$ and $[L_0, L]$ for any Kauffman triple $(L_0, L_+, L_-)$ and pair $(L_0, L)$.

- The Kauffman skein $R$-module $K(M, \mathcal{R})$ is by definition the quotient

$$K_g(M, \mathcal{R})/K_r(M, \mathcal{R}).$$



$L_0$ $\qquad$ $L_+$ $\qquad$ $L_-$

Figure 1: Kauffman triple

## 2.2 Compatibility with embeddings

This definition is compatible with embeddings in the following sense: given $i : M \to N$ an oriented embedding between two manifolds, one has a functorial morphism $i_*$ from $K(M, \mathcal{R})$ to $K(N, \mathcal{R})$. It is induced by the obvious map from $K_g(M, \mathcal{R})$ to $K_g(N, \mathcal{R})$ mapping $[L]$ to $[i(L)]$. It factors through the quotient as the image by $i$ of a Kauffman triple (resp. pair) in $M$ is a Kauffman triple (resp. pair) in $N$.

Given two manifolds $M$ and $N$, we also have a natural isomorphism

$$K(M, \mathcal{R}) \otimes_R K(N, \mathcal{R}) \simeq K(M \amalg N, \mathcal{R})$$

mapping $[L] \otimes [L']$ to $[L \amalg L']$.

In the case when $M$ has a product structure, one can endow $K(M, \mathcal{R})$ with an algebra structure. More precisely, we have the following simple proposition.

**Proposition 2.1.** *Let $S$ be a surface with or without boundary. The inclusion $i : S \times [0,1] \amalg S \times [0,1] \to S \times [0,1]$ mapping the first factor $(x, t)$ to $(x, (1+t)/2)$ and the second to $(x, t/2)$ induces an algebra structure on $K(S \times [0,1], \mathcal{R})$ (generally non-commutative).*

Let $M$ be a manifold and $B \subset M$ be a finite union of closed disjoint balls embedded in $M$. Set $N = M \setminus \mathrm{Int}(B)$ and denote by $i : N \to M$ the inclusion.

**Proposition 2.2.** *The map $i_* : K(N, \mathcal{R}) \to K(M, \mathcal{R})$ is an isomorphism.*

*Proof.* It is well-known that the map $i$ induces a bijection between isotopy classes of banded links in $N$ and $M$. Moreover, it is the same for Kauffman triples and pairs, which proves the proposition. □

## 2.3 Kauffman modules of thickened surfaces

Let $S$ be a compact oriented surface with possibly empty boundary. We call *multicurve* on $S$ a 1-dimensional submanifold of $S$ without components bounding a disc embedded in $S$.

One has the following fundamental result.

**Theorem 2.3.** *The Kauffman module $K(S \times [0,1], \mathcal{R})$ is a free $R$-module with basis given by the isotopy classes of multicurves in $S$.*

*Proof.* Write $K = K(S^1 \times [0,1], \mathcal{R})$ and denote by $K'$ be the free $R$-module generated by isotopy classes of multicurves in $S$. Suppose that $S$ is endowed with a smooth structure. Any banded link is isotopic to a smooth submanifold $L$ of $S \times [0,1]$ diffeomorphic to a finite union of copies of $S^1 \times [0,1]$. By an isotopy, one can make the band arbitrarily thin and suppose that the banded link is in generic position with respect to the projection on the first factor $S \times [0,1] \to S$. That is, $L$ can be recovered from a diagram $D$ in $S$, that is a 1-submanifold of $S$ with double points where the two branches meeting at a double point are distinguished (upper and lower branch). For such a diagram, we will denote by $\hat{D}$ the corresponding banded link obtained by taking a tubular neighborhood of $D$ in $S \times \{1/2\}$, and separating the branches at the double points.

The well-known Reidemeister theorem states that two diagrams $D_1$ and $D_2$ such that $\hat{D}_1$ and $\hat{D}_2$ are isotopic are related by a sequence of Reidemeister moves presented in Figure 2. Given a diagram $D$, we can form a Kauffman triple at each

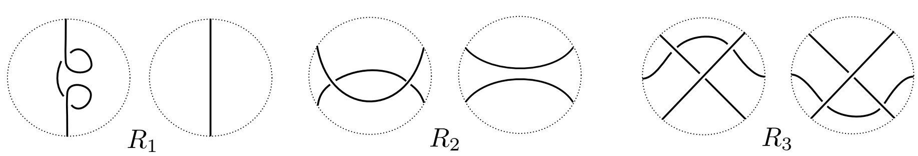



Figure 2: Reidemeister moves

crossing and hence the equality $[D_0] = t[D_+] + t^{-1}[D_-]$ holds in $K$ where $D_+$ and $D_-$ are diagrams with one crossing less. Repeating this operation, one gets only diagrams without crossings. Applying the second Kauffman relation, one can remove the components of $D$ bounding a disc. This implies that multicurves generate $K$.

In order to show that they form a basis, we do the following construction. Let $D$ be a diagram and $X$ be the set of crossings of $D$. For any $\xi : X \to \{\pm 1\}$

we define $D_\xi$ as the diagram obtained by smoothing the crossing $x \in X$ with the sign $\xi(x)$ (given as in Figure 1). Let $c(\xi) = \sum_x \xi(x)$, $n(\xi)$ be the number of curves in $D_\xi$ bounding a disc and $D_\xi^-$ be the diagram $D_\xi$ with these bounding curves removed. We set

$$\Phi(D) = \sum_\xi t^{c(\xi)}(-t^2 - t^{-2})^{n(\xi)}[D_\xi^-] \in K'$$

In order to show that this map is a well defined linear map $\Phi : K_g(S \times [0,1], \mathcal{R}) \to K'$ we need to show that it is invariant by the three Reidemeister moves. This amounts to a simple local computation: we present the case of the second Reidemeister move in Figure 3. The fact that $\Phi$ factors through Kauffman

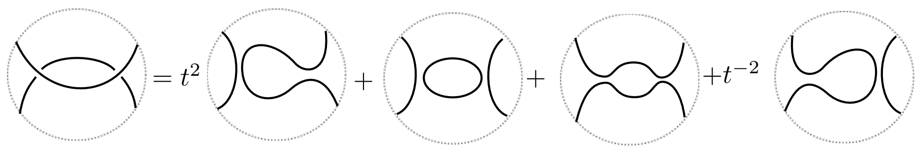



Figure 3: Invariance by the second Reidemeister move

relations is due to the fact that - up to isotopy - any Kauffman triple (or pair) can be projected as in Figure 1. The definition of $\Phi$ makes it invariant by the relation.

For any multicurve $D$, we define a map $\Psi : K' \to K$ by setting $\Psi([D]) = [\hat{D}]$. One has by construction that $\Psi = \Phi^{-1}$ which proves the theorem. $\square$

### 2.4 Knot invariants

By Proposition 2.2, one has the isomorphism $K(S^3, \mathcal{R}) \simeq K(B^3, \mathcal{R})$. Writing $B^3 = D^2 \times [0,1]$ and using Proposition 2.3, we get the isomorphism $K(B^3, \mathcal{R}) \simeq R[\emptyset]$. We will denote by $\langle \cdot \rangle : K(S^3, \mathcal{R}) \to R$ the unique isomorphism such that

$$[L] = \langle L \rangle [\emptyset] \in K(S^3, \mathcal{R}).$$

This defines a topological invariant of banded links in $S^3$ called Kauffman bracket, equivalent to the Jones polynomial.

By cabling operations, one can get easily new invariants as we explain it now. Let $A$ be the annuli $S^1 \times [0,1]$. Any multicurve in $A$ is isotopic to some subset of the form $S^1 \times X$ where $X$ has cardinality $n$. For any integer $n$, we denote by $z^n \in K(A \times [0,1], \mathcal{R})$ the corresponding basis provided by Theorem 2.3. It follows that we have a $R$-module isomorphism

$$K(A \times [0,1], \mathcal{R}) \simeq R[z].$$

One can see easily that this is indeed an isomorphism of $R$-algebras.

Given a banded knot $K$ in a manifold $M$, a tubular neighborhood gives an embedding $i : A \times [0,1] \to M$ such that $K = i(A \times \{1/2\})$. Moreover, this embedding is unique up to isotopy and pre-composition by the automorphism $\sigma$ of $A$ defined by $(z,t) \mapsto (\overline{z}, 1-t)$. This automorphism acts trivially on $z^n$ and hence on the skein module so that we can forget about this indeterminacy.

One deduce that there is a well-defined map $i_* : K(A\times[0,1],\mathcal{R}) \to K(M,\mathcal{R})$ given a banded knot $K$ in $M$. Let $T_n \in \mathbb{Z}[z]$ be the sequence of Chebyshev-like polynomials given by

$$T_0 = 0, \quad T_1 = 1, \quad T_{n+1} + zT_n + T_{n-1}, \quad n \in \mathbb{Z}.$$

We define the $n$-th colored Jones polynomial of $K$ in $S^3$ by the formula

$$J_n^K = \langle i_*(T_n)\rangle \in R$$

The introduction of Chebyshev polynomials is justified by the following lemma:

**Lemma 2.4.** *We define the following three operations on $K(A \times [0,1], R)$:*

1. *Let $\tau : A \times [0,1] \to A \times [0,1]$ be be a Dehn twist along the meridian disc $D = \{1\} \times [0,1]^2$.*
2. *Let $c : K(A \times [0,1], R) \to K(A \times [0,1], R)$ the map induced by mapping a banded link $[L]$ to $[L \cup \partial D]$*
3. *Let $d$ be the map mapping $[L]$ to $[L \cup \delta]$ where $\delta$ is homologous to the sum of $\partial D$ and a component of $\partial A \times \{1/2\}$.*

*Setting $\mu_n = (-1)^{n+1}t^{n^2-1}$ and $\lambda_n = -(t^{2n} + t^{-2n})$ we have:*

$$\tau_*(T_n) = \mu_n T_n, \quad c(T_n) = \lambda_n T_n, \quad d(T_n) = t^{2n+1}T_{n+1} + t^{-2n+1}T_{n-1}$$

*It follows that the colored Jones polynomial $J_n^K$ depends on the banding of $K$ only up to multiplication by an invertible element in $R$.*

*Proof (sketch).* Take $\mathcal{R} = \Lambda$. The two maps $\tau_*$ and $c$ commute and a simple computation using Kauffman relations show that $\tau_*(z^n) = \mu_n z^n + \cdots$ and $c(z^n) = \lambda_n z^n + \cdots$ where dots stand for lower order terms. Considering $\Lambda' = (\mathbb{Q}(t), t)$ one can find a basis $e_n$ of $K(A\times[0,1],\Lambda')$ which satisfy the proposition. A delicate induction shows that the $e_n$ satisfy the recurrence equation of Chebyshev polynomials, see [BHMV1]. One can prove the lemma more quickly once we have introduced Jones-Wenzl idempotents, see for instance [Li97]. $\square$

Consider the trivial example of the unknot $U$. If $i : A \times [0,1] \to S^3$ is the standard embedding, we get using the second Kauffman relation $\langle i_*(z^n)\rangle = (-t^2 - t^{-2})^n$. A simple induction shows that we have:

$$J_n^U = T_n(-t^2 - t^{-2}) = \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}}$$

Notice that it is hard to provide a closed formula for the colored Jones polynomials of a non trivial knot. One has such formulas for torus knots and twist knots, see [Mo95, Ha01, Ma03].

# 3 Kauffman modules and boundary

Let $M$ be a manifold with non empty boundary. We denote by $BM$ a compact tubular neighborhood of $\partial M$ in $M$ hence homeomorphic to $\partial M \times [0,1]$.

Considering the natural map $i_* : K(BM, \mathcal{R}) \to K(M, \mathcal{R})$ induced by the inclusion $i : BM \to M$, we define:

- the peripheral ideal as $\mathcal{P}(M, \mathcal{R}) = \ker i_*$: it is a left ideal in the algebra $K(BM, \mathcal{R})$.
- the boundary module $\mathcal{B}(M, \mathcal{R}) = \operatorname{im} i_*$: it is a $K(BM, \mathcal{R})$-module.
- the relative skein module $\mathcal{L}(M, \mathcal{R}) = K(M, \mathcal{R})/\mathcal{B}(M, \mathcal{R})$.

## 3.1 Universal coefficients

We first notice that the Kauffman module behaves very simply when changing coefficients. In that sense, it is sufficient to compute it with coefficients in $\Lambda$ as for any pair $\mathcal{R} = (R, t)$ there is a natural morphism $\Lambda \to \mathcal{R}$.

**Proposition 3.1.** *(Universal coefficients) Let $M$ be a manifold and $\varphi : \mathcal{R} \to \mathcal{R}'$ be a morphism. There is a natural isomorphism of $R'$-modules*

$$K(M, \mathcal{R}) \otimes R' \to K(M, \mathcal{R}')$$

*sending $f[L] \otimes 1$ to $\varphi(f)[L]$ for any banded link $L$ and coefficient $f \in R$.*

*Proof.* Consider the exact sequence defining the Kauffman module:

$$0 \to K_r(M, \mathcal{R}) \to K_g(M, \mathcal{R}) \to K(M, \mathcal{R}) \to 0$$

Applying the functor $\otimes_R R'$ which is right-exact we get the exact sequence

$$K_r(M, \mathcal{R}) \otimes_R R' \to K_g(M, \mathcal{R}) \otimes_R R' \to K(M, \mathcal{R}) \otimes_R R' \to 0$$

As $K_g(M, \mathcal{R})$ is a free $R$-module, the natural map $\Phi : K_g(M, \mathcal{R}) \otimes_R R' \to K_g(M, \mathcal{R}')$ is an isomorphism of $R'$-modules. It maps $K_r(M, \mathcal{R}) \otimes_R R'$ onto $K_r(M, \mathcal{R}')$. A look at the following commutative diagram shows that $\Phi$ induces the isomorphism of the proposition.

$$\begin{tikzcd}
K_r(M, \mathcal{R}) \otimes_R R' \arrow[r] \arrow[d, two heads] & K_g(M, \mathcal{R}) \otimes_R R' \arrow[r] \arrow[d, "\Phi"', "\sim"] & K(M, \mathcal{R}) \otimes_R R' \arrow[r] \arrow[d, "\sim"] & 0 \\
K_r(M, \mathcal{R}') \arrow[r] & K_g(M, \mathcal{R}') \arrow[r] & K(M, \mathcal{R}') \arrow[r] & 0
\end{tikzcd}$$

□

This simple statement generalizes to the relative case in the following way:

**Proposition 3.2.** *(Universal coefficients, relative case)*
*Let $\varphi : \mathcal{R} \to \mathcal{R}'$ be a morphism. We have the following isomorphism and two exact sequences:*

$$\mathcal{L}(M,\mathcal{R}) \otimes_R R' \simeq \mathcal{L}(M,\mathcal{R}')$$

$$0 \to \mathcal{D} \to \mathcal{B}(M,\mathcal{R}) \otimes_R R' \xrightarrow{\sim} \mathcal{B}(M,\mathcal{R}') \to 0$$

$$0 \to \mathrm{Tor}_1^R(\mathcal{B}(M,\mathcal{R}), R') \to \mathcal{P}(M,\mathcal{R}) \otimes_R R' \to \mathcal{P}(M,\mathcal{R}') \to \mathcal{D} \to 0.$$

*where* $\mathcal{D} = \mathrm{coker}(u)$ *and $u$ is the natural map*

$$u : \mathrm{Tor}_1^R(K(M,\mathcal{R}), R') \to \mathrm{Tor}_1^R(\mathcal{L}(M,\mathcal{R}), R').$$

*Proof.* The first isomorphism is obtained by a proof completely parallel to the proof of Proposition 3.1 where the module of relations has to be replaced by $i_* K_g(BM,\mathcal{R}) + K_r(M,\mathcal{R})$.

From the exact sequence $0 \to \mathcal{B}(M,R) \to K(M,\mathcal{R}) \to \mathcal{L}(M,\mathcal{R}) \to 0$ we get by applying the functor $\otimes_R R'$ the following exact sequence from which we get the second statement:

$$\begin{array}{ccccccc}
 & & \mathrm{Tor}_1^R(K(M,\mathcal{R}), R') & \longrightarrow & \mathrm{Tor}_1^R(\mathcal{L}(M,\mathcal{R}), R') & \longrightarrow & \\
\mathcal{B}(M,\mathcal{R}) \otimes R' & \longrightarrow & K(M,\mathcal{R}') & \longrightarrow & \mathcal{L}(M,\mathcal{R}') & \longrightarrow & 0
\end{array}$$

For the last statement, we apply the functor $\otimes_R R'$ to the exact sequence: $0 \to \mathcal{P}(M,\mathcal{R}) \to K(BM,\mathcal{R}) \to \mathcal{B}(M,\mathcal{R}) \to 0$. It gives:

$$\begin{array}{ccccccc}
 & & 0 & \longrightarrow & \mathrm{Tor}_1^R(\mathcal{B}(M,\mathcal{R}), R') & \longrightarrow & \\
\mathcal{P}(M,\mathcal{R}) \otimes \mathcal{R}' & \longrightarrow & K(BM,\mathcal{R}) \otimes_R R' & \longrightarrow & \mathcal{B}(M,\mathcal{R}) \otimes_R R' & \longrightarrow & 0
\end{array}$$

We used the fact that $K(BM,\mathcal{R})$ is a free $R$-module and hence has no Tor. One has the following diagram:

$$\begin{tikzcd}
 & \mathcal{P}(M,\mathcal{R}) \otimes \mathcal{R}' \arrow[r] \arrow[d, "\alpha"] & K(BM,\mathcal{R}) \otimes_R R' \arrow[r] \arrow[d, "\beta", "\sim"'] & \mathcal{B}(M,\mathcal{R}) \otimes_R R' \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \mathcal{P}(M,\mathcal{R}') \arrow[r] & K(BM,\mathcal{R}') \arrow[r] & \mathcal{B}(M,\mathcal{R}') \arrow[r] & 0
\end{tikzcd}$$

Using the fact that $\beta$ is an isomorphism, the snake lemma gives the isomorphism $\mathrm{coker}\,\alpha \simeq \ker\gamma$. A simple diagram chasing shows the exactness of the second sequence of the proposition. □

**Definition 3.3.** We will say that a manifold $M$ is boundary regular with respect to the pair $(\mathcal{R},\mathcal{R}')$ if we have

$$\mathcal{P}(M,\mathcal{R}) \otimes R' = \mathcal{P}(M,\mathcal{R}').$$

**Remark 3.4.** By Proposition 3.2, it is implied by the following hypothesis: the submodule $\mathcal{B}(M,\mathcal{R})$ is a free factor of $K(M,\mathcal{R})$.

We mean that $\mathcal{B}(M,\mathcal{R})$ is a free $R$-module and there is a submodule $N \subset K(M,\mathcal{R})$ such that $K(M,\mathcal{R}) = \mathcal{B}(M,\mathcal{R}) \oplus N$. The module $N$ gives a splitting of the exact sequence $0 \to \mathcal{B}(M,\mathcal{R}) \to K(M,\mathcal{R}) \to \mathcal{L}(M,\mathcal{R}) \to 0$.

We will show that this assumption will imply half of the AJ-conjecture when $M$ is the complement of a knot and the pair $(\mathcal{R},\mathcal{R}')$ is equal to the pair $(\Lambda,\varepsilon)$.

### 3.2 Non-degeneracy

Let $M$ be a manifold and $N$ a codimension 0 submanifold. We will say that the inclusion of $N$ in $M$ is non degenerate if the natural pairing

$$K(N,\mathcal{R}) \times \mathcal{B}(M \setminus N,\mathcal{R}) \to K(M,\mathcal{R})$$

is non-degenerate on the right by which we mean that $\langle x,y\rangle = 0$ for all $x \in K(N,\mathcal{R})$ implies that $y = 0$.

Consider a standard solid torus $N = A \times [0,1]$ in $M = S^3$ and let us show that it is non-degenerate: In that case, $M \setminus N$ is also a solid torus and we have $\mathcal{B}(M\setminus N,\mathcal{R}) = K(M\setminus N,\mathcal{R})$. The latter is isomorphic to $K(A' \times [0,1],\mathcal{R})$ where $A'$ is a banded link making a Hopf link with $A$. An easy computation shows that

$$\langle T_n, T_m\rangle = \frac{t^{2mn} - t^{-2mn}}{t^2 - t^{-2}}.$$

This pairing is non degenerate provided that $t$ is not a root of unity.

If $A \times [0,1]$ is embedded in a non trivial way, we have no reason to believe that the embedding is still non-degenerate, however this hypothesis will imply the other half of the AJ-conjecture.

## 4 Quantum torus and $q$-differential equations

### 4.1 The skein algebra of the torus

Let $S$ be a closed torus. The aim of this section is to propose an alternative description of the algebra $K(S \times [0,1],\mathcal{R})$.

We observe that the set of multicurves in $S$ is in bijection with $H_1(S,\mathbb{Z})$ where connected multicurves correspond to primitive elements.

Let $\mathcal{T}(S,\mathcal{R})$ be the following algebra called quantum torus.

$$\mathcal{T}(S,\mathcal{R}) = R\left\langle X_\gamma, \gamma \in H_1(S,\mathbb{Z}) | X_\gamma X_\delta = t^{-\gamma\cdot\delta} X_{\gamma+\delta}\right\rangle.$$

The notation $R\langle X_i, i \in I|F\rangle$ mean the ring of non-commutative polynomials with formal variables indexed by $I$ and coefficients in $R$ modulo the two-sided ideal generated by $F$. Given standard generators $m$ and $l$ of $H^1(S,\mathbb{Z})$ such that $m \cdot l = 1$, this algebra has the simpler presentation

$$\mathcal{T}(S,\mathcal{R}) = R\left\langle X_m, X_l | X_l X_m = t^2 X_m X_l\right\rangle.$$

With this isomorphism, if $\gamma = m^a l^b$ we have $X_\gamma = t^{ab} X_m X_l$.

Consider the automorphism of $\mathcal{T}(S,\mathcal{R})$ defined by $\overline{X_\gamma} = X_{-\gamma}$. We denote by $\mathcal{T}(S,\mathcal{R})^+$ the set of invariants by this automorphism. We have the following result:

**Proposition 4.1.** *The map $\Upsilon : K(S \times [0,1], \mathcal{R}) \to \mathcal{T}(S,\mathcal{R})^+$ defined by*

$$\Upsilon([\gamma]) = X_\gamma + X_{-\gamma}$$

*for any simple curve $\gamma$ is an isomorphism of algebras.*

*Moreover, for any $\alpha \in H^1(S,\mathbb{Z}_2)$, the map $\alpha : X_\gamma \mapsto \alpha(\gamma) X_\gamma$ is an automorphism of $K(S \times [0,1], \mathcal{R})$.*

*Proof.* We refer to [FG00, S00] for a proof. Let us illustrate the proposition with the following example. Take $\gamma$ and $\delta$ two simple curves such that $\gamma \cdot \delta = 1$. By stacking $\gamma$ above $\delta$ and resolving the crossing we get the following formula in $K(S \times [0,1], \mathcal{R})$

$$[\gamma] \cdot [\delta] = t[\gamma - \delta] + t^{-1}[\gamma + \delta]$$

As $\Upsilon([\gamma + \varepsilon\delta]) = t^{-\varepsilon}(X_{\gamma+\varepsilon\delta} + X_{-\gamma-\varepsilon\delta})$ for $\varepsilon = \pm 1$, we check directly that $\Upsilon$ is a morphism of algebras. □

## 4.2 $q$-differential relations and the quantum torus

Let $\mathcal{R} = (R, t)$ be a pair and $\mathcal{S}(\mathcal{R})$ be the set of sequences $f : \mathbb{Z} \to R$. Consider a standard torus $S = (S^1)^2$ and set $m = S^1 \times \{1\}$ and $l = \{1\} \times S^1$. By puting

$$(X_m f)_n = t^{2n} f_n \text{ and } (X_l f)_n = f_{n+1}$$

we define a representation of the quantum torus $\mathcal{T}(S,\mathcal{R})$ on $\mathcal{S}(\mathcal{R})$.

One can compare the action of $\mathcal{T}(S,\mathcal{R})$ on $\mathcal{S}(\mathcal{R})$ and the action of $K(S \times [0,1], \mathcal{R})$ on $K(A \times [0,1], \mathcal{R})$. Here we have identified $S$ with $\partial\big(A \times [0,1]\big)$ by sending $m$ to the boundary of the meridian disc and $l$ to a component of $\partial A \times \{1/2\}$.

Denoting by $M^*$ the dual of the $R$-module $M$, we define

$$\Upsilon : \begin{cases} K(A \times [0,1], \mathcal{R})^* & \to & \mathcal{S}(\mathcal{R}) \\ \lambda & \to & \Upsilon(\lambda)_n = \lambda(T_n) \end{cases}$$

We observe that it is a bijection onto the space of odd sequences as $(T_n)_{n>0}$ is a basis of $K(A \times [0,1], \mathcal{R})$.

**Lemma 4.2.** *For any $x \in K(A \times [0,1], \mathcal{R})$ and $\gamma \subset S$ we have*

$$\Upsilon([\gamma]^*.\lambda) = \Upsilon_\alpha([\gamma]).\Upsilon(\lambda)$$

*where $\alpha \in H^1(S,\mathbb{Z}_2)$ takes non trivial values on $m$ and $l$ and*

$$\Upsilon_\alpha = \alpha \circ \Upsilon : K(S \times [0,1], \mathcal{R}) \to \mathcal{T}(S,\mathcal{R}).$$

*In other terms, the map $\Upsilon : K(A \times [0,1], \mathcal{R})^* \to \mathcal{S}(\mathcal{R})$ is a morphism of $K(S \times [0,1], \mathcal{R})$-modules where the module structure on $\mathcal{S}(\mathcal{R})$ is given by the map $\Upsilon_\alpha$.*

*Proof.* The equation of the lemma is equivalent to $\lambda([\gamma]T_n) = (\Upsilon_\alpha([\gamma]).\Upsilon(\lambda))_n$. It is sufficient to prove it for $\gamma$ belonging to a generating set for $K(S\times[0,1],\mathcal{R})$. Such a generating set is given by $\{m,l,m+l\}$. The action of $[l]$ is the multiplication by $z$ in $K(A\times[0,1])$ whereas $[m]$ and $[m+l]$ correspond to the endomorphisms $c$ and $d$ of Lemma 2.4. Hence, we have:

$$[m]T_n = -(t^{2n}+t^{-2n})T_n, \quad [l]T_n = -T_{n+1} - T_{n-1}$$

$$[m+l]T_n = t^{2n+1}T_{n+1} + t^{-2n+1}T_{n-1}$$

Recall the formulas $\Upsilon_\alpha([m]) = -X_m - X_m^{-1}$, $\Upsilon_\alpha([l]) = -X_l - X_l^{-1}$ and $\Upsilon_\alpha([m+l]) = tX_mX_l + tX_m^{-1}X_l^{-1}$. The compatibility is checked by a direct comparison. $\square$

## 4.3 The peripheric ideal and $q$-differential relations

Given some $f\in\mathcal{S}(\mathcal{R})$, we define its annihilator by

$$\mathcal{A}(f) = \{P\in\mathcal{T}(S,\mathcal{R}), P.f = 0\}.$$

We will say that $f$ is $t$-holonomic if $\mathcal{A}(f)\neq\{0\}$.

Let $K$ be a banded knot in $S^3$. Denote by $i: A\times[0,1]\to S^3$ the embedding associated to $K$ as in Section 2.4. Set $M = S^3\setminus\mathrm{Im}(i)$ the complement of an open tubular neighborhood of $K$. There is a homeomorphism $S\simeq\partial\big(A\times[0,1]\big) = \partial M$ given in the preceding section. It is unique up to isotopy and allows to identify $K(BM,\mathcal{R})$ with $K(S\times[0,1],\mathcal{R})$.

**Theorem 4.3.** *Let $K$ and $M$ be as above. Denote by $J^K\in\mathcal{S}(\mathcal{R})$ the sequence $n\mapsto J_n^K$. We have the inclusion:*

$$\Upsilon_\alpha(\mathcal{P}(M,\mathcal{R}))\subset\mathcal{A}(J^K)$$

*Proof.* Consider the decomposition $S^3 = A\times[0,1]\cup S\times[0,1]\cup M$. It induces a trilinear map denoted by $(x,y,z)\mapsto\langle x|y|z\rangle$

$$\langle\cdot|\cdot|\cdot\rangle : K(A\times[0,1],\mathcal{R})\times K(BM,\mathcal{R})\times K(M,\mathcal{R})\to K(S^3,\mathcal{R})\simeq R$$

Let $y\in K(BM,\mathcal{R})$. We have $y\in\mathcal{P}(M,\mathcal{R})$ if and only if $|y|\emptyset\rangle = 0$.

By definition, one has $J_n^K = \langle T_n|\emptyset|\emptyset\rangle$ and $J^K = \Upsilon(\lambda^K)$ where $\lambda^K(x) = \langle i_*(x)\rangle = \langle x|\emptyset|\emptyset\rangle$. We then compute:

$$\langle x|y|\emptyset\rangle = \lambda^K(y.x) = (y^*.\lambda^K)(x) = 0.$$

Hence $y^*.\lambda^K = 0$. By Lemma 4.2, we get $\Upsilon_\alpha(y)J^K = 0$ and $\Upsilon_\alpha(y)\in\mathcal{A}(J^K)$. $\square$

As an illustration, let us compute $\mathcal{A}(J^U)$ where $U$ is the unknot and $\mathcal{R}=\Lambda$.

**Proposition 4.4.** *Define the following polynomials:*

$$P_1 = (X_m - X_m^{-1})X_l - (t^2 X_m - t^{-2} X_m^{-1}) \text{ and } P_2 = X_l^2 - (t^2 + t^{-2})X_l + 1.$$

*Then $\mathcal{A}(J^U)$ is the left ideal of $\mathcal{T}(S, \Lambda)$ generated by $P_1$ and $P_2$.*

*Proof.* A simple computation shows that $P_1$ and $P_2$ belong to $\in \mathcal{A}(J^U)$ hence $\mathcal{A}(J^U)$ contains the left ideal they generate. By noncommutative euclidean division in $\mathcal{T}(S, \Lambda)$, one can write any $x \in \mathcal{T}(S, R)$ in the form

$$xX_l^m = QP_2 + R_0(X_m) + R_1(X_m)X_l$$

where $m$ is a non-negative integer, and $R_0, R_1$ are polynomials in $\Lambda[X_m]$.

As $R_0(X_m) + R_1(X_m)X_l \in \mathcal{A}(J^U)$, we have:

$$R_0(t^{2n})(t^{2n} - t^{-2n}) + R_1(t^{2n})(t^{2n+2} - t^{-2n-2}) = 0, \quad \forall n \in \mathbb{Z}.$$

A polynomial $P(t, t^n)$ is 0 for any $n$ if and only if $P = 0$. Hence, $R_0(X)(X - X^{-1}) + R_1(X)(t^2 X - t^{-2} X^{-1}) = 0$ from which we deduce that $P_1$ divides $R_0(X_m) + R_1(X_m)X_l$. This proves the proposition. $\square$

For general general knots, it is difficult to find $q$-differential relations for the colored Jones polynomials due to the lack of formulas. It was done by Hikami in [Hi04] for torus knots, by S. Garoufalidis and X. Sun in [GS10] for twist knots using the computer. Such equations for the figure eight knot were derived "by hand" in [CM11a].

# 5 Skein modules and character varieties

In this section, we will consider skein modules of the form $K(M, \varepsilon)$. For this particular choice of coefficients, we see in Figure 1 that the class of a banded link $[L] \in K(M, \varepsilon)$ does not change if we change a crossing. Moreover, it does not change if we twist a band of $L$: this implies that $[L]$ depends only on the homotopy class of the (unbanded) link $L$.

This observation implies that there is a natural algebra structure on $K(M, \varepsilon)$ given by the rule $[L_1] \cdot [L_2] = [L_1 \cup i(L_2)]$ where $i$ is an isotopy of $M$ satisfying $i(L_2) \cap L_1 = \emptyset$. The class of the resulting link does not depend on $i$ thanks to the remark above.

## 5.1 The main theorem

Let $M$ be a connected compact oriented 3-manifold $M$ with base point $x$ and $k$ be any field of characteristic 0. Given a representation $\rho : \pi_1(M, x) \to \mathrm{SL}_2(k)$, we define a linear map $\chi_\rho : K_g(M, \varepsilon) \to k$ in the following way: given a banded link $L$, we consider a family $\gamma_1, \ldots, \gamma_n \in \pi_1(M, x)$ which are freely homotopic to the components of $L$. Then we set

$$\chi_\rho([L]) = \prod_{i=1}^{n} \Big( - \mathrm{Tr}(\rho(\gamma_i)) \Big) \tag{1}$$

**Theorem 5.1.** *For any connected manifold $M$ with base point $x$ and any representation $\rho : \pi_1(M,x) \to \mathrm{SL}_2(k)$, the map $\chi_\rho : K_g(M,\varepsilon) \to k$ induces an algebra morphism $K(M,\varepsilon) \to k$.*

*Proof.* We first show that $\chi_\rho : K_g(M,\varepsilon) \to k$ vanishes on the submodule $K_r(M,\varepsilon)$.

If $(L_0, L)$ is a Kauffman pair, $\chi_\rho([L_0]) = \chi_\rho([L])(-\operatorname{Tr}\rho(1)) = -2\chi_\rho([L])$ which is compatible with $[L_0, L] = [L_0] + 2[L]$ (recall that $t$=-1).

Let $(L_0, L_+, L_-)$: we would like to show that $\chi_\rho([L_0]) + \chi_\rho([L_+]) + \chi_\rho([L_-]) = 0$. We can factor all components which are not involved in the crossing and hence suppose that the singular link $L$ where the crossing is replaced by a base point $x$ is connected. There are three cases depending on how the strands of $L$ are connected outside the ball shown in Figure 1.

1. The left strands are connected together, the right strands also. Let $\gamma_L \in \pi_1(M,x)$ (resp. $\gamma_R$) be the path obtained by following $L$ from $x$ starting from the lower left strand to the upper left strand (resp. right). The knot $L$ is homotopic to $\gamma_L\gamma_R$, the link $L_+$ has two components homotopic to $\gamma_L$ and $\gamma_R$ and the knot $L_-$ is homotopic to $\gamma_L\gamma_R^{-1}$.

   Writing $A = \rho(\gamma_L)$ and $B = \rho(\gamma_R)$ we get the identities $\chi_\rho([L_0]) = -\operatorname{Tr}(AB)$, $\chi_\rho([L_+]) = \operatorname{Tr}(A)\operatorname{Tr}(B)$ and $\chi_\rho([L_-]) = -\operatorname{Tr}(AB^{-1})$. The result follows from the well-known formula, consequence of the Cayley-Hamilton identity:

   $$\operatorname{Tr}(A)\operatorname{Tr}(B) = \operatorname{Tr}(AB) + \operatorname{Tr}(AB^{-1}), \quad A, B \in \mathrm{SL}_2(k). \tag{2}$$

2. The lower left strand is connected to the upper right strand through a path $\gamma_L$ and the lower right strand is connected to the upper left strand through a path $\gamma_R$. Then $L_0, L_+$ and $L_-$ are respectively homotopic to $\gamma_L \amalg \gamma_R, \gamma_L\gamma_R$ and $\gamma_L\gamma_R^{-1}$. We are in the same situation as above with permuted terms.

3. The upper strands are connected together, as the lower strands. We are again in the first case after a rotation by 90 degrees.

It follows that $\chi_\rho$ factors to a linear map $K(M,\varepsilon) \to k$. The fact that it is an algebra morphism is obvious from Equation (1). □

**Theorem 5.2.** *For any connected manifold $M$ with base point $x$ and any algebraically closed field $k$, any algebra morphism $K(M,\varepsilon) \to k$ has the form $\chi_\rho$ for some representation $\rho : \pi_1(M,x) \to \mathrm{SL}_2(k)$.*

*Proof.* We first reduce the proof to the case when $M$ is a handlebody. We remove a ball from each closed component of $M$: this does not change $K(M,\varepsilon)$ by Proposition 2.2 nor $\pi_1(M,x)$. Hence, we can suppose that $M$ collapses to a 2-dimensional cell complex. Taking a tubular neighborhood of the 1-skeleton, we get a handlebody $H$ containing the base point $x$. The manifold $M$ is obtained by gluing 2-handles to $H$. More formally,

$$M = H \amalg \Big(\coprod_{i=1}^{n} D^2 \times [0,1]^{(i)}\Big)/(f_i)$$

where $f_i : \partial D^2 \times [0,1]^{(i)} \to \partial H$ are embeddings with disjoint images reversing the orientation.

Let $\varphi : K(M,\varepsilon) \to k$ be a algebra morphism. The inclusion $i : H \to M$ induces an algebra morphism $i_* : K(H,\varepsilon) \to K(M,\varepsilon)$. Applying the theorem for $\varphi \circ i_*$ we get a representation $\rho : \pi_1(H,x) \to \mathrm{SL}_2(k)$ such that $\varphi \circ i_* = \chi_\rho$. Let $\gamma_1,\ldots,\gamma_n \in \pi_1(H,x)$ be loops freely homotopic to the boundary of the 2-handles. We have $\pi_1(M,x) = \pi_1(H,x)/N$ where $N$ is the normal subgroup generated by $\gamma_1,\ldots,\gamma_n$. For any $\gamma \in \pi_1(H,x)$ and any $i \in \{1,\ldots,n\}$, we must have $\varphi([\gamma]) = \varphi([\gamma\gamma_i])$ as $\gamma$ and $\gamma\gamma_i$ are homotopic in $M$. Replacing $\varphi$ by $\chi_\rho$ we get

$$\mathrm{Tr}\,\rho(\gamma) = \mathrm{Tr}\,\rho(\gamma\gamma_i). \tag{3}$$

Writing $A_i = 1 - \rho(\gamma_i)$ we get $\mathrm{Tr}(\rho(\gamma)A_i) = 0$ for all $\gamma \in \pi_1(H,x)$.

Let $M$ be the sub-algebra of $\mathrm{M}_2(k)$ generated by $\rho(\gamma)$ for $\gamma \in \pi_1(H,x)$ and $M^\circ$ be its orthogonal for the form $(A,B) \mapsto \mathrm{Tr}(AB)$. The equation (3) shows that $A_i \in M \cap M^\circ$. The subspace $M \cap M^\circ$ has dimension less than 2 and exactly two if $M = M^\circ$ but $1 \in M \setminus M^\circ$. Hence $\dim M \cap M^\circ \leq 1$.

- If $M \cap M^\circ = \{0\}$ then $A_i = 0$ for all $i$. Then $\rho$ vanishes on $N$ and defines a representation $\tilde\rho : \pi_1(M,x) \to \mathrm{SL}_2(k)$. It follows that $\varphi = \chi_{\tilde\rho}$.

- If $\dim M \cap M^\circ = kH$ then $\mathrm{Tr}(H) = \mathrm{Tr}(H^2) = 0$. Given $v \in k^2$ such that $Hv \neq 0$, we find that the matrix of $H$ in the basis $(Hv, v)$ is $U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. If $M$ contains the matrix $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0$, then it contains $(V-a)U/c = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and consequently $M$ has dimension 4 and $M^\circ = 0$. Hence, $M$ contains only upper triangular matrices. This implies that the representation $\rho'$ with the upper right entry removed is still a representation with the same traces. In that case, $M$ has dimension 2, $M \cap M^\circ = \{0\}$ and we proceed as in the first case.

It remains to prove the theorem when $M$ is a handlebody: this is a difficult task as it requires to describe precisely the polynomial identities satisfied by trace functions in $M$. For the proof, we refer to [BH91, Bul97] and explain the result for a handlebody of genus 1 and 2. Let $A_n$ be a disc with standard $n$ discs removed. Fix a base point $x$ and denote by $\gamma_i$ the loops going around the removed discs. The group $\pi_1(M,x)$ is the free group generated by $\gamma_1,\ldots,\gamma_n$.

The manifold $A_n \times [0,1]$ is a handlebody of genus $n$. By Theorem 2.3, $K(M,\varepsilon)$ is the free abelian group generated by multicurves on $A_n$. Let $\varphi : K(M,\varepsilon) \to k$ be an algebra morphism.

We want to show that there is a representation $\rho : \pi_1(M,x) \to \mathrm{SL}_2(k)$ such that $\varphi = \chi_\rho$. Finding $\rho$ is equivalent to find $A_i = \rho(\gamma_i)$.

1. If $n = 1$, denoting by $x$ the class of $\gamma_1$ in $K(M,\varepsilon)$, we have $K(M,\varepsilon) = \mathbb{Z}[x]$. Let $A_1$ be a matrix satisfying $\mathrm{Tr}(A_1) = -\varphi(x)$: it defines a representation $\rho$ such that $\varphi = \chi_\rho$.

2. If $n = 2$, denote by $x, y, z$ the classes of $\gamma_1, \gamma_2, \gamma_1\gamma_2$ in $K(M,\varepsilon)$. As monomials in $x, y, z$ are represented by non intersecting multicurves, they are linearly independent and $K(M,\varepsilon) = \mathbb{Z}[x,y,z]$. In particular, $\varphi$ is described by the values it takes on $x, y, z$. Suppose that $A_1 = \begin{pmatrix} u & 0 \\ 1 & u^{-1} \end{pmatrix}$ and $A_2 = \begin{pmatrix} v & w \\ 0 & v^{-1} \end{pmatrix}$. We have the equations

$$-\varphi(x) = u + u^{-1}, -\varphi(y) = v + v^{-1}, -\varphi(z) = uv + u^{-1}v^{-1} + w$$

   for which we can find a triple of solutions by solving quadratic equations in $u$ and $v$ and a linear equation in $w$. This proves the result in the genus 2 case.

□

The theorem we have proven has the following interpretation from algebraic geometry. Given $M$ and $k$, we define the character variety

$$\mathcal{M}(M, \mathrm{SL}_2(k)) = \mathrm{Hom}(\pi_1(M), \mathrm{SL}_2(k))//\mathrm{SL}_2(k).$$

The quotient has to be understood in the sense of geometric invariant theory: two orbits of $\mathrm{SL}_2(k)$ acting by conjugation on $\mathrm{Hom}(\pi_1(M), \mathrm{SL}_2(k))$ are identified if their Zariski closures intersect themselves. More concretely, two representations $\rho, \rho'$ are equivalent if and only if $\mathrm{Tr}\,\rho(\gamma) = \mathrm{Tr}\,\rho'(\gamma)$ for all $\gamma \in \pi_1(M)$, see [CS83].

Denote by $\varepsilon_k$ the pair $(k, -1)$ where $k$ is an algebraically closed field of characteristic 0: the result

$$\mathrm{Hom}_{\mathrm{alg}}(K(M,\varepsilon_k), k) = \mathcal{M}(M, \mathrm{SL}_2(k))$$

can be rephrased by saying that the set of points of $K(M,\varepsilon_k)$ is in one-to-one correspondence with $\mathcal{M}(M, \mathrm{SL}_2(k))$. In particular, an element of $K(M,\varepsilon_k)$ can be thought of as a function on $\mathcal{M}(M, \mathrm{SL}_2(k))$ by setting $f(\varphi) = \varphi(f)$ but one has to take care of the following subtlety, some functions take only the value 0 but are non-zero. Indeed, if $f \in K(M,\varepsilon_k)$ is such that $f(\varphi) = 0$ for all $\varphi$ then $f$ is in the intersection of all maximal ideals of $K(M,k)$. By Hilbert Nullstellensatz theorem applied to the finitely generated algebra $K(M,\varepsilon_k)$, $f$ is nilpotent.

Till now, we do not know any 3-manifold with non-reduced skein algebra but there is no reason why they should not exist.

## 5.2 Poisson structures on surfaces

We observe in this section that given a surface $S$, the algebra structure on $K(S\times[0,1],\Lambda)$ can be viewed as a deformation of the commutative algebra $K(S\times[0,1],\varepsilon)$. Let us be more precise.

From the following exact sequence of $\mathbb{Z}[t^{\pm 1}]$-modules:

$$0 \longrightarrow \mathbb{Z}[t^{\pm 1}] \xrightarrow{\times(t+1)} \mathbb{Z}[t^{\pm 1}] \longrightarrow \mathbb{Z} \longrightarrow 0$$

we get the following exact sequence by tensoring with the free module $K(S\times[0,1],\Lambda)$:

$$0 \longrightarrow K(S\times[0,1],\Lambda) \xrightarrow{\times(t+1)} K(S\times[0,1],\Lambda) \xrightarrow{\sigma} K(S\times[0,1],\varepsilon) \longrightarrow 0$$

Given $f,g\in K(S\times[0,1],\varepsilon)$, there are $F,G$ in $K(S\times[0,1],\Lambda)$ with $\sigma(F)=f$ and $\sigma(G)=g$. We observe that $\sigma(FG-GF)=0$ because the algebra $K(S\times[0,1],\varepsilon)$ is commutative. Hence, $FG-GF$ is divisible by $(t+1)$. We then check easily that the following formula defines a Poisson bracket on $K(S\times[0,1],\varepsilon)$:

$$\{f,g\}=\sigma\Big(\frac{FG-GF}{t+1}\Big).$$

This Poisson structure was defined by Turaev in [TU91]. Goldman showed in [G86] that this structure was the same as the Atiyah-Bott symplectic structure on the representation spaces of surfaces, see [AB83].

Suppose now that $M$ is a manifold with boundary which is regular with respect to the pair $(\Lambda,\varepsilon)$ (see Definition 3.3). We show the following

**Proposition 5.3.** *The ideal $\mathcal{P}(M,\varepsilon)$ of $K(BM,\varepsilon)$ is a coisotropic ideal in the sense that*

$$\{f,g\}\in\mathcal{P}(M,\varepsilon)\quad \forall f,g\in\mathcal{P}(M,\varepsilon).$$

*Proof.* By paraphrasing the beginning of the section and using the regularity assumption we get the exact sequence:

$$0 \longrightarrow \mathcal{P}(M,\Lambda) \xrightarrow{\times(t+1)} \mathcal{P}(M,\Lambda) \xrightarrow{\sigma} \mathcal{P}(M,\varepsilon) \longrightarrow 0$$

Taking $F$ and $G$ as before, $FG-GF$ belongs to $\mathcal{P}(M,\Lambda)$ by the ideal property. The result follows. □

**Remark 5.4.** If $M$ is a symplectic manifold and $N$ is a submanifold, the ideal $I$ of smooth functions on $M$ vanishing on $N$ is coisotropic iff $N$ is coisotropic (this means that for all $p\in N$, $T_pN$ contains its orthogonal for the symplectic form). The statement above is reminiscent from the fact that the restriction map $r:\mathcal{M}(M,\mathrm{SL}_2(k))\to\mathcal{M}(BM,\mathrm{SL}_2(k))$ is a Lagrangian immersion (with singularities). In particular, $\mathcal{P}(M,\varepsilon)$ is the ideal defining the algebraic variety of representations of $BM$ extending to $M$.

# 6 Around the AJ-conjecture

The aim of this section is to formulate the AJ conjecture for knots in $S^3$ and to relate it to the constructions involved in this course: this conjecture was first formulated by S. Garoufalidis in 2004, see [G04].

Let $K \subset S^3$ be a knot, $M$ be the complement of a tubular neighborhood of $K$ and $S$ be the peripheral torus.

Let $\mathcal{A}(J^K)$ be the ideal of $\mathcal{T}(S,\Lambda)$ made of $q$-differential relations satisfied by the Jones polynomial. Denote by $\sigma$ the operation of tensoring with $\mathbb{Z}$ over $\Lambda$ (sending $t$ to -1). We get an ideal $\sigma\mathcal{A}(J^K) \subset \mathcal{T}(S,\varepsilon)$. On the other hand $\mathcal{P}(M,\varepsilon)$ is the ideal of $K(S\times[0,1],\varepsilon)$ defining the representations of $S$ which extend to $M$. For an ideal $I \subset \mathcal{T}(S,\varepsilon)$, we write $I^+ = I \cap \mathcal{T}(S,\varepsilon)^+$.

**Conjecture 1** (AJ)**.** *For any knot $K$ in $S^3$, we have*

$$\Upsilon_\alpha(\mathcal{P}(M,\varepsilon)) = \sigma\mathcal{A}(J^K)^+$$

**Proposition 6.1.** *If $M$ is boundary regular with respect to the pair $(\Lambda,\varepsilon)$, we have the direct inclusion in the conjecture. If the embedding $A\times[0,1] \subset S^3$ given by $K$ is non-degenerate, we have the reverse inclusion.*

*Proof.* Notice that by Theorem 4.3, we have $\Upsilon_\alpha(\mathcal{P}(M,\Lambda)) \subset \mathcal{A}(J^K)$. Applying $\sigma$, we get $\Upsilon_\alpha(\sigma\mathcal{P}(M,\Lambda)) \subset \sigma\mathcal{A}(J^K)$. Supposing that $M$ is boundary regular with respect to the pair $(\Lambda,\varepsilon)$, we get that $\sigma\mathcal{P}(M,\Lambda) = \mathcal{P}(M,\varepsilon)$ and hence the first inclusion.

Suppose that the embedding of the solid knot $K$ in $S^3$ is non-degenerate and pick $x \in \mathcal{A}(J^K)^+$. There is a unique $y \in K(S\times[0,1],\Lambda)$ such that $x = \Upsilon_\alpha(y)$. Using the trilinear map introduced in Theorem 4.3, we have $\langle z|y|\emptyset\rangle$ for all $z \in K(A\times[0,1],\Lambda)$. By the non-degeneracy assumption, this implies that $|y|\emptyset\rangle = 0$ hence, $y \in \mathcal{P}(M,\Lambda)$. Applying $\sigma$, we get the reverse inclusion. □

As an application, suppose that some knot $K$ satisfies $J^K = J^U$ where $U$ is the unknot. Then we obviously have $\mathcal{A}(J^K) = \mathcal{A}(J^U)$. From Proposition 4.4, we compute that $\sigma\mathcal{A}(J^K)$ is the sub-algebra generated by $\sigma P_2 = (X_l - 1)^2$ and $\sigma P_1 = (X_m - X_m^{-1})(X_l - 1)$. Hence, any homomorphism from $\sigma\mathcal{A}(J^K)$ to $\mathbb{C}$ has the form $\varphi_\alpha(X_l) = 1$ and $\varphi_\alpha(X_m) = \alpha \in \mathbb{C}^*$. Moreover, the morphisms $\varphi_\alpha$ and $\varphi_{\alpha^{-1}}$ induce the same character on $\sigma\mathcal{A}(J^K)^+$.

Suppose that the reverse inclusion in the AJ conjecture holds. Then, it implies that for any morphism $\varphi : \mathcal{P}(M,\varepsilon) \to \mathbb{C}$, its restriction to $\sigma\mathcal{A}(J^K)$ has the form $\varphi_\alpha$. This contradicts the non-triviality of the $A$-polynomial of $K$, or said differently, the existence of representations of $M$ whose restriction to the longitude $l$ is non trivial, see [GD04]. In particular, this implies that a non trivial knot does not have the same colored Jones polynomials as the trivial knot.

# 7 Knot state asymptotics

The aim of this section is to explain the application of the AJ conjecture to TQFT. A detailed account of these ideas can be found in [CM11a, CM11b].

## 7.1 Theta functions

We start describing theta functions in a very concise way, we refer to [Mu83] for the general theory. Consider on $\mathbb{C}$ the complex coordinate $z = x + iy$, the symplectic form $\omega = 4\pi dx \wedge dy$ or equivalently $\omega(z_1, z_2) = 4\pi \operatorname{Im}(\overline{z_1} z_2)$.

Introduce the symplectic potential $\alpha = 2\pi(xdy - ydx)$ and endow the trivial bundle $L = \mathbb{C} \times \mathbb{C}$ with the connection $d + \frac{\alpha}{i}$. There is a natural holomorphic structure on $L$ which makes the section $t(z) = \exp(2i\pi zy)$ holomorphic.

Fix an integer $k$, the Heisenberg group at level $k$ is by definition the product $\mathbb{C} \times U(1)$ with the following multiplication:

$$(z_1, u_1).(z_2, u_2) = (z_1 + z_2, u_1 u_2 \exp(\frac{ik}{2}\omega(z_1, z_2))$$

The same law defines an action of this group on $L^k$. For any $z \in \mathbb{C}$, we denote by $T_z^*$ the pull-back by the action of $(z, 1)$. In formulas,

$$T_z^* \Psi(w) = \exp(-i\frac{k}{2}\omega(z, w))\Psi(z + w)$$

Let $\mathbb{Z}^2$ be the standard lattice in $\mathbb{C}$: we identify it to the subgroup $\mathbb{Z}^2 \times \{1\}$ of the Heisenberg group. We denote by $\mathcal{H}_k$ the space of $\mathbb{Z}^2$-invariant holomorphic sections of $L^k$ and call this space the space of theta functions.

More explicitely, any element of $\mathcal{H}_k$ has the form $f(z)t(z)$ where $f$ is a holomorphic function satisfying $f(z + 1) = f(z)$ and $f(z + i) = \exp(2k\pi - 4i\pi kz)f(z)$.

One can construct an element $\Psi_0 \in \mathcal{H}_k$ by averaging $t^k$ and normalizing it. In formulas:

$$\Psi_0 = \Big(\frac{k}{2\pi}\Big)^{1/4} \sum_{n\in\mathbb{Z}} T_{ni}^* t^k = \Big(\frac{k}{2\pi}\Big)^{1/4} \Theta(z) t^k$$

where $\Theta(z) = \sum_{n\in\mathbb{Z}} \exp(4i\pi nkz - 2\pi kn^2)$.

This space has a Hermitian product provided by the formula

$$\langle \Psi_1, \Psi_2 \rangle = \int_D \Psi_1 \overline{\Psi_2} \omega$$

where $D = \{x + iy | x, y \in [0, 1]\}$. We have the following standard proposition:

**Proposition 7.1.** *For any $\ell \in \mathbb{Z}/2k\mathbb{Z}$ we set $\Psi_\ell = T_{i\ell/2k}^* \Psi_0$.*

*The family $(\Psi_\ell)_{\ell\in\mathbb{Z}/2k\mathbb{Z}}$ is an Hermitian basis of $\mathcal{H}_k$. Moreover it satisfies*

$$T_{1/2k}^* \Psi_\ell = \exp(\frac{i\ell\pi}{k})\Psi_\ell, \quad T_{i/2k}^* \Psi_\ell = \Psi_{\ell+1}, \quad \forall \ell \in \mathbb{Z}/2k\mathbb{Z}$$

The inversion $I(x) = -x$ on $\mathbb{C}$ extends trivially to $L$ and to $\mathcal{H}_k$. It satisfies $I(\Psi_\ell) = \Psi_{-\ell}$. We will denote by $\mathcal{H}_k^{\mathrm{alt}}$ the subspace of alternate sections, i. e. the set of $\Psi \in \mathcal{H}_k^{\mathrm{alt}}$ satisfying $I(\Psi) = -\Psi$.

## 7.2 The knot state

Let $S$ be a standard solid torus and $M$ be a manifold bounding $S$. For any integer $k$, topological quantum field theory (here with gauge group $\mathrm{SU}_2$) associates to $M$ a vector $Z_k(M)$ in some Hermitian space $V_k(S)$ associated to $S$. We will use the following facts, see [BHMV2, CM11a, CM11b].

- $V_k(S)$ is naturally isomorphic to $\mathcal{H}_k^{\mathrm{alt}}$ as a Hermitian vector space.
- If we modify the identification of the boundary of $M$ by some element of $\mathrm{SL}_2(\mathbb{Z})$, the corresponding vector $Z_k(M)$ is modified by the discrete metaplectic representation, see [CM11a].
- If $M$ is the complement of a tubular neighborhood of a knot $K$ in $S^3$ and $S$ is its boundary (parametrized as usual), then

  $$Z_k(M) = \frac{\sin(\pi/k)}{\sqrt{k}} \sum_{\ell \in \mathbb{Z}/2k\mathbb{Z}} J_\ell^K(-e^{i\pi/2k})\Psi_\ell.$$

- If $M$ and $N$ are two manifolds bounding $S$, the scalar product

  $$Z_k(M \cup_S (-N)) = \langle Z_k(M), Z_k(N) \rangle$$

  is equal to the Witten-Reshetikhin-Turaev invariant of $M \cup (-N)$.
- Actually, there is a framing anomaly which forces to add some extra structure (2-framings, $p_1$-structure, or Lagrangians and weights).

Let us give two simple examples: if $U$ is the unknot, $M_1 = S^3 \setminus U \times [0,1]$ is a solid torus and

$$Z_k(M_1) = k^{-1/2} \sum_{\ell \in \mathbb{Z}/2k\mathbb{Z}} \sin\big(\frac{\pi\ell}{k}\big)\Psi_\ell$$

If $M_2$ is simply $U \times [0,1]$, we have $Z_k(M_2) = 2^{-1/2}(\Psi_1 - \Psi_{-1})$. Gluing these two manifolds, we get the sphere $S^3$. We obtain the well-known formula:

$$Z_k(S^3) = \langle Z_k(M_1), Z_k(M_2) \rangle = \sqrt{\frac{2}{k}} \sin\big(\frac{\pi}{k}\big).$$

## 7.3 Semi-classical description of the knot state

In [CM11b], we conjecture a precise asymptotic description of the knot states. For the simplicity of this lecture, we will only deal with the notion of microsupport.

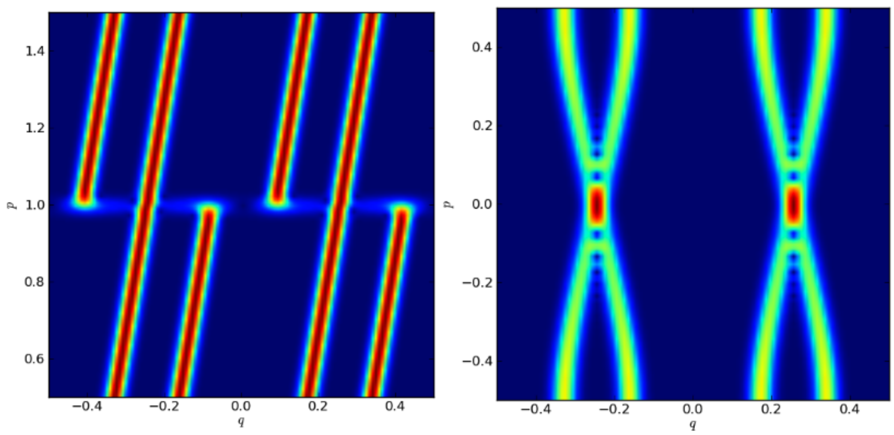



Figure 4: The pointwise norm of the knot state for the trefoil and the figure eight knot at level $k = 200$

**Definition 7.2.** Let $(\Phi_k \in \mathcal{H}_k)_{k\in\mathbb{N}}$ be a family of states.

1. It will be called admissible if there exists a positive $C$ and an integer $m$ such that
$$||\Phi_k||_{\mathcal{H}_k} \leq Ck^m$$

2. Let $U \subset \mathbb{C}$ be an open set. We will say that an admissible state $\Phi_k$ is in $O(k^{-\infty})$ on $U$ if for any $x \in U$ there is a neighborhood $V$ of $x$ included in $U$ and a sequence $C_m$ of positive numbers such that for any $m$ and $k$ we have
$$|\Phi_k(y)| \leq C_m k^{-m}, \quad \forall y \in V.$$

3. The microsupport of $(\Phi_k)$ is the smallest closed subset $F$ of $\mathbb{C}$ such that $(\Phi_k)$ is in $O(k^{-\infty})$ on $\mathbb{C} \setminus F$. We will denote it by $\mathrm{MS}(\Phi_k)$. It is $\mathbb{Z}^2$ invariant and $I$ invariant if the $\Phi_k$ are alternate sections.

It is easy to show that the states $Z_k(M)$ are admissible. Roughly, the microsupport of $Z_k(M)$ is the set where the knot state concentrates as it is shown in Figure 7.3. Let us interpret geometrically this subset.

Let $\pi : \mathbb{C} \to \mathcal{M}(S, \mathrm{SU}_2)$ defined by

$$\pi(z)(\gamma) = \exp\Big(\frac{i}{2}\omega(\gamma, x)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Big). \tag{4}$$

In this formula, we identified $\gamma \in \pi_1(S)$ with its class in $H_1(S,\mathbb{R})$ and we identified $H_1(S,\mathbb{R})$ with $\mathbb{C}$ by setting $a[m] + b[l] = a + ib$. We have the following conjecture:

**Conjecture 2.** *For any manifold $M$ bounding $S$, we have:*

$$\mathrm{MS}(Z_k(M)) = \pi^{-1}\big(r(\mathcal{M}(M, \mathrm{SU}_2))\big).$$

This conjecture was proved for the figure eight knot in [CM11a] and for the torus knots in [C11].

### 7.4 From the AJ-conjecture to the microsupport

Let $\zeta_k = -\exp(i\pi/2k)$ and $\mathcal{P}$ be the set of sequences of polynomials $f : \mathbb{Z} \to \mathbb{C}[t^{\pm 1}]$ which satisfy for all $n \in \mathbb{Z}$ and $k \in \mathbb{N}$ the equality

$$f_{n+2k}(\zeta_k) = f_n(\zeta_k).$$

For such a sequence, we can define a family $(Z_k(f) \in \mathcal{H}_k)_{k\in\mathbb{N}}$ by the formula

$$Z_k(f) = \frac{\sin(\pi/k)}{\sqrt{k}} \sum_{\ell \in \mathbb{Z}/2k\mathbb{Z}} f_\ell(\zeta_k)\Psi_\ell.$$

For any knot $K$, we have clearly $Z_k(E_K) = Z_k(J^K)$ where $E_K$ is the exterior of $K$. A simple computation using Proposition 7.1 shows that

$$Z_k(X_m f) = T^*_{1/2k} Z_k(f), \quad Z_k(X_l f) = T^*_{-i/2k} Z_k(f).$$

This implies that by setting $\rho_k(X_m) = T^*_{1/2k}$ and $\rho_k(X_l) = T^*_{-i/2k}$ we define a representation $\rho_k$ of $\mathcal{T}(S, \zeta_k)$ on $\mathcal{H}_k$. Here $\zeta_k$ means the pair $(\mathbb{C}, \zeta_k)$.

It follows that any $P \in \mathcal{A}(J^K)$ will satisfy the crucial equation

$$\rho_k(P) Z_k(E_K) = 0.$$

At this point, we will start using Toeplitz theory, we refer to [C03b, C06, C03a] for references. Recall that $\mathcal{H}_k$ is the set of holomorphic $\mathbb{Z}^2$-invariant sections of $L^k$. It is a subspace of the space $L^2_{\mathbb{Z}^2}(\mathbb{C}, L^k)$ of locally $L^2$ and $\mathbb{Z}^2$-invariant sections of $L^k$.

We denote by $\Pi_k : L^2_{\mathbb{Z}^2}(\mathbb{C}, L^k) \to \mathcal{H}_k$ the orthogonal projection. A Toeplitz operator is a sequence $(T_k \in \mathrm{End}(\mathcal{H}_k))_{k\in\mathbb{N}}$ such that there exists a sequence $f_k$ of smooth $\mathbb{Z}^2$-invariant functions on $\mathbb{C}$ satisfying

$$||T_k - \Pi_k \circ M_{f_k}||_{\mathrm{End}(\mathcal{H}_k)} = O(k^{-\infty}).$$

In this formula, $M_f$ stands for the multiplication operator by $f$. Moreover, $f_k$ is required to have an asymptotic expansion into powers of $1/k$ and we will call principal symbol the first term of this expansion.

In [CM11a], we recall the following standard results of Toeplitz theory:

**Proposition 7.3.** - *The operators $T^*_{1/2k}$ and $T^*_{-i/2k}$ are Toeplitz operators on $\mathbb{C}/\mathbb{Z}^2$ with principal symbols $\exp(2i\pi y)$ and $\exp(-2i\pi x)$ respectively.*

- *The product of two Toeplitz operators $(T_k)$ and $(S_k)$ with respective principal symbols $f$ and $g$ is a Toeplitz operator with principal symbol $fg$.*

- *If $\Phi_k$ is an admissible state in $\mathcal{H}_k$ and $T_k$ is a Toeplitz operator with principal symbol $f$ such that $T_k\Phi_k = 0$, then*

$$\mathrm{MS}(\Phi_k) \subset \{f = 0\}$$

**Remark 7.4.** These properties formalize the rough idea that the operators $T_k$ converge when $k$ goes to infinity to the multiplication operator $M_f$.

Using these properties, we obtain the following final proposition:

**Proposition 7.5.** *Let $K$ be a knot such that Conjecture 1 holds (direct inclusion). Then,*

$$\mathrm{MS}(Z_k(E_K)) \subset \pi^{-1}\Big(r\big(\mathcal{M}(E_K, \mathrm{SL}_2(\mathbb{C}))\big) \cap \mathcal{M}(S, \mathrm{SU}_2)\Big)$$

*Proof.* Choose $f$ in $\mathcal{P}(E_K, \varepsilon))$. Thanks to the AJ-conjecture, there exists a (symmetric) $q$-differential relation $P \in \mathcal{T}(S, \Lambda)$ such that $\Upsilon(f) = \sigma P$. Denote by $P(X_m, X_l, t)$ this non-commutative polynomial and set

$$S_k = P(T^*_{1/2k}, T^*_{-i/2k}, \zeta_k) \in \mathrm{End}(\mathcal{H}_k).$$

By the results stated above, this is a Toeplitz operator with principal symbol $g(x + iy) = P(e^{2i\pi y}, e^{-2i\pi x}, -1)$ which satisfies $S_k Z_k(E_k) = 0$. Hence, $\mathrm{MS}(Z_k(E_k)) \subset \{g = 0\}$.

Recall that $\mathcal{P}(E_K, \varepsilon)$ is the ideal of functions on $\mathcal{M}(S, \mathrm{SL}_2(\mathbb{C}))$ vanishing on $r(\mathcal{M}(E_K, \mathrm{SL}_2(\mathbb{C})))$. This means that for $\rho \notin r(\mathcal{M}(E_K, \mathrm{SL}_2(\mathbb{C})))$, there is some $f \in \mathcal{P}(E_K, \varepsilon)$ with $f(\rho) \neq 0$. Let $\varphi : K(S \times [0, 1], \varepsilon) \to \mathbb{C}$ the character associated to $\rho$: we have $\varphi(f) \neq 0$. Set $z = \varphi(X_m)$ and $w = \varphi(X_l)$. If $\rho$ takes its values in $\mathrm{SU}_2$, this implies that $|z| = |w| = 1$. From Formula (4), we find that $r(\rho) = \pi(x + iy)$ where $z = \exp(2i\pi y)$ and $w = \exp(-2i\pi x)$. This shows that $g(x + iy) \neq 0$, and hence $x + iy \notin \mathrm{MS}(Z_k(E_K))$. □

Let us conclude with some motivations: if $M$ and $N$ are two manifolds bounding $S$ and if $\mathrm{MS}(Z_k(M))$ and $\mathrm{MS}(Z_k(N))$ are transverse (singular) submanifolds, then scalar product

$$\langle Z_k(M), Z_k(N)\rangle = \int_D Z_k(M)(x)\overline{Z_k(N)(x)}\omega(x).$$

concentrates on $\mathrm{MS}(Z_k(M)) \cap \mathrm{MS}(Z_k(N))$ that is a discrete set of points.

Moreover, if Conjecture 2 holds, these intersection points correspond to representations $\rho \in \mathcal{M}(S, \mathrm{SU}_2)$ which extend both to $M$ and $N$, that is representations of $M \cup (-N)$. If we were able to describe $Z_k(M)$ and $Z_k(N)$ at first order around these points, we could find the leading asymptotics of $Z_k(M \cup (-N))$, that is we would prove the Witten asymptotic expansion conjecture for $M \cup (-N)$. This is exactly the strategy followed in [CM11b] to derive this conjecture for most Dehn fillings on the figure eight knot. However, it is necessary to control the asymptotics up to the two first orders. This requires to introduce more subtle objects as half-form bundles and subprincipal symbols. We refer to [CM11a, CM11b] for these questions.

# References


[AB83] Michael F. Atiyah et Raoul Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A*, **308** (1505), (1983), 523-615.

[BHMV1] C. Blanchet, N. Habegger, G. Masbaum and P. Vogel. Three-manifold invariants derived from the Kauffman bracket. *Topology* **31** (1992), 685-699.

[BHMV2] C. Blanchet, N. Habegger, G. Masbaum, et P. Vogel. Topological quantum field theories derived from the Kauffman bracket. *Topology*, **34**, vol.4, (1995), 883-927.

[BH91] G. W. Brumfiel and H. M. Hiden. $\mathrm{SL}_2$ representations of finitely generated groups *Contemporary Mathematics*, **187**, American Mathematical Society, 1991.

[Bul97] Doug Bullock. Rings of $\mathrm{SL}_2(\mathbf{C})$-characters and the Kauffman bracket skein module. *Comment. Math. Helv.*, **72**, no.4, (1997), 521-542.

[BP00] D. Bullock and J. H. Przytycki Multiplicative structure of Kauffman bracket skein module quantizations. *Proc. Amer. Math. Soc.* **128**, no.3, (2000), 923-931.

[C03a] L. Charles. Berezin-Toeplitz operators, a semi-classical approach, *Comm. Math. Phys.*, **239**, no. 1-2, (2003), 1-28.

[C03b] L. Charles. Quasimodes and Bohr-Sommerfeld conditions for the Toeplitz operators, *Comm. Partial Differential Equations*, **28**, Vol. 9-10, (2003), 1527-1566.

[C06] L. Charles. Symbolic calculus for Toeplitz operators with half-forms. *Journal of Symplectic Geometry*, **4**, Vol 2, (2006), 171-198.

[C11] L. Charles Torus knot state asymptotics arXiv:1107.4692

[CM11a] L. Charles et J. Marché. Knot state asymptotics I. Abelian representations and the A-J conjecture.

[CM11b] L. Charles et J. Marché. Knot state asymptotics II. Irreducible representations and the Witten conjecture.

[CS83] M. Culler et P. Shalen. Varieties of group representations and splittings of 3-manifolds. *Annals of Math.*, **117**, (1983), 109-146.

[G04] S. Garoufalidis. On the characteristic and deformation varieties of a knot. Proceedings of the Casson Fest, *Geometry and Topology Monographs*, **7**, (2004), 91-309.

[GD04] S. Garoufalidis and N. Dunfield. Nontriviality of the A-polynomial for knots in $S^3$, *Algebraic and Geometric Topology*, **4**, (2004) 1145-1153.

[GL05] S. Garoufalidis et T. T. Q. Le. The colored Jones function is q-holonomic. *Geom. Topol*, **9**, (2005), 1253-1293.

[GS10] S. Garoufalidis and X. Sun. The non-commutative A-polynomial of twist knots. *Journal of Knot Theory and its Ramifiations*, **19** (2010), 1571-1595.

[FG00] C. Frohman and R. Gelca. Skein modules and the noncommutative torus. *Trans Amer. Math. Soc.* **352**, (2000), 4877-4888.

[FGL02] C. Frohman, R. Gelca and W. Lofaro. The $A$-polynomial from the noncommutative viewpoint. *Trans Amer. Math. Soc.* **354**, (2002), 735-747.

[G86] W. M. Goldman. Invariant functions on Lie groups and hamiltonian flows of surface group representations. *Invent. Math.*, **85**, (1986), 263-302.

[Ha01] K. Habiro. On the quantum sl(2) invariants of knots and integral homology spheres. In: Invariant of knots and 3-manifolds (Kyoto 2001), *Geometry and Topology Monographs*, Vol. 4, (2002), 55-68.

[Hi04] K. Hikami. Difference equation of the colored Jones polynomial for torus knot. *Internat. J. Math.*, **15**, no.9, (2004), 959-965.

[HP92] Hoste J. and Przytycki J. H. A survey of skein module of 3-manifolds *Knots 90* De Gruyter- Berlin (1992), 363-379.

[Le06] T. Q. T. Le. The Colored Jones Polynomial and the A-Polynomial of Knots. *Adv. in Math.*, **207**, (2006), 782-804.

[LT11] T.Q.T. Le and A.T. Tran. On the AJ conjecture for knots. arXiv:1111.5258.

[Li97] W. B. Lickorish. An introduction to knot theory Graduate Text in Mathematics, Springer (1997).

[Ma03] G. Masbaum. Skein-theoretical derivation of some formulas of Habiro. *Algebraic and Geometric Topology*, **3**, (2003), 537-556.

[Mo95] H. C. Morton. The coloured Jones function and Alexander polynomial for torus knots. *Math. Proc. Cambridge Philos. Soc.*, **117**, no.1, (1995), 129-135.

[Mu83] D. Mumford. Tata lectures on theta. I. Progress in Mathematics, **28**, Birkhäuser Boston, Inc., Boston, MA, (1983).

[PS00] Józef H. Przytycki et Adam S. Sikora. On skein algebras and $\mathrm{Sl}_2(\mathbf{C})$-character varieties. *Topology*, **39** no.1, (2000), 115-148.

[S00] P. Sallenave. On the Kauffman bracket skein algebra of parallelized surfaces. *Ann. Sci. de l'ENS*, **33** no. 5, (2000), 593-610.

[Si09] Adam Sikora. Character varieties. To appear in *Trans. Amer. Soc.*, ArXiv:0902.2589, 2009.

[TU88] Turaev V. G. The Conway and Kauffman modules of a solid torus. (translation) *J. Soviet Math.* 52(1):2799–2805, 1990.

[TU91] Turaev V. G. Skein quantization of Poisson algebras of loops on surfaces. *Ann. Sci. École Norm. Sup.* **24**, no.4, (1991), 635-704.