Configurations of flags and representations of surface groups in complex hyperbolic geometry

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Abstract

In this work, we describe a set of coordinates on the PU(2,1)-representation variety of the fundamental group of an oriented punctured surface Σ with negative Euler characteristic. The main technical tool we use is a set of geometric invariants of a triple of flags in the complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$. We establish a bijection between a set of decorations of an ideal triangulation of Σ and a subset of the PU(2,1)-representation variety of $\pi_1(\Sigma)$.

1 Introduction

The main object studied in this paper is the representation variety of the fundamental group of a Riemann surface with cusps in PU(2,1), the group of holomorphic isometries of the complex hyperbolic plane. Although important works in the field were published since in the eighties, by Goldman and Toledo among others (see [7, 16]), many natural questions have not yet received a complete answer. Apart from general results about rigidity and flexibility (see [7, 13, 16]) most of the results are dealing with examples or families of examples of discrete and faithful representations (see [1, 9, 19, 20]). The only infinite group of finite type for which all the discrete and faithful representations in PU(2,1) are known is the modular group PSL(2,$\mathbb{Z}$) (see [2]). In the case of closed surfaces, Parker and Platis have described in [12] coordinates analogous to Fenchel-Nielsen coordinates in the setting of PU(2,1). We are dealing in this work with the case where the surface is non-compact, and the main result of the article is the description of a coordinate system on a finite cover of an open subset of the representation variety to be specified below.

Although we do not address that question here, a long term and still out of reach goal would be to describe the set of discrete and faithful representations. The starting point of our work is to remark that for such representations, the image of any boundary component generically preserves a *flag* (see below for definitions). In the spirit of Thurston's *shear coordinates* on the Teichmüller space, the main idea in constructing the coordinate system is to exploit the various configurations of flags associated to a choice of an ideal triangulation of the surface. We summarize now the idea of our construction.

Throughout this article, we will use the following notation. Let $\Sigma_{g,p}$ be a genus g surface with p punctures $x_1, \dots, x_p$, assuming $p > 0$ and $2 - 2g - p < 0$. We denote by $\pi_{g,p}$ its fundamental group and use the following standard presentation where the c_i's are homotopy classes of curves enclosing the x_i's:

$$\pi_{g,p} = \langle a_1, b_1, \dots a_g, b_g, c_1, \dots, c_p | \prod_{i=1}^{g} [a_g, b_g] \prod_{j=1}^{p} c_j \rangle.$$

Let $\widehat{\Sigma}_{g,p}$ be the universal cover of $\Sigma_{g,p}$. The surface $\widehat{\Sigma}_{g,p}$ may be seen as a topological disk with an action of $\pi_{g,p}$ and an invariant family X of boundary points projecting onto the x_i's. We will call *flag* of $\mathbf{H}^2_{\mathbb{C}}$ any pair (C, p), where C is a complex line of $\mathbf{H}^2_{\mathbb{C}}$ (see Definition 2) and p is a boundary point of C.

We consider $\mathfrak{R}_{g,p}$, the space of PU(2,1)-classes of pairs (ρ, F), where ρ is a representation of $\pi_{g,p}$ in PU(2,1) and F is an equivariant map from X to the space of flags, that is, for all g in $\pi_{g,p}$ and x in X one has $F(g.x) = \rho(g).F(x)$.

This definition is convenient for many formal aspects, but it should be noted that due to the equivariance property, the data F is equivalent to a family of p flags associated to each boundary component. More precisely, the curve c_i determines a preferred lift of x_i in X which we denote by $\widehat{x_i}$. Given an equivariant map F, we just set $F_i = F(\widehat{x_i})$. As c_i acts trivially on $\widehat{x_i}$, its image $\rho(c_i)$ must preserve the flag F_i. Reciprocally, from such a p-tuple $(F_1, \dots, F_p)$ of flags, we construct a map F by setting $F(\widehat{x_i}) = F_i$ and extend it by the equivariance property.

It is natural to consider representations and flags rather than just representations for the following reasons. The fundamental group $\pi_{g,p}$ is torsion free, therefore so is any discrete and faithful representation ρ of it. This implies that none of the $\rho(c_j)$'s is an elliptic isometry. Now, a non-elliptic isometry of PU(2,1) generically preserves a flag. More precisely,

- a loxodromic isometry has exactly two fixed points on the boundary $\partial\mathbf{H}^2_{\mathbb{C}}$, and thus preserves exactly two flags (C, p) and (C, q), where p and q are its fixed points, and C the (unique) complex line containing it.
- A parabolic isometry preserves a flag unless it is 3-step unipotent. Note that there is only one 3-step unipotent conjugacy class in PU(2,1) (see [8]).

As a consequence, for any generic discrete and faithful representation ρ, there exists such a ρ-equivariant mapping F. Note that this mapping is not uniquely defined: for any c_j which is mapped to a loxodromic isometry, there is an order two indeterminacy since it preserves two flags. As a consequence, $\mathfrak{R}_{g,p}$ is a 2^p-ramified cover over the set of representations ρ such that $\rho(c_j)$ preserves a flag for all j.

To build the coordinate system, we start from an ideal triangulation T of $\Sigma_{g,p}$ (see Definition 15). Each triangle Δ of T lifts to $\widehat{\Sigma}_{g,p}$ as a triangle whose vertices are denoted by $x, y, z \in X$. Given a pair (ρ, F), the triple of flags $(F(x), F(y), F(z))$ is well defined up to isometry. In Definition 10, we introduce a notion of genericity for a triple of flags. We will say that a pair (ρ, F) is *generic with respect to* T if the triple of flags associated to any triangle of T is generic. We will denote by $\mathfrak{R}_{g,p}^T$ the subset of $\mathfrak{R}_{g,p}$ (depending on T) containing those classes of pairs (ρ, F) that are generic with respect to T. We then introduce a set of geometric invariants to classify generic triples of flags up to isometry. The precise definition and study of these invariants is a crucial point of the article, and is summarized in Theorem 1. We postpone the precise statement of this theorem to Section 3.3, and give instead the following simplified statement.

Theorem 1(simplified)**.** *The set of isometry classes of generic triples of flags of* $\mathbf{H}_{\mathbb{C}}^2$ *is parametrised by a (real) 7 dimensional submanifold of* $\mathbb{C}^7$*.*

Combining all the invariants associated to the triangles in T, we are able to provide a coordinate system. To represent a geometric configuration, the invariants associated to adjacent triangles must satisfy compatibility relations. We will call *decoration* of a triangulation T the following data: a family of invariants for each triangle of T such that the compatibility conditions are satisfied (see Definition 14).

The main result of the article is the following

Theorem 2. *Let* T *be an ideal triangulation of* $\Sigma_{g,p}$*. There is a bijection between* $\mathfrak{R}_{g,p}^T$ *and* $\mathcal{X}(T)$*, the set of decorations of* T*.*

Unfortunately the compatibility equations are difficult to solve. However, by introducing a simplified decoration, we are able to prove that they always have solutions.

The main technical tool used in this work is the construction of the invariants which parametrize generic triples of flags. These invariants are very close in their definitions to the Koranyi-Reimann cross-ratio on the boundary of $\mathbf{H}_{\mathbb{C}}^2$. To give a flavor of what will be done here, let us recall a convenient way to define the latter. We use the projective model of $\mathbf{H}_{\mathbb{C}}^2$, that is, we view it as the projectification in $\mathbb{C}P^2$ of the negative cone of a signature (2,1) Hermitian form on $\mathbb{C}^3$ denoted by $\langle \cdot, \cdot \rangle$ (see Section 2 for details). Then points on $\partial \mathbf{H}_{\mathbb{C}}^2$ correspond to null vectors in $\mathbb{C}^3$, and the Koranyi-Reimann $\mathbf{X}(a, b, c, d)$ of a quadruple of distinct boundary points is defined as

$$\mathbf{X}(a, b, c, d) = \frac{\langle \mathbf{c}, \mathbf{a} \rangle \langle \mathbf{d}, \mathbf{b} \rangle}{\langle \mathbf{c}, \mathbf{b} \rangle \langle \mathbf{d}, \mathbf{a} \rangle},$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are lifts of a, b, c and d. This quantity is invariant under PU(2,1) and has been studied and used for instance in [3, 12, 18].

We introduce two new invariants constructed in this way.

- An invariant m which classifies pairs of flags in generic position.
- We describe an invariant δ, associated to the data of one flag and two complex lines

Combining m, δ and the classical invariants for pairs and triples of complex lines, we obtain in Theorem 1 a classification of generic triples of flags. The genericity conditions we are mentioning here are made clear in section 3.

We introduce the notion of *standard position* of an ordered triple of flags. The six standard configurations corresponding to the various orderings of a given triple of flags are related by *elementary isometries* which we compute explicitly from our invariants (see Section 4). The representation can be recovered from the decoration by taking suitable products of matrices representing elementary isometries.

A natural question would be to compute the coordinate change induced by changing the triangulation. As such a change may be decomposed as a finite sequence of so-called flips (or Whitehead moves), the question reduces to describing the effect on the coordinates of such a flip. Unfortunately, we were not able to find any usable formula. A simple formula would be a valuable tool to attack discreteness questions from this perspective.

The article is organised as follows:

- The section 2 is devoted to the exposition of notions of complex hyperbolic geometry. We describe totally geodesic subspaces of the complex hyperbolic plane and introduce invariants of pairs and triples of complex lines.
- In section 3, we describe the main technical tools which are the invariants m and δ. The main result of this section is the Theorem 1 which classifies triples of flags up to isometry.
- In section 4, we define the *standard configuration* of a flag and a complex line. Using the invariants described in the previous section, we provide two explicit matrices that are the elementary pieces necessary to construct the representations from the invariants. These matrices may be useful for numerical applications.
- The section 5 is devoted to the definition of the decoration space and to the proof of Theorem 2.
- We prove in section 6 that the compatibility equations involved in the decoration space always have solutions. The main tool is the lemma 4: it shows that once the Φ and m invariants of a triple of flag are known, there exist generically 2 possible triples of δ invariants, which correspond to the fixed points of an antiholomorphic isometry in the boundary of a disk. As a consequence of this lemma, we obtain in Proposition 11 that $\mathfrak{R}^T_{g,p}$ is a 2^N ramified cover of a simpler space denoted by $\mathcal{M}^T_{g,p}$ where N is the number of triangles in T. The latter space is an auxiliary decoration space of the triangulation T, given by the Φ and m invariants.
- We give in section 7 some indications about how to control the isometry type of the images of the boundary curves by the representation constructed from a decorated triangulation of $\Sigma_{g,p}$. We first deal with the case of an arbitrary punctured surface, and move then to the case of the 1-punctured torus.

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2 Complex hyperbolic geometry

2.1 Generalities

Consider the Hermitian form of signature $(2,1)$ in $\mathbb{C}^3$ given by the formula $\langle v,w\rangle = v^T J\overline{w}$ where J is the matrix given by

$$J = \begin{bmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0 \end{bmatrix}.$$

We define the following subsets of $\mathbb{C}^3$:

$$\begin{aligned} V_0 &= \{v \in \mathbb{C}^3 \setminus \{0\}, \langle v,v\rangle = 0\}\\ V_- &= \{v \in \mathbb{C}^3 \setminus \{0\}, \langle v,v\rangle < 0\}\\ V_+ &= \{v \in \mathbb{C}^3 \setminus \{0\}, \langle v,v\rangle > 0\} \end{aligned}$$

Let $\mathbf{P} : \mathbb{C}^3 \setminus \{0\} \to \mathbb{C}P^2$ be the canonical projection onto the complex projective space.

Definition 1. The complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$ is the set $\mathbf{P}(V_-)$ equipped with the Bergman metric.

The boundary of $\mathbf{H}^2_{\mathbb{C}}$ is $\mathbf{P}(V_0)$. The distance function associated to the Bergman metric is given in terms of Hermitian product by

$$\cosh^2\left(\frac{d(m,n)}{2}\right) = \frac{\langle \mathbf{m},\mathbf{n}\rangle\langle \mathbf{n},\mathbf{m}\rangle}{\langle \mathbf{m},\mathbf{m}\rangle\langle \mathbf{n},\mathbf{n}\rangle}, \tag{1}$$

where $\mathbf{m}$ and $\mathbf{n}$ are lifts of m and n to $\mathbb{C}^3$. It follows from (1) that U(2,1), the unitary group associated to J, acts on $\mathbf{H}^2_{\mathbb{C}}$ by holomorphic isometries. The full isometry group of $\mathbf{H}^2_{\mathbb{C}}$ is generated by PU(2,1) and the complex conjugation. The usual trichotomy of isometries for PSL(2,$\mathbb{R}$) holds here also: an isometry is elliptic if it has a fixed point inside $\mathbf{H}^2_{\mathbb{C}}$, parabolic if it has a unique fixed point on $\partial\mathbf{H}^2_{\mathbb{C}}$, and loxodromic if it has exactly two fixed points on $\partial\mathbf{H}^2_{\mathbb{C}}$, and this exhausts all possibilities.

2.2 Subspaces of $\mathbf{H}^2_{\mathbb{C}}$.

There are two types of maximal totally geodesic subspaces of $\mathbf{H}^2_{\mathbb{C}}$, which are both of (real) dimension 2: complex lines and $\mathbb{R}$-planes. We give now a few indications about these. More details may be found in [8].

2.2.1 Complex lines

Definition 2. We call *complex line* in $\mathbf{H}^2_{\mathbb{C}}$ the intersection with $\mathbf{H}^2_{\mathbb{C}}$ of the projectivization of a 2-dimensional subspace of $\mathbb{C}^3$ which intersects V_-. Such a subspace is orthogonal to a one-dimensional subspace contained in V_+: we call *polar vector* of the complex line any generator of this subspace.

Note that a complex line is an isometric embedding of the complex hyperbolic line $\mathbf{H}^1_{\mathbb{C}}$. To any complex line C is associated a unique holomorphic involution fixing pointwise C, which we shall refer to as the *complex symmetry* with respect to C. The group PU(2,1) acts transitively on the set of complex lines of $\mathbf{H}^2_{\mathbb{C}}$.

Definition 3. We call *flag* a pair (C, p) where C is a complex line and p is a point in $C \cap \partial \mathbf{H}^2_{\mathbb{C}}$.

Lemma 1. *PU(2,1) acts transitively on the set of flags of* $\mathbf{H}^2_{\mathbb{C}}$.

2.2.2 $\mathbb{R}$-planes

Definition 4. An $\mathbb{R}$-plane is the intersection with $\mathbf{H}^2_{\mathbb{C}}$ of the projection of a Lagrangian vector subspace of $\mathbb{C}^{2,1}$.

Every $\mathbb{R}$-plane P is fixed pointwise by a unique antiholomorphic isometric involution I_P, which is the projectivization of the Lagrangian symmetry with respect to any lift of P as a Lagrangian of $\mathbb{C}^{2,1}$. We will refer to I_P as the *Lagrangian reflection about* P. The standard example is the set of points of $\mathbf{H}^2_{\mathbb{C}}$ with real coordinates, which is fixed by the complex conjugation. We will refer to this $\mathbb{R}$-plane as $\mathbf{H}^2_{\mathbb{R}} \subset \mathbf{H}^2_{\mathbb{C}}$. It is an embedding of the real hyperbolic plane into $\mathbf{H}^2_{\mathbb{C}}$.

As a consequence, we obtain

Proposition 1. *Let Q be an $\mathbb{R}$-plane. There exists a matrix $M_Q \in$ SU(2,1) such that*

$$M_Q\overline{M_Q} = 1, \text{ and } I_Q(m) = \mathbf{P}(M_Q \cdot \overline{\mathbf{m}}) \text{ for any } m \text{ in } \mathbf{H}^2_{\mathbb{C}} \text{ with lift } \mathbf{m}. \tag{2}$$

Proof. Let $\mathbf{Q}$ be a lift of Q to $\mathbb{C}^{2,1}$, and choose $\mathbb{R}^3 \subset \mathbb{C}^{2,1}$ as a lift of $\mathbf{H}^2_{\mathbb{R}}$. Since the group U(2,1) acts transitively on the Lagrangian Grassmanian of $\mathbb{C}^{2,1}$, there exists a matrix $A \in$ U(2,1) such that $A\mathbb{R}^3 = \mathbf{Q}$. The matrix $M_Q = A\overline{A}^{-1}$ belongs to SU(2,1) and satisfies the condition (2). ☐

Remark 1. If I_1 and I_2 are two Lagrangian reflections with associated matrices $\mathbf{M}_1$ and $\mathbf{M}_2$, then their composition, which is a holomorphic isometry, admits the matrix $\mathbf{M}_1\overline{\mathbf{M}_2}$ as a lift to SU(2,1).

2.3 Classical invariants

2.3.1 Invariant of two complex lines

Definition 5. Let C_1 and C_2 be two complex lines of $\mathbf{H}^2_{\mathbb{C}}$, with polar vectors $\mathbf{c}_1$ and $\mathbf{c}_2$. We set

$$\varphi(C_1, C_2) = \frac{|\langle \mathbf{c}_1, \mathbf{c}_2 \rangle|^2}{\langle \mathbf{c}_1, \mathbf{c}_1 \rangle \langle \mathbf{c}_2, \mathbf{c}_2 \rangle}.$$

Clearly, $\varphi(C_1, C_2)$ does not depend on the choice of the lift in the pair of polar vectors, and is PU(2,1)-invariant. We recall the geometric interpretation of φ, and we refer to [8] for details:

- $\varphi(C_1, C_2) > 1$ if C_1 and C_2 are disjoint in $\mathbf{H}^2_{\mathbb{C}}$. In this case, the distance d between C_1 and C_2 is given by the formula $\varphi(C_1, C_2) = \cosh^2(d/2)$.
- $\varphi(C_1, C_2) = 1$ if C_1 and C_2 are either identical or asymptotic, by which we mean that they meet in $\partial \mathbf{H}^2_{\mathbb{C}}$.
- $\varphi(C_1, C_2) < 1$ if C_1 and C_2 intersect. The angle θ of their intersection is given by the relation $\varphi(C_1, C_2) = \cos^2(\theta)$.

Note that two complex lines are orthogonal if and only if $\varphi(C_1, C_2) = 0$. The φ-invariant classifies pairs of distinct complex lines up to isometries.

Proposition 2. *Let C_1, C_2, D_1, D_2 be 4 complex lines such that $C_1 \neq C_2$ and $D_1 \neq D_2$. There exists an isometry $g \in PU(2,1)$ such that $D_1 = gC_1$ and $D_2 = gC_2$ if and only if $\varphi(C_1, C_2) = \varphi(D_1, D_2)$.*

Proof. It is clear that if the two pairs are isometric, their invariant is the same. Reciprocally, choose polar vectors $\mathbf{c}_1, \mathbf{c}_2, \mathbf{d}_1, \mathbf{d}_2$ with norm 1 and such that $\langle \mathbf{c}_1, \mathbf{c}_2 \rangle$ and $\langle \mathbf{d}_1, \mathbf{d}_2 \rangle$ are in $\mathbb{R}_{\geq 0}$.

As $C_1 \neq C_2$, the vectors $\mathbf{c}_1$ and $\mathbf{c}_2$ are independent and the same is true for $\mathbf{d}_1$ and $\mathbf{d}_2$. With the assumption on the φ-invariant, the Gram matrices of $(\mathbf{c}_1, \mathbf{c}_2)$ and $(\mathbf{d}_1, \mathbf{d}_2)$ are identical. It means that one can find an isometry which maps $\mathbf{c}_1$ on $\mathbf{d}_1$ and $\mathbf{c}_2$ on $\mathbf{d}_2$. This ends the proof. $\square$

We will need the following lemma.

Lemma 2. *Let C_1 and C_2 be two non orthogonal distinct complex lines, and p_1 a point in $\partial \mathbf{H}^2_{\mathbb{C}} \cap C_1$ which is not in C_2. Except for the identity, no isometry preserves C_1 and C_2 and fixes p_1.*

Proof. Pick $\mathbf{c}_1$ and $\mathbf{c}_2$ two vectors polar to C_1 and C_2 of norm 1 and $\mathbf{p}_1$ a lift of p_1. Writing $\langle \mathbf{p}_1, \mathbf{c}_2 \rangle = a$ and $\langle \mathbf{c}_1, \mathbf{c}_2 \rangle = b$, the Hermitian form has the following matrix in the basis $(\mathbf{p}_1, \mathbf{c}_1, \mathbf{c}_2)$

$$H = \begin{bmatrix} 0 & 0 & a \\ 0 & 1 & b \\ \bar{a} & \bar{b} & 1 \end{bmatrix}$$

An isometry having the requested property has a diagonal lift to SU(2,1) in this basis. The result is obtained by writing the isometry condition ${}^t\overline{M}HM = H$, and by using the fact that b is non zero since C_1 and C_2 are not orthogonal. $\square$

2.3.2 Invariants of three complex lines

Let C_1, C_2 and C_3 be three complex lines in $\mathbf{H}^2_{\mathbb{C}}$. We will say that they are in *generic position* if their polar vectors form a basis of $\mathbb{C}^3$. There are three invariants of the triple (C_1, C_2, C_3) given by the φ-invariant of all pairs of complex lines. We will need a fourth one (given in the following definition) to classify all triples up to isometry.

Definition 6. Let C_1, C_2, C_3 be three complex lines in $\mathbf{H}^2_{\mathbb{C}}$ with respective polar vectors $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$. Then we set

$$\Phi(C_1, C_2, C_3) = \frac{\langle \mathbf{c}_1, \mathbf{c}_2 \rangle \langle \mathbf{c}_2, \mathbf{c}_3 \rangle \langle \mathbf{c}_3, \mathbf{c}_1 \rangle}{\langle \mathbf{c}_1, \mathbf{c}_1 \rangle \langle \mathbf{c}_2, \mathbf{c}_2 \rangle \langle \mathbf{c}_3, \mathbf{c}_3 \rangle}.$$

The importance of this invariant should be clear from the following two propositions (see [14]):

Proposition 3. *Let C_1, C_2, C_3 be three complex lines of $\mathbf{H}^2_{\mathbb{C}}$ in generic position. For simplicity, we denote by φ_{ij} the φ-invariant of C_i and C_j and by Φ_{ijk} the Φ-invariant of C_i, C_j, C_k. These invariants enjoy the following properties.*

1. *For all distinct $i, j, k \in \{1, 2, 3\}$, the following relations are satisfied.*

$$\varphi_{ij} = \varphi_{ji}, \Phi_{ijk} = \Phi_{jki} = \overline{\Phi_{ikj}} \text{ and } \Phi_{ijk}\Phi_{ikj} = \varphi_{ij}\varphi_{jk}\varphi_{ki}. \tag{3}$$

2. *The invariants satisfy the inequality*

$$1 - \varphi_{12} - \varphi_{23} - \varphi_{31} + \Phi_{123} + \Phi_{132} < 0. \tag{4}$$

Proof. 1. These relations are straightforward from the definitions of the invariants φ and Φ.

2. Let C_1, C_2, C_3 be three complex lines in generic position. Let $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ be three polar vectors associated to these lines. Let G be the Gram matrix of the basis $(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$. A direct computation shows that the left-hand side of relation (4) is equal to

$$\Delta(C_1, C_2, C_3) = \frac{\det G}{\langle \mathbf{c}_1, \mathbf{c}_1 \rangle \langle \mathbf{c}_2, \mathbf{c}_2 \rangle \langle \mathbf{c}_3, \mathbf{c}_3 \rangle}.$$

This number is an invariant of the triple (C_1, C_2, C_3). As the Gram matrix represents the Hermitian form, it has signature $(2, 1)$ and its determinant is negative.

□

Proposition 4. *Consider non negative real numbers φ_{ij} and complex numbers Φ_{ijk} satisfying the relations of Proposition 3. There exists a triple C_1, C_2, C_3 in generic position, unique up to isometry, such that for all distinct i, j, k in $\{1, 2, 3\}$ the relations $\varphi(C_i, C_j) = \varphi_{ij}$ and $\Phi_{ijk} = \Phi(C_i, C_j, C_k)$ hold.*

Proof. Consider real numbers φ_{ij} and complex numbers Φ_{ijk} satisfying the relations (3) and (4), and $\mathbb{C}^3$ with its canonical basis (e_1, e_2, e_3). We define a Hermitian form h on it by setting

$$h(e_i, e_i) = 1 \text{ for all } i, \qquad h(e_1, e_2) = \sqrt{\varphi_{12}},$$

$$h(e_2, e_3) = \sqrt{\varphi_{23}}, \qquad h(e_3, e_1) = \frac{\Phi_{123}}{\sqrt{\varphi_{12}\varphi_{23}}}.$$

The matrix of h in the basis (e_1, e_2, e_3) has unit diagonal entries – thus positive trace– and according to the relation (4), it has negative determinant. As a consequence, h has signature $(2, 1)$. By the classification of Hermitian forms, this model is conjugate to the standard one, and the vectors e_1, e_2, e_3 map to the polar vectors of the desired complex lines. Moreover, the few choices we made disappear projectively, hence the triple of complex lines is unique up to isometry. □

3 Invariants of flags

3.1 Invariant of two flags

Definition 7. Let (C_1, p_1) and (C_2, p_2) be two flags in $\mathbf{H}^2_{\mathbb{C}}$. We will say that they are in *generic position* if p_1 does not belong to C_2, p_2 does not belong to C_1 and C_1 is not orthogonal to C_2.

Remark 2. The condition of non-orthogonality of C_1 and C_2 will be needed to define the elementary isometries associated to a triple of flags in a unique way (see Propositions 7 and 8).

Definition 8. Let (C_1, p_1) and (C_2, p_2) be two flags in generic position. Let $\mathbf{c}_1, \mathbf{c}_2$ be polar vectors of C_1, C_2 and $\mathbf{p}_1, \mathbf{p}_2$ be representatives of p_1, p_2. We set

$$m[(C_1, p_1), (C_2, p_2)] = \frac{\langle \mathbf{c}_1, \mathbf{c}_2 \rangle \langle \mathbf{p}_1, \mathbf{p}_2 \rangle}{\langle \mathbf{c}_1, \mathbf{p}_2 \rangle \langle \mathbf{p}_1, \mathbf{c}_2 \rangle}.$$

This invariant is a complex generalization of the φ-invariant of two complex lines. Its properties are summarized in the following proposition.

Proposition 5. *Let (C_1, p_1) and (C_2, p_2) be two flags in generic position, and m_{12} their invariant $m[(C_1, p_1), (C_2, p_2)]$.*

1. *The two invariants $\varphi(C_1, C_2)$ and m_{12} are linked by the relation*

$$\varphi(C_1, C_2) = \left| \frac{m_{12}}{m_{12} - 1} \right|^2. \tag{5}$$

2. *For any complex number $m_{12} \in \mathbb{C} \setminus \{0, 1\}$ there exists a pair of flags $(C_1, p_1), (C_2, p_2)$ in generic position such that $m[(C_1, p_1), (C_2, p_2)] = m_{12}$. This pair is unique up to isometry.*

Proof. 1. Let (C_1, p_1) and (C_2, p_2) be two flags in generic position, $\mathbf{c}_1, \mathbf{c}_2$ be polar vectors of C_1, C_2 and $\mathbf{p}_1, \mathbf{p}_2$ be representatives of p_1, p_2. The family of vectors $(\mathbf{c}_1, \mathbf{c}_2, \mathbf{p}_1, \mathbf{p}_2)$ is linearly dependent, hence the determinant of its Gram matrix vanishes. Computing this determinant and dividing by non vanishing factors, we obtain the relation $|m_{12} - 1|^2 \varphi_{12} = |m_{12}|^2$. From that relation we see that m_{12} cannot be equal to 1 since the two flags are in generic position and therefore φ_{12} is non-zero. This proves relation (5).

2. In order to prove the second part of the proposition, we make the following observation: given two flags (C_1, p_1) and (C_2, p_2) in generic position, there is a unique complex line C_3 joining p_1 and p_2. Following Proposition 4, the triple (C_1, C_2, C_3) is determined by its φ-invariants, hence, we can classify pairs of flags using φ-invariants. More precisely, as C_1 and C_3 are asymptotic (they meet on $p_1 \in \partial \mathbf{H}^2_{\mathbb{C}}$), their φ invariant φ_{13} equals 1. For the same reason, $\varphi_{23} = 1$. As a consequence of relation (3), we obtain the equality $|\Phi_{123}|^2 = \varphi_{12}$. Plugging these values into the relation (4) yields

$$\Delta_{123} = -1 - \varphi_{12} + \Phi_{123} + \Phi_{132} = -|1 - \Phi_{123}|^2. \tag{6}$$

Suppose that we have $\Phi_{123} \neq 1$, then the complex lines C_1, C_2, C_3 are in generic position. Let $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ be polar vectors of these lines. They form a basis of $\mathbb{C}^3$ and the linear forms $\langle \cdot, \mathbf{c}_1 \rangle, \langle \cdot, \mathbf{c}_2 \rangle, \langle \cdot, \mathbf{c}_3 \rangle$ form a linear basis of the dual of $\mathbb{C}^3$. One can find a unique anti-dual basis $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$ such that for all i, j in $\{1, 2, 3\}$ one has $\langle \mathbf{d}_i, \mathbf{c}_j \rangle = \delta_{ij}$. A direct computation shows that the Gram matrix of the Hermitian form in the basis $(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$ is the inverse of the Gram matrix of $(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$. Moreover $\mathbf{d}_1$ being orthogonal to $\mathbf{c}_2$ and $\mathbf{c}_3$, is a representative of p_2 and $\mathbf{d}_2$ is a representative of p_1. Using these representatives we get (see Remark 3 below)

$$\begin{aligned} m_{12} &= \langle \mathbf{c}_1, \mathbf{c}_2\rangle\langle \mathbf{d}_1, \mathbf{d}_2\rangle = \frac{\langle \mathbf{c}_1, \mathbf{c}_2\rangle \left(\langle \mathbf{c}_3, \mathbf{c}_1\rangle\langle \mathbf{c}_2, \mathbf{c}_3\rangle - \langle \mathbf{c}_2, \mathbf{c}_1\rangle\langle \mathbf{c}_3, \mathbf{c}_3\rangle\right)}{\langle \mathbf{c}_1, \mathbf{c}_1\rangle\langle \mathbf{c}_2, \mathbf{c}_2\rangle\langle \mathbf{c}_3, \mathbf{c}_3\rangle\Delta_{123}} = \frac{\Phi_{123} - \varphi_{12}}{\Delta_{123}} \\ &= \frac{\Phi_{123} - |\Phi_{123}|^2}{-|1 - \Phi_{123}|^2} = \frac{\Phi_{123}}{\Phi_{123} - 1}. \end{aligned}$$

This proves that $m_{12} = 1$ if and only if $\Phi_{123} = 1$ and that m_{12} classifies pairs of flags.

□

Remark 3. Note that this anti-dual basis is usually used in the literature under a slightly different form, using the so-called Hermitian cross-product. The vector $\mathbf{d}_2$ is proportional to the Hermitian cross-product of $\mathbf{c}_1$ and $\mathbf{c}_3$, denoted by $\mathbf{c}_1 \boxtimes \mathbf{c}_3$. It is a simple computation using Hermitian cross-product to check that $\langle \mathbf{d}_1, \mathbf{d}_2\rangle$ equals $(\langle \mathbf{c}_3, \mathbf{c}_1\rangle\langle \mathbf{c}_2, \mathbf{c}_3\rangle - \langle \mathbf{c}_2, \mathbf{c}_1\rangle\langle \mathbf{c}_3, \mathbf{c}_3\rangle)\cdot (\langle \mathbf{c}_1, \mathbf{c}_1\rangle\langle \mathbf{c}_2, \mathbf{c}_2\rangle\langle \mathbf{c}_3, \mathbf{c}_3\rangle\Delta_{123})^{-1}$. See [8] for details.

3.2 Invariant of a flag and two complex lines

Definition 9. Let (C_1, p_1) be a flag and C_2, C_3 be two complex lines such that the three complex lines C_1, C_2 and C_3 are in generic position and such that p_1 does not belong to C_2 nor C_3. Take $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ three vectors polar to C_1, C_2, C_3 and $\mathbf{p}_1$ a representative of p_1. We set:

$$\delta[(C_1, p_1), C_2, C_3] = \delta^1_{23} = \frac{\langle \mathbf{c}_2, \mathbf{c}_3\rangle\langle \mathbf{p}_1, \mathbf{c}_2\rangle}{\langle \mathbf{c}_2, \mathbf{c}_2\rangle\langle \mathbf{p}_1, \mathbf{c}_3\rangle}. \tag{7}$$

This invariant may be seen as a coordinate of p_1 knowing C_1, C_2 and C_3. Its main properties are summarized in the following proposition.

Proposition 6. *Let (C_1, p_1) be a flag and C_2, C_3 be two complex lines such that the three complex lines C_1, C_2 and C_3 are in generic position and such that p_1 belongs neither to C_2 nor to C_3. The invariants δ^1_{23} and δ^1_{32} satisfy the following equations:*

$$\varphi_{23} = \delta^1_{23}\delta^1_{32} \tag{8}$$

$$0 = (1 - \varphi_{13})|\delta^1_{23}|^2 + 2\mathrm{Re}\left[(\Phi_{132} - \varphi_{23})\delta^1_{23}\right] + \varphi_{23}(1 - \varphi_{12}) \tag{9}$$

Reciprocally, take C_1, C_2, C_3 three complex lines in generic position. Any non zero value of δ^1_{23} which satisfies the second equation corresponds to a unique point p_1 in C_1 which is neither on C_2 nor on C_3.

Proof. The first equation is a direct consequence of the definition. For the second one, let $(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$ be a basis of $\mathbb{C}^3$ formed by polar vectors for C_1, C_2, C_3. Let $(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$ be its anti-dual basis. We will use the latter basis to prove relation (9). We recall that the matrix of the Hermitian form in the basis $(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$ is the inverse of the Gram matrix of $(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$.

As p_1 belongs to C_1, its representative is a linear combination of $\mathbf{d}_2$ and $\mathbf{d}_3$, and we may thus write $\mathbf{p}_1 = a\mathbf{d}_2 + b\mathbf{d}_3$. The coordinates a and b can be recovered by computing the Hermitian products $\langle \mathbf{p}_1, \mathbf{c}_2\rangle = a$ and $\langle \mathbf{p}_1, \mathbf{c}_3\rangle = b$. In particular, this implies

$$\delta^1_{23} = \frac{\langle \mathbf{c}_2, \mathbf{c}_3\rangle a}{\langle \mathbf{c}_2, \mathbf{c}_2\rangle b}. \tag{10}$$

By expressing that $\mathbf{p}_1$ is in the isotropic cone of the Hermitian form, we obtain the relation (9).

On the other hand, if we know δ^1_{23}, then according to relation (10), we know projective coordinates for $\mathbf{p}_1$. If δ^1_{23} satisfies (9), the vector $\mathbf{p}_1$ must be on the cone of the quadratic form. It proves that δ^1_{23} determines the position of p_1 on C_1 as asserted. ☐

3.3 Summary : invariants of three flags

In the remaining part of the article, we will be interested in the space of configurations of three flags. Let us summarize what are the relevant invariants for such configurations.

Definition 10. We will say that three flags $(C_i, p_i)_{i=1,2,3}$ are in *generic position* if they are pairwise in generic position, and if the triple of complex lines (C_1, C_2, C_3) is also in generic position, that is, if

- any two of the complex lines are distinct and non-orthogonal,
- any triple of vectors polar to the C_i's is a basis of $\mathbb{C}^3$.

We classify now the triples of flags up to PU(2,1).

Theorem 1. *Let (C_1, p_1), (C_2, p_2) and (C_3, p_3) be three flags in generic position. The configuration of these flags modulo holomorphic isometry is classified by the invariants φ_{ij}, Φ_{ijk} and δ^i_{jk} for all distinct i, j, k in $\{1, 2, 3\}$. These invariants satisfy the following equations for all i, j, k:*

$$(3) \qquad \varphi_{ij} = \varphi_{ji} = \overline{\varphi_{ij}} > 0,\ \Phi_{ijk} = \Phi_{jki} = \overline{\Phi_{ikj}} \text{ and } \Phi_{ijk}\Phi_{ikj} = \varphi_{ij}\varphi_{jk}\varphi_{ki}.$$

$$(4) \qquad \Delta_{ijk} = 1 - \varphi_{ij} - \varphi_{jk} - \varphi_{ki} + \Phi_{ijk} + \Phi_{ikj} < 0.$$

$$(8) \qquad \delta^i_{jk}\delta^i_{kj} = \varphi_{ij}.$$

$$(9) \qquad (1 - \varphi_{ik})|\delta^i_{jk}|^2 + 2\mathrm{Re}\left[(\Phi_{ikj} - \varphi_{jk})\delta^i_{jk}\right] + \varphi_{jk}(1 - \varphi_{ij}) = 0.$$

The space of solutions is a manifold of dimension 7. Moreover, the invariants m_{ij} attached to pairs of flags are expressed in terms of the other invariants as follows :

$$\begin{aligned} m_{ij}\Delta_{ijk}\varphi_{ik}\varphi_{jk} &= \varphi_{ik}\varphi_{jk}(\Phi_{ijk} - \varphi_{ij}) + \varphi_{ik}(\varphi_{ij}\varphi_{jk} - \Phi_{ijk})\delta^i_{kj} \\ &\quad + \varphi_{jk}(\varphi_{ij}\varphi_{ik} - \Phi_{ijk})\overline{\delta^j_{ki}} + \Phi_{ijk}(1 - \varphi_{ij})\delta^i_{kj}\overline{\delta^j_{ki}} \end{aligned} \tag{11}$$

Proof. The first part of the proof is nothing but a summary of the preceding sections. Let us now compute m_{12}. The two other m-invariants are obtained in the same way. Choose $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ polar vectors of C_1, C_2, C_3 and let $(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$ be the anti-dual basis as usual. Then, using relation (10), one can find explicit coordinates for representatives of p_1 and p_2 in the basis $(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$. Precisely, we can choose

$$\begin{cases} \mathbf{p}_1 = \langle \mathbf{c}_3, \mathbf{c}_2\rangle \mathbf{d}_2 + \langle \mathbf{c}_3, \mathbf{c}_3\rangle \delta^1_{32}\mathbf{d}_3 \\ \\ \mathbf{p}_2 = \langle \mathbf{c}_3, \mathbf{c}_1\rangle \mathbf{d}_1 + \langle \mathbf{c}_3, \mathbf{c}_3\rangle \delta^2_{31}\mathbf{d}_3. \end{cases}$$

To obtain a formula for m_{12}, we just need to replace $\mathbf{p}_1$ and $\mathbf{p}_2$ in the definition of m_{12} by the expressions above. We obtain the relation (11) after a computation. $\square$

Remark 4. The set of generic flags is actually a manifold as triples of generic flags forms an open set in the product of 3 copies of the flag variety. Since PU(2,1) acts freely on this open set, the set of isometry classes of generic triples of flags is a manifold.

4 Elementary isometries associated to a triple of flags

4.1 $\mathbb{R}$-planes associated to a triple of flags and elementary isometries

In this paragraph, we define the elementary isometries associated to a triple of flags. More precisely, we prove the

Proposition 7. *Let $F_i = (C_i, p_i)$ for $i = 1, 2, 3$ be a triple of flags in generic position such that any two complex lines are not asymptotic.*

1. *For any pair (i, j) with $i \neq j$, there exists a unique isometry E_{ij} exchanging C_i and C_j, and mapping p_j to p_i. It is called the exchange isometry associated to the pair of flags F_i and F_j.*
2. *There exists a unique isometry T^i_{jk} fixing p_i and preserving C_i which maps C_k to a complex line C'_k satisfying $R_{C'_k}(p_i) = R_{C_j}(p_i)$, where R_C is the complex symmetry with respect to the complex line C. It is called the transfer isometry associated to the ordered triple of flags (F_i, F_j, F_k).*

We will give a geometric proof of this proposition, showing that the exchange and transfer isometry are obtained as products of Lagrangian reflections which are canonically associated to a triple of flags satisfying the assumption of Proposition 7.

Proposition 8. *Let C_1 and C_2 be two complex lines which are neither orthogonal nor asymptotic.*

1. *Let p_1 be a point in ∂C_1. There exists a unique $\mathbb{R}$-plane P such that I_P, the inversion in P, preserves both C_1 and C_2, and fixes p_1.*
2. *Let p_2 be a point in ∂C_2. There exists a unique $\mathbb{R}$-plane Q such that I_Q, the inversion in Q, swaps C_1 and C_2 and maps p_1 to p_2.*
3. *Let m and n be two points in the boundary of $\mathbf{H}^2_{\mathbb{C}}$, not belonging to ∂C_1. There exists a unique Lagrangian reflection preserving C_1 and swapping m and n.*
4. *Let p_1, p_2 and p_3 be three points of $\partial \mathbf{H}^2_{\mathbb{C}}$, not contained in the boundary of a complex line. There exists a unique Lagrangian reflection fixing p_1 and swapping p_2 and p_3.*

Proof. Let $\mathbf{c}_k$ be a polar vector for C_k normalized so that $\langle \mathbf{c}_k, \mathbf{c}_k \rangle = 1$. Let $\mathbf{p}_1$ be a lift of p_1. Rescaling if necessary, we may assume that both $a = \langle \mathbf{p}_1, \mathbf{c}_2 \rangle$ and $b = \langle \mathbf{c}_1, \mathbf{c}_2 \rangle$ are real (in fact $\langle \mathbf{c}_1, \mathbf{c}_2 \rangle$ is equal to $\sqrt{\varphi_{12}}$).

1. The Hermitian form admits in the basis $(\mathbf{p_1}, \mathbf{c}_1, \mathbf{c}_2)$ the matrix
$$H = \begin{bmatrix} 0 & 0 & a \\ 0 & 1 & b \\ a & b & 1 \end{bmatrix}$$
The Hermitian product b is non-zero since C_1 and C_2 are non-orthogonal. In this basis, any lift of a Lagrangian reflection fixing p_1 and preserving C_1 and C_2 must be diagonal. It follows after writing the isometry condition $M^*HM = H$ that there is only one such reflection, given in this basis by $\mathbf{m} \longrightarrow \overline{\mathbf{m}}$.

2. This time, we use the basis $(\mathbf{c}_1, \mathbf{c}_2, \mathbf{d})$, where $\mathbf{d}$ a vector orthogonal to $\mathbf{c}_1$ and $\mathbf{c}_2$ with norm $b^2 - 1$ (indeed, we are setting $\mathbf{d} = \mathbf{c_1} \boxtimes \mathbf{c_2}$, see Remark 3). The Hermitian form is given by the matrix
$$H = \begin{bmatrix} 1 & b & 0 \\ b & 1 & 0 \\ 0 & 0 & b^2 - 1 \end{bmatrix} \quad (|b| = 1 \text{ iff } C_1 \text{ and } C_2 \text{ are asymptotic})$$
We may choose the lifts of p_1 and p_2 as follows :
$$\mathbf{p}_1 = \begin{bmatrix} -b \\ 1 \\ e^{i\theta_1} \end{bmatrix} \text{ and } \mathbf{p_2} = \begin{bmatrix} 1 \\ -b \\ e^{i\theta_2} \end{bmatrix} \text{ with } \theta_i \in \mathbb{R}.$$
The fact that I_Q exchanges C_1 and C_2 implies that any matrix for I_Q has the form
$$\begin{bmatrix} 0 & \alpha & 0 \\ \beta & 0 & 0 \\ 0 & 0 & \gamma \end{bmatrix}.$$
Writing the isometry condition and the fact that $I_Q(p_1) = p_2$, provides relations determining α, β and γ. The result follows.

3. We may choose lifts $\mathbf{m}$ and $\mathbf{n}$ of m and n such that $\langle \mathbf{m}, \mathbf{n} \rangle = 1$ and a unit vector $\mathbf{c}$ polar to the complex line containing m and n. In the basis $(\mathbf{m}, \mathbf{c}, \mathbf{n})$, where the Hermitian form has matrix J, the complex line C_1 is polar to some vector $\mathbf{c}_1 = \begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix}^T$. It is a direct computation to check that a Lagrangian reflection swapping m and n and preserving C_1 lifts to the matrix below. Hence, it exists and is unique.
$$\begin{bmatrix} 0 & 0 & \alpha/\bar{\gamma} \\ 0 & \beta/\bar{\beta} & 0 \\ \gamma/\bar{\alpha} & 0 & 0 \end{bmatrix}$$

4. This part of the proposition is classical, and we refer the reader to [8] (Lemma 7.17, page 215).

□

Proof of Proposition 7. 1. Let h_1 and h_2 be two isometries having the requested properties. Then $h_2^{-1} \circ h_1$ preserves both C_i and C_j, and fixes p_i. According to the lemma 2, this implies that h_2 and h_1 are equal. This proves the uniqueness. To prove the existence part, we apply Proposition 8, *(1)* and *(2)*.

- There exists a unique Lagrangian reflection I_2 preserving C_i and C_j and fixing p_i (this follows from Proposition 8 *(1)*).
- There exists a unique Lagrangian reflection I_1 swapping C_i and C_j, and exchanging p_i and $I_2(p_j)$. This Proposition 8 *(2)*, which may be applied since $I_2(p_j)$ belongs to C_j.

The isometry $E_{ij} = I_1 \circ I_2$ has the requested properties.

2. The uniqueness is proved in the same way as for *1*. To prove the existence, we apply Proposition 8 *(3)* and *(4)*.

- The two points $R_{C_3}(p_1)$ and $R_{C_2}(p_1)$ do not belong to ∂C_1 since the three complex lines are non-asymptotic. Thus, there exists a unique Lagrangian reflection I_3 preserving C_1 and swapping $R_{C_3}(p_1)$ and $R_{C_2}(p_1)$ (this follows from part *3* of Proposition 8). Note that I_3 does dot fix p_1.
- The three points p_1, $I_3(p_1)$ and $R_{C_2}(p_1)$ do not belong to a common complex line, for else C_1 and C_2 would be asymptotic. Thus we may apply the fourth part of Proposition 8 to obtain a (unique) Lagrangian reflection I_4 fixing $R_{C_2}(p_1)$, and swapping p_1 and $I_3(p_1)$.

The isometry $I_4 \circ I_3$ has the requested properties (note that since I_4 swaps p_1 and $I_3(p_1)$ which both belong to C_1, it preserves C_1).

□

4.2 Standard position of a triple of flags and elementary isometries

Definition 11. - Let (C_1, p_1) be a flag and C_2 be a complex line. We will say that they are in *generic position* if p_1 does not belong to C_2, and if C_1 and C_2 are distinct and non-orthogonal.

- We say that (C_1, p_1) and C_2 are in *standard position* if p_1, C_1 and C_2 are respectively represented by the following vectors:

$$\mathbf{p}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix}a\\\sqrt{2}\\1\end{bmatrix} \text{ for } a \in (-1, +\infty).$$

Remark 5. If (C_1, p_1) and p_2 are in standard position, then the $\mathbb{R}$-plane provided by Proposition 8 *(1)* is $\mathbf{H}^2_{\mathbb{R}}$. The inversion in that plane is associated to the identity matrix. The condition on C_2 is equivalent to saying that $R_{C_2}(p_1)$ is represented by the vector $\begin{bmatrix}-1 & \sqrt{2} & 1\end{bmatrix}^T$. The motivation for this definition is the following proposition:

Proposition 9. *Let (C_1, p_1) be a flag and C_2 be a complex line in generic position.*

- *There exists a unique pair in standard position which is isometric to $((C_1, p_1), C_2)$.*
- *The parameter a is given by $\varphi(C_1, C_2) = (1 + a)^{-1}$*

Proof. Since $PU(2, 1)$ acts transitively on the set of flags of $\mathbf{H}^2_{\mathbb{C}}$, we can assume that $\mathbf{p}_1$ and $\mathbf{c}_1$ are in standard position. The isometries g in PU(2,1) stabilizing the standard flag admit lifts to SU(2,1) of the following form :

$$\mathbf{g} = \begin{bmatrix} \lambda & 0 & it\lambda \\ 0 & \overline{\lambda}/\lambda & 0 \\ 0 & 0 & 1/\overline{\lambda} \end{bmatrix} \text{ with } \lambda \in \mathbb{C} \setminus \{0\} \text{ and } t \in \mathbb{R}.$$

Note that λ is well-defined up to multiplication by a cubic root of 1. Now, a generic polar vector for C_2 and its image by g are given by

$$\mathbf{c}_2 = \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \text{ and } \mathbf{g}\mathbf{c}_2 \sim \begin{bmatrix} |\lambda|^2(a+it) \\ \overline{\lambda}^2 b/\lambda \\ 1 \end{bmatrix} \text{ with } |b|^2 + 2\mathrm{Re}\,(a) > 0.$$

The assumption that $\mathbf{c}_1$ and $\mathbf{c}_2$ are not orthogonal, implies that $b \neq 0$. This means that there is only one isometry which stabilizes the standard flag and maps $\mathbf{c}_2$ in standard position. Namely, we have to set $t = -\mathrm{Im}\,(a)$ and solve $\overline{\lambda}^2 b = \sqrt{2}\lambda$. This equation has three solutions in λ which represent the same element in $PU(2,1)$. The value of $\varphi(C_1, C_2)$ is given by a straightforward computation. ☐

Remark 6. Given three flags (C_1, p_1), (C_2, p_2), (C_3, p_3), we can decide to put (C_1, p_1) and C_2 in standard position. However, we could have chosen (C_1, p_1) and C_3 or (C_2, p_2) and C_1. All these configurations can be obtained one from the other by applying elementary isometries to the configuration.

Proposition 10. *Let (C_1, p_1), (C_2, p_2), (C_3, p_3) be a triple of flags in generic position and $\Theta : \mathbb{C} \to \mathbb{C}$ be the map defined by $\Theta(\rho e^{i\theta}) = \rho e^{i\theta/3}$ for $\rho \in [0, +\infty)$ and $\theta \in (-\pi, \pi]$. Assume that (C_1, p_1) and C_2 are in standard position.*

1. *The transfer isometry T^1_{23} is given by its lift to SU(2,1):*

$$\mathbf{T}^1_{23} = \begin{bmatrix} \mu & 0 & it\mu \\ 0 & \overline{\mu}/\mu & 0 \\ 0 & 0 & 1/\overline{\mu} \end{bmatrix} \text{ where } \mu = \Theta(\frac{\delta^1_{23}\varphi_{13}}{\Phi_{123}}) \text{ and } t = \mathrm{Im}\left(\frac{2\delta^1_{23}(\varphi_{23} - \Phi_{132})}{\varphi_{12}\varphi_{23}}\right)$$

2. *The exchange isometry E_{12} is given by its lift to SU(2,1):*

$$\mathbf{E_{12}} = \begin{bmatrix} \frac{\lambda(z - \overline{z} - |z|^2)}{4|z(z-1)|^2} & \frac{\sqrt{2}\overline{z}\lambda(z - \overline{z} - |z|^2)}{4|z(z-1)|^2} + \frac{\lambda}{\sqrt{2}(z-1)} & \frac{\lambda}{1-z} + \frac{\lambda(z - \overline{z} - |z|^2)^2}{4|z(z-1)|^2} \\ \frac{\overline{\lambda}}{\sqrt{2}\lambda(\overline{z}-1)} & \frac{\overline{\lambda}}{\lambda(\overline{z}-1)} & \frac{\overline{\lambda}(|z|^2 - z - \overline{z})}{\lambda(\overline{z}-1)\sqrt{(2)}} \\ \frac{1}{\overline{\lambda}} & \frac{\sqrt{2}\overline{z}}{\overline{\lambda}} & \frac{-|z|^2 + z - \overline{z}}{\overline{\lambda}} \end{bmatrix}$$

where $z = 1/\overline{m_{12}}$ and $\lambda = 2\Theta(z(z-1))$.

Proof. 1. Suppose that the triple of flags (C_1, p_1), (C_2, p_2) and (C_3, p_3) is in generic position as it is specified in the proposition, and suppose moreover that (C_1, p_1) and C_2 are in standard position. We can choose polar vectors $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ and representatives $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ such that

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 1/\varphi_{12} - 1 \\ \sqrt{2} \\ 1 \end{bmatrix}.$$

The matrix we are interested in stabilizes C_1 and p_1 and sends C_3 to a standard complex line with polar vector

$$\mathbf{c}'_3 = \begin{bmatrix} 1/\varphi_{13} - 1 \\ \sqrt{2} \\ 1 \end{bmatrix}.$$

Call $\mathbf{g}$ the inverse of the expected matrix, and compute the image of $\mathbf{c}'_3$ by $\mathbf{g}$:

$$\mathbf{g} = \begin{bmatrix} \lambda & 0 & it\lambda \\ 0 & \overline{\lambda}/\lambda & 0 \\ 0 & 0 & 1/\overline{\lambda} \end{bmatrix} \text{ and } \mathbf{c}_3 = \mathbf{g}\mathbf{c}'_3 = \begin{bmatrix} \lambda(1/\varphi_{13} - 1 + it) \\ \overline{\lambda}\sqrt{2}/\lambda \\ 1/\overline{\lambda} \end{bmatrix}.$$

Computing explicit expressions for $\delta^1_{23}, \varphi_{23}$ and Φ_{123} yields equations for λ and t. A direct computation gives the formulas of the proposition.

2. The second matrix is obtained in three steps: let (C_1, p_1) and (C_2, p_2) be two flags in generic position such that (C_1, p_1) and C_2 are in standard position. We look for a transformation which sends (C_2, p_2) and C_1 to a standard position. We find explicitly a first transformation which sends p_2 to p_1. Then we compose it with a Heisenberg translation (see Remark 7 below) which sends the image of C_2 by the first transformation to C_1. It remains to find a matrix as in the first part which stabilize the standard flag (C_1, p_1) and sends the image of C_1 by the two first transformations to a standard complex line. The composition of these matrices gives the formula of the proposition.

□

Remark 7. A Heisenberg translation is a unipotent parabolic isometry, given by the matrix

$$\begin{bmatrix} 1 & -\overline{w}\sqrt{2} & -|w|^2 + i\tau \\ 0 & 1 & w\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \text{ with } w \in \mathbb{C} \text{ and } \tau \in \mathbb{R}.$$

It is an element of the maximal unipotent subgroup of PU(2,1) fixing the vector $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, which is a copy of the Heisenberg group of dimension 3.

Remark 8. As a consequence of Remark 5 and Proposition 7, the associated exchange isometry admits a lift of the form $M_1 \circ \overline{Id} = M_1$ where M_1 is the matrix of a Lagrangian reflection. This shows that $\mathbf{E}_{12}\overline{\mathbf{E}_{12}} = 1$.

5 Decorated triangulations and representations of $\pi_{g,p}$

In this section, we will prove the Theorem 2 stated in the introduction.

We denote by $\pi_{g,p}$ be the fundamental group of $\Sigma_{g,p}$, a surface of genus g with p punctures $x_1, \dots x_p$, assuming $p > 0$. Recall that $\widehat{\Sigma}_{g,p}$ is the universal covering of $\Sigma_{g,p}$. Provided that the inequality $2 - 2g - p < 0$ is satisfied, the surface $\widehat{\Sigma}_{g,p}$ is homeomorphic to a topological disk and the punctures lift to a $\pi_{g,p}$-invariant subset X of the boundary of $\widehat{\Sigma}_{g,p}$.

Definition 12. We set

$$\mathfrak{R}_{g,p} = \{(\rho, F)\} / PU(2,1),$$

where ρ is a morphism from $\pi_{g,p}$ to PU(2,1) and F is a map from X to the set of flags in $\mathbf{H}^2_{\mathbb{C}}$ such that for any $x \in X$ and $g \in \pi_{g,p}$ one has $F(g.x) = \rho(g).F(x)$. The group $PU(2,1)$ acts on F by isometry on the target and acts on ρ by conjugation: this action corresponds to changing the base point in $\pi_{g,p}$.

For convenience, let us recall what will be called a triangulation of Σ, which is sometimes referred to as an ideal triangulation. A triangulation of Σ is an oriented finite 2-dimensional quasi-simplicial complex T with an homeomorphism h from the topological realization $|T|$ of T to Σ which maps vertices to punctures. By quasi-simplicial, we mean that two distinct triangles of T can share the same vertices. By a slight abuse of notation, we will nevertheless refer to a 2-simplex by its vertices.

Given a triangulation T of Σ, we can lift it to a triangulation of $\widehat{\Sigma}_{g,p}$. We thus obtain a triangulation of a disk with vertices on the boundary. Such a triangulation is isomorphic to the Farey triangulation which is a very nice and visual object (see [5]). We may think that any triangulated surface is a quotient of the Farey triangulation. Given a pair (ρ, F) and a triangle Δ of T, we can pick a lift of Δ which has three vertices x, y and z in X.

Definition 13. We will say that the pair (ρ, F) is *generic with respect to* T if for any lifts of triangles of T with vertices x, y and z, the triple of flags $(F(x), F(y), F(z))$ is generic in the sense of Definition 10. We denote by $\mathfrak{R}^T_{g,p}$ the subset of $\mathfrak{R}_{g,p}$ made of pairs which are generic with respect to T.

Definition 14. Let T be a triangulation of Σ. We denote by $\mathcal{X}(T)$ the set of triples (φ, Φ, δ) where :

- φ is an $\mathbb{R}_{>0}$-valued function defined on the set of unoriented edges of T,
- Φ and δ are $\mathbb{C}$-valued functions defined on the set of ordered faces of T.

From these data, we define auxiliary invariants in the following way. For any ordered face (i, j, k) of T, we set:

$$\begin{aligned}
\Delta_{ijk} &= 1 - \varphi_{ij} - \varphi_{jk} - \varphi_{ik} + \Phi_{ijk} + \Phi_{ikj} && (12)\\
m^k_{ij} &= \frac{1}{\Delta_{ijk}\varphi_{ik}\varphi_{jk}} \left[\varphi_{ik}\varphi_{jk}(\Phi_{ijk} - \varphi_{ij}) + \right. \\
&\qquad \left. \varphi_{ik}(\varphi_{ij}\varphi_{jk} - \Phi_{ijk})\delta^i_{kj} + \varphi_{jk}(\varphi_{ij}\varphi_{ik} - \Phi_{ijk})\overline{\delta^j_{ki}} + \Phi_{ijk}(1 - \varphi_{ij})\delta^i_{kj}\overline{\delta^j_{ki}}\right] && (13)
\end{aligned}$$

The maps φ, Φ and δ must satisfy the following relations for all ordered face (i, j, k) in T:

$$\begin{aligned}
|\Phi_{ijk}|^2 &= \varphi_{ij}\varphi_{jk}\varphi_{ki} && (14)\\
\Phi_{ijk} &= \Phi_{jki} = \overline{\Phi_{ikj}} && (15)\\
\Delta_{ijk} &< 0 && (16)\\
0 &= |\delta^i_{jk}|^2 (1 - \varphi_{ik}) + 2\mathrm{Re}\left[\delta^i_{jk}\left(\Phi_{ikj} - \varphi_{jk}\right)\right] + \varphi_{jk}\left(1 - \varphi_{ik}\right) && (17)
\end{aligned}$$

Moreover, for any edge (i, j) belonging to the faces (i, j, k) and (i, j, l), we impose the relation

$$m^k_{ij} = m^l_{ij}. \qquad (18)$$

We will call *decorated triangulation* the data of an ideal triangulation and three mappings φ, Φ and δ as above.

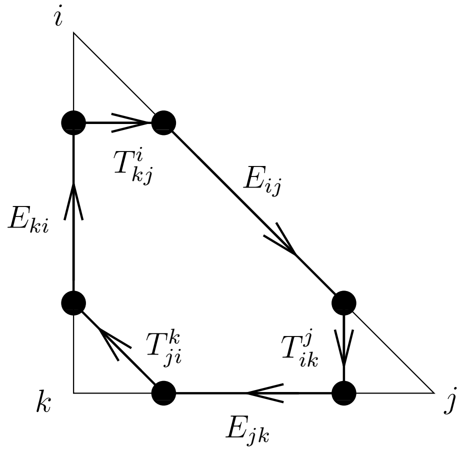

Figure 1: A hexagon associated to the triangle (x, y, z), and elementary matrices associated to its sides.

Before starting the proof of Theorem 2, let us give some useful constructions:

Definition 15. Let T be a triangulation of $\Sigma_{g,p}$. We will call *hexagonation* of T and denote by HT the subcomplex of T obtained by truncating all the corners in the triangles of T, as shown in Figure 1. The cells of HT consist in hexagons.

Let T be a decorated triangulation of $\Sigma_{g,p}$. We will define from these data a 1-cocycle A in $Z^1(HT,\mathrm{PU}(2,1))$. Let s be an oriented edge of HT. Associate to s an elementary matrix A_s as follows:

- If s lies in the edge of T connecting the vertex i to the vertex j we set $A_s = E_{ij}$.
- If s lie in the corner of the vertex i and connects the edges ij and ik of T, then we set $A_s = T^i_{jk}$.

Lemma 3. *Let T be a decorated triangulation of $\Sigma_{g,p}$. The mapping $s \longrightarrow A_s$ is a 1-cocycle of HT with values in PU(2,1).*

Proof. If (i, j, k) is a face of T and if $s_1 \cdots s_6$ are the sides of the associated hexagon, the product $\prod_{i=1}^{6} A_{s_i}$ corresponds to an isometry of $\mathbf{H}^2_{\mathbb{C}}$ stabilizing a flag and a complex line. Hence, it is the identity map of $\mathbf{H}^2_{\mathbb{C}}$ (see lemma 2). □

We now go to the proof of Theorem 2.

Proof of Theorem 2. We can finally prove the Theorem by describing two mappings inverse one of each other. For this purpose, fix a triangulation T of $\Sigma_{g,p}$.

First, we associate to a decoration of T a representation of $\pi_{g,p}$ in PU(2,1) and an equivariant map F. Assume that T is equipped with a decoration (φ, Φ, δ) and choose a vertex v of HT as base point for the fundamental group of $\Sigma_{g,p}$.

Any loop l of $\pi_1(\Sigma_{g,p}, v)$ is homotopic to a sequence $s = s_1, \ldots s_k$ of oriented edges of HT. One can associate to l the element of PU(2,1) corresponding to the product

$$A_{s_k} \cdots A_{s_1}.$$

Because of the cocycle condition given in lemme 3 above, this isometry does not depend on the choice of the simplicial path homotopic to l. This gives rise to a representation ρ of $\pi_1(\Sigma_{g,p}, v)$ into PU(2,1). Let us now construct the map F. Suppose that v lies on the edge ij of T, and at the corner i. The choice of base point gives naturally a preferred lift of i and j in $\widehat{\Sigma}_{g,p}$ that we denote by $\widehat{i}$ and $\widehat{j}$ respectively. We choose $F(\widehat{i})$ and $F(\widehat{j})$ such that they are in standard position. Next, any element x of X is parametrized by a path from v to a vertex of HT. We can suppose that this path γ is simplicial. In that way, we set $F(x) = A_\gamma^{-1}.F_0$ where F_0 is the standard flag given by the vectors $\mathbf{c}_1 = \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T, \mathbf{p}_1 = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$. One checks easily that this map F is equivariant and generic with respect to T and hence, the pair (ρ, F) gives an element of $\mathfrak{R}_{g,p}^T$.

Conversely, given a pair (ρ, F) generic with respect to T, we obtain an element of $\mathcal{X}(T)$ by the following construction. For all edges $[i,j]$ which lift to $[\widehat{i},\widehat{j}]$ we set $\varphi_{i,j} = \varphi(F(\widehat{i}), F(\widehat{j}))$ and for all triangles $[i,j,k]$ which lift to $[\widehat{i},\widehat{j},\widehat{k}]$ we define the Φ and δ invariants of i,j,k as being equal to the corresponding invariants of the triple $\left(F(\widehat{i}), F(\widehat{j}), F(\widehat{k})\right)$. These data fit by construction as an element of $\mathcal{X}(T)$. The two maps we have constructed are inverse one of the other. This ends the proof.

□

6 Solving the equations

The goal of this section is to show how to construct solutions of the equations involved in $\mathcal{X}(T)$ in a systematic way. The key lemma is the following:

Lemma 4. *Let m_{12}, m_{23}, m_{31} be three complex number different from 0 and 1. From these numbers, define $\varphi_{i,j} = |m_{ij}/(m_{ij}-1)|^2$ for all i,j. For any family $(\Phi_{ijk})_{i,j,k}$ of complex numbers satisfying the conditions*

$$\begin{aligned}
\Phi_{ijk} &= \Phi_{jki} = \overline{\Phi_{ikj}}\\
|\Phi_{ijk}| &= \sqrt{\varphi_{ij}\varphi_{jk}\varphi_{kj}}\\
\Delta_{ijk} &= 1 - \varphi_{ij} - \varphi_{jk} - \varphi_{ki} + \Phi_{ijk} + \overline{\Phi_{ijk}} < 0
\end{aligned}$$

the following set of equations

$$\begin{aligned}
\delta^i_{jk}\delta^i_{kj} &= \varphi_{jk}\\
m^k_{ij} &= \frac{1}{\Delta_{ijk}\varphi_{ik}\varphi_{jk}}(\varphi_{ik}\varphi_{jk}(\Phi_{ijk} - \varphi_{ij}) +\\
&\quad \varphi_{ik}(\varphi_{ij}\varphi_{jk} - \Phi_{ijk})\delta^i_{kj} + \varphi_{jk}(\varphi_{ij}\varphi_{ik} - \Phi_{ijk})\overline{\delta^j_{ki}} + \Phi_{ijk}(1-\varphi_{ij})\delta^i_{kj}\overline{\delta^j_{ki}})\\
0 &= |\delta^i_{jk}|^2(1-\varphi_{ik}) + 2\mathrm{Re}\,[\delta^i_{jk}(\Phi_{ikj} - \varphi_{jk})] + \varphi_{jk}(1-\varphi_{ik})
\end{aligned}$$

have two distinct solutions in the variables δ^i_{jk} provided that $\varphi_{ij} \neq 1$ for all i and j in $\{1,2,3\}$.

Proof. Geometrically, the lemma has the following interpretation: let C_1, C_2, C_3 be three complex lines in generic position. Their position is parametrized by the invariants φ_{ij} and Φ_{ijk}. The hypothesis on these invariants means that any two complex lines are neither orthogonal nor asymptotic.

The invariant m_{ij} specifies a Lagrangian reflexion I_{ij} swapping C_i and C_j in the following way: let p_1 and p_2 be two points in ∂C_1 and ∂C_2 respectively such that $m[(C_1, p_1), (C_2, p_2)] = m_{12}$. Then the second part of lemma 8 tells us that there is a unique Lagrangian involution swapping C_1 and C_2 and sending p_1 on p_2. This involution depends on p_1 and p_2 only through the data of m_{12}. In some sense, the involution I_{12} is the geometric realization of the invariant m_{12}.

A solution of the equations is equivalent to a triple of points p_1, p_2, p_3 lying respectively in $\partial C_1, \partial C_2$ and ∂C_3 such that for all i, j, $I_{ij} p_i = p_j$. Fixing a reference complex line, say C_1, we see that a solution of the equations is given by a fixed point of the product $I_{31} I_{23} I_{12}$. This product is an anti-holomorphic isometry of C_1 preserving the boundary, hence it has two distinct fixed points on this circle. This proves the lemma.

If some of the φ_{ij} are equal to one, then the corresponding complex lines are asymptotic. The same argument as above applies but the points p_i may lie at the intersection of two complex lines which is not allowed in our settings. Hence, there are less than 2 admissible solutions but there are still some degenerate ones.

□

To obtain solutions of all the equations involved by a decoration, we apply Lemma 4 for each triangle of a triangulation at the same time. More precisely, let T be an ideal triangulation of $\hat{\Sigma}_{g,p}$. We define a decoration space of T which is related to $\mathcal{X}(T)$ but which is somewhat simpler: let $\mathcal{M}(T)$ be the set of triple (φ, Φ, m) where:

- φ and m are functions defined on the set of oriented edges satisfying the following relations:
$$\varphi_{ij} = \left| \frac{m_{ij}}{m_{ij} - 1} \right|^2 \text{ and } m_{ji} = \overline{m_{ij}}.$$
Note that the φ invariant is redundant as it is a function of m but we keep it for the coherence of the notation.
- Φ is a $\mathbb{C}$-valued function defined on ordered faces of T satisfying the following equations for all ordered faces (i, j, k):
$$\Phi_{ijk} = \Phi_{jki} = \overline{\Phi_{ikj}} \text{ and } \Delta_{ijk} < 0$$

We denote by $\mathcal{X}^{nd}(T)$ (resp. $\mathcal{M}^{nd}(T)$) the non-degenerate part of $\mathcal{X}(T)$ (resp. $\mathcal{M}(T)$) by which we mean the open set of triples (φ, Φ, δ) such that $\varphi_{ij} \neq 1$ for all i and j (resp. the triples (φ, Φ, m) such that $\varphi_{ij} \neq 1$ for all i, j).

The following proposition is a direct consequence of the preceding Lemma 4.

Proposition 11. *The natural map $\mathcal{X}^{nd}(T) \to \mathcal{M}^{nd}(T)$ sending (φ, Φ, δ) to (φ, Φ, m) is a covering of order 2^N where N is equal to the number of triangles in T.*

This proposition explains that we can solve the equations in a simple way: we fix arbitrarily the φ and m invariants, and then solve (with a computer) the remaining equations in δ. The important point given by the proposition is that we are sure to obtain 2^N solutions in the non-degenerate case. The simple structure of the map from $\mathcal{X}^{nd}(T)$ to $\mathcal{M}^{nd}(T)$ should in principle allow to describe precisely the representation space.

7 Controlling the holonomy of the cusps

7.1 The general case

Consider a pair $(\rho, F) \in \mathfrak{R}_{g,p}$ and denote as usual by c_i the curve in $\Sigma_{g,p}$ enclosing x_i. Since $\rho(c_i)$ stabilizes a flag $F_i = (C_i, p_i)$, it might be either

- loxodromic, in which case its second fixed point belongs to C_i,
- parabolic, in which case p_i is its unique fixed point,
- a complex reflection, in which case its restriction to C_i is the identity.

We wish to determine the type of $\rho(c_i)$ in terms of the invariants φ, Φ and δ. The loop c_i encloses the vertex point x_i. It may be written $c_i = \gamma\nu\gamma^{-1}$, where γ is a path connecting the base point to one of the vertices of the hexagonation which is adjacent to the point x_i, and ν is a loop around x_i which is composed of a succession of edges of the hexagonation connecting two edges of the triangulation. As a consequence $\rho(c_i)$, may be written $M_\gamma N M_\gamma^{-1}$, where N is a product of elementary matrices which are all of transfer type (see Proposition 10). Write

$$N = \mathbf{T}_k \dots \mathbf{T}_j \dots \mathbf{T}_1,$$

where $\mathbf{T}_j$ is a matrix of transfer type:

$$\mathbf{T}_j = \begin{bmatrix} \mu_j & 0 & it_j\mu_j \\ 0 & \bar{\mu}_j/\mu_j & 0 \\ 0 & 0 & 1/\bar{\mu}_j \end{bmatrix},$$

and the μ_j's and t_j's are written in terms of the invariants φ, Φ and δ as in Proposition 10. Computing the product, we obtain

$$N = \begin{bmatrix} \mu & 0 & K \\ 0 & \bar{\mu}/\mu & 0 \\ 0 & 0 & 1/\bar{\mu} \end{bmatrix},$$

where $\mu = \prod \mu_j$, and

$$K = i\sum_{j=1}^{k} t_j \frac{\prod_{l=j}^{k} \mu_l}{\prod_{l=1}^{j-1} \bar{\mu}_l}.$$

We obtain thus that

- N is loxodromic if and only if $|\mu| \neq 1$,
- N is parabolic if and only if $|\mu| = 1$ and $K \neq 0$,
- N is a complex reflection if and only if $|\mu| = 1$ and $K = 0$.

7.2 Type preserving representations of the 1-punctured torus

In this section, we focus on the special case of $\Sigma_{1,1}$, the 1-punctured torus. We first summarize the existing results about this case. We denote the fundamental group of $\Sigma_{1,1}$ by

$$\pi_{1,1} = \langle a, b, c \,|\, [a, b] \cdot c = 1 \rangle.$$

Recall that a representation of $\pi_{1,1}$ is said to be *type preserving* if and only if $\rho(c)$ is a parabolic isometry. Note that there are two main types of parabolic isometries (see [8, 17] for more details):

1. screw parabolic isometries. These parabolic elements preserve a complex line, and thus a flag.
2. horizontal parabolic isometries, which preserves an $\mathbb{R}$-plane containing their fixed point. These isometries are also called non-vertical Heisenberg translations. No such parabolic isometry appears within the frame of the present work.

As a consequence, type-preserving representations of $\pi_{1,1}$ fall into two types, according to whether $\rho(c)$ is screw parabolic or horizontal parabolic.

1. If $\rho(c)$ preserves a complex line, then it is in the frame of this work. In this case, all the examples known of a discrete, faithful and type preserving representation are obtained by passing to an index 6 subgroup in a discrete, faithful and type preserving representation of the modular group PSL(2,$\mathbb{Z}$). The latter representations have all been described by Falbel and Parker in [2]. This family of examples consists up to PU(2,1) of 6 topological components, 4 of which are points, and the two other are segments.
2. If $\rho(c)$ does not preserve a complex line, then it is a consequence of [18] that there exists a unique triple of Lagrangian reflections (I_1, I_2, I_3) such that $\rho(a) = I_1 I_2$ and $\rho(b) = I_3 I_2$. In [17, 19], all these type preserving representations of $\pi_{1,1}$ are described, and, among them, a 3-dimensional family of discrete and faithful representations is identified.

We will now give necessary and sufficient conditions for $\rho(c)$ to be parabolic, in the case it preserves a flag.

A triangulation of a 1-punctured torus is made of two triangles, which we call α and β as on figure 2. We label the vertices as on figure 2. The decoration of this triangulation is given by :

- Triangle α (p_1, p_2, p_3): φ_{12}, φ_{23}, φ_{31}, Φ_{123}, δ^1_{23}, δ^2_{31} and δ^3_{12}.
- Triangle β (p_1, p_2, p_4) : φ_{12}, φ_{24}, φ_{41}, Φ_{124}, δ^1_{24}, δ^2_{41} and δ^4_{12}.

Note that because of the identification of the opposite sides of the square, the following relations hold:

$$\varphi_{23} = \varphi_{14} \text{ and } \varphi_{13} = \varphi_{24}.$$

We choose as a basepoint the vertex of the hexagonation marked by B on figure 2. Let us call ν^i_{jk} the oriented edge of the hexagonation turning around the vertex p_i from the edge $p_i p_j$ edge to the edge $p_i p_k$. As an example, ν^1_{24} is the oriented segment starting from the point B (see figure 2) and connecting the diagonal to the vertical side $p_1 p_4$. The homotopy class c is represented by the following sequence of edges, $\nu^1_{23}\nu^4_{21}\nu^2_{31}\nu^2_{14}\nu^3_{12}\nu^1_{42}$, to which correspond the product of transfer type matrices $\mathbf{T} = \mathbf{T}^1_{42}\mathbf{T}^3_{12}\mathbf{T}^2_{14}\mathbf{T}^2_{31}\mathbf{T}^4_{21}\mathbf{T}^1_{23}$. Denote by μ^i_{jk} and t^i_{jk} the two parameters in the matrix $\mathbf{T}^i_{jk}$ given by Proposition 10. The matrix $\mathbf{T}$ is upper triangular, and according to Proposition 10, its top left coefficient is

$$\mu = \Theta\left(\frac{\delta^1_{23}\varphi_{13}}{\Phi_{123}}\frac{\delta^4_{21}\varphi_{14}}{\Phi_{421}}\frac{\delta^2_{31}\varphi_{12}}{\Phi_{231}}\frac{\delta^2_{14}\varphi_{24}}{\Phi_{214}}\frac{\delta^3_{12}\varphi_{32}}{\Phi_{312}}\frac{\delta^1_{42}\varphi_{12}}{\Phi_{142}}\right). \tag{19}$$

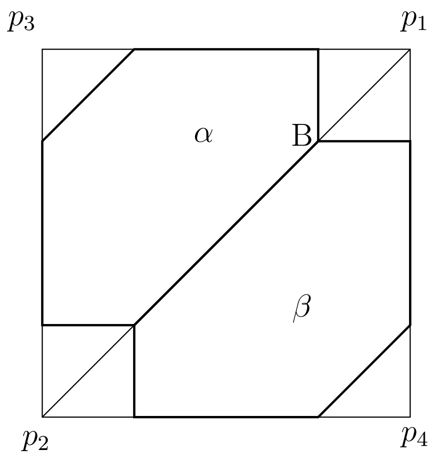

Figure 2: Ideal triangulation and hexagonation of a 1-punctured torus. The opposite sides of the square are identified.

We simplify this relation using the relations between the invariants ($\varphi_{ij} = \varphi_{ji}$, $|\Phi_{ijk}|^2 = \varphi_{ij}\varphi_{jk}\varphi_{ki}$, and $\delta^i_{jk}\delta^i_{ki} = \varphi_{ij}$). This yields:

$$|\mu|^2 = \frac{|\delta^1_{23}\delta^2_{31}\delta^3_{12}|^2}{|\delta^1_{42}\delta^2_{14}\delta^4_{12}|^2}.$$

We obtain as a direct consequence the following

Proposition 12. *Let (φ, Φ, δ) be a decorated triangulation of the punctured torus. The holonomy of a loop around the puncture is parabolic or a complex reflexion if and only if*

$$|\delta^1_{23}\delta^2_{31}\delta^3_{12}| = |\delta^1_{42}\delta^2_{14}\delta^4_{12}| \tag{20}$$

Moreover, the representation associated to the decoration is type preserving if and only if the relation (20) is satisfied and the following relation holds

$$\begin{aligned} 0 \;\neq\; & \mu^1_{42}\mu^3_{12}\mu^2_{14}\mu^2_{31}\mu^4_{21}\mu^1_{23}\, t^1_{23} + \frac{\mu^1_{42}\mu^3_{12}\mu^2_{14}\mu^2_{31}\mu^4_{21}}{\mu^1_{23}}\, t^4_{21} + \frac{\mu^1_{42}\mu^3_{12}\mu^2_{14}\mu^2_{31}}{\mu^4_{21}\mu^1_{23}}\, t^2_{31} \\ & + \frac{\mu^1_{42}\mu^3_{12}\mu^2_{14}}{\mu^2_{31}\mu^4_{21}\mu^1_{23}}\, t^2_{14} + \frac{\mu^1_{42}\mu^3_{12}}{\mu^2_{14}\mu^2_{31}\mu^4_{21}\mu^1_{23}}\, t^3_{12} + \frac{\mu^1_{42}}{\mu^3_{12}\mu^2_{14}\mu^2_{31}\mu^4_{21}\mu^1_{23}}\, t^1_{42} \end{aligned} \tag{21}$$

The relation (21) is just an explicit version of $K \neq 0$, with K as in the previous section.

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