1er Cours Bonn

Curves and vector bundles in p-adic Hodge theory

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Joint work with J.M.-Fontaine

Prepublication (under revision): "Curves and vector bundles in p-adic Hodge theory"

Review articles without any proof { Publications: "Factorization of analytic functions in mixed characteristic"
"Vector bundles and p-adic Galois representations"

→ http://www-irma.u-strasbg.fr/~fargues

Holomorphic functions in a punctured disk after Lazard (Lazard, IHES)

F complete non-archimedean field with car. p residue field. $|\cdot|$ = absolute value, v = valuation, $|\cdot| = p^{-v(\cdot)}$

Consider $\mathbb{D}^* = \{0 < |z| < 1\} \subset \mathbb{A}^1_F$ as a rigid analytic space

~~[illegible]~~

Set $\left[B := \mathcal{O}(\mathbb{D}^*) = \left\{ \underset{f}{\underbrace{\sum_{n\in\mathbb{Z}} a_n z^n}} \,/\, a_n \in F,\ \forall 0<\rho<1,\ \lim_{|n|\to+\infty} |a_n|\rho^n = 0 \right\} \right]$

$\Updownarrow$ (exercise)

$$\begin{cases} \liminf_{n\to+\infty} \dfrac{v(a_n)}{n} \geq 0 \\ \lim_{n\to+\infty} \dfrac{v(a_{-n})}{n} = +\infty \end{cases}$$

For $\rho \in]0,1[$ set $|f|_\rho = \sup_{n\in\mathbb{Z}} \{|a_n|\rho^n\}$

If $\rho = p^{-r}$ with $r>0$, $|f|_\rho = p^{-v_r(f)}$

$$\left[v_r(f) = \inf_{n\in\mathbb{Z}} \{v(a_n) + nr\} \right]$$

$|\cdot|_\rho$ = Gauss supremum norm on the annulus $\{|z|=\rho\}$

= multiplicative norm

v_r = valuation ← (translates into)

$\left(B, \left(|\cdot|_\rho\right)_{\rho\in]0,1[}\right)$ = Frechet algebra

Topology = Uniform C.V. on affinoid domains of $\mathbb{D}^*$
(Compact subsets of the associated Berkovich space if you want)

* Other spaces of holomorphic functions:

$$B^b = \{f \in B \mid \exists N, z^N f \text{ bounded holomorphic on } \mathbb{D}\}$$

$$= \left\{\sum_{n>>-\infty} a_n z^n \;/\; \exists C, \forall n, |a_n| \leq C\right\} \underset{\text{dense}}{\subset} B$$

↑ f meromorphic at 0

⇓

[Can find back B from B^b via Completion]

If $f \in B$, $\|f\|_1 = \lim_{\rho \to 1} \|f\|_\rho \in [0, +\infty]$ exists

Set $B^+ := \{ f \in B \mid \|f\|_1 \leq 1 \}$ = Closed sub $\mathcal{O}_F$-algebra of B

dense $\bigcup = \left\{ \sum_{n \in \mathbb{Z}} a_n z^n \;\middle|\; |a_n| \leq 1, \lim_{n \to +\infty} \frac{v(a_{-n})}{n} = +\infty \right\}$

$\Big\downarrow \; B^{b,+} := B^b \cap B^+ = \left\{ \sum_{n \gg -\infty} a_n z^n \;\middle|\; \forall n, |a_n| \leq 1 \right\}$

Can find back B^+ from $B^{b,+}$ via completion.

$*$ $I \subset]0,1[$ compact interval

$B_I :=$ holomorphic functions on $\{ |z| \in I \}$

$=$ Banach algebra

$B = \varprojlim_{I \subset]0,1[} B_I$ as a Frechet algebra.

Maximum modulus principle

$\Rightarrow$ the topology induced by the norms $(|\cdot|_\rho)_{\rho \in I}$ equals the topology defined by the norm $\sup\{ |\cdot|_{\rho_1}, |\cdot|_{\rho_2} \}$ if $I = [\rho_1, \rho_2]$

Zeros / growth of holomorphic functions

Recall / $\mathbb{C}$: f holomorphic on $\{|z|<1\}$ (to simplify)

$0<R<1$ $$M(R) = \sup_{|z|=R} |f(z)|$$

Hadamard: $\left[R \longmapsto -\log M(R) \text{ is concave of } \log R \right]$

Jensen formula:

$$-\log|f(0)| = \sum_{i=1}^{n} \left(-\log|f(a_i)|\right) - nR - \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta$$

if $f(0)\neq 0$, f has no zeros on $\{|z|=R\}$ and $(a_1,\ldots,a_n)$ = zeros on $\{|z|<R\}$

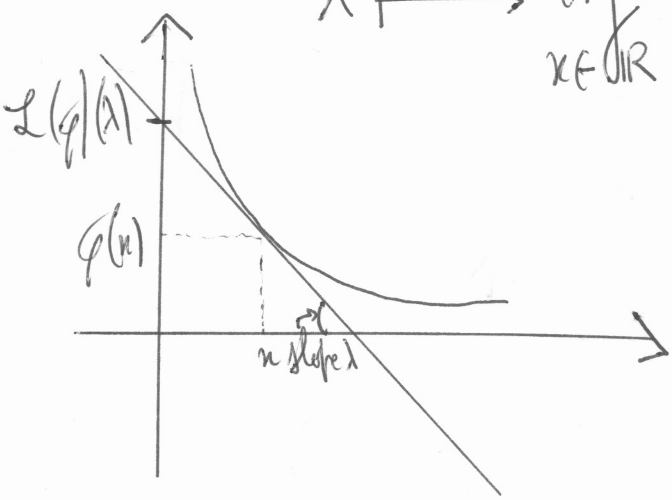
$$\Rightarrow \left[-\log|f(0)| \geq \sum_{i=1}^{n} \left(-\log|a_i|\right) - nR - \log M(R) \right]$$

Exact formula in the p-adic setting

Legendre transform

* $\varphi : \mathbb{R} \to]0,+\infty]$ convexe decreasing function

$$\left[\begin{array}{l}\mathcal{L}(\varphi) : \,]0,+\infty] \longrightarrow [-\infty,+\infty[\qquad \underline{\text{Concave}} \\ \qquad\qquad \lambda \longmapsto \inf\limits_{x\in\mathbb{R}} \{\varphi(x)+\lambda x\}\end{array}\right]$$



Slope = opposite of derivative
(want slopes = valuations of roots)

find back φ via inverse Legendre:

$$\left[\varphi(x) = \underbrace{\sup_{\lambda\in\mathbb{R}}\{\mathcal{L}(\varphi)(\lambda)-\lambda x\}}_{\mathcal{L}^{-1}(\mathcal{L}(\varphi))(x)}\right]$$

If $(\varphi_1 * \varphi_2)(x) = \inf\limits_{a+b=x}\{\varphi_1(a)+\varphi_2(b)\}$ for φ_1, φ_2 as before

then $\mathcal{L}(\varphi_1 * \varphi_2) = \mathcal{L}(\varphi_1) + \mathcal{L}(\varphi_2)$

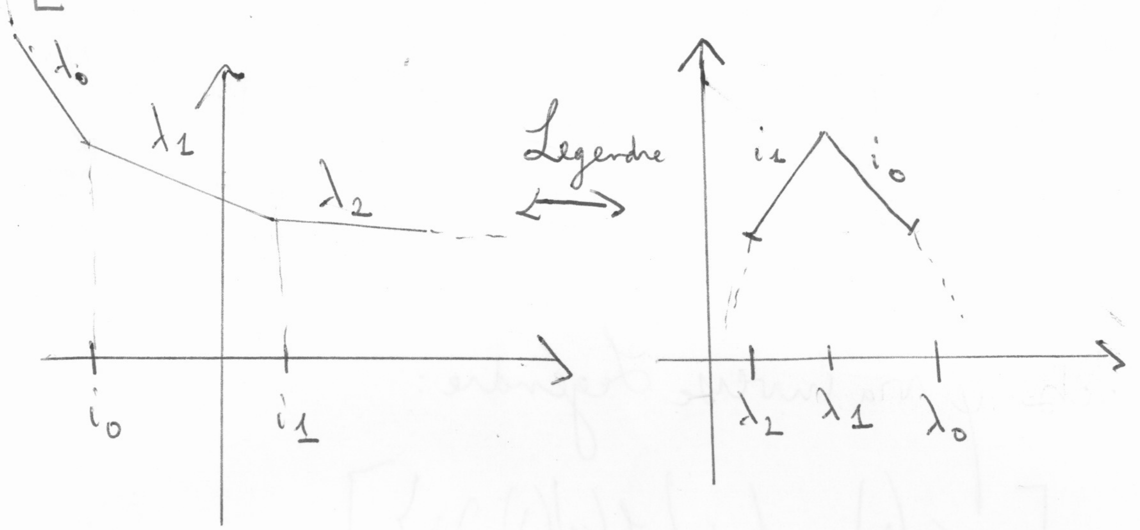
$\Big($ $(\mathbb{R},+,\times) \xrightarrow{\text{tropicalization}} (\mathbb{R},\inf,+)$

Laplace transform $\rightsquigarrow$ Legendre transform

$*$ $\rightsquigarrow$ tropical $*$ (just defined) $\Big)$

If φ = polygon (i.e. piecewise affine) then $\mathcal{L}(\varphi)$ = polygon

[<u>duality</u>: slopes $\underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}}$ x-coordinnate of breakpoints]



$$\mathcal{L}(\varphi_1 * \varphi_2) = \mathcal{L}(\varphi_1) + \mathcal{L}(\varphi_2) \Longrightarrow$$ slopes of $\varphi_1 * \varphi_2$ = concatenation of the slopes of φ_1 and φ_2

* If $f \in B$, $f = \sum_{n \in \mathbb{Z}} a_n z^n$ $\quad$ $\mathrm{Newt}(f)$ = Convex decreasing hull of $\{(n, v(a_n))\}_{n \in \mathbb{Z}}$

= polygon wt. integral x-coordinate breakpoints

~~Slopes of~~ $r \longmapsto v_r(f)$ is the Legendre transform of $\mathrm{Newt}(f)$

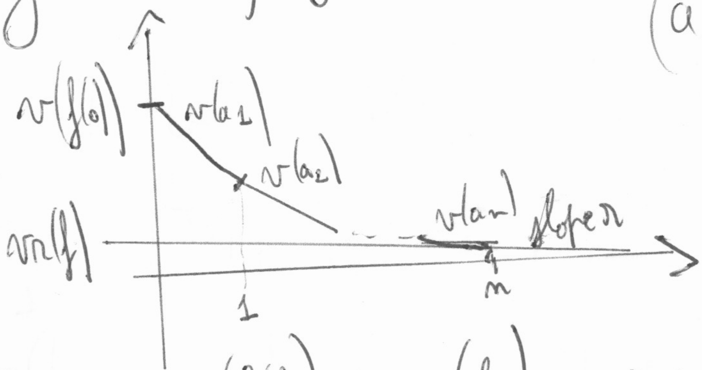
= polygon wt. integral slopes and x-coordinate breakpoints are the slopes of $\mathrm{Newt}(f)$

Slopes of $\mathrm{Newt}(f)$ = valuations of the zeros of f (with multiplicities)

p-adic Jensen formula

Ex: $f \in \mathcal{O}(\mathbb{D})$, $f(0) \neq 0$.

$(a_1, \dots, a_n)$ = zeros of f in $\{v(z) \geq r\}$



Jensen: $v(f(0)) = v_r(f) - nr + \sum_{i=1}^{n} v(a_i)$ exact formula.

$*$ $B^b = \{ f \in B \mid \exists A \in \mathbb{R},\ \mathrm{Newt}(f)_{|]-\infty, A]} = +\infty$ and $\mathrm{Newt}(f)$ is bounded below $\}$

$B^+ = \{ f \in B \mid \mathrm{Newt}(f) \subset$ upper half plane $\}$

<u>Weierstraß products</u>

$*$ $I \subset]0,1[$ compact interval $\quad \mathbb{D}_I = \{ |z| \in I \}$

Suppose $I = [\rho_1, \rho_2]$ with $\rho_1, \rho_2 \in \sqrt{|F^\times|}$

then B_I is a P.I.D. and $|\mathbb{D}_I| = \mathrm{Spm}(B_I)$.

If F is alg. closed then any element of B_I can be written uniquely $\underset{\substack{\cap\\ B_I^\times}}{\text{unit}} \times \underbrace{\prod_{i=1}^{n} (z - a_i)}_{\text{irreducible element of } B_I}$

$a_i \in F$, $|a_i| \in I$.

* Set $\mathrm{Div}^+(\mathbb{D}^*) = \left\{ \sum_{x \in |\mathbb{D}^*|} m_x [x] \;\middle|\; \begin{array}{l} m_x \in \mathbb{N},\ \forall I \subset]0,1[\text{ Compact} \\ \{x \mid m_x \neq 0 \text{ and } |x| \in I\} \text{ is finite} \end{array} \right\}$

$=$ Monoïd of $\underbrace{\text{locally}}_{\text{on the associated Berkovich space}}$ finite effective divisors on $\mathbb{D}^*$

<u>Question</u>: for each $D \in \mathrm{Div}^+(\mathbb{D}^*)$ does there exist $f \in B$ s.t. $\mathrm{div}(f) = D$? $\left(\Leftrightarrow \mathrm{Pic}(\mathbb{D}^*) \overset{?}{=} 0 \right)$

* If $\mathrm{supp}(D)$ is finite this is trivial. Suppose thus $\mathrm{supp}(D)$ is infinite.
* F alg. closed (discrete valuation case is easier but this is not the case we are interested in)

Suppose first $\exists \rho_0 \in]0,1[$ s.t. $\mathrm{Supp}(D) \subset \{0 < |z| \leq \rho_0\}$

$\Rightarrow$ Can write $D = \sum_{i \geq 0} [a_i]$ with $a_i \in F$, $0 < |a_i| < 1$

and $\lim_{i \to +\infty} |a_i| = 0$.

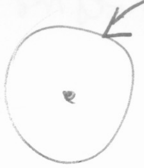
Set $f = \prod_{i=0}^{+\infty} \left(1 - \frac{a_i}{z}\right)$ C.V. in B. (for the Fréchet algebra top.)

then $\mathrm{div}(f) = D$.

$\Rightarrow$ we are reduced to the case $\mathrm{Supp}(D) \subset \{\rho_0 \leq |z| < 1\}$

for some $\rho_0 \in]0,1[$ i.e. $D = \sum_{i \geq 0} [a_i]$, $0 < |a_i| < 1$, $\lim_{i \to +\infty} |a_i| = 1$.

["$\prod_{i \geq 0} \left(1 - \frac{z}{a_i}\right)$" or "$\prod_{i \geq 0} \left(1 - \frac{a_i}{z}\right)$" do not C.V. in B]



(zeros accumulate on the exterior boundary of D^*)

Analogous problem/$\mathbb{C}$: find $f \in \mathcal{O}(\mathbb{C})$ s.t. $\mathrm{div}(f) = \sum_{n \in \mathbb{N}} [-n]$

"$z \prod_{n \geq 1} \left(1 + \frac{z}{n}\right)$" does not C.V.

Solution: renormalization factors

$$z \prod_{n \geq 1} \left[\left(1 + \frac{z}{n}\right) \underbrace{e^{-\frac{z}{n}}}_{\text{has no zeros}} \right] \text{ C.V.} = \frac{1}{e^{\gamma z} \Gamma(z)} \quad \text{and}$$

has the same divisor as "$z \prod_{n \geq 1} \left(1 + \frac{z}{n}\right)$"

Lazard: if F is spherically complete then $\exists (h_i)_{i \geq 0}$ sequence of $B^\times$

s.t. $$\underset{\underset{f}{\parallel}}{\prod_{i \geq 0} \left[\left(1 - \frac{z}{a_i}\right) h_i \right]} \quad \text{C.V. in } B \Rightarrow \mathrm{div}(f) = D.$$

Sometimes do not need renormalization factors using the following Euler's "regrouping" trick

* Example/$\mathbb{C}$: f s.t. $\mathrm{div}(f) = \sum'_{n\in\mathbb{Z}} [n]$

$$z \prod_{n\in\mathbb{Z}\setminus\{0\}} \left(1-\frac{z}{n}\right)$$ does not C.V.

but $$z \prod_{n>0} \left[\left(1-\frac{z}{n}\right)\left(1+\frac{z}{n}\right)\right]$$ C.V. $= \frac{\sin \pi z}{\pi}$

* $E/\mathbb{Q}_p$ finite $\mathcal{LT}$ = Lubin-Tate group law $/\mathcal{O}_E$ associated to E

$\log_{\mathcal{LT}} \in E[[T]]$ its logarithm

then $\log_{\mathcal{LT}} \in \mathcal{O}(\mathbb{D})$ with zeros the torsion points of E.

$$\mathcal{LT}[\pi^\infty](\bar{E}) = \{x \in \mathfrak{m}_{\bar{E}} \,/\, \exists n,\ [\pi^n]_{\mathcal{LT}}(x) = 0\}$$

$\log_{\mathcal{LT}} : \mathbb{D} \to \mathbb{A}^1$ de Jong covering with group $\mathcal{LT}[\pi^\infty](\bar{E}) / \mathbb{C}_p$.

$$T. \prod_{\zeta \in \mathcal{LT}[\pi^\infty](\bar{E})\setminus\{0\}} \left(1-\frac{T}{\zeta}\right)$$ does not C.V. since $|\zeta| \underset{\zeta\in\mathcal{LT}[\pi^\infty]}{\longrightarrow} 1$

(torsion points of L.T. accumulate to $|z|=1$)

but $$T. \prod_{n\geq 1} \left[\prod_{\zeta \in \mathcal{LT}[\pi^n]\setminus\mathcal{LT}[\pi^{n-1}]} \left(1-\frac{T}{\zeta}\right)\right]$$ C.V. in $\mathcal{O}(\mathbb{D})$

$= \log_{\mathcal{LT}}$

$$\left(\Longleftrightarrow \lim_{n\to+\infty} \frac{1}{\pi^n}[\pi^n]_{LT} = \log_{LT} \text{ in } \mathcal{O}(\mathbb{D}) \right)$$

Analytic functions in mixed characteristic

The rings B and B^+

$[E:\mathbb{Q}_q]<+\infty$, π uniformizing element of E, $\mathbb{F}_q$ residue field.

$F|\mathbb{F}_q$ complete extension $v: F \to \mathbb{R}\cup\{+\infty\}$ (non trivial)

perfect. (in particular the valuation v is not discrete)

$\mathcal{E}|E$ unique complete unramified extension of E with residue field F, $\mathcal{O}_{\mathcal{E}}/\pi\mathcal{O}_{\mathcal{E}} = F$.

$[\cdot]: F \to \mathcal{O}_{\mathcal{E}}$ Teichmüller lift

$$\mathcal{E} = \left\{ \sum_{n \gg -\infty} [x_n]\pi^n \;/\; x_n \in F \right\} \quad \text{(unique writting)}$$

$\circlearrowleft$ φ Frobenius $\quad \varphi\left(\sum_n [x_n]\pi^n\right) = \sum_n [x_n^q]\pi^n.$

$$\mathcal{E} = W_{\mathcal{O}_E}(F)\left[\frac{1}{\pi}\right] = W(F) \otimes_{W(\mathbb{F}_q)} E$$

$\circlearrowleft$

$\varphi = F^f \otimes \mathrm{Id}$ if $q = p^f$

Rem: * $\mathcal{E}$ is an analog of $F((z))$. Even more: if E is a char. p local field $E = \mathbb{F}_q((\pi))$ and $\mathcal{E}$ is the unique unramified extension of E with residue field F then $\mathcal{E} = F((\pi))$, $\pi = z$.

In this characteristic p case all the rings I will define are the same as the one I spoke before $\rightarrow$ everything works the same and is much easier $\rightarrow$ leads to a "curve" in the Hartl-Pink framework.

$$* \sum_{n\geq 0} [x_n]\pi^n + \sum_{n\geq 0} [y_n]\pi^n = \sum_{n\geq 0} \left[P_n(x_0,\dots,x_n,y_0,\dots,y_n)\right]\pi^n$$

$$P_n \in \mathbb{F}_q\left[X_0^{q^{-n}},\dots,X_{n-1}^{q^{-1}},X_n,Y_0^{q^{-n}},\dots,Y_n\right]$$

generalized polynomial

Same for the multiplication of Witt vectors

Def: $B^b = \left\{\sum_{n\gg-\infty}[x_n]\pi^n \in \mathcal{E} \;/\; \exists C, \forall n\ |x_n| \leq C\right\}$

$\cup$

$B^{b,+} = \left\{\sum_{n\gg-\infty}[x_n]\pi^n \in \mathcal{E} \;/\; \forall n, |x_n| \leq 1\right\} = W_{\mathcal{O}_E}(\mathcal{O}_F)\left[\frac{1}{\pi}\right]$

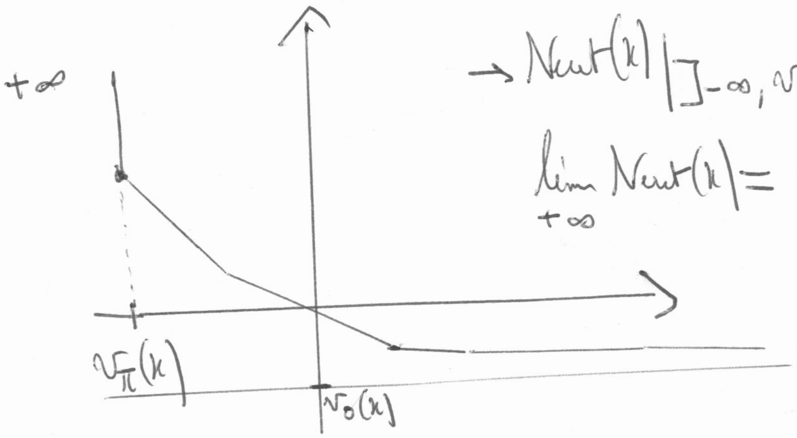
$B^b = B^{b,+}\left[\frac{1}{[a]}\right]$ for any $a \in F^\times$ with $v(a) > 0$.

Def: * For $r \geq 0$ and $x = \sum_{n \gg -\infty} [x_n] \pi^n \in B^b$ set

$$\boxed{v_r(x) = \inf_{n \in \mathbb{Z}} \{ v(x_n) + n r \}}$$

always reached if $r > 0$.

* Set $\mathrm{Newt}(x)$ = decreasing convex hull of $\{(n, v(x_n))\}_{n \in \mathbb{Z}}$



$\rightarrow \mathrm{Newt}(x)|_{]-\infty, v_\pi(x)[} = +\infty$

$\lim_{+\infty} \mathrm{Newt}(x) = v_0(x) = \inf_{n \in \mathbb{Z}} \{ v(x_n) \}$] Not reached in general

the limit may not be reached since the valuation of F is not discrete.

[**Prop:** * $r \longmapsto v_r(x)$ is the Legendre transform of $\mathrm{Newt}(x)$

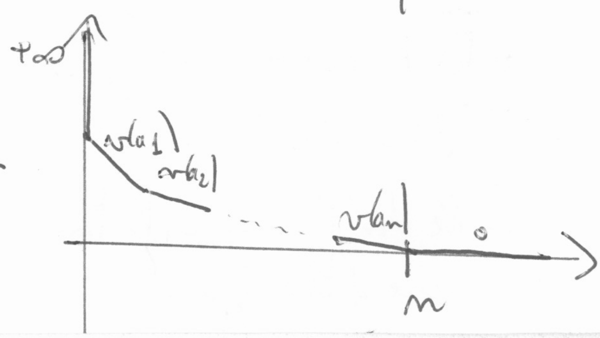
* v_r is a valuation] Have to work a little bit]

$$\Longrightarrow \quad \mathrm{Newt}(xy) = \underbrace{\mathrm{Newt}(x) * \mathrm{Newt}(y)}_{\text{Concatenation of slopes}}$$

$\llcorner$ $\forall r$, $v_r(xy) = v_r(x) + v_r(y)$ + inverse Legendre

Ex: $a_1, \dots, a_n \in m_F \setminus \{0\}$, $v(a_1) \geq \dots \geq v(a_n)$

$$\mathrm{Newt}\big((\pi - [a_1]) \dots (\pi - [a_n]) \big) =$$



Def: Set $B =$ Completion of B^b w.r.t. $(v_r)_{r>0}$
$B^+ =$ " " $B^{b,+}$ w.r.t. $(v_r)_{r>0}$ } E-Frechet algebras

$I \subset]0,1[$ Compact interval

$\underbrace{B_I}$ = " " B^b " " $(v_r)_{q^{-r} \in I}$

E-Banach algebra since if $I = [q^{-r_1}, q^{-r_2}]$

then $B_I =$ Completion of B^b w.r.t. $\inf\{v_{r_1}, v_{r_2}\}$

(since $r \mapsto v_r(x)$ is concave the "maximum modulus principle" applies: if $r_2 \leq r \leq r_1$
then $v_r(x) \geq \inf\{v_{r_1}(x), v_{r_2}(x)\}$)

$$B = \varprojlim_{I \subset]0,1[} B_I \text{ as a Frechet algebra.}$$

⚠: If $(x_n)_{n\in\mathbb{Z}} \in \mathcal{O}_F^{\mathbb{Z}}$ satisfies $\lim_{n\to+\infty} \frac{v(x_n)}{n} = +\infty$ then

$$\sum_{n\in\mathbb{Z}} [x_n]\pi^n \text{ C.V. in } B^+$$

But: * We don't know if each element of B^+ is of this form) Probably true

we think it's false (* For such elements of B^+ we don't know if such a writting is unique

* We don't know if the sum or products of elements of such type are of this type.