2ème Cours Bonn

Analytic functions in mixed characteristic

The rings B and B^+

$[E:\mathbb{Q}_p]<+\infty$, π uniformizing element of E, $\mathbb{F}_q$ residue field

$F|\mathbb{F}_q$ Complete extension $v : F \to \mathbb{R}\cup\{+\infty\}$ (non trivial)

F perfect (in particular v is not discrete)

$\mathcal{E}|E$ unique Complete unramified extension of E with residue field F, $\mathcal{O}_\mathcal{E}/\pi\mathcal{O}_\mathcal{E}=F$.

$[-]:F\to\mathcal{O}_\mathcal{E}$ Teichmüller lift

$$\mathcal{E}=\left\{\sum_{n>>-\infty}[x_n]\pi^n \;\middle|\; x_n\in F\right\} \quad \text{(unique writting)}$$

$\circlearrowleft$

φ Frobenius $\quad \varphi\left(\sum_n [x_n]\pi^n\right)=\sum_n [x_n^q]\pi^n$

$$\mathcal{E}=W_{\mathcal{O}_E}(F)\left[\tfrac{1}{\pi}\right]=W(F)\underset{W(\mathbb{F}_q)}{\otimes}E$$

$\circlearrowleft$

$\varphi=F^f\otimes Id$ if $q=p^f$

Rem: * $\mathcal{E}$ is an analog of $F((z))$. Even more than an analog:

if E is a char. p local field $E = \mathbb{F}_q((\pi))$ and $\mathcal{E}$ is the unique unramified extension of E with residue field F then $\mathcal{E} = F((\pi))$, $\pi = z$.

In this char. p case all the rings I will define are the same as the one I spoke before (1st Course) $\rightarrow$ ~~ev~~ everything works the same and is much easier $\Rightarrow$ leads to a curve in the Hartl-Pink framework.

* $$\sum_{n\geq 0}[x_n]\pi^n + \sum_{n\geq 0}[y_n]\pi^n = \sum_{n\geq 0}\left[P_n(x_0,\dots,x_n,y_0,\dots,y_n)\right]\pi^n$$

$$P_n \in \mathbb{F}_q\left[X_0^{q^{-n}},\dots,X_{n-1}^{q^{-1}},X_n,Y_0^{q^{-n}},\dots,Y_n\right]$$

generalized polynomial.

Same for the multiplication of Witt vectors.

Def: $$B^b = \left\{\sum_{n\gg-\infty}[x_n]\pi^n \in \mathcal{E} \,/\, \exists C, \forall n \ |x_n| \leq C\right\}$$

$$\cup$$

$$B^{b,+} = \left\{\sum_{n\gg-\infty}[x_n]\pi^n \in \mathcal{E} \,/\, \forall n, |x_n| \leq 1\right\} = W_{\mathcal{O}_E}(\mathcal{O}_F)\left[\frac{1}{\pi}\right]$$

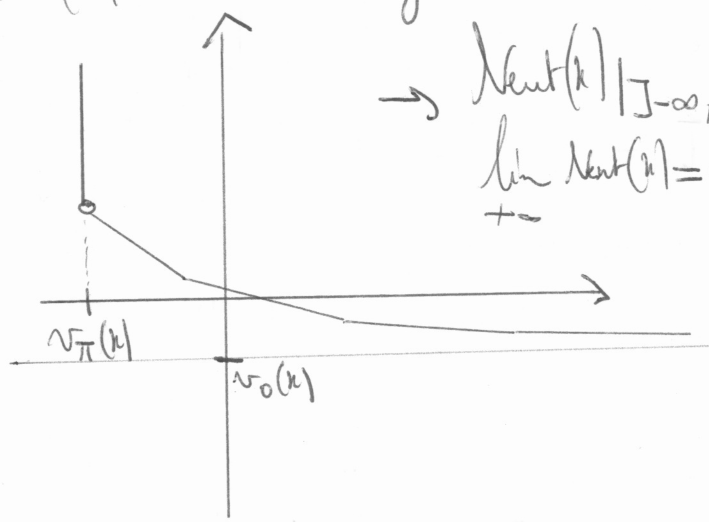
$B^b = B^{b,+}\left[\frac{1}{[a]}\right]$ for any $a \in F^\times$ with $v(a) > 0$.

Def: * For $r \geq 0$ and $x = \sum_{n >> 0} [x_n] \pi^n \in B^b$ set

$$\boxed{v_r(x) = \inf_{n \in \mathbb{Z}} \{ v(x_n) + n r \}}$$

always reached if $r > 0$

* $\mathrm{Newt}(x) =$ decreasing convex hull of $\{(n, v(x_n))\}_{n \in \mathbb{Z}}$



$\rightarrow$ $\mathrm{Newt}(x)|_{]-\infty, v_\pi(x)[} = +\infty$

$\lim_{+\infty} \mathrm{Newt}(x) = v_0(x) = \inf_{n \in \mathbb{Z}} \{ v(x_n) \}$] Not reached in general

the limit may not be reached since the valuation of F is not discrete

[**Prop:** * $r \mapsto v_r(x)$ is the Legendre transform of $\mathrm{Newt}(x)$

* v_r is a valuation] have to work a little bit

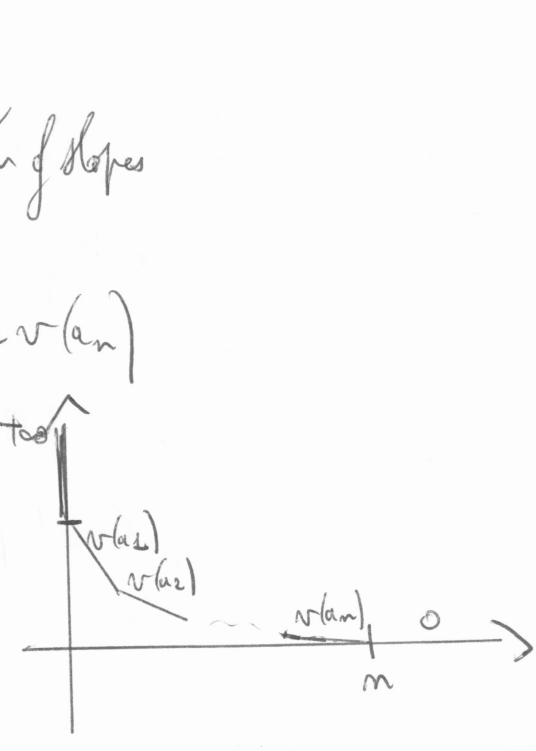
$$\Longrightarrow \mathrm{Newt}(xy) = \underbrace{\mathrm{Newt}(x) * \mathrm{Newt}(y)}_{\text{Concatenation of slopes}}$$

$\Longleftarrow$ [$\forall r \geq 0$ $v_r(xy) = v_r(x) + v_r(y)$

+ inverse Legendre

Ex: $a_1, \dots, a_n \in \mathfrak{m}_F \setminus \{0\}$, $v(a_1) \geq \dots \geq v(a_n)$

$$\mathrm{Newt}\big((\pi - [a_1]) \dots (\pi - [a_n])\big) =$$



<u>Def</u>: Set $B =$ Completion of B^b w.r.t. $(v_r)_{r>0}$ } E-Frechet algebras

$B^+ =$ " " $B^{b,+}$ w.r.t. $(v_r)_{r>0}$ }

$I \subset]0,1[$ compact interval: $\underbrace{B_I}$ = " " B^b " " $(v_r)_{q^{-r} \in I}$

E-Banach algebra since if $I = [q^{-r_1}, q^{-r_2}]$

then $B_I =$ Completion of B^b w.r.t. $\inf\{v_{r_1}, v_{r_2}\}$

($r \mapsto v_r(x)$ is concave $\Rightarrow$ "the maximum modulus principle" applies:

if $r_2 \leq r \leq r_1$ then $v_r(x) \geq \inf\{v_{r_1}(x), v_{r_2}(x)\}$.)

$$B = \varprojlim_{I \subset]0,1[} B_I \quad \text{as a Frechet algebra}$$

⚠: If $(x_n)_{n\in\mathbb{Z}} \in \mathcal{O}_F^{\mathbb{Z}}$ satisfies $\lim_{n\to+\infty} \frac{v(x_n)}{n} = +\infty$ then

$$\sum_{n\in\mathbb{Z}} [x_n]\pi^n \quad \text{C.V. in } B^+$$

<u>But</u>: * We don't know if each element of B^+ is of this form) probably true

We think it's false (* For such element of B^+ we don't know if such a writting is unique

* We don't know if the sum or products of elements of this type are of this type.

(Same for B)

There is a sub E-vector space of B^+ where the preceding remark does not apply. Fontaine has defined for any ring R the group of Witt bivectors $BW(R)$ (see his Astérisque). Elements of $BW(R)$ have infinite Teichmüller expansions on the left and on the right. Can define $BW_{\mathcal{O}_E}(R)$ for any $\mathcal{O}_E$-algebra R (works the same).

One has

$$BW_{\mathcal{O}_E}(\mathcal{O}_F) := \varprojlim_{\substack{\mathfrak{a}\subset \mathcal{O}_F \\ \text{non zero ideal}}} BW_{\mathcal{O}_E}(\mathcal{O}_F/\mathfrak{a}) = \left\{ \sum_{n\in\mathbb{Z}} V_\pi^n[x_n] \;\Big/\; \liminf_{n\to-\infty} v(x_n) > 0 \right\}$$

Can define bivectors for complete rings

$$= \left\{ \sum_{n\in\mathbb{Z}} [x_n]\pi^n \;\Big/\; \liminf_{n\to-\infty} q^n v(x_{-n}) > 0 \right\}$$

embedding via this writing

B^+

The point is that if

$$\sum_{b\geq 0}[x_b]\pi^b + \sum_{b\geq 0}[y_b]\pi^b = \sum_{b\geq 0}[P_b(x_0,\dots,x_b,y_0,\dots,y_b)]\pi^b$$

$$P_b \in \mathbb{F}_q[X_0^{q^{-b}},\dots,X_b,Y_0^{q^{-b}},\dots,Y_b]$$ (Universal generalized polynomials giving the addition of the Witt vectors)

then in $BW_{\mathcal{O}_E}(\mathcal{O}_F)$

$$\sum_{n\in\mathbb{Z}}[x_n]\pi^n + \sum_{n\in\mathbb{Z}}[y_n]\pi^n = \sum_{n\in\mathbb{Z}}\underbrace{\Big[\lim_{b\to+\infty} P_b(x_{n-b},\dots,x_n,y_{n-b},\dots,y_n)\Big]}_{\text{exists because of}}\pi^b$$

does not exist without this condition

Periods of p.-div. groups $\in \underbrace{BW(\mathcal{O}_F)}$

Captures all periods with Dieudonné-Manin slopes $\in [0,1]$ but not the other one $(t \in BW(\mathcal{O}_F), t^2 \notin BW(\mathcal{O}_F))$

In equal char. (Hartl-Pink setting) no such restriction $\Rightarrow$ no restriction on slopes of formal $\mathcal{O}$-modules.

Don't have a good understanding of the additive structure of B... but a good one of the multiplicative one

Frobenius: $v_r(\varphi(x)) = q\, v_{\frac{r}{q}}(x)$ $\quad B, B^+ \supset \varphi$ extending φ on B^b and $B^{b,+}$

$$\varphi\left(B_{[\rho_1,\rho_2]}\right) = B_{[\rho_1^q,\, \rho_2^q]}$$

Rem: φ = arithmetic Frobenius: in equal char. $\varphi\left(\sum_{n\in\mathbb{Z}} a_n z^n\right) = \sum_{n\in\mathbb{Z}} a_n^q z^n$

$\left(\text{not } \sum_{n\in\mathbb{Z}} a_n z^{qn}, \text{ geometric Frob.}\right)$

The particular case of B^+: If $r \geq r' > 0$ and $x = \sum_{n\gg-\infty} [x_n]\pi^n \in B^{b,+}$, $x_n \in \mathcal{O}_F$

$$\forall n \quad v(x_n) + n r' = \frac{r'}{r}\left[\frac{r}{r'}\underbrace{v(x_n)}_{\geq 0} + n r\right] \geq \frac{r'}{r}\left(v(x_n) + n r\right)$$

$$\Rightarrow \boxed{v_{r'}(x) \geq \frac{r'}{r}\, v_r(x)} \Rightarrow \text{any lucky sequence for } v_r \text{ is lucky for } v_{r'}.$$

$\Rightarrow$ If $\rho = q^{-r}$, $\rho' = q^{-r'}$, $\rho \leq \rho'$ then $B^+_{[\rho,\rho']} = \underset{\substack{\| \\ B^+_{]0,\rho]}}}{B^+_\rho} \subset B^+_{\rho'}$

$$\Rightarrow \left[\begin{aligned} B^+ &= \bigcap_{\rho>0} B^+_\rho \quad , \quad B^+_\rho \circlearrowleft \varphi \text{ since } \varphi(B^+_\rho) = B^+_{\rho^p} \subset B^+_\rho \\ &= \boxed{\bigcap_{n\geq 0} \varphi^n\left(B^+_{\rho_0}\right)} \text{ for any } \rho_0 \in]0,1[\\ &= \text{biggest subring of } B^+_{\rho_0} \text{ where } \varphi \text{ is bijective} \end{aligned}\right.$$

Moreover for $E = \mathbb{Q}_p$ and $\rho \in |F| \cap]0,1[$ set

$$B^+_{\mathrm{cris},\rho} = \widehat{\underbrace{W(\mathcal{O}_F)\left[\frac{[a]^n}{n!}\right]}_{\text{P.D. hull of the ideal } [a]W(\mathcal{O}_F)}}\left[\frac{1}{p}\right] \quad \text{(} p\text{-adic completion)}$$

where $a \in F$, $|a| = \rho$.

One checks the unit ball of $B^{b,+}$ for v_r is $W(\mathcal{O}_F)\left[\frac{[b]}{p}\right]$ if $v(b) = r$.

$$\Rightarrow \quad \boxed{B^+_{\rho^p} \subset B^+_{\mathrm{cris},\rho} \subset B^+_{\rho^{p-1}}}$$

$$\Rightarrow \quad \boxed{B^+ = \bigcap_{n\geq 0} \varphi^n\left(B^+_{\mathrm{cris},\rho_0}\right)} \text{ for any } \rho_0 \in]0,1[$$

Newton polygon of elements of B

Each element of B may not be uniquely written $\sum_{n\in\mathbb{Z}}' [x_n]\pi^n$ ....
how to define the Newton polygon of elements of B?

Prop: $\underset{\substack{\cap \\ B^b}}{x_n} \xrightarrow[n\to+\infty]{} x \in B\setminus\{0\}$ then $\forall I \subset]0,1[$ compact

$\exists N, n \geq N$ and $q^{-r} \in I \Rightarrow v_r(x_n) = v_r(x)$

$\to$ Consequence of the following Dini type theorem: $\varphi_n :]0,+\infty[\to \mathbb{R}$ concave, $\varphi_n \xrightarrow[n\to+\infty]{} \varphi$ pointwise. Then the C.V. is uniform on each compact of $]0,+\infty[$.

(not true on all $]0,1[$)

Def: $x \in B$. $\mathrm{Newt}(x)$ = Inverse Legendre transform of $r \mapsto v_r(x)$
= polygon with integral x-coordinate breakpoints.

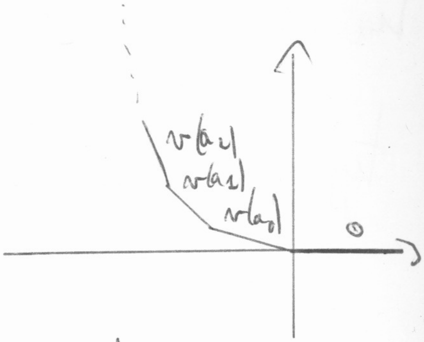
If $(\lambda_i)_{i\in\mathbb{Z}}$ slopes of $\underset{(x\neq 0)}{\mathrm{Newt}(x)}$, $\lambda_i \geq \lambda_{i+1}$ and $\lim_{i\to+\infty} \lambda_i = 0$
$\lim_{i\to-\infty} \lambda_i = +\infty$.

λ_i = slope on $[i,i+1]$

Ex: * $\mathrm{Newt}\left(\sum_{n\in\mathbb{Z}} [x_n]\pi^n\right) =$ What you think

* $(a_n)_{n\geq 0}$ sequence of $F^\times$ s.t. $(v(a_n))_n$ is increasing and $a_n \xrightarrow[n\to+\infty]{} 0$.

$$\mathrm{Newt}\left(\prod_{n\geq 1}\left(1-\frac{[a_n]}{\pi}\right)\right) =$$



Prop: * $B^+ = \{x \in B \,/\, \mathrm{Newt}(x) \subset \text{upper half plane}\}$

$v_0(x) = \lim_{+\infty} \mathrm{Newt}(x)$, v_0 = valuation on B^+

* $B^b = \{x \in B \,/\, \exists A,\ \mathrm{Newt}(x)|_{]-\infty, A]} = +\infty$ and $\mathrm{Newt}(x)$ bounded below$\}$

Applications: * $B^\times = (B^b)^\times = \left\{\sum_{n\geq n_0} [x_n]\pi^n \,/\, x_{n_0}\in F^\times,\ n_0\in\mathbb{Z},\ \forall n\geq n_0,\ v(x_n)\geq v(x_{n_0})\right\}$

* For $d<0$ $B^{\varphi=\pi^d} = 0$

* $B^{\varphi=Id} = E$

* For $d\geq 0$, $B^{\varphi=\pi^d} = (B^+)^{\varphi=\pi^d}$

Exo.

(use $\mathrm{Newt}(\varphi(x)) = q\,\mathrm{Newt}(x)$

$\mathrm{Newt}(\pi x)(\cdot) = \mathrm{Newt}(x)(\cdot - 1)$ + preceding prop.

+ growing properties of Newt

$\lambda_i \xrightarrow[i\to-\infty]{} +\infty$, $\lambda_i \xrightarrow[i\to+\infty]{} 0$)

Fontaine's ring $\mathcal{R}$

Fix $Q \in \mathcal{O}_E[X]$, $Q \equiv X^q \bmod \pi$. Set $Q_n = \underbrace{Q \circ \dots \circ Q}_{n\text{-times}}$

typically $Q = \underbrace{[\pi]}_{\text{Lubin-Tate group law}}{}_{LT}$, $Q_n = [\pi^n]_{LT}$.

A $\mathcal{O}_E$-algebra. Set $\mathcal{R}_Q(A) = \left\{ (x^{(n)})_{n \geq 0} \;/\; x^{(n)} \in A,\ Q(x^{(n+1)}) = x^{(n)} \right\}$

$\underline{\text{Ex}}$: $E = \mathbb{Q}_p$, $Q = X^p$, $\mathcal{R}(A)$ = usual definition by Fontaine

Prop: A π-~~algebra~~ adic. $I \subset A$ closed ideal s.t. A is $I + (\pi)$-adic. Then

(adic = separated and complete)

$$\mathcal{R}_Q(A) \xrightarrow{\sim} \mathcal{R}_Q(A/I)$$

with inverse given by

$$(x^{(n)})_{n \geq 0} \longmapsto \left(\lim_{k \to +\infty} Q_k\Big(\underbrace{\widehat{x^{(n+k)}}}_{\text{any lift of } x^{(n+k)} \text{ to } A} \Big) \right)_{n \geq 0}$$

In particular $\mathcal{R}_Q(A) \xrightarrow{\sim} \mathcal{R}_Q(A/\pi A) = \varprojlim_{\mathbb{N}} A/\pi A$, Frob_q = transition map. $= \mathcal{R}(A)$

$$\begin{tikzcd}
\mathcal{R}_Q : \pi\text{-adic } \mathcal{O}_E\text{-algebras} \arrow[rr, "\text{canonical factorization}"'] \arrow[dr] & & \text{perfect } \mathbb{F}_q\text{-algebras} \arrow[dl] \\
& \text{Sets} \arrow[ul, bend left, "\text{by def.}"] &
\end{tikzcd}$$

If Q_1, Q_2 satisfy preceding hypothesis canonical iso. $\mathcal{R}_{Q_1} \xrightarrow{\sim} \mathcal{R}_{Q_2}$.

<u>Twisted Teichmüller</u>: * By def. the ramified Witt vectors = functor

$$W_{O_E} : O_E\text{-alg} \to O_E\text{-alg}$$

+ Teichmüller multiplicative $[-] : (-) \to W_O(-)$

+ F Frobenius endomorphism

+ for each unif. element $\pi \in O_E$ an operator V_π satisfying $FV_\pi = \pi$,

$W_{O_E}(A)$ is V_π-adic and

$$W_{O_E}(A) = \left\{ \sum_{n\geq 0} V_\pi^n [a_n] \;/\; a_n \in A \right\} \quad \text{unique writting}$$

* $[xy] = [x].[y] \Rightarrow [-]$ is well adaptated to $\mathbb{G}_m$

Want a Teichmüller well adaptated to a L.T. group$/O_E$.

If $w_n(X_0, .., X_n) = X_0^{q^n} + \pi X_1^{q^{n-1}} + \ldots + \pi^{n-1} X_{n-1}^q + \pi^n X_n$

by def.
$$w_n : W_{O_E} \longrightarrow \mathbb{G}_a \quad \text{is a group morphism.}$$
$$\sum_{n\geq 0} V_\pi^n [x_n] \longmapsto w_n(x_0, .., x_n)$$

[Prop: $\exists!$ application $[-]_Q : (-) \to W_{O_E}(-)$ s.t. $\forall n$, $w_n([x]_Q) = Q_n(x)$

$$W_O(A) = \left\{ \sum_{n\geq 0} V_\pi^n [a_n]_Q \;/\; a_n \in A \right\} \quad \text{unique writting}$$

$$Q([x]_Q) = [Q(x)]_Q$$

If A = perfect $\mathbb{F}_q$-algebra $\boxed{[x]_Q = \lim_{n\to+\infty} Q_n\left([x^{q^{-n}}]\right)}$.

* Suppose $Q = [\pi]_{LT}$ $\quad LT =$ Lubin-Tate groupe law
↳ i.e. $Q(0) = 0$ and $Q'(0) = \pi$

$A =$ perfect adic $\mathbb{F}_q$-algebra. $A^{\circ\circ} =$ top. nilpotent elements.

If I is an ideal of definition of A then $W_O(A)$ is $\underbrace{(\pi, [a])_{a \in I}}_{\{\sum_{n\geq 0} [a_n]\pi^n \,/\, a_0 \in I\}}$ - adic

and $W_O(A)^{\circ\circ} = \left\{ \sum_{n \geq 0} [a_n] \pi^n \,/\, a_0 \in A^{\circ\circ} \right\}$.

$\Rightarrow \left(W_O(A^{\circ\circ}), \underset{LT}{+} \right) = O_E$-module

Then $\left(A^{\circ\circ}, \underset{LT}{+} \right) \hookrightarrow \left(W_O(A^{\circ\circ}), \underset{LT}{+} \right)$ is an O_E-modules morphism.
$a \longmapsto [a]_Q$

$$\boxed{[x]_Q \underset{LT}{+} [y]_Q = \left[x \underset{LT}{+} y \right]_Q .}$$

Ex: * $Q(X) = X^q$, $[x]_Q = [x]$
* $E = \mathbb{Q}_p$, $Q(X) = (1+X)^p - 1$, $[x]_Q = [1+x] - 1$

Back to R_Q: $A =$ perfect $\mathbb{F}_q$-algebra

$$A \xrightarrow{\sim} R_Q(W_{O_E}(A))$$
$$a \longmapsto \left([a^{q^{-n}}]_Q \right)_{n \geq 0}$$

Prop: The functors

left adjoint

$$\text{Perfect } \mathbb{F}_q\text{-algebras} \underset{R_Q(-)}{\overset{W_O(-)}{\rightleftarrows}} \pi\text{-adic } O_E\text{-alg.}$$

right adjoint

are adjoints. The adjunction morphisms are

$$\begin{cases} A \xrightarrow{\sim} R_Q(W_O(A)) \\ a \longmapsto \left([a^{q^{-n}}]_Q \right)_{n \geq 0} \end{cases}$$

Fontaine's θ-morphism.

$$\begin{cases} W_O(R_Q(A)) \xrightarrow{\theta} A \\ \sum\limits_{n \geq 0} [x_n]_Q \pi^n \longmapsto \sum\limits_{n \geq 0} x_n^{(0)} \pi^n \end{cases}$$