The space $|Y|$

$E|\mathbb{Q}_q$, π unif. $F|\mathbb{F}_q$ complete

$v : F \to \mathbb{R} \cup \{+\infty\}$

F algebraically closed (see the articles for the general case of any perfect field)

Def: $x = \sum_{n\geq 0} [x_n]\pi^n \in W_{\mathcal{O}}(\mathcal{O}_F)$ is primitive (may speak about it later)

if $x_0 \neq 0$ and $\exists\, n,\ x_n \in \mathcal{O}_F^\times$ (smallest n)

$\Updownarrow$

$x \notin \pi W_{\mathcal{O}}(\mathcal{O}_F)$ and $\tilde{x} \in W_{\mathcal{O}}(k_F)$ satisfies $\tilde{x} \neq 0$.

Set $\deg(x) = v_\pi(\tilde{x})$ for x primitive.

* degree 0 primitive elements $= W_{\mathcal{O}}(\mathcal{O}_F)^\times \curvearrowright \{\text{primitive elements}\}$

Th: I ideal generated by a degree 1 primitive element.

$A = W_{\mathcal{O}}(\mathcal{O}_F)/I$, $\theta : W_{\mathcal{O}}(\mathcal{O}_F) \twoheadrightarrow A$. Then:

1) $A[\frac{1}{\pi}]$ is a complete alg. closed extension of E with ring of integers A

2) If $R(A) = \{(x^{(n)})_{n\geq 0} \,/\, x^{(n)} \in A,\ (x^{(n+1)})^q = x^{(n)}\}$ = Can. p perfect ring

$$(x+y)^{(n)} = \lim_{k\to+\infty} \left(x^{(n+k)} + y^{(n+k)}\right)^{q^k}$$

$$(xy)^{(n)} = x^{(n)}.y^{(n)}$$

Then $\mathcal{O}_F \xrightarrow{\sim} R(A)$

$$x \longmapsto \left(\theta\left([x^{q^{-n}}]\right)\right)_{n\geq 0}$$

3) $\mathcal{O}_F \longrightarrow A$ is surjective

$$x \longmapsto \theta([x])$$

4) If $\underline{\pi} \in R(A)$ is such that $\underline{\pi}^{(0)} = \pi$ then $I = (\pi - [\underline{\pi}])$.

$\Longrightarrow$ * If $\deg(x) = 1$ (x primitive) then $x = \text{unit} \times (\pi - [a])$ with $a \in \mathfrak{m}_F \setminus \{0\}$ (Consequence of 4))

(Weierstrass factorization)

* If $y \in W_{\mathcal{O}}(\mathcal{O}_F)$ $\exists b \in \mathcal{O}_F$ $\exists z \in W_{\mathcal{O}}(\mathcal{O}_F)$ $y = zx + [b]$ (Consequence of 3))

(Weierstrass division)

⚠ : a and b are not unique!

Def: $|Y| = \{\text{primitive degree 1 elements}\} / W_{\mathcal{O}}(\mathcal{O}_F)^{\times}$

One checks if x is primitive of degree 1

$$W_{\mathcal{O}}(\mathcal{O}_F)[\tfrac{1}{\pi}] \twoheadrightarrow W_{\mathcal{O}}(\mathcal{O}_F)[\tfrac{1}{\pi}]/(x)$$

$\downarrow$ B $\nearrow$ extends with kernel Bx.

Thus $\boxed{|Y| \subset \mathrm{Spm}(B)}$.

If $\mathfrak{m} \in |Y|$, $C_{\mathfrak{m}} = B/\mathfrak{m}$, $\mathcal{O}_{C_{\mathfrak{m}}} = W_{\mathcal{O}}(\mathcal{O}_F)/\mathfrak{m} \cap W_{\mathcal{O}}(\mathcal{O}_F)$

$\theta_{\mathfrak{m}} : B \twoheadrightarrow C_{\mathfrak{m}}$

$v_{\mathfrak{m}}$ = unique valuation on $C_{\mathfrak{m}}$ s.t. $v_{\mathfrak{m}}(\theta_{\mathfrak{m}}([a])) = v(a)$

$(C_{\mathfrak{m}}, v_{\mathfrak{m}})$ = complete alg. closed extension of E.

⚠: $v_{\mathfrak{m}}$ does not extend v_π on E, $v_{\mathfrak{m}}(\pi) = v(a)$ if $\mathfrak{m} = (\pi - [a])$.

Set $\boxed{\|\mathfrak{m}\| = q^{-v_{\mathfrak{m}}(\pi)} \in]0,1[}$.

$\|\cdot\| : |Y| \to]0,1[$ (distance at the origin in the punctured disk)

<u>Parametrization of $|Y|$</u>: $\mathfrak{m} \in |Y|$, $\mathfrak{m} = (\pi - [a])$ but a is not unique

moreover, given $a, b \in \mathfrak{m}_F \setminus \{0\}$ it is difficult to decide wether $(\pi - [a]) = (\pi - [b])$ or not.

* $\mathcal{LT}$ = Lubin-Tate group law$/\mathcal{O}_E$

$\mathcal{G}$ = associated formal group

For $x \in \mathfrak{m}_F$ set $[x]_Q = \lim\limits_{n \to +\infty} [\pi^n]_{\mathcal{LT}}([x^{q^{-n}}]) \in W_{\mathcal{O}}(\mathfrak{m}_F)$

(twisted Teichmüller lift, more adapted to $\mathcal{LT}$ groups than the usual one $\leadsto \mathcal{G}_m$)

$[\cdot]_Q : (\mathfrak{m}_F, +_{\mathcal{LT}}) \hookrightarrow (W_{\mathcal{O}}(\mathfrak{m}_F), +_{\mathcal{LT}})$

Def: $\varepsilon \in \mathfrak{m}_F \setminus \{0\}$ set $u_\varepsilon = \dfrac{[\varepsilon]_Q}{[\varepsilon^{1/q}]_Q} \in W_O(O_F)$

primitive of degree 1.

Th: $$\left(\mathcal{G}(O_F) \setminus \{0\}\right) / O_E^\times \xrightarrow{\sim} |Y|$$
$$\mathfrak{m}_F \setminus \{0\} \cong \qquad \varepsilon . O_E^\times \longmapsto (u_\varepsilon)$$

inverse of the map.

$\rightarrow$ If $m \in |Y|$ choose $\tilde{\varepsilon}$ a generator of $T_\pi(\mathcal{G}) = \varprojlim_n \mathcal{G}[\pi^n](O_{C_m})$

then $\varepsilon = \tilde{\varepsilon} \bmod \pi \in R(O_{C_m}/\pi O_{C_m}) = R(O_{C_m}) = O_F$ and

$m = (u_\varepsilon)$.

Factorization

Th: $x \in W_O(O_F)$ primitive of degree d. Then $\exists y_1, \dots, y_d$ primitive of degree 1, $\exists u \in W_O(O_F)^\times$ s.t. $x = u \prod_{i=1}^{d} y_i$

$$\Rightarrow \boxed{x = \text{unit} \times (\pi - [a_1]) \cdots (\pi - [a_d])}.$$

Main theorem

Th: $f \in B$. For each slope $\lambda > 0$ of $\mathrm{Newt}(f)$ $\exists\, m \in |Y|$ s.t. $\|m\| = q^{-\lambda}$ and $\theta_m = 0 \Longrightarrow f = (\pi - [a]) \cdot g$ with $v(a) = \lambda$, $g \in B$.

$\longrightarrow$ any holomorphic function $f \in B$ has a zero in $|Y|$ with prescribed valuation by a slope of $\mathrm{Newt}(f)$.

Divisor of an element of B

Def: $\mathrm{Div}^+(Y) = \left\{ \sum_{m \in |Y|} a_m [m] \;/\; a_m \in \mathbb{N},\ \forall I \subset]0,1[\text{ compact},\ \{m \,/\, a_m \neq 0 \text{ and } \|m\| \in I\} \text{ is finite} \right\}$

($\mathrm{Div}^+(Y)$: monoïd)

* $m \in |Y|$, $\theta_m : B \twoheadrightarrow C_m$

Set $B^+_{dR,m}$ = Completion of B w.r.t. m
$= \text{D.V.R.} = \text{“}\widehat{\mathcal{O}_{Y,m}}\text{”}$

$\mathrm{ord}_m : B^+_{dR,m} \longrightarrow \mathbb{N} \cup \{+\infty\}$ associated valuation

For $f \in B$, $\mathrm{ord}_m(f) > 0 \Longleftrightarrow \theta_m(f) = 0$.

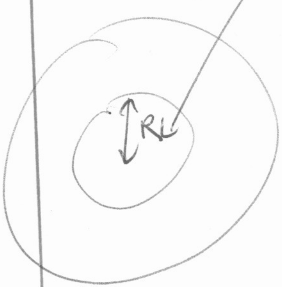
* Set for $f \in B$, $\mathrm{div}(f) = \sum_{m \in |Y|} \mathrm{ord}_m(f)\,[m] \in \mathrm{Div}^+(Y)$

Corollary: Slopes of $\mathrm{Newt}(f) = \{ -\log_q \|m\| \;/\; m \in \mathrm{supp}(\mathrm{div} f) \}$ (w.t. multiplicity $\mathrm{ord}_m(f)$).

Weierstrass products

Th: $f \in B$, $0 < R < 1$, can write

remove the zeros.



$$f = \underset{\in B}{g} \cdot \underbrace{\prod_{i=1}^{+\infty}\left(1 - \frac{[z_i]}{\pi}\right)}_{\text{C.V. in } B}, \quad \lim_{i \to +\infty} z_i = 0$$

where $\mathrm{Supp}(\mathrm{div}(g)) \subset \{ \mathfrak{m} \in |Y| \,/\, \|\mathfrak{m}\| \geq R \}$

$\Rightarrow \exists A \in \mathbb{R}, \ \mathrm{Newt}(g)_{|]-\infty, A]} = +\infty$

$\Rightarrow g \in \left\{ \sum_{n \gg -\infty} [x_n] \pi^n \in W(F)[\frac{1}{\pi}] \,\middle/\, \liminf_{n \to +\infty} \frac{v(x_n)}{n} \geq 0 \right\}$

In particular, if $f \in B^+$ then $g \in B^{b,+} = W_{\mathcal{O}}(\mathcal{O}_F)[\frac{1}{\pi}]$.

Problem: We don't know if one can expand such a g as an infinite Weierstrass product (lack of good renormalization factors)

Th: $\forall f,g \in B\setminus\{0\}$, $\mathrm{div}(f) \geq \mathrm{div}(g) \Leftrightarrow f \in (g)$

$*$ $B\setminus\{0\}/B^\times \xhookrightarrow{\mathrm{div}} \mathrm{Div}^+(Y)$

↑ injection of monoïds — don't know the image

$*$ $\forall I \subset]0,1[$ B_I is a P.I.D. with maximal ideals $\{\mathfrak{m} B_I \,/\, \mathfrak{m} \in |Y|,\ \|\mathfrak{m}\| \in I\}$

The curve: Let's say we want to study φ-modules with coefficients in B

That is to say φ-equivariant vector bundles on Y

$\Updownarrow$ — vector bundles on $Y/\varphi^{\mathbb{Z}}$

Y ↓ "$Y = \mathcal{M}(B)$" with classical points $|Y|$

What is $Y/\varphi^{\mathbb{Z}}$?: Easy to classify φ-modules of rank 1 with coeff. in B.

Parametrized by integer $n \in \mathbb{Z}$, to n corresponds (M_n, φ) where $M_n = B.e$ and $\varphi(e) = \pi^n e$.

Thus $\mathbb{Z} \xrightarrow{\sim} \mathrm{Pic}(Y/\varphi^{\mathbb{Z}})$ where $\mathcal{L}$ is such that $\mathbb{V}(\mathcal{L}) = (Y \times \mathbb{A}^1)/\mathbb{Z}$ ($\mathbb{A}^1$: acts via powers of π; Y: acts via powers of φ)

$n \mapsto \mathcal{L}^{\otimes n}$

One has $\forall d \in \mathbb{Z}$, $H^0(Y/\varphi^{\mathbb{Z}}, \mathcal{L}^{\otimes d}) = B^{\varphi = \pi^d}$.

Kedlaya, Hartl-Pink: $M = \varphi$-module/Robba ring $\Rightarrow M^{\varphi=\pi^d} \neq 0$ for $d \gg 0$

(first step in their proof of their classification theorem)

$\rightsquigarrow$ "L sample"

[Decrate L sample: set $P = \bigoplus_{d \geq 0} B^{\varphi=\pi^d}$ and look at $\mathrm{Proj}(P)$!!]

~~Multiplicative~~ Structure of the graded algebra P

$P_d = B^{\varphi=\pi^d}$, $d \geq 0$, homogenous elements of degree d

Newton polygon considerations: $B^{\varphi=\pi^d} = 0$ if $d < 0$

$P_0 = E$

$$\boxed{B^{\varphi=\pi^d} = (B^+)^{\varphi=\pi^d}}$$

<u>Def</u>: $\mathrm{Div}^+(Y/\varphi^{\mathbb{Z}}) = \{ D \in \mathrm{Div}^+(Y) \,/\, \varphi^* D = D \}$

free monoid on $|Y|/\varphi^{\mathbb{Z}} \hookrightarrow \mathrm{Div}^+(Y/\varphi^{\mathbb{Z}})$

$$m \bmod \varphi^{\mathbb{Z}} \longmapsto \sum_{n \in \mathbb{Z}} [\varphi^n(m)]$$

Th: The morphism of monoids

$$\left(\bigcup_{d\geq 0} P_d \setminus \{0\}\right) \Big/ \underset{\underset{P_0^\times}{\wr}}{E^\times} \xrightarrow{\ \mathrm{div}\ } \mathrm{Div}^+\left(Y/\varphi^{\mathbb{Z}}\right)$$

is an isomorphism.

Corollary: P is graded factorial with irreducible elements of degree 1

Injectivity: Allready done: $f, g \in B$, $\mathrm{div}(f) = \mathrm{div}(g) \Leftrightarrow (f) = (g)$

But if $f, g \in P_d \setminus \{0\}$, $\mathrm{div}(f) = \mathrm{div}(g) \Rightarrow f = u.g$, $u \in B^\times$

and $\varphi(u) = u \Rightarrow u \in E^\times$.

Surjectivity: [Weierstrass products]

$x \in W_O(O_F)$ primitive of degree $d > 0$. Suppose moreover $\tilde{x} \in W_O(k_F)$ satisfies $\tilde{x} = \pi^d$ (for example $x = \prod_{i=1}^{d} (\pi - [a_i])$, $a_i \in \mathfrak{m}_F \setminus \{0\}$).

If $\mathfrak{m} = (x)$ we are looking for $f \in P_d \setminus \{0\}$ such that

$$\left[\mathrm{div}(f) = \sum_{n \in \mathbb{Z}} [\varphi^n(\mathfrak{m})]\right] = \sum_{n\geq 0} (-) + \sum_{n<0} (-)$$

cut the division in two parts.

$$\Pi(x) = \underbrace{\prod_{n\geq 0}\left(\frac{\varphi^n(x)}{\pi^d}\right)}_{\text{C.V. in } B^+ =: \Pi^+(x)} \cdot \text{``}\underbrace{\prod_{n<0}\varphi^n(x)}_{\Pi^-(x) \text{ does not C.V.}}\text{''}$$

One has formally $\varphi(\Pi(x)) = \pi^d . \Pi(x)$ and moreover $\mathrm{div}(\Pi^+(x)) = \sum_{n\geq 0}$

<u>How to define $\Pi^-(x)$?</u>

$a = x \bmod \pi$, $a \in \mathfrak{m}_F \setminus \{0\}$

$$\text{``}\underbrace{\prod_{n<0}\varphi^n(a)}_{} = a^{\frac{1}{q}+\frac{1}{q^2}+\cdots} = a^{\frac{1}{q-1}}\text{''}$$

Can write solutions of Kümmer as ∞ not C.V. products.

does not C.V. but you can define it as a solution of $X^{q-1} = a$

Idem for $\Pi^-(x)$: We know thanks to Newton polygon considerations that $\Pi^-(x)$ should lie in $B^{b,+} = W_O(O_F)[\frac{1}{\pi}]$

Moreover, formally $\varphi(\Pi^-(x)) = x . \Pi^-(x)$

[<u>Prop</u>: Up to an $E^\times$-multiple there is a unique solution in $B^{b,+}$ of the functional equation $\varphi(y) = xy$.

Set $\Pi^-(x) =$ this solution (well defined up to an $E^\times$-multiple)

$\varphi(\Pi^-(x)) = x . \Pi^-(x) \Rightarrow \mathrm{div}(\Pi^-(x)) = \sum_{n<0} [\varphi^n(m)]$

Now set $\Pi(x) = \Pi^+(x) \cdot \Pi^-(x)$

Then $\operatorname{div}(\Pi(x)) = \sum_{n \in \mathbb{Z}} [\varphi^n(m)]$

□

The case of degree 1 elements

If we take $\varepsilon \in \mathfrak{m}_F \setminus \{0\}$, $u_\varepsilon = \dfrac{[\varepsilon]_Q}{[\varepsilon^{1/q}]_Q}$

$$\Pi^+(u_\varepsilon) = \prod_{n \geq 0} \frac{\varphi^n(u_\varepsilon)}{\pi}$$

$$= \frac{1}{\pi[\varepsilon^{1/q}]_Q} \underbrace{\lim_{n \to +\infty} \pi^{-n} [\pi^n]_{LT} ([\varepsilon]_Q)}_{\log_{LT}([\varepsilon]_Q)}$$

~~~~

$\Pi^-(u_\varepsilon) =$ solution ~~~~ y to $\varphi(y) = u_\varepsilon \cdot y$

$\rightarrow$ Can take $y = \pi[\varepsilon^{1/q}]_Q =: \Pi^-(u_\varepsilon)$

$$\Rightarrow \boxed{\Pi(u_\varepsilon) = \log_{LT}([\varepsilon]_Q)}$$
~~~~

$\Rightarrow$ the Weierstrass product $\Pi(u_\varepsilon)$ ~~is given by~~ is the Weierstrass product

$$\log_{\mathcal{LT}} = T \cdot \prod_{n\geq 1} \left[\prod_{\zeta \in \mathcal{LT}[\pi^n] \setminus \mathcal{LT}[\pi^{n-1}]} \left(1 - \frac{T}{\zeta}\right) \right]$$

applied to $[\varepsilon]_Q$.

Banach space for the top. induced by the one of B.

Th.: $\mathcal{G}(G_F) \xrightarrow{\sim} (B^+)^{\varphi=\pi} = P_1$ Isomorphism of Banach spaces.

$\varepsilon \longmapsto \log_{\mathcal{LT}}([\varepsilon]_Q)$

Moreover: $[\varepsilon]_Q = \lim_{n\to+\infty} [\pi^n]_{\mathcal{LT}}([\varepsilon^{q^{-n}}])$

$\Rightarrow \log_{\mathcal{LT}}([\varepsilon]_Q) = \lim_{n\to+\infty} \pi^n \log_{\mathcal{LT}}([\varepsilon^{q^{-n}}])$

If $\log_{\mathcal{LT}} = \sum_{n\geq 0} \frac{T^{q^n}}{\pi^n}$ then the preceding iso. is

$$\boxed{\begin{array}{l} (m_{F,\mathcal{LT}}, +) \xrightarrow{\sim} (B^+)^{\varphi=\pi} \\ x \longmapsto \sum_{n\in\mathbb{Z}} [x^{q^{-n}}]\pi^n \end{array}}$$

"Theta functions of weight 1 can be either written as infinite Σ or infinite Π"

$(B^+)^{\varphi=\pi} = H^0(Y/\varphi^{\mathbb{Z}}, \mathcal{L})$ $\forall$ $(\mathcal{L} = (Y \times \mathbb{A}^1)/\mathbb{Z}$

$\mathcal{L}$ given by the automorphy factor

$\mathbb{Z} \to B^\times$, $n \mapsto \pi^n$ $\in Z^1(\varphi^{\mathbb{Z}}, B^\times)$.

Case $E = \mathbb{Q}_p$:

$$H = \exp\left(\sum_{n\geq 0} \frac{T^{p^n}}{p^n}\right) = \text{Artin-Hasse exponential}$$

$$\log_{\mathcal{LT}} = \sum_{n\geq 0} \frac{T^{p^n}}{p^n}$$

$$H : \mathcal{LT} \xrightarrow{\sim} \widehat{\mathbb{G}}_m$$

$$\varepsilon \longmapsto \sum_{n\in\mathbb{Z}} [\varepsilon^{p^{-n}}]\, p^n$$

$$(\mathfrak{m}_F, +_{\mathcal{LT}}) \xrightarrow{\ \sim\ } B^{\varphi=p}$$

$$\simeq \downarrow H \qquad \simeq$$

$$(1+\mathfrak{m}_F, \times) \longrightarrow B^{\varphi=p}$$

$$\varepsilon \longmapsto \log([\varepsilon]) = \sum_{n\geq 1} (-1)^n \frac{([\varepsilon]-1)^n}{n}$$

$$= \text{Fontaine's } t.$$

More precisely, if $\mathfrak{m} = (u_\varepsilon)$ and $\rho = \|\mathfrak{m}\|$ then Fontaine's ring B^+_{cris} associated to $\mathbb{C}_p$ is $B^+_{cris,\rho}$ and $B^+ = \bigcap_{n\geq 0} \varphi^n(B^+_{cris,\rho})$

$$\Rightarrow B^{\varphi=p} = (B^+)^{\varphi=p} = (B^+_{cris,\rho})^{\varphi=p} \ni t.$$