Cours n°4

<u>Recall from last time:</u> $E|\mathbb{Q}_p$, π unif. element, $\mathbb{F}_q$ residue field
$F|E$ complete alg. closed

B = E-Fréchet algebra obtained by completion of

$\circlearrowleft$ φ automorphism $\quad B^b = \left\{ \sum_{n \gg -\infty} [x_n]\pi^n \,/\, x_n \in F, \exists C\ |x_n| \leq C \right\}$

$$|Y| \subset \mathrm{Spm}(B), \quad \|\cdot\| : |Y| \to]0,1[, \quad \|\varphi(m)\| = \|m\|^q$$

$\| \quad \left\{ \text{ideals generated by } \sum_{n \geq 0} [x_n]\pi^n \in W_{\mathcal{O}}(\mathcal{O}_F) \,/\, x_0 \in m_F,\ x_1 \in \mathcal{O}_F^\times \right\}$ (what I called a primitive degree 1 element)

$\forall m \in |Y|$, $C_m = B/m$ = Complete alg. closed extension of E

$B^+_{dR,m}$ = m-adic completion of B = D.V.R.

$\mathrm{ord}_m : B^+_{dR} \to \mathbb{N} \cup \{+\infty\}$ valuation

$$f \in B \setminus \{0\}, \quad \mathrm{div}(f) = \sum_{m \in |Y|} \mathrm{ord}_m(f)\,[m] \in \mathrm{Div}^+(Y)$$

Want to study the graded algebra $P = \bigoplus_{d \geq 0} \underbrace{B^{\varphi = \pi^d}}_{P_d}$.

<u>Multiplicative structure of P</u>

<u>Def.</u> $\mathrm{Div}^+(Y/\varphi^{\mathbb{Z}}) = \left\{ D \in \mathrm{Div}^+(Y) \,/\, \varphi^* D = D \right\}$

$= $ free monoid on $|Y|/\varphi^{\mathbb{Z}} \hookrightarrow \mathrm{Div}^+(Y/\varphi^{\mathbb{Z}})$

$$m \bmod \varphi^{\mathbb{Z}} \longmapsto \sum_{n \in \mathbb{Z}} [\varphi^n(m)]$$

If $f \in P_d$ $\quad \varphi(f) = \pi^d f \Rightarrow \mathrm{div}(\varphi(f)) = \mathrm{div}(\pi^d f) = \mathrm{div}(f)$

$$\mathrm{div}(\varphi(f)) = \varphi^* \mathrm{div}(f)$$

$$\Longrightarrow \mathrm{div}(f) \in \mathrm{Div}^+(Y/\varphi^{\mathbb{Z}}).$$

<u>Th</u>: The morphism of monoids

$$\Big(\underbrace{\bigcup_{d \geq 0} P_d \setminus \{0\}}_{\text{monoid of homogeneous non zero elements of } P}\Big) \Big/ \underbrace{E^\times}_{P_0^\times} \xrightarrow{\ \mathrm{div}\ } \mathrm{Div}^+(Y/\varphi^{\mathbb{Z}})$$

is an isomorphism.

<u>Corollary</u>: P is graded factorial with irreducible elements of degree 1.

<u>Proof</u>: <u>Injectivity</u>: Recall from last time: $f, g \in B \setminus \{0\}$,

$\mathrm{div}(f) = \mathrm{div}(g) \Longleftrightarrow (f) = (g)$.

<u>Here</u>: if $f \in P_d \setminus \{0\}$, $g \in P_{d'} \setminus \{0\}$, $\mathrm{div}(f) = \mathrm{div}(g) \Rightarrow f = ug$, $u \in B^\times \subset B^b$

$$\varphi(u) = \pi^{d-d'} u \text{ but } (B^b)^{\varphi = \pi^{d-d'}} = \begin{cases} 0 & \text{if } d \neq d' \\ E & \text{if } d = d' \end{cases} \Rightarrow u \in E^\times.$$

<u>Surjectivity</u>: <u>Weierstrass products</u>

$x \in W_{\mathcal{O}}(\mathcal{O}_F)$ primitive of degree $d > 0$.

Suppose moreover $\tilde{x} \in W_{\mathcal{O}}(k_F)$ is such that $\tilde{x} = \pi^d$.

that is to say $x \in \pi^d + W_{\mathcal{O}}(\mathfrak{m}_F)$

(one can always suppose this up to multiplying x by a unit)

We are looking for $f \in P_d \setminus \{0\}$ s.t. $\mathrm{div}(f) = \sum_{n\in\mathbb{Z}} \varphi^n(D)$

$$= \sum_{n\geq 0} - + \sum_{n<0} -$$

(split the divisor in two parts)

$$\Pi(x) = \underbrace{\prod_{n\geq 0}\left(\frac{\varphi^n(x)}{\pi^d}\right)}_{\text{C.V. in } B \text{ because } \frac{x}{\pi^d}\in 1+W_{\mathcal{O}}(\mathfrak{m}_F)[\frac{1}{\pi}]} \cdot \text{"}\underbrace{\prod_{n<0}\varphi^n(x)}_{\Pi^-(x) \text{ does not C.V.}}\text{"}$$

formally $\varphi(\Pi(x)) = \pi^d \cdot \Pi(x)$ and $\mathrm{div}(\Pi(x)) = \sum_{n\in\mathbb{Z}}\varphi^n(D)$

How to define $\Pi^-(x)$? $\quad a = x \bmod \pi \in \mathfrak{m}_F \setminus \{0\}$

$$\text{"}\prod_{n<0}\varphi^n(x) \bmod \pi = \underbrace{\prod_{n<0}}_{\text{does not C.V. but you can define it as a solution of } X^{q-1}=a} a^{q^{-n}} = a^{\frac{1}{q}+\frac{1}{q^2}+\dots} = a^{\frac{1}{q-1}}\text{"}$$

Can write solution of Kummer as ∞ not C.V. products

Idem for $\Pi^-(x)$: We know thanks to Newton polygon considerations that $\Pi^-(x)$ should live in $B^{b,+} = W_{\mathcal{O}}(\mathcal{O}_F)[\frac{1}{\pi}]$

Moreover formally $\varphi(\Pi^-(x)) = x \cdot \Pi^-(x)$

[<u>Prop</u>: Up to an $E^\times$-multiple there is a unique non zero solution in $B^{b,+}$ of the functional equation $\varphi(y) = x y$.

→ easy approximation argument + solve Kummer and Artin-Schreier F.

Set $\Pi^-(x)$ = this solution (well defined up to an $E^\times$-multiple)

$$\varphi(\Pi^-(x)) = x.\Pi^-(x) \Rightarrow \mathrm{div}(\Pi^-(x)) = \sum_{n<0} \varphi^n(D)$$

Now set $\Pi(x) = \Pi^+(x).\Pi^-(x)$

then $\mathrm{div}(\Pi(x)) = \sum_{n\in\mathbb{Z}} \varphi^n(D)$ $\square$

The case of degree 1 elements

If we take $\varepsilon \in \mathfrak{m}_F \setminus \{0\}$, $u_\varepsilon = \dfrac{[\varepsilon]_Q}{[\varepsilon^{1/q}]_Q}$ primitive degree 1

$$\Pi^+(u_\varepsilon) = \prod_{n\geq 0} \frac{\varphi^n(u_\varepsilon)}{\pi}$$

$$= \frac{1}{\pi[\varepsilon^{1/q}]_Q} \underbrace{\lim_{n\to+\infty} \pi^{-n} \underbrace{[\varepsilon^{q^n}]_Q}_{[\pi^n]_{LT}([\varepsilon]_Q)}}_{\log_{LT}([\varepsilon]_Q)}$$

$\Pi^{-}(u_\varepsilon) =$ solution y to $\varphi(y) = u_\varepsilon \cdot y$

$\longrightarrow$ Can take $y = \pi[\varepsilon^{1/q}]_Q =: \Pi^{-}(u_\varepsilon)$

$$\Longrightarrow \boxed{\Pi(u_\varepsilon) = \log_{\mathcal{LT}}([\varepsilon]_Q)}$$

$\Longrightarrow$ the Weierstrass product $\Pi(u_\varepsilon)$ is the Weierstrass product

$$\log_{\mathcal{LT}} = T \cdot \prod_{n \geq 1} \left[\prod_{\zeta \in \mathcal{LT}[\pi^n] \setminus \mathcal{LT}[\pi^{n-1}]} \left(1 - \frac{T}{\zeta}\right) \right]$$

applied to $[\varepsilon]_Q$.

$*$ If $\log_{\mathcal{LT}} = \sum_{n \geq 0} \frac{T^{q^n}}{\pi^n}$ then the preceding iso. is

$$\boxed{\begin{array}{ccc} (\mathfrak{m}_F, +_{\mathcal{LT}}) & \xrightarrow{\sim} & (B^+)^{\varphi = \pi} \\ x & \longmapsto & \sum_{n \in \mathbb{Z}} [x^{q^{-n}}] \pi^n \end{array}} \in BW_{\mathcal{O}_E}(\mathcal{O}_F)$$

In fact, $[\varepsilon]_Q = \lim_{n \to +\infty} [\pi^n]_{\mathcal{LT}}([\varepsilon^{q^{-n}}])$ (by definition)

$\Rightarrow \log_{\mathcal{LT}}([\varepsilon]_Q) = \lim_{n \to +\infty} \pi^n \log_{\mathcal{LT}}([\varepsilon^{q^{-n}}])$

<u>Case $E = \mathbb{Q}_p$:</u>

$$H = \exp\left(\sum_{n\geq 0} \frac{T^{p^n}}{p^n}\right) = \text{Artin-Hasse exponential}$$

$$\log_{\mathcal{LT}} = \sum_{n\geq 0} \frac{T^{p^n}}{p^n}$$

$$H : \mathcal{LT} \xrightarrow{\sim} \widehat{\mathbb{G}}_m$$

$$\begin{tikzcd}
\varepsilon \arrow[r, mapsto] & \displaystyle\sum_{n\in\mathbb{Z}} [\varepsilon^{p^{-n}}]\, p^n \\
(\mathfrak{m}_F, +_{\mathcal{LT}}) \arrow[r, "\sim"] \arrow[d, "H"', "\simeq"] & B^{\varphi = p} \\
(1+\mathfrak{m}_F, \times) \arrow[ur, "\simeq"] & \varepsilon \arrow[r, mapsto] & \log([\varepsilon]) = \displaystyle\sum_{n\geq 1} (-1)^n \frac{([\varepsilon]-1)^n}{n}
\end{tikzcd}$$

$= \text{Fontaine's } t$

More precisely, if $\mathfrak{m} = (u_\varepsilon)$ and $\rho = \|\mathfrak{m}\|$ then Fontaine's ring B^+_{cris} associated to $C_{\mathfrak{m}} = B/\mathfrak{m}$ is $B^+_{cris,\rho}$ and $B^+ = \bigcap_{n\geq 0} \varphi^n(B^+_{cris,\rho})$ and $\prod(u_\varepsilon) = \text{Fontaine's } t \in B^+_{cris,\rho}$.

↑ depends on $\mathfrak{m}$.

Back to the curve:

Set $X = \mathrm{Proj}(P)$

$P = \bigoplus_{d\geq 0} B^{\varphi=\pi^d}$ graded factorial with irreducible elements of degree 1.

$\Rightarrow$ generated by degree 1 elements and if $t \in P_1 \setminus \{0\}$

$$D^+(t) = \mathrm{Spec}\big(\underbrace{P[\tfrac{1}{t}]_0}_{B[\frac{1}{t}]^{\varphi=Id}}\big)$$

$B\left[\frac{1}{t}\right]^{\varphi=Id}$ = factorial ring

Purpose now: Prove that this is a PID that is to say all irreducible elements generate a maximal ideal.

Fundamental exact sequence

Th: $t_1,\dots,t_n \in P_1\setminus\{0\}$, $i\neq j \Rightarrow t_i \notin E t_j$. $a_1,\dots,a_n \in \mathbb{N}_{\geq 1}$, $d = \sum_i a_i$. For $1\leq i\leq n$ let $m_i \in |Y|$ be such that $\mathrm{div}(t_i) = \sum_{n\in\mathbb{Z}} [\varphi^n(m_i)]$.

Then there is an exact sequence

$$0 \to E.\prod_{i=1}^n t_i^{a_i} \longrightarrow B^{\varphi=\pi^d} \xrightarrow{\ u\ } \prod_{i=1}^n B^+_{dR,m_i}/m_i^{a_i} B^+_{dR,m_i} \longrightarrow 0$$

Proof: $x \in B^{\varphi=\pi^d}\setminus\{0\}$ s.t. $u(x)=0$. This means $\forall i$, $x \in m_i^{a_i}$

$\Rightarrow \mathrm{div}(x) \geq \sum_{i=1}^n a_i [m_i]$. But $\varphi(x) = \pi^d x \Rightarrow \mathrm{div}(x) = \varphi(\mathrm{div}(x))$

$$\Rightarrow \mathrm{div}(x) \geq \sum_{i=1}^n a_i \underbrace{\sum_{n\in\mathbb{Z}} [\varphi^n(m_i)]}_{\mathrm{div}(t_i)} = \mathrm{div}\left(\prod_{i=1}^n t_i^{a_i}\right)$$

$\Rightarrow x = z \cdot \prod_{i=1}^{n} t_i^{a_i}$, $z \in B$ but $\varphi(x) = \pi^{d} x \Rightarrow z \in B^{\varphi = \mathrm{Id}} = E$.

$\uparrow$ preceding : $x, y \in B$, $\mathrm{div}(x) \geq \mathrm{div}(y) \Leftrightarrow x \in (y)$.

* <u>Surjectivity of u:</u> Easily reduced to the case $d=1$ that is to say for $m \in |Y|$, $\partial_m : B^{\varphi=\pi} \xrightarrow{\partial_m} C_m$ is surjective.

Note $C = C_m$, $\partial = \partial_m$.

<u>Recall</u>: $\mathcal{L}_T$ = Lubin-Tate groupe law$/\mathcal{O}_E$

$\mathcal{G}$ = associated formal group.

$$(m_F, +_{\mathcal{L}_T}) \xrightarrow{\sim} B^{\varphi=\pi}$$

$$x \longmapsto \lim_{n\to+\infty} \pi^n \log_{\mathcal{L}_T}\left([x^{q^{-n}}]\right)$$

For A = adic $\mathcal{O}_E$-algebra set

$$X(\mathcal{G})(A) = \{(x_n)_{n\in\mathbb{Z}} \mid x_n \in \mathcal{G}(A) \text{ and } \pi x_{n+1} = x_n\}$$

<u>Then</u>: $X(\mathcal{G})(\mathcal{O}_C) \xrightarrow{\sim} X(\mathcal{G})(\mathcal{O}_C/\pi\mathcal{O}_C) = X(\mathcal{G}_0)(\mathcal{O}_C/\pi\mathcal{O}_C)$ where $\mathcal{G}_0 = \mathcal{G} \otimes \mathbb{F}_q$.

$$\Big(\lim_{b\to+\infty} \pi^b \underbrace{\widehat{x_{n+b}}}_{\text{any lift of } x_{n+b}}\Big)_{n\geq 0} \longleftarrow\!\!\shortmid \ (x_n)_{n\in\mathbb{Z}}$$

Isomorphism of projective systems

$$\begin{array}{ccccccc}
\mathcal{G}_0 & \xleftarrow{\times\pi} & \mathcal{G}_0 & \xleftarrow{\times\pi} & \mathcal{G}_0 & \xleftarrow{\times\pi} & \cdots \\
\| & & \downarrow V_\pi & & \downarrow V_\pi & & \\
\mathcal{G}_0 & \xleftarrow{F} & \mathcal{G}_0 & \xleftarrow{F} & \mathcal{G}_0 & \xleftarrow{F} & \cdots
\end{array}$$

that induces an isomorphism

$$X(\mathcal{G}_0)(\mathcal{O}_C/\pi\mathcal{O}_C) \xrightarrow{\sim} \varprojlim_{\mathbb{N},F} \mathcal{G}_0(\mathcal{O}_C/\pi\mathcal{O}_C) = \mathcal{G}_0\big(R(\mathcal{O}_C/\pi\mathcal{O}_C)\big)$$

$$\mathcal{G}_0\big(R(\mathcal{O}_C/\pi\mathcal{O}_C)\big) = \mathcal{G}_0(\mathcal{O}_F)$$

via $\mathcal{O}_F \xrightarrow{\sim} R(\mathcal{O}_C) \xrightarrow{\sim} R(\mathcal{O}_C/\pi\mathcal{O}_C)$

Then

$$\begin{tikzcd}
\mathcal{G}_0(\mathcal{O}_F) = (\mathfrak{m}_F, +_{LT}) \arrow[r, "\sim"] & B^{\varphi=\pi} \arrow[r, "\theta"] & C \\
X(\mathcal{G})(\mathcal{O}_C) \arrow[u, "\simeq"] \arrow[urr] & & \\
(x_n)_{n\in\mathbb{N}} \arrow[rr, mapsto] & & \log_{LT}(x_0)
\end{tikzcd}$$

surjective since C is alg. closed

$\Rightarrow \log_{LT} : \mathcal{G}(\mathcal{O}_C) \to C$ is surjective

and $\mathcal{G}(\mathcal{O}_C) \xrightarrow{\times\pi} \mathcal{G}(\mathcal{O}_C)$ " " $\square$

The Curve

Th: The scheme $X = \mathrm{Proj}(P)$ is an integral noetherian regular scheme of dimension 1 (i.e. the gluing of a finite number of spectra of Dedekind rings)

Moreover:

1) $\forall t \in P_1 \setminus \{0\}$, $D^+(t) = \mathrm{Spec}\left(P\left[\frac{1}{t}\right]_0\right)$ with $P\left[\frac{1}{t}\right]_0 = B\left[\frac{1}{t}\right]^{\varphi = Id}$ is a P.I.D.

2) $\forall t \in P_1 \setminus \{0\}$, $V^+(t) = \{\infty_t\}$ one closed point and if $\mathrm{div}(t) = \sum_{n \in \mathbb{Z}} [\varphi^n(m)]$ there is a canonical identification

$$\widehat{\mathcal{O}}_{X, \infty_t} = B^+_{dR, m}.$$

3) The application

$$P_1 \setminus \{0\} / E^\times \longrightarrow |X| \quad \text{is a bijection.}$$
$$E^\times . t \longmapsto \infty_t$$

(closed points)

Proof: 1) We have allready seen that $B_e := B\left[\frac{1}{t}\right]^{\varphi = Id}$ is a U.F.D. with irreducible elements $\left\{ \frac{t'}{t} \,/\, t' \in P_1 \setminus E.t \right\}$. For such a t' let $m' \in |Y|$ be such that $\mathrm{div}(t') = \sum_{n \in \mathbb{Z}} [\varphi^n(m')]$. Then since $t' \notin E.t$, $\vartheta_{m'}(t) \neq 0$ and thus $\vartheta_{m'}$ induces a morphism

$$\begin{cases} B_e \xrightarrow{\tilde{\partial}_{m'}} C_{m'} \\ \dfrac{x}{t^d} \longmapsto \partial_{m'}(t)^{-d}\partial_{m'}(x) \end{cases}$$

Using the fundamental exact sequences associated to the powers of t' one checks (exo.) that $\begin{cases} \ker(\tilde{\partial}_{m'}) = B_e \cdot \frac{t'}{t} \\ \tilde{\partial}_{m'} \text{ is surjective.} \end{cases}$

$\Rightarrow B_e \cdot \frac{t'}{t}$ is a maximal ideal.

2) Let $t \in P_1 \setminus \{0\}$, $\mathrm{div}(t) = \sum_{n \in \mathbb{Z}} [\varphi^n(m)]$, $C = C_m$, $\partial = \partial_m$.

$V^+(t) = \mathrm{Proj}(P/tP)$

One has $P/tP = E \oplus \bigoplus_{d \geq 1} P_d / tP_{d-1}$

Let $A = \{ f \in C[X] \,/\, f(0) \in E \}$. Then, there is a morphism of graded E-algebras
graded by the degree of f.

$$P/tP = E \oplus \bigoplus_{d\geq 1} P_d/tP_{d-1} \longrightarrow A$$

$$(\lambda, (x_d \bmod tP_{d-1})_{d\geq 1}) \longmapsto \lambda + \sum_{d \geq 1} \partial(x_d)$$

Fundamental exact sequence $\Rightarrow$ this is an iso.

One computes $\mathrm{Proj}(P/tP) = \mathrm{Proj}(A) =$ one element set given by the ideal (0) of P/tP.

$\Rightarrow V^+(t) = \{\infty_t\}$ one closed point of X.

Moreover $\underbrace{\mathcal{O}_{X,\infty_t}}_{\text{D.V.R.}} = \left\{ \frac{x}{y} \;\middle|\; d \geq 0,\ x \in P_d \text{ and } y \in P_d \setminus t P_{d-1} \right\} \subset \mathrm{Frac}(P)$

with uniformizing element $\frac{t}{t'}$ for any $t' \in P_1 \setminus E.t$.

Then $\mathcal{O}_{X,\infty_t} \hookrightarrow B^+_{dR,m}$ and this induces an isomorphism

$\widehat{\mathcal{O}_{X,\infty_t}} \xrightarrow{\sim} B^+_{dR,m}$ since $\frac{t}{t'}$, $t' \in P_1 \setminus E.t$, is a unif. element of $B^+_{dR,m}$ and by the fundamental exact sequence they have the same residue field.

$\square$