The Curve

$$P = \bigoplus_{d \geq 0} \underbrace{B^{\varphi = \pi^d}}_{P_d} \qquad P_0 = E$$

Th: The scheme $X = \mathrm{Proj}(P)$ is an integral noetherian regular scheme of dimension 1 (i.e. the gluing of a finite number of spectra of Dedekind rings)

Moreover:

1) $\forall t \in P_1 \setminus \{0\}$, $D^+(t) = \mathrm{Spec}\left(P\left[\frac{1}{t}\right]_0\right)$ with

$P\left[\frac{1}{t}\right]_0 = B\left[\frac{1}{t}\right]^{\varphi = \mathrm{Id}}$ is a PID

2) $\forall t \in P_1 \setminus \{0\}$, $V^+(t) = \{\infty_t\}$ one closed point and if $\mathrm{div}(t) = \sum_{n \in \mathbb{Z}} [\varphi^n(m)]$, $m \in |Y|$, there is a canonical identification $\widehat{\mathcal{O}_{X,\infty_t}} = B^+_{dR,m}$

3) The application

$$P_1 \setminus \{0\} / E^\times \longrightarrow |X| \quad \text{(closed points)} \quad \text{is a bijection}$$

$$E^\times \cdot t \longmapsto \infty_t$$

and thus $\boxed{|Y|/\varphi^{\mathbb{Z}} \xrightarrow{\sim} |X|}$
$m \bmod \varphi^{\mathbb{Z}} \longmapsto V^+(t_m)$ where $\mathrm{div}(t_m) = \sum_{n \in \mathbb{Z}} [\varphi^n(m)]$

Proof: Consequence of the fundamental exact sequence (or you can prove it directly by the same proof we used for the fundamental exact sequence): $t \in P_1 \setminus \{0\}$, $\mathrm{div}(t) = \sum_{n \in \mathbb{Z}} [\varphi^n(m)]$

if $1 \leq d' \leq d$ $\quad 0 \to t^{d'} P_{d-d'} \to P_d \to B^+_{dR,m} / t^{d'} B^+_{dR,m} \to 0$

In particular $\boxed{d \geq 1,\ \theta_m : P_d / t P_{d-1} \xrightarrow{\sim} C_m}$ (*)

1) We have allready seen that $B_e := B\left[\frac{1}{t}\right]^{\varphi = Id}$ is a U.F.D. with irreducible elements $\left\{ \frac{t'}{t} \;/\; t' \in P_1 \setminus E.t \right\}$. For such a t' let $m' \in |Y|$ be such that $\operatorname{div}(t') = \sum_{n \in \mathbb{Z}} [\varphi^n(m')]$.

$t' \notin E.t \Rightarrow \theta_{m'}(t) \neq 0$ and thus $\theta_{m'} : B \twoheadrightarrow C_{m'}$ induces a morphism

$$B_e \xrightarrow{u} C_{m'}$$

* Since $\theta_{m'} : P_1 \twoheadrightarrow C_{m'}$ is surjective,

$$\begin{aligned} P_1 \hookrightarrow B_e &\xrightarrow{u} C_{m'} \text{ is surjective} \Rightarrow u \text{ is surjective.} \\ x &\longmapsto \frac{x}{t} \end{aligned}$$

* If $\frac{x}{t^d} \in \ker u$, $x \in P_d$, $\theta_{m'}(x) = 0 \Rightarrow x \in t' P_{d-1}$

$\uparrow$ consequence of fund. exact sequence

$$\Rightarrow \frac{x}{t^d} \in \frac{t'}{t} \cdot B_e$$

$$\Rightarrow \ker(u) = \frac{t'}{t} \cdot B_e$$

$\Rightarrow \frac{t'}{t}$ generates a maximal ideal of B_e

$\Rightarrow B_e$ is a PID. $\Rightarrow X$ is a noetherian regular ~~[illegible]~~ scheme of dim. 1.

2)

(2)

2) Consequence of the fundamental exact sequence:

iso. of graded algebra

$$P/tP \xrightarrow{\sim} \{ f \in C_m[X] \,/\, f(0) \in E \} =: A$$

$$\bar{x} \longmapsto \theta_m(x) X^d$$

$\cap$

P_d/tP_{d-1}

One checks easily $\mathrm{Proj}(P/tP) = \mathrm{Proj}(A) = \{ \text{ideal } (0) \text{ of } P/tP \}$

$\Rightarrow V^+(t) = \{\infty_t\}$ one closed point.

One then has $\mathcal{O}_{X,\infty_t} = \left\{ \frac{x}{y} \in \mathrm{Frac}(P) \,/\, x \in P_d,\ y \in P_d \setminus tP_{d-1} \text{ for some } d \in \mathbb{N} \right\}$

$\Updownarrow$

$\theta_m(y) \neq 0 \Leftrightarrow y \in (B^+_{dR})^\times$

Thus $\mathcal{O}_{X,\infty_t} \hookrightarrow B^+_{dR,m}$. Moreover this injection of DVR induces an iso on residue fields and unif. element $\mapsto$ unif. element

$\Rightarrow \widehat{\mathcal{O}_{X,\infty_t}} \xrightarrow{\sim} B^+_{dR,m}$ iso. on completion $\square$

$*$ Thus, concretely, if $K = \mathcal{O}_{X,\eta} = \left\{ \frac{x}{y} \,/\, x \in P_d,\ y \in P_d \setminus \{0\} \text{ for some } d \geq 0 \right\}$

$\subset \mathrm{Frac}(P)$

is the field of rational functions on X one has $\forall x \in |X|$ a valuation

$$v_x : K \longrightarrow \mathbb{Z} \cup \{+\infty\}$$

S.t. ~~Open~~ non-empty open subsets of X = Complementary of finite sets

and $\forall U \subset X$ non empty open $\Gamma(U, \mathcal{O}_X) = \{f \in K \,/\, \forall x \in U,\ v_x(f) \geq 0\}$.

More on the structure of Be:

$t \in P_1 \setminus \{0\}$, $\infty = \infty_t$, $B_e = B[\frac{1}{t}]^{\varphi = Id} = \Gamma(X \setminus \{\infty\}, \mathcal{O}_X)$

$B_{dR}^+ = \widehat{\mathcal{O}}_{X,\infty}$

v_∞ = valuation of $\mathcal{O}_{X,\infty}$ associated to ∞

via $B_e \subset B_{dR} = B_{dR}^+[\frac{1}{t}]$

Set $\deg = -v_\infty : B_e \longrightarrow \mathbb{Z} \cup \{-\infty\}$

and $\forall i \in \mathbb{Z}$, $Fil_i B_e = \{\deg \leq i\} = B_e \cap t^{-i} B_{dR}^+$

$= \{\frac{x}{t^i} \,/\, x \in P_i\}$

Then:
$$\begin{cases} Fil_i B_e = 0 \text{ if } i < 0 \quad \text{i.e. } \deg : B_e \to \mathbb{N} \cup \{-\infty\} \\ Fil_0 B_e = E = \Gamma(X, \mathcal{O}_X) \\ \text{If } i \geq 0,\ Fil_{i+1} B_e / Fil_i B_e \xrightarrow{\sim} t^{-i-1} B_e / t^{-i} B_e \end{cases}$$

But $Fil_0 B_e \hookrightarrow B_{dR}^+ / t B_{dR}^+ = C$

not surjective

$\rightarrow$ different from $\mathbb{P}^1_k$, $B_e = k[T] = \Gamma(\mathbb{P}^1 \setminus \{\infty\}, \mathcal{O}_{\mathbb{P}^1})$, $\deg = -v_\infty$

Line bundles on X

$$E(X) = \mathcal{O}_{X,\eta} \overset{\text{non trivial}}{=} \underbrace{\mathrm{Frac}(B)^{\varphi = \mathrm{Id}}}_{\text{"meromorphic functions on } Y \text{ invariant under } \varphi\text{"}}$$

Def: $f \in E(X)^\times$, $\mathrm{div}(f) = \sum_{x \in |X|} v_x(f)[x] \in \underbrace{\mathrm{Div}(X)}_{\text{free ab. group on } |X|}$

$$\underset{\text{Weil divisors}}{\mathrm{Div}(X)} \simeq \underset{\text{Cartier divisors}}{\{(\mathcal{L}, s)\}/\sim}$$

$\mathcal{L}$: line bundle
$s \in \mathcal{L}_\eta \setminus \{0\}$ generic trivialisation

$D \longmapsto (\mathcal{O}_X(D), 1)$ where $\Gamma(U, \mathcal{O}_X(D)) = \{ f \in E(X) \,/\, (\mathrm{div} f + D)_{|U} \geq 0 \}$

$\mathrm{div}(s) \longleftarrow (\mathcal{L}, s)$

$$\mathrm{Div}(X)/\sim \ \xrightarrow{\sim} \mathrm{Pic}(X)$$

principal divisors

Lemma: $\forall f \in E(X)^\times$, $\deg(\mathrm{div} f) = 0$

$\to$ If $t \in P_1 \setminus \{0\}$, $B_e = \Gamma(X \setminus V^+(t), \mathcal{O}_X)$

$\forall x \in B_e \setminus \{0\}$, $x = \underset{E^\times}{\underset{\cap}{u}} \prod_{i=1}^{n} \left(\frac{t'_i}{t}\right)$ $\quad t'_i \in P_1 \setminus Et$

then $\mathrm{div}(x) = \sum_{i=1}^{n} [\infty_{t'_i}] - n[\infty_t]$ $\square$

Thus: $\boxed{\mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z}}$

Def: $d \in \mathbb{Z}$ set $\mathcal{O}_X(d) := \widetilde{P\underbrace{[d]}_{\text{shift by } d}}$ line bundle on X.

$$\mathcal{O}_X(d) = \mathcal{O}_X(1)^{\otimes d}$$

One has $H^0(X, \mathcal{O}_X(d)) = \begin{cases} 0 & \text{if } d < 0 \\ P_d & \text{if } d \geq 0 \end{cases}$

* Moreover, if $t \in H^0(X, \mathcal{O}_X(1))$, $\mathrm{div}(t) = [\infty_t]$ $\quad (\{\infty_t\} = V^+(t))$

Thus $t : \mathcal{O}_X([\infty_t]) \xrightarrow{\sim} \mathcal{O}_X(1)$

$\Rightarrow \deg(\mathcal{O}_X(1)) = 1$ and $\boxed{\forall d \in \mathbb{Z},\ \deg(\mathcal{O}_X(d)) = d.}$

$$\begin{tikzcd}
H^0(X,\mathcal{O}_X) \arrow[r, "\times t"] \arrow[d, equal] & H^0(X,\mathcal{O}_X(1)) \arrow[r, "\times t"] \arrow[d, equal] & H^0(X,\mathcal{O}_X(2)) \arrow[r] \arrow[d, equal] & {} \\
P_0 \arrow[r, hook, "\times t"] \arrow[d, equal] & P_1 \arrow[r, hook, "\times t"] \arrow[d, "\simeq"', "x \mapsto x/t"] & P_2 \arrow[r, hook] \arrow[d, "\simeq", "x \mapsto x/t^2"'] & \cdots \\
\mathrm{Fil}_0 B_e \arrow[r, phantom, "\subset"] \arrow[d, equal] & \mathrm{Fil}_1 B_e \arrow[r, phantom, "\subset"] \arrow[d, equal] & \mathrm{Fil}_2 B_e \arrow[r, phantom, "\subset"] \arrow[d, equal] & \cdots \\
H^0(X,\mathcal{O}_X) \arrow[r, hook] & H^0(X,\mathcal{O}_X([\infty_t])) \arrow[r, phantom, "\subset"] & H^0(X,\mathcal{O}_X(2[\infty_t])) \arrow[r, phantom, "\subset"] & \cdots
\end{tikzcd}$$

Prop: $\deg : \mathrm{Pic}(X) \xrightarrow{\sim} \mathbb{Z}$ with inverse $d \mapsto [\mathcal{O}_X(d)]$

$\rightarrow$ B_e is a PID $\Rightarrow \mathrm{Pic}(X \setminus \{\infty\}) = 0$

$$\Rightarrow \mathrm{Pic}(X) \simeq \underbrace{B_e^\times}_{E^\times} \backslash B_{dR}^\times / (B_{dR}^+)^\times = B_{dR}^\times / (B_{dR}^+)^\times \xrightarrow[\sim]{v_\infty} \mathbb{Z}$$

$$\mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z}$$

$\square$

Fix $\infty \in |X|$, $\{\infty\} = V^+(t)$

$$B_e = \Gamma(X \setminus \{\infty\}, \mathcal{O}_X) \subset B_{dR} = B_{dR}^+\left[\tfrac{1}{t}\right]$$

* Bun_X = Category of vector bundles on X

$\mathcal{C}$ = Category of couples (M, W)

W: free B_{dR}^+-module of finite rank

M: sub-B_e-module of $\underbrace{W\left[\tfrac{1}{t}\right]}_{B_{dR}\text{-v.s.}}$ such that $M \underset{B_e}{\otimes} B_{dR} = W$.

B_e is a PID $\Rightarrow$

$$\boxed{\begin{array}{ccc} \mathrm{Bun}_X & \xrightarrow{\sim} & \mathcal{C} \\ \mathcal{E} & \longmapsto & \left(H^0(X \setminus \{\infty\}, \mathcal{E}),\ \widehat{\mathcal{E}}_\infty\right) \end{array}}$$

* If $\mathcal{E} \longleftrightarrow (M, W)$ one has

$$R\Gamma(X, \mathcal{E}) \simeq \Big[\underset{0}{M \oplus W} \longrightarrow \underset{+1}{W\left[\tfrac{1}{t}\right]}\Big]$$
$$(x, y) \longmapsto x - y$$

In particular
$$\begin{cases} H^0(X, \mathcal{E}) = M \cap W \\ H^1(X, \mathcal{E}) = W\left[\tfrac{1}{t}\right] / M + W \end{cases}$$

* One has $\mathcal{O}_X(d)$ $(B_e,\ t^{-d} B_{dR}^+)$

$\mathcal{O}_X(d) = \mathcal{O}_X(d[\infty])$

via the preceding equivalence

Thus: $H^1(X, \mathcal{O}_X(d)) = B_{dR} / t^{-d}B^+_{dR} + B_e$

[**Prop**: $H^1(X, \mathcal{O}_X(d)) \simeq \begin{cases} 0 & \text{if } d \geq 0 \\ B^+_{dR} / t^{-d}B^+_{dR} + E \neq 0 & \text{if } d < 0 \end{cases}$ $\simeq C/E$

In particular $H^1(X, \mathcal{O}_X) = 0$ and $H^1(X, \mathcal{O}_X(-1)) \neq 0$.

like $\mathbb{P}^1$ — different from $\mathbb{P}^1$]

→ * $(B_e, \deg)$ almost euclidean $\Rightarrow$ if $\frac{x}{y} \in \mathrm{Frac}(B_e)$ then $\exists a, b \in B_e$

$x = ay + b$, $\deg(b) \leq \deg(y)$ thus $\frac{x}{y} = a + \underbrace{\frac{b}{y}}_{\deg \leq 0} \Rightarrow \frac{b}{y} \in B^+_{dR}$

$$\Rightarrow B_{dR} = B^+_{dR} + B_e \Rightarrow H^1(X, \mathcal{O}_X) = 0$$

and $H^1(X, \mathcal{O}_X(d)) = 0$ if $d \geq 0$.

* $d < 0$

$$B_{dR} / t^{-d}B^+_{dR} + B_e = B^+_{dR} + B_e / t^{-d}B^+_{dR} + B_e$$

$$= B^+_{dR} / t^{-d}B^+_{dR} + \underbrace{B_e \cap B^+_{dR}}_{E} \qquad \square$$

$\boxed{H^1(X, \mathcal{O}_X(-1)) \neq 0 \text{ precisely because } (B_e, \deg) \text{ is not euclidean}}$

For $\mathbb{P}^1$, $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$ because $(k[T], \deg)$ is euclidean.