Tower of curves

* F fixed $E|\mathbb{Q}_p$, $\mathbb{F}_q \subset F$ residue field $\rightsquigarrow B_E \supset \varphi_E$

π_E unif.

$$P_{E,\pi_E} = \bigoplus_{d\geq 0} B_E^{\varphi_E = \pi_E^d}$$

$X_E = \mathrm{Proj}(P_{E,\pi_E})$ does not depend canonically on the choice of π_E - $\mathcal{O}_{X_E}(1)$ does depend (two choices give isomorphic line bundles but no canonical isomorphism)

If $E'|E$ canonical isomorphism $\boxed{X_{E'} \xrightarrow{\sim} X_E \otimes_E E'}$

Particular case: $E_h|E$ unramified of degree h with residue field $\mathbb{F}_{q^h} = F^{\mathrm{Frob}_q^h = \mathrm{Id}}$. Then: $B_{E_h} = B_E$, $\boxed{\varphi_{E_h} = \varphi_E^h}$, $\pi_{E_h} = \pi_E$

$$P_{E_h,\pi_E} = \bigoplus_{d\geq 0} B_E^{\varphi_E^h = \pi_E^d}$$

$B_E^{\varphi_E = \pi_E^d} \hookrightarrow B_E^{\varphi_E^h = \pi_E^{hd}}$ induces $P_{E,\pi_E,\bullet} \hookrightarrow P_{E_h,\pi_E,h\bullet}$ (morphism of graded rings)

induces $X_{E_h} = X_E \otimes_E E_h$.

* Set now $X = X_E$, $X_h = X_{E_h}$

$(X_h)_{h\geq 1}$ = pro-Galois covering of $X = X_1$ wt. group $\widehat{\mathbb{Z}}$.

* Note $\pi_h : \boxed{X_h \longrightarrow X}$ galois with group $\mathbb{Z}/h\mathbb{Z}$

totally decomposed: $\forall x \in |X|$, $\# \pi^{-1}(x) = h$.

If $\mathcal{E} \in \mathrm{Bun}_X$ $\begin{cases} \deg \pi_h^* \mathcal{E} = h . \deg \mathcal{E} \\ \mathrm{rk}\, \pi_h^* \mathcal{E} = \mathrm{rk}\, \mathcal{E} \end{cases}$ for ex. $\pi_h^* \mathcal{O}_X(d) = \mathcal{O}_{X_h}(hd)$

If $\mathcal{E} \in \mathrm{Bun}_{X_h}$ $\begin{cases} \deg \pi_{h*} \mathcal{E} = \deg \mathcal{E} \\ \mathrm{rk}\, \pi_{h*} \mathcal{E} = h . \mathrm{rk}\, \mathcal{E} \end{cases}$

[Def: $\lambda \in \mathbb{Q}$, $\lambda = \frac{d}{h}$, $d \in \mathbb{Z}$, $h \in \mathbb{N}_{\geq 1}$, $(d,h) = 1$. Set

$$\mathcal{O}_X(\lambda) = \pi_{h*} \mathcal{O}_{X_h}(d)$$

$\boxed{\mu(\mathcal{O}_X(\lambda)) = \lambda}$ where $\mu = \frac{\deg}{\mathrm{rk}}$ (Harder-Narasimhan slope)

Some properties: * $\mathcal{O}(\lambda) \otimes \mathcal{O}(\mu) = \bigoplus_{\text{finite}} \mathcal{O}(\lambda + \mu)$

* $\mathcal{O}(\lambda)^\vee = \mathcal{O}(-\lambda)$

* $H^0(\mathcal{O}(\lambda)) = 0$ if $\lambda < 0 \Rightarrow \Big[\mathrm{Hom}(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = \bigoplus_{\text{finite}} H^0(\mathcal{O}(\mu - \lambda))$

$= 0$ if $\mu < \lambda \Big]$

$$H^1(\mathcal{O}(\lambda)) = 0 \text{ if } \lambda \geq 0 \Rightarrow \left[\operatorname{Ext}^1(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = \bigoplus_{\text{finite}} H^1(\mathcal{O}(\mu - \lambda)) = 0 \text{ if } \mu \geq 1 \right]$$

$$\left[\text{if } \lambda = \frac{d}{h},\ (d,h) = 1,\ H^1(X, \mathcal{O}_X(\lambda)) = H^1(X, \pi_{h*}\mathcal{O}_{X_h}(d)) = H^1(X_h, \mathcal{O}_{X_h}(d)) = 0 \text{ if } d \geq 0 \right.$$

$\rightarrow$ need the preceding results (vanishing of $H^1(\mathcal{O}(d))$ for $d \geq 0$) for all curves X_h, $h \geq 1$.

<u>Th.</u> 1) The semi-stable vector bundles of slope λ are the direct sums of $\mathcal{O}_X(\lambda)$

2) The HN filtration of a vector bundle splits

3) The application

$$\{\lambda_1 \geq \dots \geq \lambda_n \,/\, n \in \mathbb{N}, \lambda_i \in \mathbb{Q}\} \longrightarrow \operatorname{Bun}_X/\sim$$

$$(\lambda_1, \dots, \lambda_n) \longmapsto \left[\bigoplus_{i=1}^{n} \mathcal{O}_X(\lambda_i)\right]$$

is a bijection

<u>Sketch of proof:</u> * 1)+2) $\Rightarrow$ 3)

* $\operatorname{Ext}^1(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = 0$ if $\lambda \leq \mu$

Thus 2) $\Rightarrow$ 1).

It remains to prove 1): one proves the statement simultaneously for all X_h, $h \geq 1$.

<u>Th</u>: The theorem is equivalent to: let $\mathcal{E}$ be a vector bundle that is an extension $0 \to \mathcal{O}_X(-\frac{1}{n}) \to \mathcal{E} \to \mathcal{O}_X(1) \to 0$, $n \geq 1$ then $H^0(X, \mathcal{E}) \neq 0$.

<u>Proof</u>: * If the ~~statement~~ $\boxed{\text{main theorem}}$ is true let $\mathcal{E}$ be an extension as in the statement. $\Downarrow$ $\mathcal{E} \simeq \bigoplus_{i=1}^{n} \mathcal{O}_X(\lambda_i)$. Then $\exists i_0$ s.t.

$$\mathrm{Hom}\left(\mathcal{O}_X\left(-\tfrac{1}{n}\right), \mathcal{O}_X(\lambda_{i_0})\right) \neq 0 \Rightarrow \lambda_{i_0} \geq -\frac{1}{n}.$$

Write $\lambda_{i_0} = \frac{d}{h}$, $(d,h) = 1$. $\mathrm{rk}(\mathcal{E}) = n+1 \Rightarrow h \leq n+1$.

* If $h = n+1$, $\mathrm{rk}(\mathcal{O}_X(\lambda_{i_0})) = \mathrm{rk}(\mathcal{E}) \Rightarrow \mathcal{E} = \mathcal{O}_X(\lambda_{i_0})$

  But $\deg \mathcal{E} = 0 \Rightarrow \lambda_{i_0} = 0 \Rightarrow \mathcal{O}_X(\lambda_{i_0}) = \mathcal{O}_X = \mathcal{E}$ _ Impossible.

* If $h \leq n$, $\frac{d}{h} \geq -\frac{1}{n} \Rightarrow \begin{cases} \frac{d}{h} \geq 0 \Rightarrow H^0(X, \mathcal{O}_X(\lambda_{i_0})) \neq 0 \\ \qquad \Rightarrow H^0(X, \mathcal{E}) \neq 0 \\ \text{or } \frac{d}{h} = -\frac{1}{n} \Rightarrow \mathcal{E} = \mathcal{O}_X\left(-\frac{1}{n}\right) \oplus \mathcal{O}_X(b) \\ \text{for some } b \in \mathbb{Z}. \text{ But } \deg \mathcal{E} = 0 \\ \Rightarrow b = 1 \Rightarrow H^0(X, \mathcal{E}) \neq 0. \end{cases}$

* In the other direction. Let $\mathcal{E}$ be a semi-stable vector bundle on X.

For $h \geq 1$, one checks: * $\mathcal{E}$ s.s. $\iff \pi_h^* \mathcal{E}$ s.s.

* $\exists \lambda, m$ s.t. $\mathcal{E} \simeq \mathcal{O}_X(\lambda)^m \iff \exists \mu, m$ s.t. $\pi_h^*\mathcal{E} \simeq \mathcal{O}_{X_h}(\mu)^m$.

But $\mu(\pi_h^*\mathcal{E}) = h.\mu(\mathcal{E})$. Thus, up to replacing X by X_h and $\mathcal{E}$ by $\pi_h^*\mathcal{E}$ for $h >> 1$ one can suppose $\boxed{\mu(\mathcal{E}) \in \mathbb{Z}}$.

Replacing $\mathcal{E}$ by $\mathcal{E}(d)$ for some $d \in \mathbb{Z}$ one can suppose $\boxed{\mu(\mathcal{E}) = 0}$.

this is why the tower of coverings of X is useful for classifying v.b. on X: kill denominators of slopes.

<u>The rank 2 case</u> (general case a little bit more complicated): $\mathcal{E}$ s.s. of slope 0 and rank 2. Let $\mathcal{L} \subset \mathcal{E}$ be a subbundle of maximal degree. $\mathcal{E}$ s.s. $\Rightarrow \deg \mathcal{L} \leq 0$, $\mathcal{L} \simeq \mathcal{O}_X(-d)$ with $d \geq 0$.

Thus $\mathcal{E}$ is an extension $0 \to \mathcal{O}_X(-d) \to \mathcal{E} \to \mathcal{O}_X(d) \to 0$

* If $d = 0$ since $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) = 0$, $\mathcal{E} \simeq \mathcal{O}_X^2$ and <u>this is finished</u>
* If $d \leq 1$, $-d+2 \leq d \Rightarrow \exists u : \mathcal{O}_X(-d+2) \longrightarrow \mathcal{O}_X(d)$ non zero

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X(-d) \arrow[r] \arrow[d, equal] & \mathcal{E}' \arrow[r] \arrow[d] & \mathcal{O}_X(-d+2) \arrow[r] \arrow[d, "u"] & 0 \\
0 \arrow[r] & \mathcal{O}_X(-d) \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{O}_X(d) \arrow[r] & 0
\end{tikzcd}$$

pullback via u

Twisting via $\mathcal{O}(d-1)$ one obtains

$$0 \to \mathcal{O}_X(-1) \to \mathcal{E}'(d-1) \to \mathcal{O}_X(1) \to 0$$

By hypothesis, $H^0(X, \mathcal{E}'(d-1)) \neq 0$.

Thus $\exists$ non zero morphism $\mathcal{O}_X(1-d) \xrightarrow{v} \mathcal{E}'$.

The composite $\mathcal{O}_X(1-d) \xrightarrow{v} \mathcal{E}' \longrightarrow \mathcal{E}$ is non zero

$\rightarrow$ Contradicts the maximality of $\deg \mathcal{L} = -d$ among sub-line bundles of $\mathcal{E}$.

$\square$

Modifications of vector bundles associated to p-divisible groups

Let $L|E$ be the completion of the maximal unramified extension of E with residue field $\overline{\mathbb{F}}_q$.

$\varphi\text{-Mod}_L$ = associated category of isocrystals.

$$\begin{array}{ccc} \varphi\text{-Mod}_L & \longrightarrow & \mathrm{Bun}_X \\ (D, \varphi) & \longmapsto & \Big(\underbrace{\bigoplus_{d \geq 0} \left(D \otimes_L B\right)^{\varphi = \pi^d}}\Big)^{\sim} \end{array}$$

graded module$/P = \bigoplus_{d \geq 0} B^{\varphi = \pi^d}$

$$\boxed{\mathcal{E}(D, \varphi) \simeq \bigoplus_{i=1}^{r} \mathcal{O}_X(\lambda_i)^{m_i}}$$

$(\lambda_i)_i$ = Dieudonné-Manin slopes of (D, φ) with multiplicities $(m_i)_i$.

For $\infty \in |X|$, $B^+_{dR} = \widehat{\mathcal{O}_{X,\infty}}$, $C = k(\infty) =$ residue field

$$\widehat{\mathcal{E}(D,\varphi)_\infty} = D \underset{L}{\otimes} B^+_{dR}. \text{ Set } D_C := D \underset{L}{\otimes} C.$$

Suppose $\mathrm{Fil}\, D_C \subset D_C$ is a ~~lin~~ sub-vector space. Then one can construct a modification $\mathcal{E}(D,\varphi,\mathrm{Fil}\, D_C)$ of $\mathcal{E}(D,\varphi)$ at ∞ st:

$$0 \to \mathcal{E}(D,\varphi,\mathrm{Fil}\, D_C) \longrightarrow \mathcal{E}(D,\varphi) \longrightarrow \underbrace{i_{\infty *}\left(D_C/\mathrm{Fil}\, D_C\right)}_{\text{skyscraper sheaf}} \longrightarrow 0$$

Reformulation of Comparison theorem:

$\underline{\text{Th}}$: If $(D,\varphi,\mathrm{Fil}\, D_C) =$ filtered covariant Dieudonné module associated to a π-divisible O_E-module H then $\mathcal{E}(D,\pi^{-1}\varphi,\mathrm{Fil}\, D_C)$ is a trivial vector bundle $\simeq V_p(H) \underset{E}{\otimes} \mathcal{O}_X$.

Applying $H^0(X,-)$ to the preceding exact sequence we obtain the usual formula $V_p(H) = \mathrm{Fil}^0(D \otimes B)^{\varphi=\pi}$.

Application:

Th. Let $0 \to \mathcal{E} \to \mathcal{O}_X\left(\frac{1}{n}\right) \to \mathcal{F} \to 0$ be a modification of degree $\underline{1}$ of $\mathcal{O}_X\left(\frac{1}{n}\right)$ i.e. $\mathcal{F}$ = torsion coherent sheaf of degree 1.

Then $\mathcal{E} \simeq \mathcal{O}_X^n$.

[$\mathcal{F} \simeq i_{x*}k(x)$ for some $x \in |X|$.

Proof: Surjectivity of the period morphism for Lubin-Tate spaces (Laffaille, Gross-Hopkins): if (D, φ) = isocrystal of a $\underline{1}$-dimensional formal π-divisible $\mathcal{O}_E$-module of height $n / \overline{\mathbb{F}_q}$ then any filtration $\mathrm{Fil}\, D_C \subset D_C$ of codimension $\underline{1}$ is the Hodge filtration of a π-divisible $\mathcal{O}_E$-module$/\mathcal{O}_C$. $\square$

$(C = k(x).)$

Dual statement using Drinfeld spaces

↳ in the sens of the isomorphism between the twin towers.

Th: Let $\mathcal{F}$ = torsion coherent sheaf of degree $\underline{1}$ and $\mathcal{E}$ a v.b. equipped with a modification

$$0 \to \mathcal{O}_X^n \to \mathcal{E} \to \mathcal{F} \to 0$$

Then $\exists\, r \in \{1, \dots, n\}$ s.t. $\mathcal{E} \simeq \mathcal{O}_X\left(\frac{1}{r}\right) \oplus \mathcal{O}_X^{n-r}$.

Proof of the theorem for rank 2 vector bundles

$$0 \to \mathcal{O}_X(-1) \to \mathcal{E} \to \mathcal{O}_X(1) \to 0$$

We ~~so~~ want to prove that $H^0(X,\mathcal{E}) \neq 0$.

Remark: If $X = \mathbb{P}^1$, choose $\mathcal{O}_X \xrightarrow{\neq 0} \mathcal{O}_X(1)$ and pull back

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_X(-1) & \to & \mathcal{E}' & \to & \mathcal{O}_X & \to & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O}_X(-1) & \to & \mathcal{E} & \to & \mathcal{O}_X(1) & \to & 0 \end{array}$$

$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$

$\Rightarrow \mathcal{E}' \simeq \mathcal{O}_X \oplus \mathcal{O}_X(-1)$

$\Rightarrow H^0(X,\mathcal{E}') \neq 0 \Rightarrow H^0(X,\mathcal{E}) \neq 0$.

Does not work here since $H^0(X,\mathcal{O}_X(-1)) \neq 0$.

Chose a non zero morphism $\mathcal{O}_X(-1)$ and push forward the preceding exact sequence

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \to & \mathcal{O}_X(-1) & \to & \mathcal{E} & \to & \mathcal{O}_X(1) & \to & 0 \\ & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \to & \mathcal{O}_X(1) & \to & \mathcal{E}' & \to & \mathcal{O}_X(1) & \to & 0 \\ & & \downarrow & & & & & & \\ & & \mathcal{F} & & & & & & \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

Split exact sequence since $\mathrm{Ext}^1(\mathcal{O}(1),\mathcal{O}(1)) = 0$

torsion Coherent of degree 2.

$\rightsquigarrow$ exact sequence $0 \to \mathcal{E} \to \underbrace{\mathcal{O}_X(1) \oplus \mathcal{O}_X(1)}_{\mathcal{E}'} \xrightarrow{u} \mathcal{F} \to 0$

that is to say $\mathcal{E} =$ degree 2 modification of $\mathcal{O}_X(1)^{\oplus 2}$

Choose a dévissage $0 \to \underbrace{\mathcal{F}'}_{\text{degree } 1} \to \mathcal{F} \to i_{x*}k(x) \to 0$ for some $x \in |X|$

Note $\vartheta : \mathcal{O}_X(1) \twoheadrightarrow i_{x*}k(x)$ inducing $B^{\varphi=\pi^{d_m}} \twoheadrightarrow C_m = k(x)$
for $m \in |Y|$ associated to $x \in |X|$

$$\begin{tikzcd} \mathcal{O}(1)\oplus\mathcal{O}(1) \arrow[r, two heads, "u"] \arrow[rr, bend right, "v = \text{Composite}"'] & \mathcal{F} \arrow[r, two heads] & i_{x*}k(x) \end{tikzcd}$$

Then $\exists \lambda, \mu \in k(x)$ s.t.

$$\begin{aligned} \mathcal{O}(1)\oplus\mathcal{O}(1) &\xrightarrow{v} i_{x*}k(x) \\ (a,b) &\longmapsto \lambda\vartheta(a)+\mu\vartheta(b) \end{aligned}$$

v surjective $\Rightarrow (\lambda,\mu) \neq (0,0)$

Then: * If $\lambda = 0$ or $\mu = 0$ $\ker v \simeq \underbrace{\mathcal{O}_X}_{=\ker\vartheta} \oplus \mathcal{O}_X(1)$

* If $\lambda \neq 0$ and $\mu \neq 0$ we have an exact sequence

$$0 \longrightarrow \underset{\ker\vartheta\oplus\ker\vartheta}{\overset{\shortparallel}{\mathcal{O}_X^2}} \longrightarrow \ker v \longrightarrow i_{x*}k(x) \longrightarrow 0$$

$$(a,b) \longmapsto \vartheta(a)$$

$$\ker v \simeq \begin{cases} \mathcal{O}_X \oplus \mathcal{O}_X(1) \\ \mathcal{O}_X\left(\tfrac{1}{2}\right) \end{cases}$$

$\Longrightarrow$
$\uparrow$ preceding theorem (Drinfeld case)

Thus at the end $\ker v \simeq \begin{cases} \mathcal{O}\oplus\mathcal{O}(1) \\ \mathcal{O}\left(\tfrac{1}{2}\right) \end{cases}$

* Now, $0 \to \mathcal{E} \to \ker v \xrightarrow{u_{|\ker v}} \mathcal{F}' \to 0$

* If $\ker v \simeq \mathcal{O}_X\left(\tfrac{1}{2}\right)$ by the preceding theorem (L.T. case)

$$\mathcal{E} \simeq \mathcal{O}_X^2 \Rightarrow H^0(X,\mathcal{E}) \neq 0.$$

* If $\ker v \simeq \mathcal{O}_X \oplus \mathcal{O}_X(1)$ one checks easily that either

$\mathcal{E} \simeq \mathcal{O}_X \oplus \mathcal{O}_X$ either $\mathcal{E} \simeq \mathcal{O}_X(1)\oplus\mathcal{O}_X(-1)$

$\Rightarrow H^0(X,\mathcal{E}) \neq 0.$ $\square$