Where does the curve and the Conjecture come from? What happened in Trieste, Orsay and Berkeley?

Starting point: HN filtration of finite flat group schemes

$K|\mathbb{Q}_p$ complete (not necessarily discrete valuation)

$G/\mathcal{O}_K$ finite flat group scheme of order a power of p

$$\begin{cases} ht\, G := \log_p |G| \\ \deg G := \sum_i v(a_i) \quad \text{if } \omega_G \simeq \bigoplus_i \mathcal{O}_K / a_i \mathcal{O}_K \end{cases}$$

$v(p) = 1$

$\mu(G) = \dfrac{\deg G}{ht\, G} \in [0,1] \rightsquigarrow$ HN filtration

$\rightsquigarrow$ began to discuss with Fontaine about HN filtrations

(Fontaine began to see HN filtrations everywhere)

* Next step: H p-div. group $/\mathcal{O}_K$

$$\begin{cases} \deg H[p^n] = n \dim H \\ \mathrm{ht}\, H[p^n] = n\, \mathrm{ht}\, H \end{cases}$$

$$\mathrm{HN}(H[p^n]) : [0, n\,\mathrm{ht} H] \longrightarrow [0, n \dim H]$$

Def (Renormalized HN polygon): $\mathrm{HN}(H)(x) := \lim_{n \to +\infty} \frac{1}{n} \mathrm{HN}(H[p^n])(nx)$

(standard procedure in dynamical system: subadditive sequence and other stuff)

$$\mathrm{HN}(H) : [0, \mathrm{ht} H] \longrightarrow [0, \dim H]$$

Concave function.

Is it a polygon?

Solution: "Linearize" the non linear objects that are finite flat group schemes using Hodge-Tate periods

Recall: $K = \hat{K} = C$ (One can always suppose this, HN filtrations satisfy Galois descent)

H.T. map

$$\alpha_H : V_p(H) \longrightarrow \omega_{H^D}\left[\tfrac{1}{p}\right]$$

$$\alpha_H \otimes 1 : V_p(H) \underset{\mathbb{Q}_p}{\otimes} C \twoheadrightarrow \omega_{H^D}\left[\tfrac{1}{p}\right]$$

surjective, part of a Hodge-Tate exact sequence

for $X = (V, W, \beta)$

V: $\mathbb{Q}_p$-v.s. (finite dimensional)

W: C-v.s.

$\beta : V_C \twoheadrightarrow W$

set $\begin{cases} \mathrm{rk}\, X := \dim_{\mathbb{Q}_p} V \\ \deg X := \dim_C \ker \beta \end{cases}$

and $\mu(X) = \dfrac{\deg X}{\mathrm{rk}\, X}$ $\rightsquigarrow$ HN filtrations in the exact category of $\{(V, W, \beta)\}$

Strict subobjects of $X \xrightarrow{\sim}$ sub $\mathbb{Q}_p$-v.s. of V

"generic fiber functor"

$\underline{\text{Th}}$: $HN(H) = HN\left(V_p(H), \omega_{H^D}[\tfrac{1}{p}], \alpha_H \otimes 1\right)$

$\longrightarrow$ uses the fact that coker$(\alpha_G \otimes 1)$ is killed by $p^{\frac{1}{p-1}}$

$\uparrow$ finite flat group scheme

"$\frac{1}{p-1} = v_p(2i\pi)$" – Renormalization process kills this bound.

$\Rightarrow HN(H) =$ polygon.

Then I wanted to prove the following theorem.

$\underline{\text{Th}}$: $HN(H) \leq \text{Newt}(\underbrace{H_{kc}})$

special fiber

$\uparrow$ Concave version of Newton polygon

$\Rightarrow HN(H)$ is a line if H_{kc} is isoclinic.

$\rightsquigarrow$ this is where I began to be interested in BC spaces and discuss them with Fontaine.

In fact, recall (Colmez original definition):

BC spaces = Functors $\left\{\begin{array}{c}\text{Sympathetic}\\ C\text{-algebras}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\mathbb{Q}_p\text{- Banach}\\ \text{spaces}\end{array}\right\}$

$\hookrightarrow$ Now (see Arthur's PHD) we know those are the perfectoid C-algebras R s.t. $H^1_{\text{pro-ét}}(\text{Spa}(R, R^\circ), \mathbb{Q}_p) = 0$.

that are of the form

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Extension of a finite dimensional C-v.s. by a finite dim. $\mathbb{Q}_p$-v.s. / finite dimensional $\mathbb{Q}_p$-v.s.

Come back to the preceding: exact sequence of BC spaces

$$0 \to V_p(H) \longrightarrow (D \otimes B^+_{cris})^{\varphi = p} \longrightarrow \mathrm{Lie}\, H \left[\frac{1}{p}\right] \to 0$$

Covariant isocrystal of H_{bc}

given by comparison theorems for H

BC spaces = abelian category + two additive functions

Strong motivation for introducing HN filtrations on BC spaces w.r.t. Fontaine

looks like

$\mathrm{Coh}_{\underset{\text{curve}}{X}}$ + deg and rk !!!!

⟹ Can define HN filtrations on BC spaces

⟹ implies easily the preceding theorem
$HN(H) \leq Newt(H)$ by applying
these two additive functions
to the preceding exact sequence of BC spaces.

(now we do this by looking at HN filtrations of
modifications of vector bundles)

* We tried with Fontaine to geometrize BC spaces

↳ question already asked by Fontaine in his article for Kato's 50

~~[illegible]~~

+ classify them using
these HN filtrations

gave rise to section 4 of our article w/ Fontaine on formal $\mathbb{Q}_p$-vector spaces where

later called universal cover by Scholze-Weinstein

we geometrize $\varprojlim_{\times p} \underbrace{H}_{\text{formal } p\text{-div. group}}$

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After some work w.t. Fontaine we could construct a functor

$$\{\text{some BC spaces}\} \longrightarrow \left\{\begin{array}{l} B_e\text{-module free of finite type} \\ B^+_{dR}\text{-module of finite type} \\ \quad + \text{"gluing datum"} \end{array}\right\}$$

Looks like Beauville-Laszlo !!!

In the meantime, Fontaine explained those things to Berger who looked at those B-pairs and proved that B_e is Bezout using Kedlaya's theorem.

Then Fontaine was able to prove it is a P.I.D. in fact.

+ he came with a proof of [weakly ad. ⇒ admissible] using those two ingredients.

[⟶ discovered the ~~ff~~ curve in Trieste.]

First as a gluing of Spec(B_e) + point at ∞ defined by the DVR B^+_{dR}.

Then we realized one could define it as
Proj(P) directly

already considered
by Fontaine in a preceding article

I conjectured immediately in Trieste that
any vector bundle on $X = \mathrm{Proj}(P)$ is a
$\bigoplus$ of $\mathcal{O}(\lambda)$ like a generalization of
Grothendieck's classification of v.b. $/\mathbb{P}^1$ and one
should look at the "stack of G-bundles/curve"
(protestations by Fontaine)
→ Began to study the curve + vector bundles
on it with Fontaine.

Structure of the curve:

We tried to understand the closed points $|X|$ of $X = \mathrm{Proj}(P)$

$\longrightarrow$ introduce those primitive elements in A_{inf}

Recall: $F|\mathbb{F}_p$ perfectoid field

$\mathbb{A} = W(\mathcal{O}_F)$

$\xi = \sum_{n\geq 0} [x_n] p^n \in \mathbb{A}$ is primitive of degree $d>0$

if $x_0 \neq 0$, $x_0, \dots, x_{d-1} \in \mathfrak{m}_F$, $x_d \in \mathcal{O}_F^\times$

$\rightarrow$ notion comes from Weierstrass theory (distinguished power series)

With Fontaine we defined

$$Y = \{\text{Irreducible primitive}\}/\mathbb{A}^\times$$
$$\downarrow$$
$$\mathrm{Spec}(\mathbb{A})$$

+ application $Y \longrightarrow \underbrace{|X|}_{\text{closed point}}$

s.t. $Y/\varphi^{\mathbb{Z}} \xrightarrow{\sim} |X|$ uniformization of closed points of X.

+ Conjectured that Y has a structure of "rigid analytic space"
(was proven later using the fact that Y is "pre-perfectoid")

+ Remarked that the associated residue field are "strictly p-perfect" (cf. article "Factorization of analytic functions in mixed char.")

Now called a perfectoid field

with taking a degree d extension of F.

Vector bundles

Proved classification results for vector bundles $/X$

$\longrightarrow$ any v.b. $/X \simeq \bigoplus_{\text{finite}} \mathcal{O}(\lambda)$, $\lambda \in \mathbb{Q}$

Naturally in the proof modifications of v.b. showed up

(1st occurrence): $0 \to \mathcal{E} \to \mathcal{O}\left(\frac{1}{2}\right) \to i_{\infty *} C \to 0$

degree-1 modification of $\mathcal{O}\left(\frac{1}{2}\right)$: prove that $\mathcal{E}$ is a trivial vector bundle

$\longrightarrow$ analog problem allready considered by Colmez-Fontaine in their proof of [wa $\Rightarrow$ a] so called fundamental lemma (called like that because Fontaine was jalous of Laumon) of p-adic Hodge theory:

$(\lambda_0, \lambda_1) \in C^2 \setminus \{(0,0)\}$

$$\left(B^+_{cris}\right)^{\varphi^2 = p} \longrightarrow C$$

$$x \longmapsto \lambda_0 \vartheta(x) + \lambda_1 \vartheta(\varphi(x))$$

quasi-logarithms of a formal group

does the fundamental lemma of p-adic Hodge theory imply the fundamental lemma of Langlands? Still unsolved question is surjective with kernel a two dimensional

$\mathbb{Q}_p$ - v.s.

I remarked that this statement is in fact a
consequence of a result by Lafaille / Gross-Hopkins

L.T. space $\longrightarrow\!\!\!\!\!\rightarrow \mathbb{P}^1$ is surjective

Gross-Hopkins period morphism

+ Comparison theorem $\Rightarrow$ results about period maps
for Rapoport-Zink spaces
$\Rightarrow$ results about modifications
of vector bundles.

Idem: To prove the classification theorem we need
to prove that if

$$0 \to \mathcal{O}_X^2 \to \mathcal{E} \to i_{\infty *} C \to 0$$

degree +1 modification of $\mathcal{O}_X^2$ is either $\mathcal{E} \simeq \mathcal{O}_X \oplus \mathcal{O}_X(1)$
or $\mathcal{E} \simeq \mathcal{O}_X\left(\frac{1}{2}\right)$

$\rightarrow$ Drinfeld space shows up in Ω
+ Hodge-Tate periods of R.Z.-spaces in $\mathbb{P}^1(\mathbb{Q}_p) = \partial\Omega$

→ First time I really saw R.Z. spaces as moduli of modifications of v.b.
Very important remark for the Hecke property in the Conjecture.

Drinfeld space shows up as a moduli of modifications of vector bundles.

(2^{nd} occurrence): In the proof of weakly admissible implies admissible

(D, φ) isocrystal $\rightsquigarrow \mathcal{E}(D, \varphi)$ vector bundle$/X$

$(D, \varphi, \mathrm{Fil}^{\cdot} D_K)$ filtered φ-module $\rightsquigarrow \mathcal{E}(D, \varphi, \mathrm{Fil}^{\cdot} D_K)$
modified vector bundle using the Hodge filtration

In fact: $K|\mathbb{Q}_p$ discrete valuation perfect residue field
$C = \widehat{\overline{K}} \circlearrowleft \Gamma = \mathrm{Gal}(\overline{K}|K)$
W = finite dimensional K-v.s.

$$\left\{ \text{Filtrations of } W \right\} \xrightarrow{\sim} \left\{ \Gamma\text{-stable lattices in } W \otimes_K B_{dR} \right\}$$

$$\mathrm{Fil}^\bullet W \longmapsto \mathrm{Fil}^0 (W \otimes_K B_{dR})$$

$\longrightarrow$ 1st occurence of the B_{dR}-affine Grassmanian
(used a lot in geometric Langlands)

$G = GL_n$

$\mathbb{G}_m = \mathbb{C}^\times \curvearrowright Gr$ = affine Grassmanian $/\mathbb{C}$

via $\lambda \in \mathbb{C}^\times$, $\lambda . t = \lambda t$

$$Gr(\mathbb{C}) = G(\mathbb{C}((t))) / G(\mathbb{C}[[t]])$$

μ = cocharacter of G up to conjugation

Gr^μ = corresponding Schubert cell

$$\begin{array}{c} Gr^\mu \\ \downarrow \\ G/P_\mu \end{array}$$

$\mathbb{C}^\times$-invariant morphism induces $(Gr^\mu)^{\mathbb{C}^\times} \xrightarrow{\sim} G/P_\mu$

$\mathbb{C}^\times \longleftrightarrow \Gamma$

$\hookrightarrow \sigma.(t) = \chi_{cyc}(\sigma)\, t.$

Fontaine's t

(I remember explaining this to Rapoport in Orsay. Later he claimed one should call this the B L. map)

Forward in time:

Scholze - Weinstein proved that if

(D, φ) = Dieudonné module of a p-divisible group$/\mathcal{O}_C$

$$\text{modifications} \quad 0 \to \mathcal{E} \to \mathcal{E}(D,\varphi) \to \mathcal{F} \to 0$$

$\mathcal{F}$: minuscule torsion coherent sheaf at ∞

$\Updownarrow$

via Comparison Theorem

p-div. groups$/\mathcal{O}_C$

(I had proved full faithfulness via rigid analytic p-div. groups)

$\Rightarrow$ RZ. spaces = moduli of modifications of vector bundles.

Hot topic perfectoid spaces at MSRI:

Preparing my talks: I remarked (someone had asked to Scholze before in a preceding talk if any perfectoid field is the tilting of a char. 0 perf. field: he said "ask to Fargues and Fontaine")

Up to power of Frobenius if $F|\mathbb{F}_p$ perfectoid then

$\{$ untilts of finite deg. extensions of $F \} \simeq \{$ closed points of the curve $\}$

I guess this was one of the motivations for Scholze to introduce $\mathrm{Spa}(\mathbb{Q}_p)^\diamond$

Modern formulation: $\mathrm{Div}^1 = \mathrm{Spa}(\mathbb{Q}_p)^\diamond / \varphi^{\mathbb{Z}}$.

Other step: Peter asks me in January if I can classify G_bundles$/X$ for any reductive G
(needed for his work with Ana)

I say yes, very probably I can prove this is the same as Kottwitz set $B(G)$.

$\rightarrow$ very important step: if I had restricted myself to GL_n I would probably never had discovered the conjecture.
(in particular the L-packet phenomenon) and the action of S_φ is fundamental

Langlands program is any G: not only GL_n.

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Special program in Berkeley:

* Peter introduces $\mathrm{Spa}(\mathbb{Q}_p)^\diamond$ + diamonds + generalizations of R.Z. spaces as moduli of modifications of G-bundles
$(G, b, \mu) \rightsquigarrow \mathcal{M}(G, b, \mu)$ = moduli of modifications $\mathcal{E}_1 \rightsquigarrow \mathcal{E}_b$ of type μ

+ announces program to do the local version of V. Lafforgue's construction of Langlands parameters by "putting together all cohomology spaces of those local Shtuka moduli spaces for all b's and μ" + "factorization sheaf property"

* I decide to upgrade everything by geometrizing and introducing Bun_G

Declic: Hecke property for GL_n $\rightsquigarrow$ Drinfeld tower shows up

+ $|B(G)| \xrightarrow{\sim} |\mathrm{Bun}_G|$
+ the fact that $\kappa: B(G) \longrightarrow \pi_1(G)_\Gamma = X^*(Z(\hat{G})^\Gamma)$

defines a decomposition

$$\mathrm{Bun}_G = \coprod_{\alpha \in \pi_1(G)_\Gamma} \mathrm{Bun}_G^{\alpha}$$

$$Z(\hat{G})^\Gamma \subset S_\varphi$$

Langlands parameter

Biggest mistery after the first version of the conjecture: how do you get back local Langlands? Usually, in Drinfeld context, take the associated trace of Frob function. Does not work here (later lead to the character sheaf property)

<u>Peter's suggestion</u>: Take the stalk of $\mathcal{F}_\varphi$ at the trivial

$\bar{G}$-bundle ⇝ representation of $G(\mathbb{Q}_p)$

I found this to <u>naive</u> at first but was immediately convinced when I remembered those Kaletha's articles about Kottwitz conjecture for extended pure inner forms.