

p-adic Twists and Shtukas

The non-archimedean case

$$E \begin{cases} \rightarrow \mathbb{F}_q((\pi)) \\ \rightarrow [E, \mathcal{O}_E] \ll +\infty \end{cases} \quad \mathbb{F}_q = \mathcal{O}_E / \pi$$

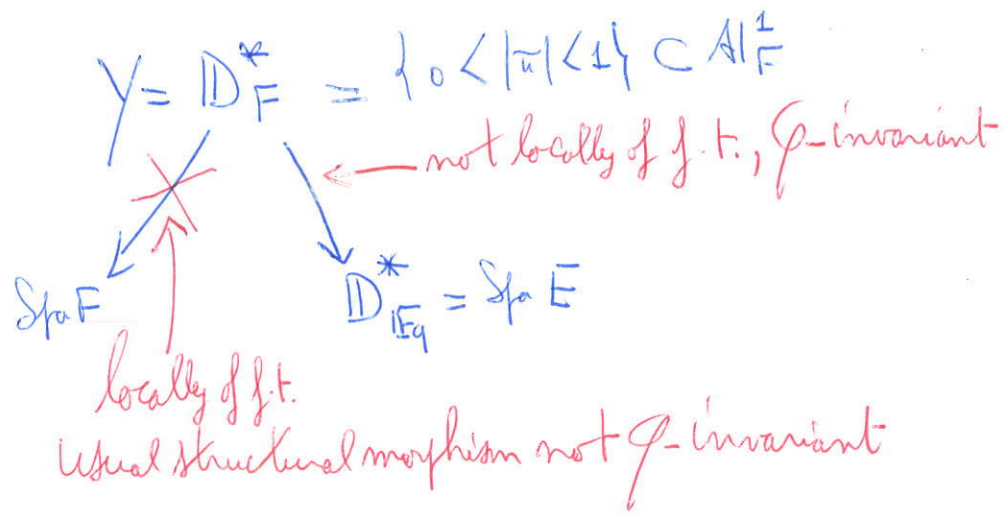
\rightsquigarrow $X = Y/\varphi^2$ E-odic curve

$$Y = \text{Spa}(A) \setminus V(\pi [\mathcal{O}_F]) \quad 0 < |\varpi_F| < 1$$

$$A = \begin{cases} \mathcal{O}_F[[\pi]] \\ W_{\mathcal{O}_E}(\mathcal{O}_F) = \left\{ \sum_{m \in \mathbb{Z}} [x_m] \pi^m / x_m \in \mathcal{O}_F \right\} \end{cases} \quad \text{Fontaine's Ainf}$$

$$\varphi \left(\sum_m [x_m] \pi^m \right) = \sum_m [x_m^q] \pi^m$$

* $E = \mathbb{F}_q((\pi))$



* Line bundle $\mathcal{O}(1)$

$$Y \times \mathbb{A}^1 \xrightarrow{\varphi} Y \times \mathbb{A}^1 \xrightarrow{\varphi^{-1}}$$

$$\downarrow \varphi^2$$

$$Y/\varphi^2$$

"ample" was schematic curve

$$\text{Proj} \left(\bigoplus_{d \geq 0} \underbrace{H^0(X, \mathcal{O}(d))}_{\mathcal{O}(Y)^{\varphi = \pi^d}} \right) = \text{Dedekind } E\text{-scheme}$$

Won't use it so much in those talks

* F alg. closed $\overline{\mathbb{F}_q} \subset F$ $L = \widehat{E}^{\text{un}} \ni \sigma = \varphi$

$$\varphi\text{-Mod}_L = \{ (\mathcal{D}, \varphi) \}$$

finite dim. L-v.s.

$$\begin{array}{ccc} \varphi\text{-Mod}_L & \longrightarrow & \text{Bun}_X \\ (\mathcal{D}, \varphi) & \longmapsto & \mathcal{E}(\mathcal{D}, \varphi) \end{array} \rightsquigarrow \begin{array}{c} Y/\varphi^2 \times D \\ \downarrow \\ Y/\varphi^2 \end{array}$$

Ex: $\mathcal{O}(1) = \mathcal{E}(L, \pi^{-1}\sigma)$

Th: $\mathcal{E}(-)$ is essentially surjective.

Rephrasing via Deligne-Mumford: Canonical $\widehat{\mathbb{Z}}$ -Cover

$$\begin{array}{c} (X_h)_{h \geq 1} \\ \downarrow \\ X_1 = X \end{array} \Bigg) \widehat{\mathbb{Z}}\text{-unfolding of Frob. cover}$$

$$\begin{aligned} X_h &= Y/\varphi^{h2} \\ &= X \otimes_E E_h \end{aligned}$$

$E_h | E$ unramified deg. h

Then if $\mathcal{O}_X(\lambda) = \pi_{h*} \mathcal{O}_{X_h}(d)$
 \parallel
 Stable w.r. to slope λ

$\lambda = \frac{d}{h}$
 $(d, h) = 1$
 $\pi_h: X_h \rightarrow X$

$\forall \mathcal{E} \in \text{Bun}_X \quad \mathcal{E} \simeq \bigoplus_i \mathcal{O}_X(\lambda_i), \lambda_i \in \mathbb{Q}$

The archimedean case: $E = \mathbb{C} \quad X = \mathbb{P}^1_{\mathbb{C}}$

$E = \mathbb{R} \quad X = \widetilde{\mathbb{P}}^1_{\mathbb{R}} = \mathbb{P}^1_{\mathbb{C}} / z \sim -\frac{1}{\bar{z}}$ *no real point*
 = Severi-Brauer associated to \mathbb{H}
 = (smooth) conic without real point

$\mathbb{P}^1_{\mathbb{C}} \xrightarrow{u} \widetilde{\mathbb{P}}^1_{\mathbb{R}}$ $\mathbb{Z}/2\mathbb{Z}$
 analogous of the preceding $\widehat{\Sigma}$ -Cover
 $\lambda \in \frac{1}{2}\mathbb{Z}, \mathcal{O}(\lambda) = \begin{cases} \text{line bundle s.t. } u^* \mathcal{O}(\lambda) = \mathcal{O}(2\lambda) & \text{if } \lambda \in \mathbb{Z} \\ u_* \mathcal{O}(2\lambda) & \text{if } \lambda \notin \mathbb{Z} \end{cases}$
nb. 2 stable slope λ

Prop: $\forall \mathcal{E} \in \text{Bun}_X \quad \mathcal{E} \simeq \bigoplus_i \mathcal{O}(\lambda_i) \quad \lambda_i \in \frac{1}{2}\mathbb{Z}$

$* \infty \in \widetilde{\mathbb{P}}^1_{\mathbb{R}} \quad u^{-1}(\infty) = \{0, \infty\}$
 $\mathbb{P}^1_{\mathbb{C}} \ni \mathbb{C}^* \xrightarrow{u} \widetilde{\mathbb{P}}^1_{\mathbb{R}} \ni \mathbb{S}^1$
 via $\lambda \cdot t = \lambda t$ if $t = \frac{1}{z}$
stabilizes ∞

$V \in \text{Vect}_{\mathbb{R}}$

$U(1)$ -equivariant modifications (at ∞)
of $V \otimes_{\mathbb{R}} \mathbb{C}_X$
 \cong
Filtrations of $V_{\mathbb{C}}$

$W = V_{\mathbb{C}}$

$G = GL(W)$

$G_{\mathbb{R}} =$ affine Grassmannian
 $G_{U(1)} = G(\mathbb{C}) = G(\mathbb{C}[[H]]) / G(\mathbb{C}[[H]])$

$G_{\mathbb{R}}^{\mu} =$ open Schubert cell

\downarrow $U(1)$ -invariant affine fibration
 G/P_{μ}

$(G_{\mathbb{R}}^{\mu})^{U(1)} \xrightarrow{\sim} G/P_{\mu}$

$\{ \text{Filtrations of } W \} \xrightarrow{\sim} \{ U(1)\text{-invariant lattices in } W((H)) \}$

$\text{Fil}^i W \longmapsto \text{Fil}^i(W((H))) = \sum_{j \in \mathbb{Z}} \text{Fil}^j W \otimes t^{-j} \mathbb{C}[[H]]$

thus $(V, \text{Fil}^i V_{\mathbb{C}}) \rightsquigarrow \mathcal{E}(V, \text{Fil}^i V_{\mathbb{C}}) = U(1)\text{-eq. modification of } V \otimes_{\mathbb{R}} \mathbb{C}_X$

Prop. $(V, \text{Fil}^i V_{\mathbb{C}})$ pure Hodge structure of weight w

$\iff \mathcal{E}(V, \text{Fil}^i V_{\mathbb{C}})$ semi-stable slope $\frac{w}{2}$ i.e. $\simeq \mathcal{O}(\frac{w}{2})^n$, $n \in \mathbb{N}$

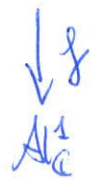
Rem. Explains why Shimura var. are uniformized at the archimedean place: only one Newton strata, the basic one.

Thus: X/\mathbb{C} proper smooth, $d \in \mathbb{N}$, Hodge filtration defines a $U(1)$ -eq. modification

$$H_{\text{Betti}}^d(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathcal{O}_X \dashrightarrow \text{Hodge } \frac{w}{2} \text{ semi-stable v.l.}$$

Deligne's Construction via λ -Connections

$X \times \mathbb{A}_{\mathbb{C}}^1$ $\lambda = \text{coordinate on } \mathbb{A}_{\mathbb{C}}^1$



$$C^\bullet = (\Omega^\bullet_{X \times \mathbb{A}_{\mathbb{C}}^1 / \mathbb{A}_{\mathbb{C}}^1}, \lambda d) \simeq [0 \xrightarrow{d} \lambda^{-1} \Omega^1 \xrightarrow{d} \lambda^{-2} \Omega^2 \xrightarrow{d} \dots]$$

$$C^\bullet|_{f^{-1}(\mathbb{A}_{\mathbb{C}}^1 \setminus \{0\})} \simeq \underbrace{\Omega^\bullet_{X/\mathbb{C}}}_{\text{usual de Rham complex}} \boxtimes \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^1 \setminus \{0\}}$$

$R^d f_* C^\bullet =$ vector bundle whose restriction to $\mathbb{A}_{\mathbb{C}}^1 \setminus \{0\}$ is $H_{\text{deR}}^d(X) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^1 \setminus \{0\}}$

\uparrow fibra at $\lambda=0$ is Hodge cohomology
 \Rightarrow rank does not jump thanks to the degeneration of Hodge \Rightarrow de Rham

$$H_{\mathbb{R}}^d = H_{\text{Betti}}^d \otimes_{\mathbb{R}} \mathbb{C}$$

\leadsto using this real structure $R^d f_* C^\bullet$ descend to a v.l. on $\widetilde{\mathbb{P}}_{\mathbb{R}}^1$
 whose restriction to $\widetilde{\mathbb{P}}_{\mathbb{R}}^1 \setminus \{0, \infty\}$ is $H_{\text{Betti}}^d \otimes_{\mathbb{R}} \mathcal{O}_{\widetilde{\mathbb{P}}_{\mathbb{R}}^1}$

Question: Make the links with Scholze and BMS Construction

Conjecture: Deligne's Construction works in a relative setting
gives the Gauss Manin Connection for $\lambda \in \mathbb{G}_m$
and a Higgs bundle at $\lambda = 0$

BMS Construction in a relative setting gives a Higgs bundle?
(written before Zhu email on Saturday)

The non-archimedean case

$F, E \text{ mod } \gamma$
 $X = Y/q^Z$
 $|Y|^{cl} \subset |Y|$ ← classical take points

{units of finite deg. ext. of F }

$y \in |Y|^{cl}$ $b(y) \in E$ perfectoid $[b(y)^b: F] \ll +\infty$

$\widehat{\mathcal{O}}_{y,y} = B_{dR}^+(b(y))$

* $E = \mathbb{Q}_p$ K/\mathbb{Q}_p d.v. perfect residue field $K_0 = W(b_K)/\mathbb{Q}$

$C = \widehat{K}$ $G_K = \text{Gal}(\overline{K}/K)$

$F = \mathbb{C}^b \text{ mod } X \text{ curve} / \mathbb{Q}_p$ $+\infty \in |X|$
 \uparrow $C = \text{units of } F$

$b(\infty) = C$ $\widehat{\mathcal{O}}_{X,\infty} = B_{dR}^+(C)$

$X \curvearrowright G_K$ stabilizes ∞

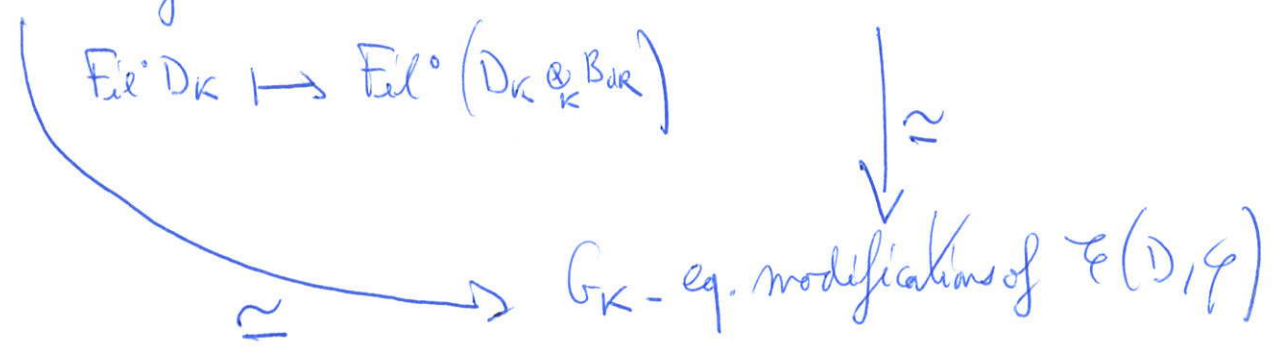
Same principle as before: $(D, \varphi) \in \varphi\text{-Mod}_{K_0}$

$$G_K \leftrightarrow U(1)$$

$$\begin{matrix} \psi \\ \sigma \end{matrix} \quad \sigma(t) = \chi_{\text{cyc}}(\sigma) t$$

Fuchsian filtered φ -modules

Filtrations of $D_K \xrightarrow{\sim} G_K$ -invariant lattices in $D \otimes_{K_0} B_{\text{dR}}$



Reformulation of Theji: X/K_K proper smooth, $d \in \mathbb{N}$

$$(D, \varphi) = H_{\text{cris}}^d(X_s/W) \left[\begin{matrix} \frac{1}{r} \\ r \end{matrix} \right]$$

Then the Hodge filtration of $D_K = H_{\text{dR}}^d(X_K)$ defines a G_K -eq. modification

$$H_{\text{ét}}^d(X_{\bar{K}}, \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \mathcal{O}_X \dashrightarrow \mathcal{E}(D, \varphi)$$

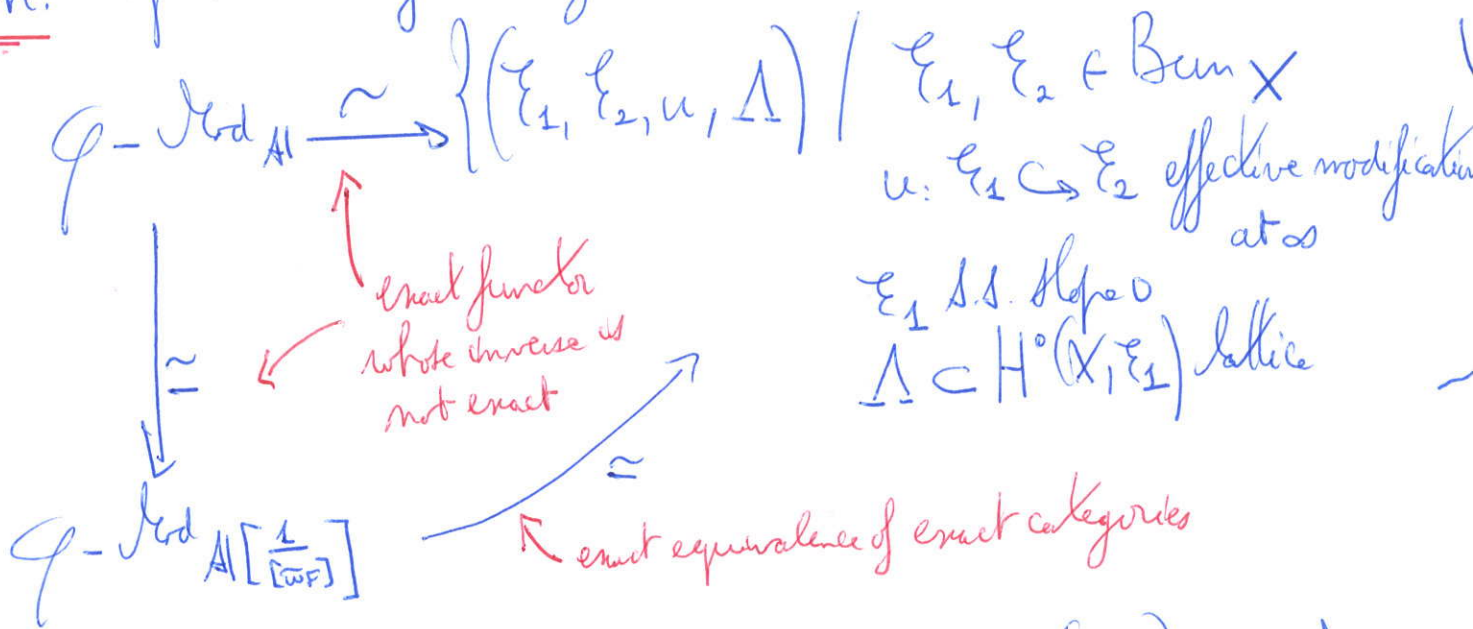
+ lattice $H_{\text{ét}}^d(X_{\bar{K}}, \mathcal{Z}_Y) / \text{torsion}$

* Any C/E alg. closed - $F = C^b$, $\infty \in |X|$

$$A^b \xrightarrow{\sigma} G_c \quad \text{base } \sigma = (\xi)$$

Def: $\mathcal{Q}\text{-Mod}_A = \left\{ (M, \varphi) \mid M \text{ free } A\text{-module and } \text{Cher } \varphi \right.$
 $\left. \text{is killed by a power of } \xi \right\}$

Th: Equivalence of Categories



Ex: * Tsuji \Rightarrow k/k_K proper smooth, d.H.N. mod $(M, \varphi) \in \mathcal{Q}\text{-Mod}_{A^b}$
 \downarrow
 G_K

$$\rightsquigarrow (M, \varphi) \otimes_A W(b_c) = F\text{-crystal}$$

What is it? B.M.S.: Integral crystalline Coh (if there is no torsion)

$$* \left\{ (M, \varphi) / \text{Cher } \varphi \text{ killed by } \xi \right\} \simeq \text{BT } G_c$$

minuscule modifications

(Lau found another proof)

Rem: * Tautologically false if C perfectoid non alg. closed

* Works with any modification: multiples (par)

$X = \text{Proj}(P)$
 $P = \bigoplus_{i \geq 0} K_{2i}(G_F/\overline{\omega}_F)[\frac{1}{T}]$
 Higher algebra analog of the curve?