p-adic Twistors and Shtukas

The non-archimedean case

$$E \begin{cases} \nearrow \mathbb{F}_q((\pi)) \\ \searrow [E:\mathbb{Q}_p] < +\infty \end{cases} \qquad \mathbb{F}_q = \mathcal{O}_E/\pi$$

$$\rightsquigarrow \boxed{X = Y/\varphi^{\mathbb{Z}}} \quad E\text{-adic curve}$$

$$Y = \mathrm{Spa}(\mathbb{A}) \setminus V(\pi[\varpi_F]) \qquad 0 < |\varpi_F| < 1$$

$$\mathbb{A} = \begin{cases} \mathcal{O}_F[[\pi]] \\ W_{\mathcal{O}_E}(\mathcal{O}_F) = \left\{ \sum_{n \in \mathbb{Z}} [x_n]\pi^n \,/\, x_n \in \mathcal{O}_F \right\} \end{cases}$$ Fontaine's A_{inf}

$$\varphi\left(\sum_n [x_n]\pi^n\right) = \sum_n [x_n^q]\pi^n$$

* $E = \mathbb{F}_q((\pi))$

$$\begin{tikzcd} & Y = \mathbb{D}^*_F = \{0 < |\pi| < 1\} \subset \mathbb{A}^1_F \arrow[dl] \arrow[dr] & \\ \mathrm{Spa}\, F & & \mathbb{D}^*_{\mathbb{F}_q} = \mathrm{Spa}\, E \end{tikzcd}$$

not locally of f.t., φ-invariant

locally of f.t.
usual structural morphism not φ-invariant

* Line bundle $\mathcal{O}(1)$

$$\begin{tikzcd} Y \times \mathbb{A}^1 \circlearrowleft \varphi = \times \pi^{-1} \arrow[d, "/\varphi^{\mathbb{Z}}"] \\ Y/\varphi^{\mathbb{Z}} \end{tikzcd}$$

"ample" ⇝ schematical curve

$$\mathrm{Proj}\Big(\bigoplus_{d\geq 0}\underbrace{H^0(X,\mathcal{O}(d))}_{\mathcal{O}(Y)^{\varphi=\pi^d}}\Big)=\text{Dedekind } E\text{-scheme}$$

Won't use it so much in these talks

* F alg. closed $\overline{\mathbb{F}_q}\subset F$ $\quad L=\widehat{E^{un}}\ni\sigma=\varphi$

$$\varphi\text{-}\mathrm{Mod}_L=\{(D,\varphi)\}$$

σ-linear auto.

finite dim. L-v.s.

$$\begin{array}{ccc}\varphi\text{-}\mathrm{Mod}_L & \longrightarrow & \mathrm{Bun}_X\\ (D,\varphi) & \longmapsto & \mathcal{E}(D,\varphi)\end{array}\quad\rightsquigarrow\quad \begin{array}{c}Y\times_{\varphi^{\mathbb{Z}}}D\\ \downarrow\\ Y/\varphi^{\mathbb{Z}}\end{array}$$

Ex: $\mathcal{O}(1)=\mathcal{E}(L,\pi^{-1}\sigma)$

Th: $\mathcal{E}(-)$ is essentially surjective.

Rephrasing via Dieudonné-Manin: Canonical $\widehat{\mathbb{Z}}$-cover

$$\left.\begin{array}{c}(X_h)_{h\geq 1}\\ \downarrow\\ X_1=X\end{array}\right)\widehat{\mathbb{Z}}\text{-unfolding of Frob. cover}$$

$$X_h=Y/\varphi^{h\mathbb{Z}}=X\otimes_E E_h$$

$E_h|E$ unramified deg. h

Then if $\mathcal{O}_X(\lambda) = \pi_{h*}\mathcal{O}_{X_h}(d)$ $\quad \lambda = \frac{d}{h}$

$\quad\quad\quad\quad \overset{\shortparallel}{\text{stable v.b. slope } \lambda}$ $\quad (d,h)=1$

$\pi_h : X_h \to X$

$\forall \mathcal{E} \in \mathrm{Bun}_X \quad \mathcal{E} \simeq \bigoplus_i \mathcal{O}_X(\lambda_i), \ \lambda_i \in \mathbb{Q}$

The archimedean case: $E = \mathbb{C}$ $\quad X = \mathbb{P}^1_{\mathbb{C}}$

$E = \mathbb{R} \quad X = \widetilde{\mathbb{P}}^1_{\mathbb{R}} = \mathbb{P}^1_{\mathbb{C}} / z \sim -\frac{1}{\bar{z}}$ no real point

$= $ Severi-Brauer associated to $\mathbb{H}$

$= $ (smooth) conic without real point

$$\begin{array}{l} \mathbb{P}^1_{\mathbb{C}} \\ \downarrow u \ \big) \mathbb{Z}/2\mathbb{Z} \\ \widetilde{\mathbb{P}}^1_{\mathbb{R}} \end{array}$$

analogous of the preceding $\hat{\mathbb{Z}}$-cover

$$\lambda \in \tfrac{1}{2}\mathbb{Z}, \ \mathcal{O}(\lambda) = \begin{cases} \text{line bundle s.t. } u^*\mathcal{O}(\lambda) = \mathcal{O}(2\lambda) & \text{if } \lambda \in \mathbb{Z} \\ \underbrace{u_*\mathcal{O}(2\lambda)}_{\text{rk } 2 \text{ stable slope } \lambda} & \text{if } \lambda \notin \mathbb{Z} \end{cases}$$

Prop: $\forall \mathcal{E} \in \mathrm{Bun}_X \quad \mathcal{E} \simeq \bigoplus_i \mathcal{O}(\lambda_i) \quad \lambda_i \in \frac{1}{2}\mathbb{Z}$

$* \ \infty \in \widetilde{\mathbb{P}}^1_{\mathbb{R}} \quad u^{-1}(\infty) = \{0, \infty\}$

$$\begin{array}{ccc} \mathbb{P}^1_{\mathbb{C}} & \circlearrowleft & \mathbb{C}^\times \\ \downarrow u & & \uparrow \\ \widetilde{\mathbb{P}}^1_{\mathbb{R}} & \circlearrowleft & U(1) \end{array}$$

via $\lambda . t = \lambda t$ if $t = \frac{1}{z}$

$U(1)$ stabilizes ∞

$V \in \mathrm{Vect}_{\mathbb{R}}$

$$\left[\begin{array}{c} U(1)\text{-equivariant modifications (at } \infty) \\ \text{of } V \otimes_{\mathbb{R}} \mathcal{O}_X \\ \wr\wr \\ \text{Filtrations of } V_{\mathbb{C}} \end{array}\right]$$

$W = V_{\mathbb{C}}$ $\qquad G = GL(W)$ $\qquad Gr$ = affine Grassmanian

$\underset{U(1)}{\circlearrowleft}$ $\quad Gr(\mathbb{C}) = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$

Gr^{μ} = open Schubert cell

$$Gr^{\mu} \xrightarrow{\quad} G/P_{\mu}$$

$\downarrow$ $U(1)$-invariant affine fibration

G/P_{μ}

$$(Gr^{\mu})^{U(1)} \xrightarrow{\sim} G/P_{\mu}$$

$$\{\text{Filtrations of } W\} \xrightarrow{\sim} \{U(1)\text{-invariant lattices in } W((t))\}$$

$$\mathrm{Fil}^{\bullet} W \longmapsto \mathrm{Fil}^{0}(W((t))) = \sum_{i \in \mathbb{Z}} \mathrm{Fil}^{i} W \otimes t^{-i} \mathbb{C}[[t]]$$

Thus $(V, \mathrm{Fil}^{\bullet} V_{\mathbb{C}}) \rightsquigarrow \mathcal{E}(V, \mathrm{Fil}^{\bullet} V_{\mathbb{C}}) = U(1)$-eq. modification of $V \otimes \mathcal{O}_X$

__Prop:__ $(V, \mathrm{Fil}^{\bullet} V_{\mathbb{C}})$ pure Hodge structure of weight w

$\Updownarrow$

$\mathcal{E}(V, \mathrm{Fil}^{\bullet} V_{\mathbb{C}})$ semi-stable slope $\frac{w}{2}$ i.e. $\simeq \mathcal{O}(\frac{w}{2})^{n}$, $n \in \mathbb{N}$

__Rem:__ Explains why Shimura var. are uniformized at the archimedean place: only one Newton strata, the basic one.

Thus: $X/\mathbb{C}$ proper smooth, dGIN, Hodge filtration defines a $U(1)$-eq. modification

$$H^d_{Betti}(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathcal{O}_X \dashrightarrow \text{slope } \frac{w}{2} \text{ semi-stable v.b.}$$

Deligne's Construction via λ-connections

$$\begin{array}{l} X \times \mathbb{A}^1_{\mathbb{C}} \\ \downarrow f \\ \mathbb{A}^1_{\mathbb{C}} \end{array} \qquad \lambda = \text{coordinate on } \mathbb{A}^1$$

$$C^\cdot = \left(\Omega^\cdot_{X \times \mathbb{A}^1/\mathbb{A}^1}, \lambda d\right) \simeq \left[\mathcal{O} \xrightarrow{d} \lambda^{-1}\Omega^1 \xrightarrow{d} \lambda^{-2}\Omega^2 \xrightarrow{d} \cdots\right]$$

$$C^\cdot|_{f^{-1}(\mathbb{A}^1 \setminus \{0\})} \simeq \underbrace{\Omega^\cdot_{X/\mathbb{C}}}_{\text{usual de Rham complex}} \boxtimes \mathcal{O}_{\mathbb{A}^1 \setminus \{0\}}$$

$R^d f_* C^\cdot$ = vector bundle whose restriction to $\mathbb{A}^1 \setminus \{0\}$ is $H^d_{dR}(X) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{A}^1 \setminus \{0\}}$

fiber at $\lambda = 0$ is Hodge cohomology
$\Rightarrow$ rank does not jump thanks to the degeneration of Hodge $\Rightarrow$ de Rham

$$H^d_{\mathbb{R}} = H^d_{Betti} \otimes_{\mathbb{R}} \mathbb{C}$$

$\rightsquigarrow$ using this real structure $R^d f_* C$ descend to a v.b. on $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$ whose restriction to $\widetilde{\mathbb{P}}^1_{\mathbb{R}} \setminus \{\infty\}$ is $H^d_{Betti} \otimes_{\mathbb{R}} \mathcal{O}_{\widetilde{\mathbb{P}}^1_{\mathbb{R}}}$

Question: Make the links with Scholze and BMS Construction

Enjeux: Deligne's Construction works in a relative setting
gives the Gauss Manin Connection for $\lambda \in \mathbb{G}_m$
and a Higgs bundle at $\lambda = 0$
BMS Construction in a relative setting gives a Higgs bundle?
(written before 2hr email on Saturday)

The non-archimedean case

$F, E \rightsquigarrow Y \qquad X = Y/\varphi^{\mathbb{Z}}$

classical Tate points

$$|Y|^{cl} \subset |Y|$$
$$\| $$
$$\{\text{untilts of finite deg. ext. of } F\}$$

$y \in |Y|^{cl} \quad k(y)|E$ perfectoid $\quad [k(y)^\flat : F] < +\infty$

$$\widehat{\mathcal{O}}_{Y,y} = B^+_{dR}(k(y))$$

$* \; E = \mathbb{Q}_p \quad K|\mathbb{Q}_p$ d.v. perfect residue field $\quad K_0 = W(k_K)_{\mathbb{Q}}$

$$C = \widehat{\overline{K}} \qquad G_K = \mathrm{Gal}(\overline{K}|K)$$

$F = C^\flat \rightsquigarrow X$ curve $/\mathbb{Q}_p \quad + \;\infty \in |X|$

$C =$ untilt of F

$$k(\infty) = C \qquad \widehat{\mathcal{O}}_{X,\infty} = B^+_{dR}(C)$$

$X \circlearrowleft G_K$ stabilizes ∞

Same principle as before: $(D,\varphi) \in \varphi\text{-Mod}_{K_0}$

$$G_K \longleftrightarrow U(1)$$
$$\sigma \qquad \sigma(t) = \chi_{cyc}(\sigma)\, t$$

$$\begin{tikzcd} \text{Filtrations of } D_K \arrow[r, "\text{Fontaine filtered } \varphi\text{-modules}", "\sim"'] \arrow[dr, bend right, "\simeq"'] & G_K\text{-invariant lattices in } D \otimes_{K_0} B_{dR} \arrow[d, "\simeq"] \\ & G_K\text{-eq. modifications of } \mathcal{E}(D,\varphi) \end{tikzcd}$$

$$\mathrm{Fil}^\bullet D_K \longmapsto \mathrm{Fil}^0(D_K \otimes_K B_{dR})$$

Reformulation of Tsuji: $\mathfrak{X}/\mathcal{O}_K$ proper smooth, $d \in \mathbb{N}$

$$(D,\varphi) = H^d_{cris}(\mathfrak{X}_s/W)\left[\tfrac{1}{p}\right]$$

Then the Hodge filtration of $D_K = H^d_{dR}(\mathfrak{X}_K)$ defines a G_K-eq. modification

$$H^d_{et}(\mathfrak{X}_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathcal{O}_X \dashrightarrow \mathcal{E}(D,\varphi)$$

+ lattice $H^d_{et}(\mathfrak{X}_{\bar{K}}, \mathbb{Z}_p)/\text{torsion}$

* Any $C|E$ alg. closed. $F = C^\flat$, $\infty \in |X|$

$\mathbb{A}^\bullet \xrightarrow{\theta} O_C$ $\ker \theta = (\xi)$

Def: $\varphi\text{-}\mathrm{Mod}_{\mathbb{A}} = \left\{ (M, \varphi) \;\middle|\; \begin{array}{l} M \text{ free } \mathbb{A}\text{-module and Coker } \varphi \\ \text{is killed by a power of } \xi \end{array} \right\}$

Th: Equivalence of Categories

$$\begin{array}{ccc} \varphi\text{-}\mathrm{Mod}_{\mathbb{A}} & \xrightarrow{\sim} & \left\{ (\mathcal{E}_1, \mathcal{E}_2, u, \Lambda) \;\middle|\; \begin{array}{l} \mathcal{E}_1, \mathcal{E}_2 \in \mathrm{Bun}\, X \\ u : \mathcal{E}_1 \hookrightarrow \mathcal{E}_2 \text{ effective modification at } \infty \\ \mathcal{E}_1 \text{ s.s. slope } 0 \\ \Lambda \subset H^0(X, \mathcal{E}_1) \text{ lattice} \end{array} \right\} \\ \downarrow \simeq & \nearrow \simeq & \\ \varphi\text{-}\mathrm{Mod}_{\mathbb{A}[\frac{1}{[\varpi_F]}]} & & \end{array}$$

exact functor whose inverse is not exact

exact equivalence of exact categories

Ex: * Tsuji $\Rightarrow$ $\mathcal{X}/O_K$ proper smooth, $d \in \mathbb{N}$ $\rightsquigarrow$ $(M, \varphi) \in \varphi\text{-}\mathrm{Mod}_{\mathbb{A}^\bullet}$ $\circlearrowleft G_K$

$$\rightsquigarrow \underbrace{(M, \varphi) \otimes_{\mathbb{A}} W(k_C)}_{\text{What is it?}} = F\text{-cristal}$$

B.M.S.: integral crystalline coho (if there is no torsion)

(Lau found another proof)

* $\{ (M, \varphi) \,/\, \underbrace{\mathrm{Coker}\, \varphi \text{ killed by } \xi}_{\text{minuscule modifications}} \} \simeq BT_{O_C}$

Rem: * Tautologicaly false if C perfectoid non alg. closed
* Works with any modification: multipattes (power)

$X = \mathrm{Proj}(P)$
$P = \bigoplus_{d \geq 0} K_{2d}(O_F/\varpi_F)[\frac{1}{?}]$
Higher algebra analog of the curve?