$\boxed{p\text{-adic Twistors and Shtukas}}$ ②

About Mark's remark: $W = \mathbb{C}$-v.s.

$$\{\text{Filtrations of } W\} \xrightarrow{\sim} \{U(1)\text{-stable lattices in } W((t))\}$$

$$\|$$

$$\{\underbrace{\text{Lie } U(1)\text{-stable}}\text{ lattices in } W((t))\}$$

$$\Big\downarrow \quad \frac{d(e^{i\theta})}{e^{i\theta}} = i\,d\theta$$

$$\nabla : W((t)) \longrightarrow W((t))dt \qquad W = W((t))^{\nabla = 0}$$

$$\underset{\mathbb{C}[[t]]\text{-lattice}}{M \subset W((t))} \quad \text{Lie } U(1)\text{-stable} \Leftrightarrow \boxed{\nabla(M) \subset M\frac{dt}{t}}$$

$$E \nearrow \mathbb{F}_q((\pi)) \qquad E \searrow [E:\mathbb{Q}_p] < +\infty$$

$C|E$ alg. closed $\quad F = C^\flat \quad 0 < |\varpi_F| < 1$

$$\overset{\circlearrowleft}{\varphi}\ Y = \mathrm{Spa}(\mathbb{A}) \setminus V(\pi[\varpi_F])$$

$$= \mathbb{D}^*_F \text{ if } E = \mathbb{F}_q((\pi))$$

$\delta : |Y| \longrightarrow\ ]0,1[$ radius function) distance to the origin $\pi = 0$.

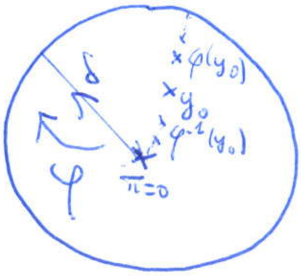
$$\delta(\varphi(y)) = \delta(y)^{1/q}$$

$\Big(\rightsquigarrow \quad |X| \longrightarrow\!\!\!\!\rightarrow S^1 = \mathbb{T} = \ ]0,1[/\varphi^{\mathbb{Z}}$ homotopy eq. $\Big)$

Berkovich spectrum

$y_0 \in |Y|^{cl}$ $\quad k(y_0) = C \; , \; \widehat{\mathcal{O}}_{Y,y_0} = B^+_{dR}(C)$

$\overset{\shortparallel}{V(\xi)}$



$y_0 \rightsquigarrow \infty \in |X|$

$\updownarrow$

$div(t) = \sum_{n\in\mathbb{Z}} [\varphi^n(y_0)] \in Div(Y) \qquad div(\mu) = \sum_{n\geq 0} [\varphi^n(y_0)]$

Def: * $Modif_X^{\geq 0} = \{ \underbrace{\mathcal{E}_1 \hookrightarrow \mathcal{E}_2}_{Bun_X}$ effective modif. at $\infty \}$

* $Sht_Y^{\geq 0} = \{ (\mathcal{E}, v) \,/\, \mathcal{E} \in Bun_Y \, , \; v : \varphi^*\mathcal{E} \overset{\text{modif. at } y_0}{\hookrightarrow} \mathcal{E}$
s.t. Coker v = torsion coherent sheaf supported at y_0 $\}$

Prop: $\exists$ antiequivalence $Sht_Y^{\geq 0} \xrightarrow{\sim} Modif_X^{\geq 0}$ s.t. if

$(\mathcal{E}, v) \longmapsto [\underbrace{\mathcal{E}_1 \hookrightarrow \mathcal{E}_2}_{\varphi\text{-eq. v.b. on } Y}]$ and $\rho = \delta(y)$

$\mathcal{E}|_{Y_{]0,\rho[}} \simeq \check{\mathcal{E}}_1|_{\underbrace{Y_{]0,\rho[}}}$ $\rightsquigarrow$ annulus $\delta^{-1}(]0,\rho[)$

$\mathcal{E}|_{Y_{]\rho,1[}} \simeq \check{\mathcal{E}}_2|_{Y_{]\rho,1[}}$

$\rightarrow (\mathcal{E}, v) \in \mathrm{Sht}_Y^{\geq 0}$

$$\cdots \hookrightarrow \varphi^{2*}\mathcal{E} \xhookrightarrow{\varphi^{*}v} \varphi^{*}\mathcal{E} \xhookrightarrow{v} \mathcal{E} \xhookrightarrow{\varphi^{-1*}v} \varphi^{-1*}\mathcal{E} \hookrightarrow \varphi^{-2*}\mathcal{E} \hookrightarrow \cdots$$

$(\varphi^{n*}\mathcal{E})_{n\in\mathbb{Z}}$ = ind/pro system of modifications

The modif. $\varphi^{n*}v$ is supported on $\varphi^{-n}(y)$ $\quad \delta(\varphi^{-n}(y)) \begin{cases} \xrightarrow{n\to+\infty} 0 \\ \xrightarrow{n\to-\infty} 1 \end{cases}$

$\Rightarrow \forall U \subset Y$ quasicompact for $|n| >> 0$ $(\varphi^{n*}v)_{|U}$ is an iso.

$\Rightarrow$ locally on Y the ind/pro system $(\varphi^{n*}\mathcal{E})_{n\in\mathbb{Z}}$ is essentially constant

$$\Rightarrow \left\{ \begin{array}{l} \mathcal{E}^{\infty} = \varinjlim_{n\leq 0} \varphi^{n*}\mathcal{E} \;\Big)\text{ union} \\ \mathcal{E}_{\infty} = \varprojlim_{n\geq 0} \varphi^{n*}\mathcal{E} \;\Big)\text{ intersection} \end{array} \right. \text{ are } \varphi\text{-eq. } \underline{\text{v.b.}} \text{ on } Y.$$

$\mathcal{E}_\infty \subset \mathcal{E}^\infty$ modification of φ-eq. v.b. on Y supported on $\varphi^{\mathbb{Z}}(y_0)$

$\rightsquigarrow \left[\check{\mathcal{E}}^{\infty} \subset \check{\mathcal{E}}_{\infty} \right] \in \mathrm{Modif}_X^{\geq 0}$

Exercise: Describe the inverse functor. $\square$

The space $\mathcal{Y}$

$$Y = \mathcal{S}pa(A) \setminus V(\pi[\varpi_F]) = D(\pi) \cap D([\varpi_F])$$

$$\begin{array}{l} \ \ \cap \\ \mathcal{Y} = \mathcal{S}pa(A)_a = \mathcal{S}pa(A) \setminus V(\pi, [\varpi_F]) = D(\pi) \cup D([\varpi_F]) \\ \ \ \| \\ Y \cup V(\pi) \cup V([\varpi_F]) = \text{Compactification of } Y \text{ by the divisors } (\pi) \text{ and } ([\varpi_F]) \end{array}$$

fixed by φ

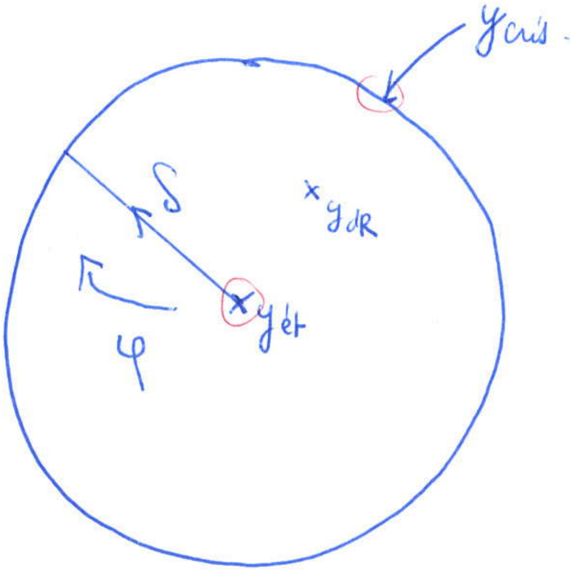
Ex: $E = \mathbb{F}_q((\pi))$ $\quad \mathcal{Y} \setminus V([\varpi_F]) = \mathbb{D}_F \subset \mathbb{D}_F^* = Y$

δ extends to $\delta : |\mathcal{Y}| \longrightarrow [0,1]$

$V(\pi) = \{y_{ét}\} = \{\delta = 0\}$

$V([\varpi_F]) = \{y_{cris}\} = \{\delta = 1\}$

$V(\xi) = \{\underset{\underset{y_0}{\shortparallel}}{y_{dR}}\}$



Extension at $y_{ét}$

$\mathcal{O}_{Y, y_{ét}} = \mathcal{O}_{\mathcal{E}^+} =$ henselian DVR
uniformizing π
residue field F

$$\widehat{\mathcal{O}_{\mathcal{E}^+}} = \begin{cases} F((\pi)) \\ W_{O_E}(F) \end{cases}$$

Th (Kedlaya): (i) expanding property of φ $\Downarrow$ Dieudonné-Manin: (unit root cristals)

$$\varphi\text{-eq. v.b. on } Y_{[0,1[} \underset{\substack{\uparrow \\ \text{stalks at } y_{ét}}}{\xrightarrow{\sim}} \varphi\text{-}\mathrm{Mod}_{\mathcal{O}_{\mathcal{E}^+}} \xrightarrow{\sim} \varphi\text{-}\mathrm{Mod}_{\widehat{\mathcal{O}_{\mathcal{E}^+}}} \overset{\downarrow}{=} \text{finite free } O_E\text{-modules}$$

(ii) If $\mathcal{E} \in \mathrm{Bun}_X$, $\mathcal{E} = \varphi$-eq. v.b. on Y

$\mathcal{E}$ s.s. slope $0 \iff \mathcal{E}$ extends to a φ-eq. v.b. on $Y_{[0,1[}$

And the set of such extensions $\xrightarrow{\sim} \{$ lattices in $H^0(X, \mathcal{E})\}$

Corollary: Via $\mathrm{Sht}_Y^{\geq 0} \xrightarrow{\sim} \mathrm{Modif}_X^{\geq 0}$

$$\mathrm{Sht}^{\geq 0}_{Y_{[0,1[}} \xrightarrow{\sim} \Big\{ \underset{\mathcal{E}_1 \text{ s.s. slope } 0}{[\mathcal{E}_1 \hookrightarrow \mathcal{E}_2] \in \mathrm{Modif}_X^{\geq 0}} + \Lambda \subset H^0(X, \mathcal{E}_1) \text{ lattice} \Big\}$$

$$\longrightarrow \underset{\substack{\cap \\ \mathrm{Sht}_Y^{\geq 0}}}{\mathscr{S}(\mathcal{E}, v)} \longmapsto \underset{\substack{\cap \\ \mathrm{Modif}_X^{\geq 0}}}{[\mathcal{E}_1 \hookrightarrow \mathcal{E}_2]}$$

near $y_{ét}$, $\mathcal{E} \underset{\substack{\uparrow \\ \text{compatible with} \\ \text{Frob. structure.}}}{\simeq} \check{\mathcal{E}}_1$

Extension at y_{cris} :

Fact: $\mathcal{O}(Y_{]0,1]}) = B^+_{rig} = \bigcap_{n\geq 0} \varphi^n(B^+_{cris})$

$$F\text{-isocrystals on } \mathrm{Spec}(\mathcal{O}_F/\varpi_F) \xrightarrow{\sim} \varphi\text{-eq. v.b. on } Y_{]0,1]}$$

$\Vert$

free B^+_{cris}-modules + semi-linear automorphism (linearization = auto)

Th: (F.-Fontaine):

$$\begin{tikzcd} \varphi\text{-eq. v.b. on } Y_{]0,1]} \arrow[r, "\sim", "\text{restriction to } Y"'] \arrow[d, "\simeq \ \text{stalk at } y_{cris}"] & \varphi\text{-eq. v.b. on } Y \\ \varphi\text{-Mod}_{\underbrace{\mathcal{O}_{Y,y_{cris}}}_{\text{Henselian local ring}}} \arrow[d, "\simeq"'] & \\ \varphi\text{-Mod}_{\overline{B}} & \end{tikzcd}$$

local wrt. max. ideal [illegible] $W_{\mathcal{O}_E}(m_F)/\mathfrak{p}$ [illegible]

$$\overline{B} = \mathcal{O}_{Y,y_{cris}} / \sqrt{([\varpi_F])} = \overbrace{W_{\mathcal{O}_E}(\mathcal{O}_F)[\tfrac{1}{\pi}]/}\ \mathfrak{p}$$

$$\mathfrak{p} = \left\{ \sum_{n\gg -\infty} [x_n]\pi^n \,/\, \sup_n |x_n| < 1 \right\}$$

But: This is not an equivalence of exact categories

$$0 \to \mathcal{E}_1 \xrightarrow{\alpha} \mathcal{E}_2 \xrightarrow{\beta} \mathcal{E}_3 \to 0 \text{ on } \mathcal{Y}_{]0,1]}$$

becomes exact when restricted to $\mathcal{Y}$ iff $\forall a \in m_F$ $\ker\beta / \mathrm{Im}\,\alpha$

and $\mathrm{Coker}\,\beta$ are killed by $[a]$.

$\rightsquigarrow$ almost exact functor extension: $\varphi\text{-}\mathrm{Mod}_{\mathcal{Y}} \xrightarrow{\sim} \varphi\text{-}\mathrm{Mod}_{\mathcal{Y}_{]0,1]}}$.

Corollary:

$$\begin{tikzcd}
\mathrm{Sht}^{\geq 0}_{\mathcal{Y}} \arrow[r, "\sim"] & \mathrm{Sht}^{\geq 0}_{\mathcal{Y}_{[0,1[}} \arrow[d, "\simeq"] \\
& \left\{ [\mathcal{E}_1 \hookrightarrow \mathcal{E}_2] \in \mathrm{Mod}^{\geq 0}_{\mathcal{Y}_X} + \Lambda \subset H^0(X, \mathcal{E}_1) \text{ lattice wt } \mathcal{E}_1 \text{ ss slope } 0 \right.
\end{tikzcd}$$

GAGA.

Th (Kedlaya): (i) $\mathrm{Bun}_{\underbrace{\mathrm{Spec}(\mathbb{A}) \setminus V(\pi, [\varpi_F])}_{\mathrm{Spec}(\mathbb{A}[\frac{1}{u}]) \cup \mathrm{Spec}(\mathbb{A}[\frac{1}{[\varpi_F]}])}} \xrightarrow[\sim]{GAGA} \mathrm{Bun}_{\mathcal{Y}}$ "$\mathrm{Spa}(\mathbb{A}) \setminus V(\pi, [\varpi_F])$"

(ii) Any vector bundle on $\mathrm{Spec}(\mathbb{A}) \setminus V(\pi, [\varpi_F])$ is trivial

$\Rightarrow$ finite free $\mathbb{A}$-modules $\xrightarrow{\sim} \mathrm{Bun}_{\mathcal{Y}}$

Corollary: $\varphi\text{-Mod}_{/\mathbb{A}} \xrightarrow{\sim} \mathrm{Sht}_{\mathcal{Y}}^{\geq 0} \xrightarrow{\sim} \{ \text{---} \}$

$\Longrightarrow$ theorem.

Moreover: If $0 \to A \to B \to C \to 0$ is a sequence in $\varphi\text{-Mod}_{/\mathbb{A}}$

it gives rise to an exact sequence of Modifications + lattice

$\Updownarrow$

its cohomology is almost zero w.r.t. to the almost structure on $\mathbb{A}$ given by $([a])_{a \in m_F}$.

Generalization: $K|E$ perfectoid, $F = K^\flat$.

$\varphi\text{-Mod}_{/\mathbb{A}} \xrightarrow{\sim} \Big\{ (\mathcal{E}_1 \hookrightarrow \mathcal{E}_2, \Lambda) \Big/ \mathcal{E}_1$ s.s. slope 0, Λ G_K-invariant lattice in the Galois rep. $H^0(X_{\hat{\bar{F}}}, \mathcal{E}_{1|X_{\hat{\bar{F}}}})$

if $\Big[$ $\mathcal{E}_2 \simeq \bigoplus_{\lambda \in \mathbb{Q}} \mathcal{O}_X(\lambda) \wedge \rho_\lambda$

$\rho_\lambda : G_K \longrightarrow GL_{D_\lambda}(W_\lambda)$ semi-linear rep.
$\uparrow D_\lambda$ v.s.

$\forall \lambda$, ρ_λ is unramified $\Big\}$

Modification of v.b. on X_F (v.b. satisfying descent w.r.t. the pro-étale covering $X_{\hat{\bar{F}}} \to X_F$)

Why the extension functor $\mathrm{Bun}_X \longrightarrow \varphi$-eq. v.b. $/ \mathcal{Y}_{]0,1]}$ is not exact

⑤

$t_1, t_2 \in H^0(X, \mathcal{O}_X(1)) = \mathcal{O}(Y)^{\varphi=\pi}$ linearly independant

$\Rightarrow V(t_1) \cap V(t_2) = \emptyset$

$\rightsquigarrow \; 0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X^2 \to \mathcal{O}_X(1) \to 0$ Koszul

$(a,b) \mapsto a t_1 + b t_2$

$c \mapsto (c t_2, -c t_1)$

But now, if $t \in B^{\varphi=\pi}$ then $t \in B^+_{rig} = \Gamma(\mathcal{Y}_{]0,1]}, \mathcal{O}_{\mathcal{Y}})$

$t = \sum_{n \in \mathbb{Z}} [\varepsilon^{q^{-n}}] \pi^n \Rightarrow t(y_{crys}) = 0 \in W(k_F)_{\mathbb{Q}} = k(y_{crys})$

$\Rightarrow$ the sequence $0 \to \mathcal{O}_{\mathcal{Y}_{]0,1]}} \xrightarrow{(t_2,-t_1)} \mathcal{O}_{\mathcal{Y},]0,1]}^2 \xrightarrow{t_1 \oplus t_2} \mathcal{O}_{\mathcal{Y},]0,1]} \to 0$

is not exact when localized at $\mathcal{O}_{\mathcal{Y}, y_{crys}}$.