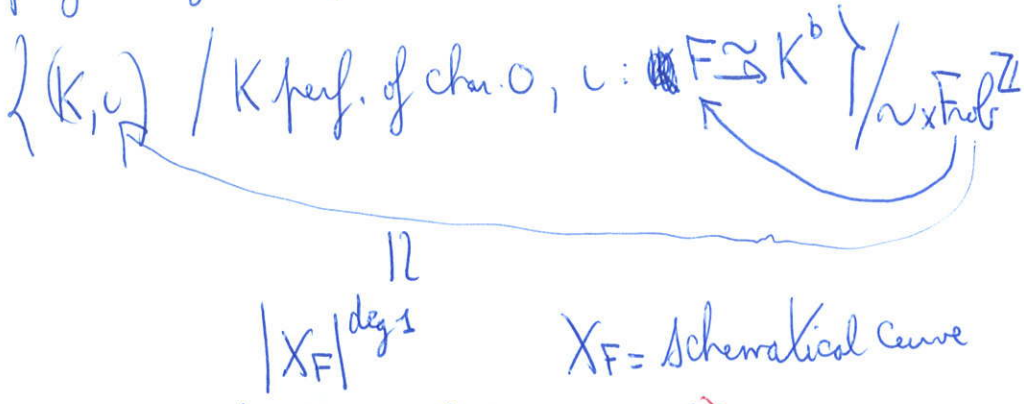


The adic curve

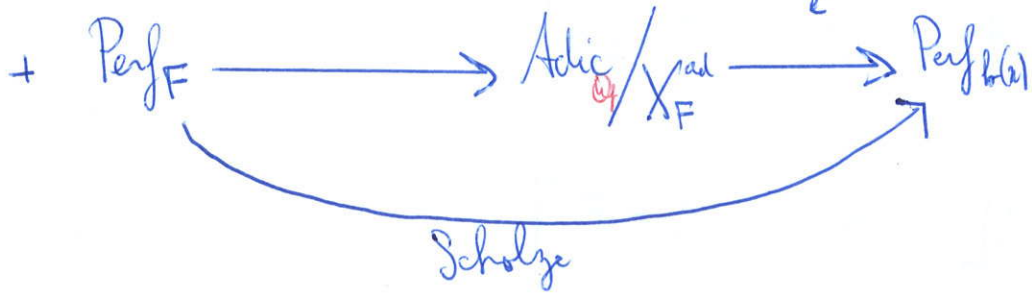
1

* K/\mathbb{Q}_p perfectoid $\rightsquigarrow K^b$
 $\text{Perf}_K \xrightarrow{\sim} \text{Perf}_{K^b}$

* F perfectoid of char. p - no canonical choice of K s.t. $F \simeq K^b$



Purpose: Construct X_F^{ad} (+ describe it) fiber at $x \in |X_F^{\text{ad}}|^{\text{classical of deg } 1} \simeq |X_F|^{\text{deg } 1}$



Recall from Fontaine's talks: \mathbb{F}_q

F/\mathbb{F}_q perfectoid
 E non-archimedean b_E/\mathbb{F}_q
 $1/p \leq |\omega_E| < 1$

For $p \in \mathbb{J}_0, 1 \leq |p|_p \leq 1$ \exists pan E s.t. $|\omega_E|_p = p$

$A = F$ -perfectoid algebra

$| \cdot | =$ spectral norm on A

* $I \subset]0, 1[$ closed interval

$\rightsquigarrow B_{A, E, I} =$ preperfectoid (Banach) E -algebra

$$\text{s.t. } B_{A, E, I} \hat{\otimes}_E E' = B_{A, E', I} \quad \text{if } E' | E$$

$$\text{if } E \text{ perfectoid } B_{A, E, I}^{\flat} = B_{A, E^{\flat}, I}$$

A perfectoid affinoid algebra $\rightsquigarrow A^+ \rightsquigarrow B_{A, E, I}^+ \subset B_{A, E, I}^{\circ}$

Def: $V_{A, E, I} = \text{Spa}(B_{A, E, I}, B_{A, E, I}^+)$

preperfectoid/ E

If $I \subset I' \subset]0, 1[$ and $\rho_1, \rho_2 \in |F^{\times}|$

$[\rho_1, \rho_2]$

$\rho_1 = |a|, \rho_2 = |b|$

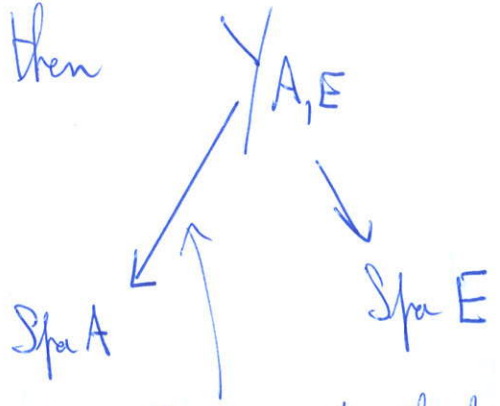
Then: $B_{A, E, I} = B_{A, E, I'} \left\langle \frac{[a]}{\omega_E}, \frac{\omega_E}{[b]} \right\rangle$

$\Rightarrow Y_I \subset Y_{I'}$ rational domain

Def: $Y_{A,E} = \varinjlim_{I \subset J \subset I'} Y_{A,E,I}$ perfect/E

$Y_{A,E}^b = Y_{A,E^b}$ if E perfectoid

Ex: Char E = p then



$B_{A,E,I} = A$ -algebra since $[E]$ additive
"Completion of $A \otimes_{\mathbb{F}_q} E$ "

$$Y_{A,E} = Y_{F,E} \times_{Spa F} Spa A$$

↓
Spa E

giving: Prop: $f_1, \dots, f_m, g \in A$ $(f_i, g)_i = A$

then $\sum_i B_I [f_i] + B_I [g] = B_I$

and $B_A \left\langle \frac{f_1, \dots, f_m}{g} \right\rangle_{E, I} = B_{A, E, I} \left\langle \frac{[f_1], \dots, [f_m]}{[g]} \right\rangle$

\rightsquigarrow Can glue $A_I \rightarrow Y_{A, E}$

to a functor

$$\begin{array}{ccc} \text{Perf}_F & \longrightarrow & \text{Perf}_E / Y_{F, E} \\ \mathbb{Z}_1 & \longrightarrow & Y_{\mathbb{Z}, E} \end{array}$$

Frobenius: $\rho \in]0, 1[$ note $\varphi(\rho) = \rho^q$.

Frob on $A \rightsquigarrow \varphi: B_{A, E, I} \xrightarrow{\sim} B_{A, E, \varphi(I)}$

$$\varphi\left(\sum_n [\lambda_n] \lambda_n\right) = \sum_n [\lambda_n^q] \lambda_n$$

"arithmetic" Frobenius: if E discrete

$$\varphi\left(\sum_{n \geq 0} [\lambda_n] \pi_E^n\right) = \sum_{n \geq 0} [\lambda_n^q] \pi_E^n$$

$\pi_E = \text{variable}$.

\rightsquigarrow $Y_{\mathbb{Z}, E}$
 \curvearrowright automorphism acts totally discontinuously
 since $\varphi(\text{Radius } \rho) = \text{Radius } \rho^{1/q}$ $\rho \in]0, 1[$

Def. $X_{\mathbb{Z}, E}^{\text{ad}} = Y_{\mathbb{Z}, E} / \varphi^{\mathbb{Z}}$

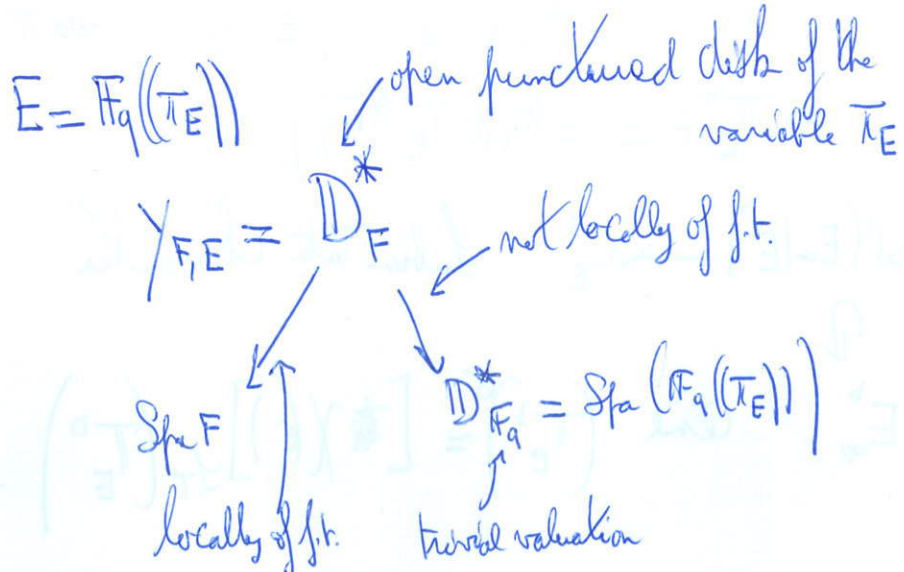
\rightsquigarrow functor $\left[\text{Perf}_F \longrightarrow \text{Preperf}_E / X_{F, E}^{\text{ad}} \right]$

Description of $X_{F, E}^{\text{ad}}$

the adic curve
Rem: $\mathbb{Z} = F$ -perfectoid space
 " \mathbb{Z} " does not exist
 $\text{Frob}_{\mathbb{Z}} \setminus$
 but $\varphi^{\mathbb{Z}} \setminus \underbrace{\mathbb{Z} \times_{\text{Spa } F} Y_{F, E}}_{Y_{\mathbb{Z}, E}}$ does

E discrete valuation

* Char $E = p$



$$X_{F,E} = \mathbb{P}^1 / \mathbb{D}_F^*$$

$$\swarrow \varphi_2 / \mathbb{P}^1 \quad \searrow \text{Spa}(\mathbb{F}_q(\pi_E))$$

$$* E = \mathbb{F}_q(\pi_E^{1/p^\infty}) \quad \gamma_{F,E} = \mathbb{D}_F^* \text{ 1/10}$$

* Char $E = 0$, $[E:\mathbb{Q}_p] = b_E = \mathbb{F}_q - \pi_E$ unif.
 L.T. = Lubin-Tate group law / \mathcal{O}_E

$$E_\infty = E(\text{LT}[\pi_E^\infty])$$

$\pi_E^b :=$ generator of $T_\pi(\text{L.T.})$

$$\pi_E^b = \left(\pi_E^{b(n)} \right)_{n \geq 0}$$

π_E^m - random point of L.T.

$$[\pi_E]_{\text{LT}} \left(\pi_E^{b(n+1)} \right) = \pi_E^{b(n)}$$

$\downarrow \text{mod } \pi_E$

$$\text{Frob}_q \left(\pi_E^{b(n+1)} \right) \equiv \pi_E^{b(n)} \text{ mod } \pi$$

$$\Rightarrow \pi_E^b \in E_\infty^b = \mathbb{F}_q(\pi_E^{1/p^\infty})$$

$\chi: \text{Gal}(E_\infty/E) \rightarrow G_E^x$ Lubin-Tate character.

\downarrow
 E_∞^b

$$\text{and } \left(\pi_E^b \right)^{\sigma} = [\chi(\sigma)]_{\text{LT}} \left(\pi_E^b \right)$$

$$\Rightarrow \mathcal{G} = \text{L.T. group} / \mathbb{F}_q$$

$$\tilde{\mathcal{G}} = \varprojlim_{x \in \mathbb{F}_q} \mathcal{G} = \mathcal{G}^{1/p^\infty} = \text{formal } E\text{-vector space}$$

$$\mathcal{E} = \left(\tilde{\mathcal{G}} \hat{\otimes}_{\mathbb{F}_q} G_F \right) \simeq \mathbb{D}_F^{* 1/p^\infty}$$

\uparrow after fixing a coordinate on \mathcal{G}

\mathcal{E} is an E -Banach space in Perf_F

$$\text{Then } \begin{array}{ccc} Y_{F, E_\infty} & \simeq & \mathcal{E} \setminus \{0\} \\ \downarrow G & & \downarrow G \\ \text{Gal}(E_\infty/E) & \xrightarrow{\chi} & G_E^x \end{array}$$

$$\Rightarrow \left| Y_{F, E} \right|_{\text{Gal}(E_0/E)} = \left| Y_{F, E_\infty} \right|_{G_E^x} \simeq \left| \mathcal{E} \setminus \{0\} \right|$$

$$\Rightarrow \left| X_{F, E}^{\text{ad}} \right|_{\text{classical points}} \simeq \left| \mathcal{E} \setminus \{0\} \right|_{G_E^x}$$

$$\left| X_{F, E} \right| \simeq \left| \mathcal{G}(G_F) \setminus \{0\} \right|_{G_E^x}$$

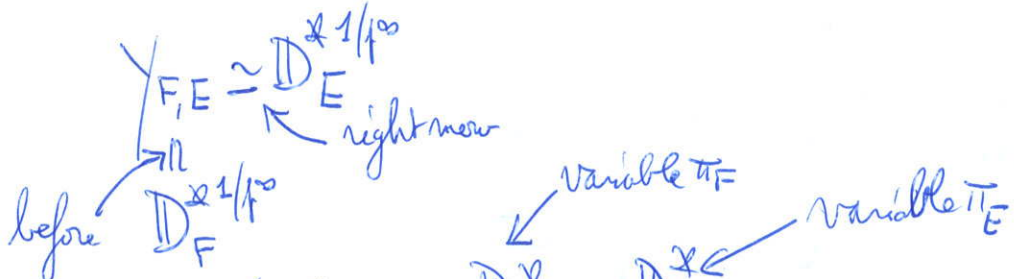
Case $F = \mathbb{F}_q((\pi_F^{1/p^\infty}))$ any E

$$\mathcal{Y}_{F,E} \simeq \mathbb{D}_E^{\times 1/p^\infty} = \{0 < |\pi| < 1\} \subset \mathcal{Y}_E(E < \pi^{1/p^\infty})$$

$T = [\pi_F]$

\uparrow \uparrow
 Radius q^{-n} Radius $q^{-1/n}$

Reason for this when $F = \mathbb{F}_q((\pi_F^{1/p^\infty}))$, $E = \mathbb{F}_q((\pi_E^{1/p^\infty}))$



Inject of $E = \mathbb{F}_q((\pi_E))$ $F = \mathbb{F}_q((\pi_F))$

$$\mathbb{D}_E^{\times} \simeq \mathbb{D}_F^{\times}$$

Radius $q^n \leftrightarrow$ Radius $q^{-1/n}$

$$\sum_{m \in \mathbb{Z}} a_m \frac{\pi_E^m}{\mathbb{F}_q((\pi_F))} = \sum_{m \in \mathbb{Z}} \left(\sum_{n \in \mathbb{Z}} a_{n,m} \pi_F^n \right) \pi_E^m$$

$$= \sum_{m \in \mathbb{Z}} \left(\sum_{n \in \mathbb{Z}} a_{n,m} \pi_E^n \right) \pi_F^m$$

\mathbb{Z} has a tors under C.V. conditions showing up in \mathbb{D}^*

$$|\cdot|_{\pi_E}^{q^{-n}} = |\cdot|_{\pi_F, q^{-1/n}}$$

Robba ring

* \mathcal{Y}_F Case $F = \mathbb{F}_q((\pi_F)) = \bigcup_i \mathbb{F}_q((\pi_i))$

\mathbb{C}_F^b

finite étale transition morphisms.

$$\Rightarrow \mathcal{Y}_{F,E} \simeq \lim_i \mathbb{D}_E^{\times 1/p^\infty}$$