The adic curve

* $K|\mathbb{Q}_p$ perfectoid $\rightsquigarrow K^\flat$

$$\text{Perf}_K \xrightarrow{\sim} \text{Perf}_{K^\flat}$$

* F perfectoid of char. p - no canonical choice of K s.t. $F \simeq K^\flat$

$$\begin{tikzcd} \{(K,\iota) \arrow[phantom, r, "/ \; K \text{ perf. of char. } 0,\ \iota : F \xrightarrow{\sim} K^\flat \}"] & & /\sim \times \text{Frob}^{\mathbb{Z}} \arrow[l, bend left] \arrow[ll, bend left=40] \end{tikzcd}$$

$$\wr\wr$$

$$|X_F|^{\deg 1} \qquad X_F = \text{schematical curve}$$

Purpose: Construct X_F^{ad} (+ describe it)

$$\begin{tikzcd} + \ \text{Perf}_F \arrow[r] \arrow[rr, bend right, "\text{Scholze}"'] & \text{Adic}_{\mathbb{Q}_4}/X_F^{ad} \arrow[r, "\text{fiber at } x \in |X_F^{ad}|^{\text{classical of deg } 1} \simeq |X_F|^{\deg 1}."] & \text{Perf}_{k(x)} \end{tikzcd}$$

Recall from Fontaine's talk : $\mathbb{F}_q$. $F|\mathbb{F}_q$ perfectoid

E non-archimedean $k_E|\mathbb{F}_q$

$|p| \leq |\varpi_E| < 1$

For $\rho \in]0,1[$ $|\cdot|_\rho$ on E s.t. $|\varpi_E|_\rho = \rho$

$A =$ F-perfectoid algebra

$|\cdot| =$ spectral norm on A

$* \ I \subset]0,1[$ closed interval

$\rightsquigarrow B_{A,E,I} =$ preperfectoid (Banach) E-algebra

s.t. $B_{A,E,I} \hat{\otimes}_E E' = B_{A,E',I}$ if $E'|E$

if E perfectoid $B^\flat_{A,E,I} = B_{A,E^\flat,I}$.

A perfectoid affinoid algebra $\rightsquigarrow A^+ \rightsquigarrow B^+_{A,E,I} \subset B^\circ_{A,E,I}$

$$\boxed{\text{Def:} \quad Y_{A,E,I} = \mathrm{Spa}\left(B_{A,E,I}, B^+_{A,E,I}\right)}$$ preperfectoid$/E$

If $\underset{\substack{\| \\ [\rho_1,\rho_2]}}{I} \subset I' \subset]0,1[$ and $\rho_1, \rho_2 \in |F^\times|$

$\rho_1 = |a|$, $\rho_2 = |b|$

Then:
$$\boxed{B_{A,E,I} = B_{A,E,I'}\left\langle \frac{[a]}{\varpi_E}, \frac{\varpi_E}{[b]} \right\rangle}$$

$\Rightarrow$ $Y_I \subset Y_{I'}$ rational domain

Def: $$Y_{A,E} = \varinjlim_{I \subset]0,1[} Y_{A,E,I} \quad \text{preperf}/E$$

$$Y^{\flat}_{A,E} = Y_{A,E^{\flat}} \text{ if } E \text{ perfectoid}$$

Ex: If Char $E = p$ then

$$\begin{tikzcd} & Y_{A,E} \arrow[dl] \arrow[dr] & \\ \mathrm{Spa}\, A & & \mathrm{Spa}\, E \end{tikzcd}$$

$B_{A,E,I}$ = A-algebra since $[-]$ additive

$\overset{\shortparallel}{\text{Completion of }} A \otimes_{\mathbb{F}_q} E$

$$\boxed{\begin{tikzcd} Y_{A,E} = Y_{F,E} \times_{\mathrm{Spa}\, F} \mathrm{Spa}\, A \\ Y_{F,E} \arrow[d] \\ \mathrm{Spa}\, E \end{tikzcd}}$$

gluing: Prop: $f_1, \dots, f_n, g \in A$ $\quad (f_i, g) = A$

Then $\sum_i B_I[f_i] + B_I[g] = B_I$

and $B_{A\langle \frac{f_1,\dots,f_n}{g}\rangle, E, I} = B_{A,E,I}\left\langle \frac{[f_1],\dots,[f_n]}{[g]}\right\rangle$

$\rightsquigarrow$ Can glue $A \longmapsto Y_{A,E}$

to a functor

$$\boxed{\begin{array}{ccc} \mathrm{Perf}_F & \longrightarrow & \mathrm{Préperf}_E / Y_{F,E} \\ \mathcal{X} & \longmapsto & Y_{\mathcal{X},E} \end{array}}$$

<u>Frobenius:</u> $\rho \in]0,1[$ note $\varphi(\rho) = \rho^q$.

Frob$_q$ on A $\rightsquigarrow$ $\varphi : B_{A,E,I} \xrightarrow{\sim} B_{A,E,\varphi(I)}$

$$\varphi\Big(\sum_n [x_n]\lambda_n\Big) = \sum_n [x_n^q]\lambda_n$$

"arithmetic" Frobenius: if E discrete

$$\varphi\Big(\sum_{n\geq 0}' [x_n]\pi_E^n\Big) = \sum_{n\geq 0}' [x_n^q]\pi_E^n$$

π_E = variable.

$\rightsquigarrow \; Y_{\mathcal{F},E}$ $\circlearrowleft$ φ automorphism acts totally discontinuously

since $\varphi(\text{Radius } \rho) = \text{Radius } \rho^{1/q}$ $\quad \rho \in \,]0,1[$

$$\boxed{\underline{\text{Def}}: \; X^{ad}_{\mathcal{F},E} := Y_{\mathcal{F},E}/\varphi^{\mathbb{Z}}}$$

$$\rightsquigarrow \text{functor } \left[\mathrm{Perf}_F \longrightarrow \mathrm{Préperf}_E / \underbrace{X^{ad}_{F,E}}_{\text{the adic curve}} \right]$$

Description of $X^{ad}_{F,E}$

Rem: $\mathcal{F} = F$- perfectoid space

"$\mathrm{Frob}_{\mathcal{F}} \backslash \mathcal{F}$" does not exist

but $\varphi^{\mathbb{Z}} \backslash \underbrace{\mathcal{F} \times_{\mathrm{Spa}\,F} Y_{F,E}}_{Y_{\mathcal{F},E}}$ does

E discrete valuation

$*$ char $E = p$ $\quad E = \mathbb{F}_q((\pi_E))$

open punctured disk of the variable π_E

not locally of f.t.

$$\begin{tikzcd} & Y_{F,E} = \mathbb{D}^*_F \arrow[dl] \arrow[dr] & \\ \mathrm{Spa}\,F & & \mathbb{D}^*_{\mathbb{F}_q} = \mathrm{Spa}(\mathbb{F}_q((\pi_E))) \end{tikzcd}$$

locally of f.t. $\quad$ trivial valuation

$$\begin{tikzcd} & X_{F,E} = \varphi^{\mathbb{Z}} \backslash \mathbb{D}_F^{*} \arrow[dl, "\times"] \arrow[dr] & \\ \cancel{\varphi^{\mathbb{Z}} \backslash \operatorname{Spa} F} & & \operatorname{Spa}(\mathbb{F}_q((\pi_E))) \end{tikzcd}$$

* $E = \mathbb{F}_q((\pi_E^{1/p^\infty}))$ $\quad$ $Y_{F,E} = \mathbb{D}_F^{* \, 1/p^\infty}$.

* Char $E = 0$, $[E : \mathbb{Q}_p]$ $\quad$ $k_E = \mathbb{F}_q$ - π_E unif.

$\mathcal{L}.T. =$ Lubin-Tate group law$/\mathcal{O}_E$

$$\hat{E}_\infty = \widehat{E(\mathcal{LT}[\pi_E^\infty])}$$

$\pi_E^\flat :=$ generator of $T_\pi(\mathcal{L}.T.)$ $\qquad$ $\pi_E^\flat = (\pi_E^{(n)})_{n \geq 0}$

$\uparrow$

π_E^n - torsion point of $\mathcal{L}.T.$

$$\begin{tikzcd} {[\pi_E]_{\mathcal{LT}}(\pi_E^{(n+1)}) = \pi_E^{(n)}} \arrow[d, "\bmod \pi_E"] \\ \operatorname{Frob}_q(\pi_E^{(n+1)}) \equiv \pi_E^{(n)} \bmod \pi \end{tikzcd}$$

$\Rightarrow \pi_E^\flat \in E_\infty^\flat = \mathbb{F}_q((\pi_E^{\flat \, 1/p^\infty}))$.

$\chi : \operatorname{Gal}(E_\infty | E) \longrightarrow \mathcal{O}_E^\times$ Lubin-Tate character.

$\circlearrowright$

$E_\infty^\flat$ and $(\pi_E^\flat)^\sigma = [\chi(\sigma)]_{\mathcal{LT}}(\pi_E^\flat)$.

$\Rightarrow$ $\mathcal{G}$ = L.T. group $/\mathbb{F}_q$

$$\widetilde{\mathcal{G}} = \varprojlim_{\substack{\times \pi_E \\ \mathrm{Frob}_q}} \mathcal{G} = \mathcal{G}^{1/p^\infty} = \text{formal } E\text{-vector space}$$

$$\mathcal{E} = \big(\widetilde{\mathcal{G}} \,\hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_F\big)_\eta \simeq \mathbb{D}_F^{*\,1/p^\infty}$$

$\uparrow$ after fixing a coordinate on $\mathcal{G}$

$\|$
E-Banach space in Perf_F

Then

$$\begin{tikzcd} Y_{F,E_\infty^\flat} \arrow[r, "\sim"] \arrow[loop below] & \mathcal{E}\setminus\{0\} \arrow[loop below] \\ \mathrm{Gal}(E_\infty|E) \arrow[r, "\chi"] & \mathcal{O}_E^\times \end{tikzcd}$$

$$\Rightarrow \quad |Y_{F,E}| = \mathrm{Gal}(E_\infty|E)\backslash |Y_{F,E_\infty}| \xrightarrow{\sim} \mathcal{O}_E^\times \backslash |\mathcal{E}\setminus\{0\}|$$

$$\Rightarrow \boxed{\begin{array}{ccc} |X^{ad}_{F,E}| & \xrightarrow{\sim} & E^\times \backslash |\mathcal{E}\setminus\{0\}| \\ \cup \text{ classical points} & & \cup \\ |X_{F,E}| & \xrightarrow{\sim} & E^\times \backslash \mathcal{G}(\mathcal{O}_F)\setminus\{0\} \end{array}}$$

Case $F = \mathbb{F}_q((\pi_F^{1/p^\infty}))$ any E

$$\underset{\substack{\uparrow \\ \text{Radius } q^{-n}}}{Y_{F,E}} \simeq \underset{\substack{\uparrow \\ \text{Radius } q^{-1/n}}}{\mathbb{D}_E^{*\,1/p^\infty}} = \{0 < |T| < 1\} \subset \mathrm{Spa}\left(E\langle T^{1/p^\infty}\rangle\right)$$

$T = [\pi_F]$

Radius $q^{-n} \longleftrightarrow$ Radius $q^{-1/n}$

$\uparrow$

reason for this when $F = \mathbb{F}_q((\pi_F^{1/p^\infty}))$, $E = \mathbb{F}_q((\pi_E^{1/p^\infty}))$

$$\begin{tikzcd} Y_{F,E} \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\wr"] & \mathbb{D}_E^{*\,1/p^\infty} \\ \mathbb{D}_F^{*\,1/p^\infty} & \end{tikzcd}$$

right now

before

In fact if $E = \mathbb{F}_q((\pi_E))$ $F = \mathbb{F}_q((\pi_F))$ $\quad \underset{\substack{\uparrow \\ \text{variable } \pi_F}}{\mathbb{D}_E^*} \simeq \underset{\substack{\uparrow \\ \text{variable } \pi_E}}{\mathbb{D}_F^*}$

Radius $q^n \Longleftrightarrow$ Radius $q^{-1/n}$

$$\sum_{n\in\mathbb{Z}}' \underset{\substack{\cap \\ \mathbb{F}_q((\pi_F))}}{a_n} \pi_E^n = \sum_{n\in\mathbb{Z}}' \left(\sum_{m\in\mathbb{Z}} a_{n,m} \pi_F^m\right) \pi_E^n$$

$$\text{“}=\text{”} \sum_{m\in\mathbb{Z}} \left(\sum_{n\in\mathbb{Z}} a_{n,m} \pi_E^n\right) \pi_F^m$$

Σ has a sens under C.V. conditions showing up in $\mathbb{D}^*$

$$|\cdot|_{\pi_E, q^{-n}} = |\cdot|_{\pi_F, q^{-1/n}} \rightsquigarrow \text{Robba ring}$$

* ~~Y/φ~~ Case $F = \widehat{\overline{\mathbb{F}_q((\pi_F))}} = \widehat{\bigcup_i \mathbb{F}_q((\pi_i))}$

$\wr\wr$

$\mathbb{C}_p^\flat$

finite étale transition morphisms.

$$\Rightarrow Y_{F,E} \sim \varprojlim_i \mathbb{D}_E^{*\,1/p^\infty}$$