From local class field to the Curve and vice versa

"G-torseurs en théorie de Hodge p-adique"

$[E:\mathbb{Q}_p] < +\infty$ $\quad \mathbb{F}_q = \mathcal{O}_E/\pi$

$F|\mathbb{F}_q$ complete alg. closed. $\quad$ Ex: $\widehat{\overline{\mathbb{F}_q((T))}} \simeq \mathbb{C}_p^\flat$

⇝ Curve $\quad X =$ one dim. regular scheme $/E$

joint work w/ Fontaine $\quad X^{ad} =$ adic space $/E$

$X^{ad} = Y/\varphi^{\mathbb{Z}}$ $\quad Y = \mathrm{Spa}\left(W_{\mathcal{O}_E}(\mathcal{O}_F)\right) \setminus V(p[\varpi_F])$ $\quad 0 < |\varpi_F| < 1$

$\varphi \circlearrowright Y$ $\quad \varphi\left(\sum_n [x_n]\pi^n\right) = \sum_n [x_n^q]\pi^n$

alg. closure of $\mathbb{F}_q$ in F

$L = W_{\mathcal{O}_E}(\overline{\mathbb{F}}_q) = \widehat{E^{un}} \circlearrowleft \sigma = \varphi$

finite dim. L-v.s.

$\varphi\text{-Mod}_L =$ isocrystals $= \{(D,\varphi)\}$

semi-linear iso.

$$\begin{array}{ccccc}
\varphi\text{-Mod}_L & \longrightarrow & \mathrm{Bun}_{X^{ad}} & \xleftarrow[\sim]{\text{GAGA}} & \mathrm{Bun}_X \\
(D,\varphi) & \longmapsto & \mathcal{E}(D,\varphi)^{ad} & \longleftarrow & \mathcal{E}(D,\varphi)
\end{array}$$

$$\mathcal{E}(D,\varphi)^{ad} = \begin{array}{c} Y \underset{\varphi^{\mathbb{Z}}}{\times} D \\ \downarrow \\ Y/\varphi^{\mathbb{Z}} \end{array}$$

Ex: $(D,\varphi) = (L, \pi^{-1}\sigma)$ $\qquad \mathcal{E}(D,\varphi)^{ad} = \mathcal{O}_{X^{ad}}(1)$

then $X = \mathrm{Proj}\left(\bigoplus_{d\geq 0} H^0\left(X^{ad}, \mathcal{O}_{X^{ad}}(d)\right)\right)$

Th (Fontaine–F.): $\mathcal{E}(-): \varphi\text{-}\mathrm{Mod}_L \to \mathrm{Bun}_X$ is essentially surjective

Via Dieudonné–Manin: $\forall\, \mathcal{E} \in \mathrm{Bun}_X$, $\mathcal{E} \simeq \bigoplus_i \mathcal{O}_X(\lambda_i)$, $\lambda_i \in \mathbb{Q}$

G reductive gp./E $\qquad B(G) := G(L)/\sigma\text{-conj.}$ Kottwitz

$$b \in G(L) \rightsquigarrow \begin{tikzcd} \mathrm{Rep}_E G \arrow[r] \arrow[rr, bend right] & \varphi\text{-}\mathrm{Mod}_L \arrow[r, "\mathcal{E}(-)"] & \mathrm{Bun}_X \end{tikzcd}$$

$$(V,\rho) \longmapsto (V_L, \rho(b)\sigma)$$

defines a G-torsor $\mathcal{E}_b/X$ (Tannakian version)

Th: This induces a bijection

$$B(G) \xrightarrow{\sim} H^1_{et}(X,G)$$
$$[b] \longmapsto [\mathcal{E}_b]$$

Nice features: Ex: * $H^1(E,G) \subset B(G)$
$\quad\quad\quad\quad\quad\quad\quad ||$
unit root G-isocrystals
induced by pullback via $X \longrightarrow \operatorname{Spec} E$

* dictionnary: Kottwitz $\longleftrightarrow$ Atiyah-Bott reduction theory

Ex: b basic $\Longleftrightarrow$ $\mathcal{E}_b$ semi-stable.

Proof technical - Speak about something linked.

Find a simpler proof for $G = T$ torus.

Kottwitz: Suffices to prove $T \longmapsto H^1(X,T)$ right exact like $T \mapsto B(T)$

lead to prove $\rightsquigarrow$ Th: $H^2(X,T) = 0$

dévissage $\rightsquigarrow$ Th: $H^2(X, \mathbb{G}_m) = 0$

Grothendieck: $\mathrm{Br}(X) \xrightarrow{\sim} H^2(X, \mathbb{G}_m)$
$\quad \| $
Azumaya algebras$/\sim$

$$\boxed{\text{Th: } \mathrm{Br}(X) = 0}$$

Proof: $\boxed{\text{Key Prop: The map } \mathrm{Br}(E) \longrightarrow \mathrm{Br}(X) \text{ is zero.}}$

$\to$ B simple algebra$/E$

local class field $\Rightarrow \exists (D, \varphi) \in \varphi\text{-Mod}_L$ isoclinic s.t. $B \simeq \mathrm{End}(D, \varphi)$

Then $B \underset{E}{\otimes} \mathcal{O}_X \xrightarrow{\sim} \mathcal{E}nd(\mathcal{E}(D, \varphi))$

$\Rightarrow [B \underset{E}{\otimes} \mathcal{O}_X] = 0 \in \mathrm{Br}(X)$ $\square$

$\to$ $\mathcal{A}$ = Azumaya alg. $/X$ = twisted form of a matrix algebra

$(\mathcal{A}^{\geq \lambda})_{\lambda \in \mathbb{Q}}$ = H.N. filtration as a v.b.

$\mathcal{A}^{\geq \lambda} \cdot \mathcal{A}^{\geq \mu} \subset \mathcal{A}^{\geq \lambda + \mu}$ ($\otimes$ of semi-stable = semi-stable)

$\uparrow$ algebra law

③

$e : \mathcal{O}_X \longrightarrow \mathcal{A}^{\geq 0}$ unit section

$G = \mathcal{A}^\times / e(\mathcal{O}_X^\times) =$ group scheme $/X$ (twisted form of PGL_n)

$\cup$

$H = (\mathcal{A}^{\geq 0})^\times / e(\mathcal{O}_X^\times)$ (smooth closed subgroup-scheme)

($\mathcal{A}^{\geq 0} \subset \mathcal{A}$ sub-algebra)

$\mathrm{Lie}\, G = \mathcal{A} / e(\mathcal{O}_X) =$ twisted form of $\mathfrak{pgl}_n$

$\Rightarrow$ Killing form $\mathrm{Lie}\, G \times \mathrm{Lie}\, G \longrightarrow \mathcal{O}_X$ is perfect

Then $(\mathrm{Lie}\, G)^{\geq \lambda, \perp} = (\mathrm{Lie}\, G)^{> -\lambda}$

$\Rightarrow (\mathrm{Lie}\, H)^\perp = \mathcal{A}^{>0} \subset \mathcal{A}^{\geq 0}/e(\mathcal{O}_X) = \mathrm{Lie}\, H$

$\Rightarrow$ H parabolic subgroup-scheme of G.

Thus $\exists\, P \subset PGL_n$ parabolic subgp. - the PGL_n-torsor has a reduction to P $=$ Atiyah-Bott canonical reduction -

$$H^1(X, P) \longrightarrow H^1(X, PGL_n)$$

inner twisting

$$\alpha \longmapsto [\mathrm{Isom}(M_n, \mathcal{A})]$$

$$P^\alpha = H$$

$$R_u P^\alpha = \mathcal{U} = 1 + \underbrace{\mathcal{A}^{>0}}_{\text{nilpotent ideal}}$$

fiber of $H^1(X, P) \xrightarrow{f} H^1(X, P/R_uP)$

over $f(\alpha) \simeq H^1(X, \mathcal{U})$

$\mathcal{U}$ filtered by $\mathcal{U}^{\geq \lambda} = 1 + \mathcal{A}^{\geq \lambda}$, $\lambda \in \mathbb{Q}_{>0}$.

$\mathcal{U}^{\geq\lambda}/\mathcal{U}^{>\lambda} \simeq \mathcal{A}^{\geq\lambda}/\mathcal{A}^{>\lambda}$ = semi-stable of slope ≥ 0

$\Rightarrow H^1(X, \mathcal{A}^{\geq\lambda}/\mathcal{A}^{>\lambda}) = 0$

$\Rightarrow H^1(X, \mathcal{U}) = \{*\}$

Thus α determined by $f(\alpha)$

$\Rightarrow \mathcal{A}$ determined by $\mathcal{A}^{\geq 0}/\mathcal{A}^{>0} = \prod$ semi-stable slope 0 Azumaya alg.

But Slope 0 s.s. vector bundles $\xrightarrow[\otimes\text{-equivalence}]{H^0(X,-)\ \sim}$ Vect_E (finite dim. E-v.s.)

$\Rightarrow$ if $\mathcal{B}$ = slope 0 s.s. Azumaya alg., $\mathcal{B} = B \otimes_E \mathcal{O}_X$
with B = simple E-algebra.

$\Rightarrow \square$ using the Key proposition.

Vice-Versa: $B(G) \xrightarrow{\sim} H^1(X, G)$

$+ \; B(GL_n) \longrightarrow\!\!\!\!\to B(PGL_n)$

$\Rightarrow \; Br(X) = 0 \Longrightarrow$ local class field.

reverse key prop. proof

Application: * Kummer exact sequence + Computation of Pic

$\Rightarrow \quad \exists \; tr : H^2(X, \mu_n) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$

s.t. $\eta_X := c_1(\mathcal{O}(1)) = cl([x]) \qquad x \in |X|$

$\underset{\text{fundamental class of the curve}}{\overset{\shortparallel}{}}$

satisfies $tr(\eta_X) = 1$

* Recall X geo. simply connected. $\Gamma = Gal(\bar{E}/E)$

discrete Γ-modules $\simeq$ étale local systems on X.

Th: M = finite discrete Γ-module
$\mathcal{F}$ = associated local system / $X_{ét}$

Then for $0 \leq i \leq 2$, $H^i(\Gamma, M) \xrightarrow{\sim} H^i_{ét}(X, \mathcal{F})$

Moreover $H^2(E, \mu_n) \xrightarrow{\sim} H^2(X, \mu_n)$

$$\begin{array}{c} H^2(E,\mu_n) \\ \shortparallel \\ Br(E)[n] \end{array}$$

fundamental class of class field $\longleftrightarrow$ η_X = fundamental class of the curve

$\Longrightarrow$ geometric interpretation of:

* Tate-Nakayama duality = Poincaré duality on X

$$H^i(X, \mathcal{F}) \times H^{2-i}(X, \check{\mathcal{F}}(1)) \longrightarrow H^2(X, \mu_n) \xrightarrow[\sim]{tr} \mathbb{Z}/n\mathbb{Z}$$

* Tate formula for Euler-Poincaré characteristic of Galois coho. = usual formula for E.P. char of local systems on curves.