①

$\overline{\mathbb{F}_q} | \mathbb{F}_q$

$$* E \nearrow \mathbb{F}_q((\pi)) \qquad \searrow [E : \mathbb{Q}_p] < +\infty \quad \mathbb{F}_q = \mathcal{O}_E / \pi$$

$* F | \mathbb{F}_q$ perfectoid field (Complete perfect for example $\widehat{\overline{\mathbb{F}_q((T))}}$, $\mathbb{F}_q((T^{1/p^\infty}))$)

$\rightsquigarrow$ joint work with Fontaine

Curve $\nearrow$ Compact p-adic Riemann surface not top. of finite type $\qquad X_F^{ad}$ = E-adic space

Curve $\searrow$ "Complete" algebraic curve" not of finite type$|E$ $\qquad X_F^{sch}$ = Dedekind E-scheme

$\forall f \in E(X)^\times \quad \deg(\operatorname{div} f) = 0$

$* S \in \mathrm{Perf}_{\mathbb{F}_q} = \mathbb{F}_q$-perfectoid spaces. Can define X_S = E-adic space

" = family of curves $(X_{k(s)})_{s \in |S|}$ "

For $E = \mathbb{F}_q((\pi))$ $\qquad X_S = \mathbb{D}_S^* / \varphi^{\mathbb{Z}}$ where $\varphi = \mathrm{Frob}_S$, $\varphi^*(\pi) = \pi$.

$$\mathbb{D}_S^* = \{0 < |\pi| < 1\} \subset \mathbb{A}_S^1$$

loc. of finite type. usual structural morphism but not φ-invariant

not loc. of finite type but φ-invariant

$$S \longleftarrow \mathbb{D}_S^* \longrightarrow \mathbb{D}_{\mathbb{F}_q}^* = \mathrm{Spa}(E)$$

trivial valuation

$\mathbb{D}_S^* \circlearrowleft \varphi$ totally discontinuous without fixed points

$\varphi(\text{Annulus with radius } \rho) = \text{Annulus with radius } \rho^{1/q}$

for $\rho \in]0,1[$

* For $E|\mathbb{Q}_p$ $\quad S = \mathrm{Spa}(R)$ $\quad R = \mathbb{F}_q$- perfectoid algebra

$X_S = Y_S/\varphi^{\mathbb{Z}}$ where Y_S constructed from

$W_{O_E}(R^0) = \left\{ \sum_{n\geq 0} [x_n]\pi^n \,/\, x_n \in R^0 \right\} \circlearrowleft \varphi = \text{Frob}$ of Witt vectors

that replaces $R^0[[\pi]]$ for $\mathbb{F}_q((\pi))$

* $L = \widehat{E^{un}}$

$$\begin{tikzcd} & \overline{\mathbb{F}_q}((\pi)) \\ L = \widehat{E^{un}} \arrow[ur] \arrow[dr] & \\ & W_{O_E}(\overline{\mathbb{F}_q})_{\mathbb{Q}} \end{tikzcd}$$

$\varphi\text{-Mod}_L = \text{Isocrystals} = \{(\underbrace{D}_{\text{finite dim. } L\text{-v.s.}}, \underbrace{\varphi}_{\sigma\text{-linear automorphism}})\}$ classified by Dieudonné-Manin

$S/\overline{\mathbb{F}_q}$

$$\begin{array}{ccc} \varphi\text{-}\mathrm{Mod}_L & \longrightarrow & \mathrm{Bun}_{X_S} \\ (D,\varphi) & \longmapsto & \mathcal{E}(D,\varphi) \end{array}$$

$\uparrow$ géo. realization

$$Y_S \times_{\varphi^{\mathbb{Z}}} D \longrightarrow Y_S/\varphi^{\mathbb{Z}} = X_S$$

Th: (F.-Fontaine): If S is a géo. point, $S = \mathrm{Spa}(F)$ F alg. closed, then $\mathcal{E}(-)$ is essentially surjective.

Via Dieudonné–Manin: $\forall \lambda \in \mathbb{Q}$ Can define
$\mathcal{O}(\lambda) =$ stable vector bundle of slope λ on X_F

Then $\forall \mathcal{E} \in \mathrm{Bun}_X$ $\quad \mathcal{E} \simeq \bigoplus_i \mathcal{O}(\lambda_i)$, $\lambda_i \in \mathbb{Q}$.

* $G =$ reductive group$/E$

$B(G) = G(L)/\sigma\text{-Conjugacy}$ (Kottwitz set)
$\quad \|$
G-isocrystals

G-isocrystal

$$\begin{tikzcd} b \in G(L) \rightsquigarrow \mathrm{Rep}_E(G) \arrow[r] \arrow[rr, bend right, "G\text{-bundle } \mathcal{E}_b"'] & \varphi\text{-}\mathrm{Mod}_L \arrow[r, "\mathcal{E}(-)"] & \mathrm{Bun}_{X_S} \quad S/\mathbb{F}_q \end{tikzcd}$$

$$(V,\rho) \longmapsto (V_L, \rho(b)\sigma)$$

G-bundle $\mathcal{E}_b$.

__Th:__ S géo. point $= \mathrm{Spa}(F)$ - X_F = schematical curve.

$\mathcal{E}_b =$ G-torsor on X_F (via GAGA $\mathrm{Bun}_{X_F^{sch}} \xrightarrow{\sim} \mathrm{Bun}_{X_F^{ad}}$)

$$B(G) \xrightarrow{\sim} H^1_{et}(X_F, G)$$

$$[b] \longmapsto [\mathcal{E}_b]$$

__Nice features:__ * b basic $\Longleftrightarrow$ $\mathcal{E}_b$ semi-stable

generalization of isoclinic
ν_b = slope morphism is central

* $H^1(E, G) \subset B(G)$ induced by pullback
via $X_F \to \mathrm{Spec}(E)$ in étale Coho.

$\{$unit root G-isocrystals$\}$ $= H^1(E,G)$

The stack Bun_G:

$\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ = perfectoid spaces + pro-étale topology.

Can define a stack $\mathrm{Bun}_G : S \mapsto$ G-bundles on X_S
on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$

Preceding theorem says $\boxed{B(G) = |\mathrm{Bun}_{G,\overline{\mathbb{F}}_q}|}$

Connected Components

Borovoi's fundamental group

Kottwitz map $\kappa : B(G) \longrightarrow \pi_1(G)_\Gamma$ $\quad \Gamma = \mathrm{Gal}(\overline{E}/E)$

Via the preceding identification:
* for $G = GL_n$ $\kappa = -$ degree of a v.b.
* In general $\kappa = -c_1^G$ = equivariant first Chern class.

Fact. κ is locally constant on $\mathrm{Bun}_{G,\overline{\mathbb{F}}_q}$

$$\Rightarrow \left[\mathrm{Bun}_{G,\overline{\mathbb{F}}_q} = \coprod_{\alpha \in \pi_1(G)_\Gamma} \underbrace{\mathrm{Bun}^{\alpha}_{G,\overline{\mathbb{F}}_q}}_{\text{open/closed substacks}} \right]$$

G quasi-split from now on

H.N. stratification:

$$HN : |\mathrm{Bun}_{G,\overline{\mathbb{F}}_q}| \longrightarrow X_*(A)^+_{\mathbb{Q}}$$

$X_*(A)^+_{\mathbb{Q}}$ ← positive Weyl chamber; A ← maximal split torus

⇝ HN-polygon = semi-continuous

s.s. ← semi-stable locus

Only thing to know: $\mathrm{Bun}_{G,\overline{\mathbb{F}}_q}^{ss}$ = open

Uniformization of semi-stable locus

* Kottwitz: $\kappa : B(G)_{basic} \xrightarrow{\sim} \pi_1(G)_\Gamma$

Geometric traduction: $\forall \alpha \in \pi_1(G)_\Gamma \quad |\mathrm{Bun}^{\alpha, ss}_{G,\overline{\mathbb{F}}_q}|$ = one point

* b basic $\quad J_b = \sigma$-centralizer of b = inner form of G.

$\quad\quad \|$

extended pure inner form of $G \supseteq$ Vogan's pure inner forms

via $H^1(E,G) \subset B(G)$

$\quad\quad \|$

all inner forms of G if Z_G connected, for $G = GL_n$ for example.

Fact: The sheaf on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$: $S \longmapsto \mathrm{Aut}(\mathcal{E}_b/X_S)$

is $\underline{J_b(E)}$

Ex. $b = 1$. $\underline{\mathrm{Aut}}(\text{trivial } G\text{-bundle}) = \underline{G(E)}$.

4

Th: b basic, $\alpha = \kappa(b)$. Then

$$x_b : \left[\mathrm{Spa}(\overline{\mathbb{F}}_q) / \underline{J_b(E)} \right] \xrightarrow{\sim} \mathrm{Bun}_{G,\overline{\mathbb{F}}_q}^{\alpha, ss}$$

induced by $\mathcal{E}_b$.

Hecke correspondences $\qquad \mu \in X_*(T)^+/\Gamma$

$$\mathrm{Spa}(E)^\diamond = \text{sheaf of untilts to } E$$

If $S^\# =$ untilt of $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ it induces a Cartier divisor

$$S^\# \hookrightarrow X_S$$

For $S = \mathrm{Spa}(R)$, $X_R =$ relative schematical curve then

$$(X_R)^\wedge_{/\mathrm{Spec}(R^\#)} = \mathrm{Spf}(\underbrace{B^+_{dR}(R^\#)})$$

Fontaine's ring B^+_{dR}.

$$
\begin{tikzcd}
 & \mathrm{Hecke}^{\leq \mu} \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\
\mathrm{Bun}_G & & \mathrm{Bun}_G \times \mathrm{Spa}(E)^\diamond
\end{tikzcd}
$$

two G-bundles + a modification bounded by μ along the Cartier divisor defined by the untilt

The Conjecture: G quasisplit$/E$ $\ell \neq p$

${}^L G = \overline{\mathbb{Q}}_\ell$ - Langlands dual

should be defined intrinsecaly via a hypothetical geometric Satake for the B_{dR}-affine Grassmannian

$\varphi : W_E \to {}^L G$ Langlands parameter

$$S_\varphi = \mathrm{Aut}(\varphi) = \{ g \in \hat{G} \,/\, g\varphi g^{-1} = \varphi \}$$
$$\cup$$
$$Z(\hat{G})^\Gamma$$

Suppose φ is discrete i.e. $S_\varphi / Z(\hat{G})^\Gamma$ is finite.

We say moreover φ is cuspidal if the associated 1-cocycle $I_E \to \hat{G}$ has finite image.

Conjecture. $\exists\, \mathcal{F}_\varphi$ perverse $\overline{\mathbb{Q}}_\ell$-Weil sheaf on $\mathrm{Bun}_{G,\overline{\mathbb{F}}_q}$ equipped with an action of S_φ such that

(1) $\forall \alpha \in \pi_1(G)_\Gamma = X^*(Z(\widehat{G})^\Gamma)$ the action of $Z(\widehat{G})^\Gamma \subset S_\varphi$ on $\mathcal{F}_\varphi|_{\mathrm{Bun}^\alpha_{G,\overline{\mathbb{F}}_q}}$ is given by α

(2) If φ is cuspidal and $j : \mathrm{Bun}_G^{ss} \overset{\text{open}}{\hookrightarrow} \mathrm{Bun}_G$ then

$$\mathcal{F}_\varphi = j_! j^* \mathcal{F}_\varphi$$

(3) $\forall b$ basic $\quad x_b : [\mathrm{Spa}(\overline{\mathbb{F}}_q)/J_b(E)] \to \mathrm{Bun}_{G,\overline{\mathbb{F}}_q}$

$$x_b^* \mathcal{F}_\varphi = \bigoplus_{\substack{\rho \in \widehat{S_\varphi} \\ \rho|_{Z(\widehat{G})^\Gamma} = \alpha(b)}} \rho \otimes \pi_{\varphi,\rho,b}$$

$\circlearrowleft$ $\underbrace{J_b(E) \times S_\varphi}_{\text{smooth action since } \ell \neq p}$

$\{\pi_{\varphi,\rho,b}\}_\rho$ = L-packet of a local Langlands correspondence for the inner form J_b

(4) (Hecke ~~...~~ eigenvector property) $\forall \mu \in X_*(T)^+/\Gamma$

$$\overrightarrow{h}_! \Big(\overleftarrow{h}^* \mathcal{F}_\varphi \otimes \underbrace{IC_\mu}_{\overline{\mathbb{Q}}_\ell(-)[-] \text{ if } \mu \text{ minuscule}} \Big) = \mathcal{F}_\varphi \boxtimes r_\mu \circ \varphi$$

$r_\mu \circ \varphi$ = Continuous rep. of $W_{\breve{E}}$ with values in $\overline{\mathbb{Q}_\ell}$

= Weil sheaf on $\mathrm{Spa}(E)^\diamond \times \mathrm{Spa}(\overline{\mathbb{F}_q})$

$$\mathrm{Spa}(E)^\diamond \times \mathrm{Spa}(\overline{\mathbb{F}_q}) = \mathrm{Spa}(\widehat{E^{un}})^\diamond$$

(5) Local/global compatibility with Caraiani/Scholze sheaf associated to Hodge type Shimura varieties

(6) Character sheaf property: T_φ = stable distribution on $G(E)$ associated to φ

$\{G(E)\}_{ell} \to B(G)_{basic}$. For $\delta \in G(E)_{ell}$, x_δ is defined $/\overline{\mathbb{F}_q}$

elliptic conjugacy classes

Then: $\mathrm{Tr}(\underbrace{\mathrm{Frob}}_{\text{acting via the Weil sheaf structure on } \mathcal{F}_\varphi}, x_\delta^* \mathcal{F}_\varphi) = T_\varphi(\delta)$ if δ = elliptic regular

Trace of Frob function on $\mathcal{F}_\varphi \leftrightarrow T_\varphi$