Geometrization of the local Langlands Correspondence

$[E:\mathbb{Q}_p] < +\infty$

$\mathbb{F}_q = \mathcal{O}_E/\pi$

$F|\mathbb{F}_q$ perfectoid $\rightsquigarrow$ Curve (joint work w/ Fontaine)

- Curve $\to$ X^{ad} = E-adic space (Compact Rieman surface)
- Curve $\to$ X = Dedekind E-scheme

$$X^{ad} = Y/\varphi^{\mathbb{Z}} \qquad Y = \mathrm{Spa}\left(W_{\mathcal{O}_E}(\mathcal{O}_F)\right) \setminus V(\pi[\varpi_F]) \qquad 0 < |\varpi_F| < 1$$

$$W_{\mathcal{O}_E}(\mathcal{O}_F) \ni \underbrace{\sum_{n\geq 0}[x_n]\pi^n}_{\text{holo. fct. variable } \pi}, \quad x_n \in \mathcal{O}_F$$

Y = "punctured disk variable π, coefficients in F"

* F alg. closed $\overline{\mathbb{F}}_q \subset F$, $L = \widehat{E^{un}} = W_{\mathcal{O}_E}(\overline{\mathbb{F}}_q)_{\mathbb{Q}} \ni \sigma = \varphi$

$\varphi\text{-}\mathrm{Mod}_L$ = Isocrystals $= \{(D,\varphi)\}$

- Isocrystals: Dieudonné-Manin
- D: finite dim. L-v.s.
- φ: semi linear auto.

$$\begin{array}{ccccc}
\varphi\text{-}\mathrm{Mod}_L & \longrightarrow & \mathrm{Bun}_{X^{ad}} & \xleftarrow[\sim]{\text{GAGA}} & \mathrm{Bun}_X \\
(D,\varphi) & \longmapsto & \mathcal{E}(D,\varphi)^{ad} & \longleftarrow\!\shortmid & \mathcal{E}(D,\varphi)
\end{array}$$

$\mathcal{E}(D,\varphi)^{ad}$:

$$\begin{array}{c} Y \times_{\varphi^{\mathbb{Z}}} D \\ \downarrow \\ Y/\varphi^{\mathbb{Z}} \end{array}$$

$\ni \varphi$ = automorphy factor.

Ex: $\mathcal{O}(1) := \mathcal{E}(L, \pi^{-1}\sigma)$ = "ample" line bundle

$$X = \mathrm{Proj}\Big(\bigoplus_{d\geq 0} \underbrace{H^0(X^{ad}, \mathcal{O}(d))}_{\mathcal{O}(Y)^{\varphi = \pi^d}}\Big)$$ automorphic function

Th (F.-Fontaine): $\varphi\text{-Mod}_L \xrightarrow{\mathcal{E}(-)} \mathrm{Bun}_X$ is essentially surjective

* G/E reductive group $\qquad B(G) = G(L)/G\text{-conj.}$ (Kottwitz)
  "iso. classes of G-isocrystals"

$\cancel{b \in G(L)} \rightsquigarrow \quad \cancel{\mathrm{Rep}(G)}$

$$\begin{tikzcd} G(L) \ni b \rightsquigarrow & \mathrm{Rep}\, G \arrow[r] \arrow[rr, bend right, "\mathcal{E}_b"'] & \varphi\text{-Mod}_L \arrow[r, "\mathcal{E}(-)"] & \mathrm{Bun}_X \end{tikzcd}$$

$(V,\rho) \mapsto (V_L, \rho(b)\sigma)$

$\mathcal{E}_b$ = G-bundle = G-torsor on X.

Th: $B(G) \xrightarrow{\sim} H^1_{et}(X, G)$

$[b] \mapsto [\mathcal{E}_b]$

Nice features: Dictionary Kottwitz $\longleftrightarrow$ Atiyah-Bott reduction theory

Ex: b basic $\Longleftrightarrow$ $\mathcal{E}_b$ semi-stable.

The stack Bun_G

$Perf_{\overline{\mathbb{F}}_q}$ = $\overline{\mathbb{F}}_q$- perfectoid spaces + pro-étale topology

$\ni$
S

X_S = E- adic space = "family of curves $(X_{b(s)})_{s \in S}$"

$Bun_G(S)$ = groupoid of G- bundles on X_S

[Bun_G = stack on $Perf_{\overline{\mathbb{F}}_q}$]

Preceding theorem $\Rightarrow$ $\boxed{B(G) \xrightarrow{\sim} |Bun_G \otimes \overline{\mathbb{F}}_q|}$

Connected components: $\kappa : B(G) \longrightarrow \pi_1(G)_\Gamma$ Kottwitz map

$\Gamma = Gal(\overline{E}|E)$

Borovoi π_1

κ locally constant on $Bun_G \otimes \overline{\mathbb{F}}_q$, for $G = GL_n$, $\pi_1(G)_\Gamma = \mathbb{Z}$ and κ = degree of a v. b.

$\Downarrow$

$$Bun_{G, \overline{\mathbb{F}}_q} = \coprod_{\alpha \in \pi_1(G)_\Gamma} \underbrace{Bun^\alpha_{G, \overline{\mathbb{F}}_q}}_{\text{Connected conjecturaly}}$$

open/closed substack

H.N. stratification:

$\{\pi_{\varphi,\rho}\}_\rho$ = L-packet for a local Langlands correspondence for the inner form J_b of G

(3) $\forall \mu \in X_*(A)^+$ $\mathcal{F}_\varphi$ is a Hecke$_\mu$ eigenvector with eigenvalue $r_\mu \circ \varphi$.

$$\overrightarrow{h}_! \left(\overleftarrow{h}^* \mathcal{F}_\varphi \otimes IC_\mu \right) \simeq \mathcal{F}_\varphi \boxtimes r_\mu \circ \varphi$$

$r_\mu \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}({}^LG)$ $\quad$ $r_\mu \circ \varphi \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(W_E)$

$\quad$ " Weil-local system on $\mathrm{Spa}(E)^\diamond$

(4) If φ is cuspidal, $j : \mathrm{Bun}_G^{s.s.} \hookrightarrow \mathrm{Bun}_G$, then

$$\mathcal{F}_\varphi = j_! j^* \mathcal{F}_\varphi$$

(5) Local global compatibility with Caraiani-Scholze sheaf on $G_{\mathbb{Q}_p}/P_\mu$ for (G, X) = Hodge type Shimura datum.

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* Conjecture implies Kottwitz conjecture describing the s.c. part of the cohomology of R.Z.-spaces in terms of local Langlands.
* True for GL_1, equivalent to local class field theory.