

Geometrization of the local Langlands Correspondence

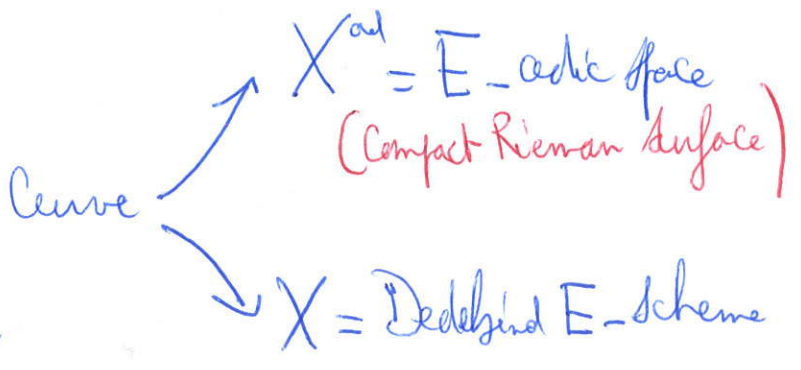
①

$[E: \mathbb{Q}_p] < \infty$

$\mathbb{F}_q = \mathcal{O}_E / \pi$

F/\mathbb{F}_q perfectoid

joint work with Fargues



$X^{\text{ad}} = Y/\mathbb{G}_m^2$

$Y = \text{Spa}(W_{\mathcal{O}_E}(\mathcal{O}_F)) \setminus V(\pi[\mathcal{O}_F]) \quad \mathcal{O} \setminus \mathcal{O}_F \setminus \mathcal{O}_F$

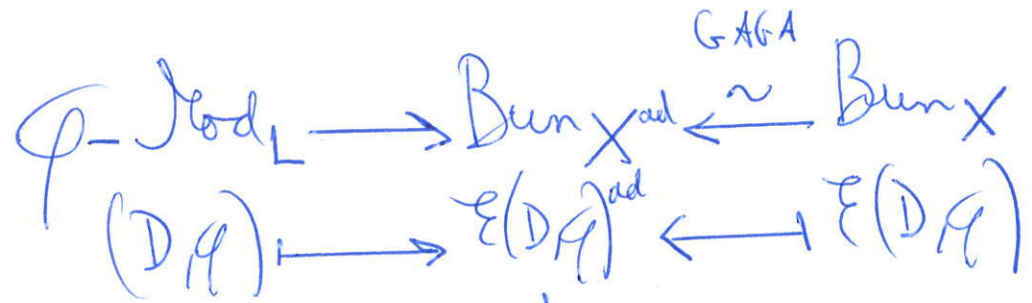
"punctured disk variable π , coefficients in F " $\rightarrow \sum_{m \geq 0} [a_m] \pi^m, a_m \in \mathcal{O}_F$

hob. fct. variable π

* Falg. closed $\overline{\mathbb{F}_q} \subset F, L = \widehat{E}^{\text{un}} = W_{\mathcal{O}_E}(\overline{\mathbb{F}_q}) \rtimes \sigma = \varphi$

$\varphi\text{-Mod}_L = \text{Isocrystals} = \left\{ \begin{matrix} (D, \varphi) \\ \uparrow \quad \uparrow \\ \text{finite dim. } L\text{-v.s.} \quad \text{semi linear auto.} \end{matrix} \right\}$

Dieudonné-Manin



$\hookrightarrow Y \times_{\mathbb{G}_m^2} D \rtimes \varphi = \text{automorphy factor.}$

\downarrow
 \mathbb{G}_m^2
 \downarrow
 Y/\mathbb{G}_m^2

Ex. $\mathcal{O}(1) := \mathcal{E}(L, \pi^{-1}\sigma) =$ "ample" line bundle

$$X = \text{Proj} \left(\underbrace{\bigoplus_{d \geq 0} H^0(X^{\text{ad}}, \mathcal{O}(d))}_{\mathcal{O}(Y)^{G = \pi^d}} \right) \text{ automorphic function}$$

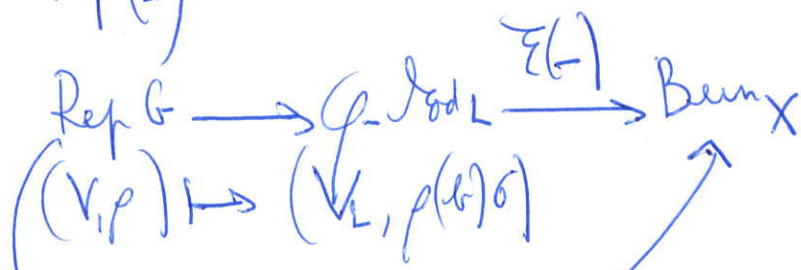
Th (F. Fontaine): $\varphi\text{-Mod}_L \xrightarrow{\mathcal{E}(\cdot)} \text{Bun}_X$ is essentially surjective

* G/E reductive group $B(G) = G(L)/G\text{-conj.}$ (Kottwitz)
 "or. classes of G -isocrystals"

~~$G(L)$ mod~~

$\text{Rep}(G)$

$G(L) \ni \rho$ mod



$\mathcal{E}_G = G\text{-bundle} = G\text{-torsor on } X.$

Th: $B(G) \xrightarrow{\sim} H_{\text{ét}}^1(X, G)$
 $[G] \mapsto [\mathcal{E}_G]$

Nice features: Dictionary Kottwitz \leftrightarrow Atiyah-Bott Reduction theory

Ex: ρ basic $\Leftrightarrow \mathcal{E}_\rho$ semi-stable.

The Kac-Bung

$\text{Perf}_{\mathbb{F}_q} = \mathbb{F}_q$ -perfectoid spaces + pro-étale topology

\Downarrow
 S

$X_S = E$ -adic space = "family of curves $(X_{b(S)})_{b \in S}$ "

$\text{Bun}_G(S) =$ groupoid of G -bundles on X_S

[$\text{Bun}_G =$ Kac on $\text{Perf}_{\mathbb{F}_q}$]

Preceding theorem \Rightarrow

$$B(G) \xrightarrow{\sim} |\text{Bun}_G \otimes \overline{\mathbb{F}_q}|$$

Connected Components:

$$K: B(G) \rightarrow \pi_1(G) \Gamma \quad \text{Kottwitz map}$$

\uparrow Borel $\pi_1 \quad \Gamma = \text{Gal}(\overline{E}/E)$

K locally constant on $\text{Bun}_G \otimes \overline{\mathbb{F}_q}$, for $G = GL_n, \pi_1(G) \Gamma = \mathbb{Z}$
and $K =$ degree of a v.l.

\Downarrow

$$\text{Bun}_{G, \overline{\mathbb{F}_q}} = \coprod_{\alpha \in \pi_1(G) \Gamma} \text{Bun}_{G, \overline{\mathbb{F}_q}}^\alpha$$

\leftarrow open/closed substack

Connected conjecture

H.N. stratification

$\{\pi_{\varphi, \rho}\}_p = L$ -packet for a local Langlands
 Correspondence for the inner form J_b of G

(3) $\forall \mu \in X_*(A)^+$ \mathcal{T}_{φ} is a Hecke μ eigenvector with
 eigenvalue $r_{\mu} \circ \varphi$.

$$h_! \left(h^* \mathcal{T}_{\varphi} \otimes IC_{\mu} \right) \simeq \mathcal{H}_{\varphi} \boxtimes r_{\mu} \circ \varphi$$

$$r_{\mu} \in \text{Rep}_{\overline{\mathbb{Q}_E}}({}^L G) \quad r_{\mu} \circ \varphi \in \text{Rep}_{\overline{\mathbb{Q}_E}}(W_E)$$

Weil-local system on $\text{Spa}(E)^{\text{an}}$

(4) If φ is cuspidal, $j: \text{Bun}_G^{\text{d.s.}} \hookrightarrow \text{Bun}_G$, then

$$\mathcal{T}_{\varphi} = j_! j^* \mathcal{T}_{\varphi}$$

(5) Local global Compatibility with Caraiani-Scholze
 sheaf on G_{an}/P_{μ} for $(G, X) = \text{Hodge type Shimura datum}$.

* Conjecture implies Kottwitz Conjecture describing the s.c.
 part of the cohomology of R.Z. spaces in terms of local Langlands.

* True for GL₁, equivalent to local class field theory.