# THE CURVE AND THE LANGLANDS PROGRAM : THE ABELIAN CASE

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Abstract. This is a review of the work of the author on his geometrization conjecture of the local Langlands correspondence in the $\mathrm{GL}_1$-case, following [3].

## 1. Background on the Picard scheme of a compact Riemann surface

1.1. **The Jacobian as a complex torus.** Let $X$ be a compact Riemann surface of genus $g$. We define

$$\mathrm{Jac}_X = H^0(X, \Omega^1_X)^* / H_1(X, \mathbb{Z})$$

where

— $H^0(X, \Omega^1_X)$ is the $g$-dimensional $\mathbb{C}$-vector space of holomorphic 1-forms, differential forms locally of the form $f(z)dz$ in a local coordinate $z$ with $f$ holomorphic, defined everywhere on $X$.

— $H_1(X, \mathbb{Z}) = \pi_1(X)^{ab}$ is the first homology group of $X$, isomorphic to $\mathbb{Z}^{2g}$.

— The embedding

$$H_1(X, \mathbb{Z}) \hookrightarrow H^0(X, \Omega^1_X)^*$$

is defined by sending the 1-homology cycle $c \in H_1(X, \mathbb{Z})$ to the linear form

$$\omega \longmapsto \int_c \omega$$

This makes $H_1(X, \mathbb{Z})$ a lattice inside $H^0(X, \Omega^1_X)^*$.

Then, $\mathrm{Jac}_X$ is not only a compact complex torus but in fact an abelian variety over $\mathbb{C}$. In fact, this is equipped with a canonical principal polarization.

1.2. **Equivalence classes of degree 0 divisors and holomorphic line bundles.** We define

$$\mathrm{Div}^0(X) = \Big\{ \sum_{x \in X} m_x[x] \;\big|\; m_x = 0 \text{ for almost all } x \text{ and } \sum_x m_x = 0 \Big\},$$

the group of degree 0 Weil divisors on $X$. Let us note

$$\mathcal{M}(X)$$

for the field of meromorphic functions on $X$. As is well known, the analytification functor induces an equivalence between smooth proper curves over $\mathrm{Spec}(\mathbb{C})$ and compact Riemann surfaces. If $X = \mathfrak{X}^{an}$ then $\mathcal{M}(X) = \mathbb{C}(\mathfrak{X})$, the field of rational functions on $\mathfrak{X}$.

For $f \in \mathcal{M}(X)^\times$ we define

$$\mathrm{div}(f) = \sum_{x \in X} \mathrm{ord}_x(f)[x] \in \mathrm{Div}^0(X)$$

where $\mathrm{ord}_x(f)$ gives the order of the zero/opposite of the order of the pole of $f$ at $x$: if $f = \sum_{n \geq k} a_n z_x^n$ is the Laurent expansion of $f$ in a local coordinate $z_x$ in a neighborhood of $x$ with $a_k \neq 0$, $\mathrm{ord}_x(f) = k$.

The group of equivalence classes of degree 0 divisors is

$$\mathrm{Div}^0(X)/\sim,$$

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where $D \sim D'$ if $D - D' = \mathrm{div}(f)$ for some non-zero $f \in \mathcal{M}(X)$. This is identified with

$$\mathrm{Pic}^0(X).$$

Here the class of the degree 0 divisor $D \in \mathrm{Div}^0(X)$ is sent to $[\mathcal{O}(D)] \in \mathrm{Pic}(X)$. In the other direction, if $\mathscr{L}$ is an holomorphic line bundle on $X$, there exists a meromorphic trivialization $\eta : \mathcal{M}(X) \xrightarrow{\sim} \mathscr{L} \otimes_{\mathcal{O}_X} \mathcal{M}(X)$. On can then define naturally $\mathrm{div}(\eta) \in \mathrm{Div}(X)$ and this defines an inverse to the map $[D] \mapsto [\mathcal{O}_X(D)]$.

1.3. **The modular interpretation of the complex torus $\mathrm{Jac}_X$.** This is the following theorem.

**Théorème 1.1** (Abel, Jacobi)**.** *The morphism*

$$\begin{array}{ccc} \mathrm{Div}^0(X) & \longrightarrow & H^0(X, \Omega^1_X)^*/H_1(X,\mathbb{Z}) \\ [x]-[y] & \longmapsto & \left[\omega \mapsto \int_y^x \omega\right] \ \mathit{mod}\ H_1(X,\mathbb{Z}) \end{array}$$

*induces an isomorphism of groups*

$$\mathrm{Pic}^0(X) = \mathrm{Div}^0(X)/\sim \xrightarrow{\sim} \mathrm{Jac}_X .$$

In the preceding the symbol $\int_y^x$ means that we chose a path $\gamma$ from $y$ to $x$, $\gamma(0) = y$ and $\gamma(1) = x$, and then $\int_y^x \omega := \int_\gamma \omega$. The choice of another path may give a different linear form on $H^0(X, \Omega^1_X)$ but this differs from the preceding by an element of $H_1(X,\mathbb{Z})$, and thus our map is well defined.

Let us note that in the preceding theorem, the injectivity part is due to Abel and the surjectivity to Jacobi.

1.4. **The full Picard scheme.** Let $\mathrm{Pic}_X$ be the Picard variety classifying holomorphic line bundles on $X$. One has an exact sequence of complex analytic groups

$$0 \longrightarrow \underbrace{\mathrm{Pic}^0_X}_{\mathrm{Jac}_X} \longrightarrow \underbrace{\mathrm{Pic}_X}_{\coprod_{d\in\mathbb{Z}} \mathrm{Pic}^d_X} \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0.$$

This sequence is split by the choice of a point $\infty \in X$ by sending $1 \in \mathbb{Z}$ to $\mathcal{O}_X([\infty])$.

The choice of such a point $\infty$ identifies $\mathrm{Pic}^0_X$ and $\mathrm{Pic}^d_X$ via the map $\mathscr{L} \mapsto \mathscr{L}(d[\infty])$. This induces an isomorphism

$$\pi_1(\mathrm{Pic}^0_X) \xrightarrow{\sim} \pi_1(\mathrm{Pic}^d_X).$$

Here we don't fix a base point in the $\pi_1$ since it is abelian and thus does not depend canonically on the choice of such a base point. Translation by a point on the complex torus $\mathrm{Jac}_X$ induces the identity at the level of the $\pi_1$ and we thus have a canonical identification

$$\pi_1(\mathrm{Pic}^0_X) = \pi_1(\mathrm{Pic}^d_X)$$

independent on the choice of $\infty$.

## 2. Geometric class field theory for compact Riemann surfaces

There is an Abel-Jacobi morphism in degree 1

$$\begin{array}{rcl} \mathrm{AJ}^1 : X & \longrightarrow & \mathrm{Pic}^1_X \\ x & \longmapsto & [\mathcal{O}([x])]. \end{array}$$

This is the zero map when $g = 0$, an isomorphism when $g = 1$, and a Zariski closed immersion

$$X \hookrightarrow \mathrm{Pic}^1_X$$

when $g > 1$.

Geometric class field theory for compact Riemann surfaces is then the following elementary statement.

**Théorème 2.1.** *The morphism*

$$\pi_1(\mathrm{AJ}^1) : \pi_1(X, x) \longrightarrow \pi_1(\mathrm{Pic}^1_X) = \pi_1(\mathrm{Jac}_X)$$

*induces an isomorphism*

$$\pi_1(X)^{ab} \xrightarrow{\sim} \pi_1(\mathrm{Jac}_X).$$

In other terms, any abelian Galois cover of $X$ comes by pullback via $\mathrm{AJ}^1$ from a cover of $\mathrm{Jac}_X$.

## 3. The geometric Langlands point of view

3.1. **Statement of the theorem.** From now on, $X$ is a smooth projective curve over the algebraically closed field $k$. We note

$$\mathrm{Jac}_X = \mathrm{Pic}^0_X$$

its Jacobian, an abelian variety over $k$.

We seek to construct a canonical isomorphism

$$\pi_1(X)^{ab} \xrightarrow{\sim} \pi_1(\mathrm{Jac}_X).$$

Here the $\pi_1$ is Grothendieck's profinite fundamental group, the profinite completion of the topological $\pi_1$ of the preceding sections in the case of compact Riemann surfaces. As before, since we are working with abelian fundamental groups, there is no need to fix a base point: if $x$ and $x'$ are two geometric points of $X$ there is a canonical isomorphism

$$\pi_1(X, x)^{ab} \xrightarrow{\sim} \pi_1(X, x')^{ab}$$

independent of the choice of a path between $x$ and $x'$.

**Remarque 3.1.** *As a consequence of the independence of the base geometric point of the abelianized $\pi_1$ of a connected Noetherian scheme $X$, for any such $X$, for any point $x \in X$, there is a canonical morphism of abelian groups*

$$Gal(k(x)^{sep}|k(x))^{ab} \longrightarrow \pi_1(X)^{ab}.$$

*In particular if $X$ is an $\mathbb{F}_q$-scheme for any closed point $x$ of $X$ one can define* $\mathrm{Frob}_x \in \pi_1(X)^{ab}$. *In general, this is only well defined as a conjugacy class $\{\mathrm{Frob}_x\}$ in $\pi_1(X)$ (which is itself only defined up to inner automorphisms if we don't fix a geometric base point).*

**Remarque 3.2.** *We use all the time the fact that if $A$ is an abelian variety over $k$ then $\pi_1(A, 0)$ is an abelian group. In fact this is equal to*

$$\prod_{\ell} T_\ell(A) = \varprojlim_{n \geq 1} A[n](k).$$

*where $\ell$ goes through the set of all prime numbers. This is a consequence of the fact if $Z_1$ and $Z_2$ are two proper $k$-schemes then $\pi_1(Z_1 \times_{Spec(k)} Z_2, (z_1, z_2)) \xrightarrow{\sim} \pi_1(Z_1, z_1) \times \pi_1(Z_2, z_2)$ (*[1, Exposé X, Corollaire 1.7]*). As a consequence, the group law of $A$ induces a law $* : \pi_1(A, 0) \times \pi_1(A, 0) \to \pi_1(A, 0)$ that is a morphism of groups and makes $(\pi_1(A, 0), *)$ a group. From this we deduce that $*$ is the group law of $\pi_1(A, 0)$ and this is abelian (same proof as the one for the $\pi_1$ of $H$-spaces in algebraic topology). As a consequence, if $B \to A$ is a connected finite étale cover, any choice of a point in $B(k)$ mapping to $0 \in A(k)$ makes $B$ an abelian variety and the morphism $B \to A$ an étale isogeny.*

Here is now what we want to prove: the Abel-Jacobi map in degree 1

$$\mathrm{AJ}^1 : X \longrightarrow \mathrm{Pic}^1_X$$

induces an isomorphism

$$\pi_1(X)^{ab} \xrightarrow{\sim} \pi_1(\mathrm{Pic}^1_X) = \pi_1(\mathrm{Jac}_X)$$

(abelianization in the category of profinite groups i.e. $\pi_1(X)^{ab} := \pi_1(X)/\overline{[\pi_1(X), \pi_1(X)]}$). In other terms: any finite étale connected abelian cover of $X$ comes by pull-back via $\mathrm{AJ}^1$, after fixing a point $\infty \in X(k)$ and thus an isomorphism $\mathrm{Jac}_X \xrightarrow{\sim} \mathrm{Pic}^1_X$, from an étale isogeny $B \twoheadrightarrow \mathrm{Jac}_X$.

We dualize the situation and look at characters of our $\pi_1$. This is then reduced to the following geometric Langlands statement for $\mathrm{GL}_1$. Here $\ell$ is any prime number.

**Théorème 3.3.** *Any rank 1 étale $\overline{\mathbb{Q}}_\ell$ local system $\mathscr{E}$ on $X$ descends along $\mathrm{AJ}^1 : X \to \mathrm{Pic}^1_X$ to a rank 1 étale $\overline{\mathbb{Q}}_\ell$-local system*

$$\mathrm{Aut}_{\mathscr{E}}$$

*on* $\mathrm{Pic}^1_X$.

Here the notation $\mathrm{Aut}_{\mathscr{E}}$ means the automorphic local system associated to $\mathscr{E}$ by the geometric Langlands program.

3.2. **Sketch of the proof of the theorem.**

3.2.1. *Symmetrization.* For $d \geq 1$ let

$$\mathrm{Div}^d_X$$

be the Hilbert scheme of degree $d$ effective divisors on $X$. There is a morphism

$$\begin{aligned} \pi_d : X^d &\longrightarrow \mathrm{Div}^d_X \\ (x_1, \dots, x_d) &\longmapsto \sum_{i=1}^d [x_i] \end{aligned}$$

that induces an isomorphism

$$X^d/\mathfrak{S}_d \xrightarrow{\sim} \mathrm{Div}^d_X$$

(categorical quotient by $\mathfrak{S}_d$). In particular we have $X = \mathrm{Div}^1_X$.

We now define for $d \geq 1$

$$\mathcal{S}_d\mathscr{E} := \Big[\pi_{d*}\mathscr{E}^{\boxtimes d}\Big]^{\mathfrak{S}_d}.$$

This is again a rank 1 local system on $\mathrm{Div}^d_X$. This is deduced for example from [1, Exposé IX, Remarque 5.6]: for any projective variety $Z$ over $k$ algebraically closed and $d > 1$, $\pi_1(Z^d/\mathfrak{S}_d) = \pi_1(Z)^{ab}$.

**Remarque 3.4.** *More generally, when $\mathscr{E}$ is a higher rank local system, $\mathcal{S}_d\mathscr{E}$ is a perverse sheaf on* $\mathrm{Div}^d$*, the so-called Laumon sheaf showing up in the geometric Langlands program.*

3.2.2. *Descent in high degree.* Let us recall the following consequence of the Riemann-Roch theorem.

**Théorème 3.5.** *For $d > 2g - 2$,*

$$\mathrm{AJ}^d : \mathrm{Div}^d_X \longrightarrow \mathrm{Pic}^d_X$$

*is a Zariski locally trivial fibration with fiber $\mathbb{P}^{d-g}_k$.*

More precisely, the choice of a point $\infty \in X(k)$ induces an identification between

$$\big[\,\mathrm{Pic}_X / \mathbb{G}_m\big]$$

and the Picard stack of $X$ i.e. the gerbe

$$\text{Picard stack} \longrightarrow \mathrm{Pic}_X = \text{coarse moduli space}$$

is split by the choice of $\infty$. There is then a universal line bundle $\mathscr{L}_d$ on

$$X \times_{\mathrm{Spec}(k)} \mathrm{Pic}^d_X\,.$$

If $p_d$ is the projection to the factor $\mathrm{Pic}^d_X$ then the perfect complex

$$Rp_{d*}\mathscr{L} \in \mathrm{Perf}^{[0,1]}(\mathscr{O}_{\mathrm{Pic}^d_X})$$

is concentrated in degree 0 when $d > 2g-2$ and is a vector bundle

$$\mathscr{H}_d = p_{d*}\mathscr{L}$$

on $\mathrm{Pic}^d_X$. One then has for $d > 2g-2$

$$\begin{tikzcd}
\mathrm{Div}^d_X \arrow[r, "\sim"] \arrow[rr, bend left, "\mathrm{AJ}^d"] & \mathbb{P}(\mathscr{H}_d) \arrow[r] \arrow[dr, "\substack{\text{loc.triv. fibration}\\ \text{in } \mathbb{A}^{d-g+1}\smallsetminus\{0\}}"'] & \mathrm{Pic}^d_X \\
& & \mathcal{P}ic^d_X \arrow[u] \arrow[u, bend right, dash, "B\mathbb{G}_m"'] \arrow[r, equal] & \text{Picard stack}
\end{tikzcd}$$

For $d > 2g-2$ we have a Serre exact sequence ([1, Exposé X, Corollaire 1.4]) for $z \in \mathrm{Div}^d_X(k)$ with image $w = \mathrm{AJ}^d(z)$

$$\pi_1\Big(\underbrace{(\mathrm{AJ}^d)^{-1}(w)}_{\simeq \mathbb{P}^{d-g}_k}, z\Big) \longrightarrow \pi_1(\mathrm{Div}^d_X, z) \longrightarrow \pi_1(\mathrm{Pic}^d_X, \mathrm{AJ}^d(z)) \longrightarrow 1.$$

The simple connectedness of $\mathbb{P}^{d-g}_k$ implies then that for $d > 2g-2$

$$\pi_1(\mathrm{Div}^d_X, z) \xrightarrow{\sim} \pi_1(\mathrm{Pic}^d_X, w).$$

As a consequence, we obtain the following result.

**Proposition 3.6.** *For $d > 2g-2$, the symmetrized local system $\mathcal{S}_d\mathscr{E}$ descends along $\mathrm{AJ}^d$ to a unique rank 1 étale $\overline{\mathbb{Q}}_\ell$-local system $\mathscr{F}_d$ on $\mathrm{Pic}^d_X$,*

$$\mathcal{S}_d\mathscr{E} = (\mathrm{AJ}^d)^*\mathscr{F}_d.$$

3.2.3. *Extension to all degrees.* We now have a rank one étale locale system on $\coprod_{d>2g-2}\mathrm{Pic}^d_X$. Suppose that we have $d, d' \geq 1$. Then following diagram is then commutative

$$\begin{tikzcd}
\mathrm{Div}^d_X \times \mathrm{Div}^{d'}_X \arrow[r] \arrow[d, "\mathrm{AJ}^d \times \mathrm{AJ}^{d'}"'] & \mathrm{Div}^{d+d'}_X \arrow[d, "\mathrm{AJ}^{d+d'}"] \\
\mathrm{Pic}^d_X \times \mathrm{Pic}^{d'}_X \arrow[r, "m_{d,d'}"] & \mathrm{Pic}^{d+d'}_X
\end{tikzcd}$$

i.e. the Abel-Jacobi morphism is compatible with the monoid law of $\mathrm{Div}_X = \coprod_{d\geq 1}\mathrm{Div}^d_X$ and the group law of the Picard scheme $\mathrm{Pic}_X$.

We deduce the following result since the pullback under $\mathrm{Div}^d_X \times \mathrm{Div}^{d'}_X \to \mathrm{Div}^{d+d'}_X$ of $\mathcal{S}_{d+d'}\mathscr{E}$ is $\mathcal{S}_d\mathscr{E} \boxtimes \mathcal{S}_{d'}\mathscr{E}$ and the descent along $\mathrm{AJ}^d \times \mathrm{AJ}^{d'}$ is unique.

**Proposition 3.7.** *For $d, d' > 2g-2$ there is a canonical identification*

$$\mathscr{F}_d \boxtimes \mathscr{F}_{d'} = m^*_{d,d'}\mathscr{F}_{d+d'}.$$

Via the canonical identification $\pi_1(\mathrm{Jac}_X) = \pi_1(\mathrm{Pic}^d_X)$ for all $d$ we deduced that the collection $(\mathscr{F}_d)_{d>2g-2}$ is given by a collection of characters $(\chi_d)_{d>2g-2}$ of $\pi_1(\mathrm{Jac}_X)$ satisfying

$$\chi_{d+d'}(\sigma + \tau) = \chi_d(\sigma).\chi_{d'}(\tau).$$

All $\chi_d$, $d > 2g-2$, are thus equal. From this we deduce that $(\mathscr{F}_d)_{d>2g-2}$ extends canonically to a rank 1 local system $\mathscr{F} = (\mathscr{F}_d)_{d\in\mathbb{Z}}$ on the Picard group scheme $\mathrm{Pic}_X$ satisfying $m^*\mathscr{F} \simeq \mathscr{F} \boxtimes \mathscr{F}$ where $m$ is the group law of $\mathrm{Pic}_X$. Let us put this in the form of a lemma.

**Lemme 3.8.** *There is an identification between*

(1) *Rank* 1 *étale* $\overline{\mathbb{Q}}_\ell$*-local systems on* $\mathrm{Jac}_X$*,*

(2) *Rank* 1 *étale* $\overline{\mathbb{Q}}_\ell$*-local systems* $\mathscr{F}$ *on the Picard scheme* $\mathrm{Pic}_X$ *satisfying* $m^*\mathscr{F} \simeq \mathscr{F} \boxtimes \mathscr{F}$ *where* $m$ *is the addition law,*

(3) *For any* $d \geq 0$*, rank* 1 *étale* $\overline{\mathbb{Q}}_\ell$*-local systems* $\mathscr{F}$ *on the monoid* $\mathrm{Pic}_X^{>d}$ *satisfying* $m^{>d*}\mathscr{F} \simeq \mathscr{F} \boxtimes \mathscr{F}$ *where* $m^{>d} : \mathrm{Pic}_X^{>d} \times \mathrm{Pic}_X^{>d} \to \mathrm{Pic}_X^{>2d}$.

We can now set

$$\mathrm{Aut}_{\mathscr{E}} := \mathscr{F}_1.$$

One has $\mathcal{S}_d\big[(\mathrm{AJ}^1)^*\mathscr{F}_1\big] = \mathcal{S}_d\mathscr{E}$ and thus $(\mathrm{AJ}^1)^*\mathscr{F}_1 = \mathscr{E}$ via the identification $\pi_1(X)^{ab} = \pi_1(\mathrm{Div}_X^d)$ for $d \geq 2$. This finishes the proof of theorem 3.3.

## 4. Geometric Langlands and class field theory for function fields

The preceding is for algebraic curves over an algebraically closed field. Let now $X$ be a smooth projective curve over the finite field $\mathbb{F}_q$. Suppose moreover that $X$ is geometrically connected i.e. $\mathbb{F}_q$ is algebraically closed inside

$$F = \mathbb{F}_q(X)$$

the field of rational functions on $X$.

### 4.1. The reciprocity map.

We use the exact sequence

$$1 \longrightarrow \pi_1^{geo}(X) \longrightarrow \pi_1(X) \longrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_q|\mathbb{F}_q) \longrightarrow 1$$

and define

$$W_X \subset \pi_1(X),$$

the Weil group of $X$. For each point $x \in X(\mathbb{F}_q)$ there is a morphism

$$W_{\mathbb{F}_q(x)} \longrightarrow W_X,$$

see remark 3.1. This defines a reciprocity map

$$\begin{tikzcd} \mathrm{Div}(X) \arrow[r, "\deg"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\simeq"', "n\mapsto \mathrm{Frob}_q^n"] & 0 \\ W_X^{ab} \arrow[r] & \mathrm{Frob}_q^{\mathbb{Z}} \arrow[r] & 1 \end{tikzcd}$$

that sens $[x]$, $x$ a closed point of $X$, to $\mathrm{Frob}_x \in W_X$, and where $\deg([x]) = [\mathbb{F}_q(x) : \mathbb{F}_q]$.

Here is the theorem we want to prove using geometric clas field theory.

**Théorème 4.1.** *The reciprocity map* $\mathrm{Div}(X) \to W_X^{ab}$ *factorizes through the group of principal divisors and induces an isomorphism*

$$\mathrm{Div}(X)/\sim\, = \mathrm{Pic}_X(\mathbb{F}_q) \xrightarrow{\sim} W_X^{ab}.$$

Formulated in adelic terms, there is a reciprocity morphism

$$\mathbb{A}_F^\times \longrightarrow W_X^{ab}$$

that is everywhere unramified, i.e. factorizes through $\mathbb{A}_F^\times / \prod_{x\in|X|} \mathcal{O}_{F_x}^\times$, and sends the uniformizing element $\pi_x$ to $\mathrm{Frob}_x$. The theorem then says that if $f \in F^\times$,

$$\prod_{x\in|X|} \mathrm{Frob}_x^{\mathrm{ord}_x(f)} = 1 \qquad \text{(reciprocity law)}$$

and this induces an isomorphism

$$\boxed{F^\times \backslash \mathbb{A}_F^\times / \prod_{x\in|X|} \mathcal{O}_{F_x}^\times \xrightarrow{\sim} W_X^{ab}.}$$

4.2. **The reciprocity law.** The exact sequence

$$1 \longrightarrow \pi_1^{geo}(X) \longrightarrow W_X \longrightarrow \mathrm{Frob}_q^{\mathbb{Z}} \longrightarrow 1$$

induces an exact sequence

$$1 \longrightarrow \operatorname{coker}\big(\pi_1^{geo}(X)^{ab} \xrightarrow{\mathrm{Frob}_q - \mathrm{Id}} \pi_1^{geo}(X)^{ab}\big) \longrightarrow W_X^{ab} \longrightarrow \mathrm{Frob}_q^{\mathbb{Z}} \longrightarrow 1.$$

Geometric class field theory for $X_{\overline{\mathbb{F}}_q}$ then gives an identification between the left term and

$$\operatorname{coker}\big(\,\mathrm{T}\,\mathrm{Jac}_X \xrightarrow{\mathrm{Frob}_q - \mathrm{Id}} \mathrm{T}\,\mathrm{Jac}_X\,\big)$$

where $\mathrm{T}\,\mathrm{Jac}_X = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathrm{Jac}_X(\overline{\mathbb{F}}_q))$ is its Tate module. We now use Lang isogeny that gives an exact sequence

$$1 \longrightarrow \mathrm{Jac}_X(\mathbb{F}_q) \longrightarrow \mathrm{Jac}_X(\overline{\mathbb{F}}_q) \xrightarrow{\mathrm{Frob}_q - \mathrm{Id}} \mathrm{Jac}_X(\overline{\mathbb{F}}_q) \longrightarrow 1.$$

Applying $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, -)$ we obtain an exact sequence

$$1 \longrightarrow \mathrm{T}\,\mathrm{Jac}_X \xrightarrow{\mathrm{Frob}_q - \mathrm{Id}} \mathrm{T}\,\mathrm{Jac}_X \longrightarrow \underbrace{\mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, \mathrm{Jac}_X(\mathbb{F}_q))}_{\mathrm{Jac}_X(\mathbb{F}_q)} \longrightarrow 1$$

We thus have a canonical exact sequence

$$1 \longrightarrow \mathrm{Jac}_X(\mathbb{F}_q) \longrightarrow W_X^{ab} \longrightarrow \mathbb{Z} \longrightarrow 1.$$

It remains to identify the deduced map $\mathrm{Div}(X)^0 \to \mathrm{Jac}_X(\mathbb{F}_q)$ with the evident quotient map to prove theorem 4.1. This is deduced from the following.

**Proposition 4.2.** *Let $A$ be an abelian variety over $\mathbb{F}_q$.*

*(1) Via the splitting of $W_A \to \mathrm{Frob}_q^{\mathbb{Z}}$ sending $\mathrm{Frob}_q^n$ to $\mathrm{Frob}_0^n$, where $0$ is the origin in $A(\mathbb{F}_q)$, one has*

$$\begin{aligned} W_A^{ab} &= \operatorname{coker}\big(\pi_1^{geo}(A) \xrightarrow{F-\mathrm{Id}} \pi_1^{geo}(A)\big) \oplus \mathbb{Z} \\ &= \operatorname{coker}\big(TA \xrightarrow{F-\mathrm{Id}} TA\big) \oplus \mathbb{Z} \\ &\underbrace{=}_{Lang} A(\mathbb{F}_q) \oplus \mathbb{Z} \end{aligned}$$

*(2) Via this identification, for any closed point $x$ of $A$ the element $\mathrm{Frob}_x \in W_A^{ab}$ corresponds to*

$$\big(\,\mathrm{Tr}_{\mathbb{F}_q(x)|\mathbb{F}_q}(x), [\mathbb{F}_q(x) : \mathbb{F}_q]\big) \in A(\mathbb{F}_q) \oplus \mathbb{Z}.$$

## 5. Abelian $\pi_1$ in number theory

5.1. **Number fields.** The topic of abelian fundamental groups in number theory is vast. Of course we have Kronecker-Weber theorem:

$$\mathbb{Q}^{ab} = \bigcup_{n\geq 1} \mathbb{Q}(\zeta_n)$$

the cyclotomic extension. In general there is no explicit formula for the maximal abelian extension of a number field in terms of a geometric object like $\mathbb{G}_m$ and its torsion points. Nevertheless class field theory holds over any number field and the analogue of theorem 4.1 says that the maximal abelian everywhere unramified extension of a number field $K$, its Hilbert class field, is a finite extension of $K$ with Galois group $\mathrm{Cl}_K$ the ideal class group of $K$. More precisely, we have the following reciprocity law. Let $L|K$ be the maximal abelian everywhere unramified extension of $K$. Then, $L|K$ is finite and the map

$$\mathbb{A}_K^\times \Big/ \prod_{v|\infty \text{ real}} K_v^{+\times} \times \prod_{v|\infty \text{ complex}} K_v^\times \times \prod_{v\nmid\infty} \mathcal{O}_{K_v}^\times \longrightarrow \mathrm{Gal}(L|K)$$

that sends

- $-1 \in K_v^\times$ to the complex complex conjugation $\mathrm{Frob}_v$ induced by $K \hookrightarrow K_v = \mathbb{R}$ if $K_v \simeq \mathbb{R}$,
- $\pi_v \in K_v^\times$ a pseudo-uniformizing element of $K_v$ to $\mathrm{Frob}_v$ the associated Frobenius element at the finite place $v$

factorizes through the group of principal ideles (reciprocity law) and induces an isomorphism

$$K^\times \Big\backslash \mathbb{A}_K^\times \Big/ \prod_{v|\infty \text{ real}} K_v^{+\times} \times \prod_{v|\infty \text{ complex}} K_v^\times \times \prod_{v\nmid\infty} \mathcal{O}_{K_v}^\times \xrightarrow{\sim} \mathrm{Gal}(L|K).$$

Let us point that there is no "geometric proof" of reciprocity laws for number fields as this is the case for function fields over finite fields. All proofs relies on Galois cohomological computations. Typically there is no geometric proof of the fact that for a number field $K$ and a class $\alpha \in \mathrm{Br}(K)$, the sum over all places $v$ of $K$ of the invariant of $\alpha_v \in \mathrm{Br}(K_v)$, as an element of $\mathbb{Q}/\mathbb{Z}$, is zero:

$$\sum_v \underbrace{\mathrm{inv}_v(\alpha_v)}_{\substack{\in \frac{1}{2}\mathbb{Z}/\mathbb{Z} \text{ if } K_v \simeq \mathbb{R} \\ 0 \text{ if } K_v \simeq \mathbb{C} \\ \in \mathbb{Q}/\mathbb{Z} \text{ if } v\nmid\infty}} = 0 \in \mathbb{Q}/\mathbb{Z}.$$

Some partial reciprocity results are known via geometric methods: the construction of algebraic Hecke characters associated to CM abelian varieties but this only concerns CM fields and not the full maximal abelian extension of any given number field.

5.2. **Local fields.** For $p$-adic fields the situation is better. The local Kronecker-Weber theorem holds,

$$\mathbb{Q}_p^{ab} = \bigcup_{n\geq 1} \mathbb{Q}_p(\zeta_n).$$

But this holds for any finite degree extension $E$ of $\mathbb{Q}_p$ up to replacing the formal group $\widehat{\mathbb{G}}_m$ by a Lubin-Tate group. More precisely, if $\pi$ is uniformizing element in $E$, $q$ is the cardinal of the residue field of $E$, and

$$f(T) = \sum_{n\geq 0} \frac{T^{q^n}}{\pi^n} \in E[\![T]\!]$$

then

$$X \underset{\mathcal{LT}}{+} Y = f^{-1}(f(X) + f(Y)) \in \mathcal{O}_E[\![X,Y]\!]$$

defines a one dimensional formal group law equipped with an action of $\mathcal{O}_E$ with logarithm $f$. This is a so-called Lubin-Tate group law. Multiplication by $\pi$ on it is finite flat of degree $q$ (this is a lift of the $q$-Frobenius) and

$$T_\pi \mathcal{LT} = \varprojlim_{n\geq 1} \mathcal{LT}[\pi^n](\mathcal{O}_{\overline{E}})$$

is a free rank 1 $\mathcal{O}_E$-module. Let

$$E^{un} = \bigcup_{(n,p)=1} E(\zeta_n)$$

be the maximal unramified extension of $E$. Then, locall class field theory says that

$$\boxed{E^{ab} = E^{un}\big(\text{torsion points of } \mathcal{LT}\big).}$$

The same holds for equal characteristic local fields, $E = \mathbb{F}_q(\!(\pi)\!)$. One has to replace the Lubin-Tate formal group by a Drinfeld module, $\widehat{\mathbb{G}}_{a/\mathcal{O}_E}$ equipped with the action of $\mathcal{O}_E$ where the action of $\mathbb{F}_q$ is the linear one and the action of $\pi$ is given by $\pi - \mathrm{Frob}_q$. In other terms, $E^{ab}$ is given by the $\bigcup_{n\geq 1}\mathbb{F}_{q^n}(\!(\pi)\!)$ where we add the roots of the polynomials $\underbrace{(\pi T - T^q)\circ\cdots\circ(\pi T - T^q)}_{n\text{-times}}$ when $n$ varies.

The Tate module of the Lubin-Tate formal group defines a Lubin-Tate character

$$\chi_{\mathcal{LT}} : \mathrm{Gal}(\overline{E}|E) \longrightarrow \mathcal{O}_E^\times.$$

We can then define

$$\chi = \chi_{\mathcal{LT}}.\pi^w : W_E \longrightarrow E^\times$$

where $w : W_E \to \mathbb{Z}$ is such that $\tau \equiv \mathrm{Frob}_q^{w(\tau)}$ on the residue field. In this context the reciprocity law says that

$$\chi : W_E^{ab} \xrightarrow{\sim} E^\times.$$

## 6. The local reciprocity law via the curve ([3]

Let $E$ be a local field with residue field $\mathbb{F}_q$ and uniformizing element $\pi$.

Let $* = \mathrm{Spd}(\overline{\mathbb{F}}_q)$ be the final object of the $v$-topos of sheaves on $\overline{\mathbb{F}}_q$-perfectoid spaces equipped with the $v$-topology ([4]). For $S$ an $\overline{\mathbb{F}}_q$-perfectoid space we can define

$$X_S$$

the relative curve parametrized by $S$ as an $E$-adic analytic space.

For $d \geq 1$ we can define a notion of a degree $d$ relative Cartier divisor on $X_S$ when $S$ varies. This defines a $v$-sheaf

$$\mathrm{Div}^d \longrightarrow *.$$

Contrary to the "classical case", $\mathrm{Div}^1$ is not the curve itself but an object that looks like the curve "seen in a mirror". For any $S$ as before, any untilt $S^\sharp$ of $S$ over $E$ defines a Carteir divisor of degree 1

$$S^\sharp \hookrightarrow X_S.$$

If $\breve{E}$ is the completion of the maximal unramified extension of $E$, $\mathrm{Spa}(\breve{E})^\diamond = \mathrm{Spa}(E)^\diamond \times_{\mathrm{Spd}(\mathbb{F}_q)} \mathrm{Spf}(\overline{\mathbb{F}}_q)$, and we thus have a morphism of $v$-sheaves

$$\mathrm{Spa}(\breve{E})^\diamond \longrightarrow \mathrm{Div}^1$$

over $*$.

**Proposition 6.1.** *The preceding morphism induces an isomorphism of $v$-sheaves*

$$\mathrm{Spa}(\breve{E})^\diamond/\varphi^{\mathbb{Z}} \xrightarrow{\sim} \mathrm{Div}^1.$$

In particular, $\mathrm{Div}^1 \to *$ is proper relatively representable in spatial diamonds. We can go further. As in the classical case, the $v$-sheaf $\coprod_{d\geq 1}\mathrm{Div}^d$ is a commutative monoid over $*$.

**Théorème 6.2.** *For any $d \geq 1$, the summation map of $d$-divisors of degree 1, $\pi_d : (\mathrm{Div}^1)^d \to \mathrm{Div}^d$, is quasi-pro-étale surjective and induces an isomorphism of pro-étale sheaves*

$$(\mathrm{Div}^1)^d/\mathfrak{S}_d \xrightarrow{\sim} \mathrm{Div}^d$$

*(quotient as pro-étale sheaves).*

We can go further an now introduce the Picard stack

$$\mathcal{P}ic \longrightarrow *$$

as a $v$-stack. One has

$$\mathcal{P}ic = \coprod_{d\in\mathbb{Z}} \mathcal{P}ic^d$$

and the choice of $\pi$ induces an identification

$$\mathcal{P}ic^d \xrightarrow{\sim} \underbrace{[\,*\,/\underline{E}^\times]}_{\substack{\text{classifying stack of}\\ \text{pro-étale } \underline{E}^\times\text{-torsors}}}$$

Here is now the main theorem of [3], see [2] too.

**Théorème 6.3.** *For $d \geq 2$ the Abel-Jacobi morphism*

$$\mathrm{AJ}^d : \mathrm{Div}^d \longrightarrow \mathcal{P}ic^d$$

*is a pro-étale locally trivial fibration in simply connected spatial diamonds.*

More precisely, this is a pro-étale fibration in punctured absolute Banach-Colmez spaces

$$\mathbb{B}^{\varphi=\pi^d} \smallsetminus \{0\}$$

and we prove those are spatial diamonds satisfying: any finite étale cover has a section when $d \geq 2$.

**Exemple 6.4.** *In the equal characteristic case, when $E = \mathbb{F}_q(\!(\pi)\!)$, one has*

$$\mathbb{B}^{\varphi=\pi^d} \smallsetminus \{0\} = \mathrm{Spa}\big(\overline{\mathbb{F}}_q[\![x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}]\!], \overline{\mathbb{F}}_q[\![x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}]\!]\big) \smallsetminus V(x_1, \dots, x_d),$$

*that is thus in this case representable by a quasi-compact quasi-separated perfectoid space. In this case the simple connectedness result is reduced after deperfectization and algebraization (GAGA) to Zariski-Nagata purity result: any finite étale cover of Spec$(\overline{\mathbb{F}}_q[\![x_1, \dots, x_d]\!]) \smallsetminus V(x_1, \dots, x_d)$ extends to a finite étale cover of Spec$(\overline{\mathbb{F}}_q[\![x_1, \dots, x_d]\!])$ when $d \geq 2$ and is thus trivial.*

The local reciprocity law is then deduced in the following way. The map

$$\text{”}\pi_1(\mathrm{Div}^1) \longrightarrow \pi_1(\mathcal{P}ic^1)\text{“}$$

(we put quotes because there is no general definition of a $\pi_1$ for such objects) is identified with

$$\chi : W_E^{ab} \longrightarrow E^\times.$$

The method of section 3 then applies to prove this is an isomorphism.

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