The curve and the Langlands program: the abelian case

Laurent Fargues (CNRS/IMJ)

Abelianized π_1 : the case of Riemann surfaces

- X = compact Riemann surface genus g
- Set

$$\mathrm{Jac}_X = \underbrace{H^0(X, \overbrace{\Omega^1_X}^{\substack{\text{hol.}\\ \text{1-forms}}})}_{\mathbb{C}\text{-v.s. dim. } g}{}^* / \underbrace{H_1(X,\mathbb{Z})}_{\pi_1(X)^{ab} \simeq \mathbb{Z}^{2g}}$$

here

$$\begin{array}{ccc} \pi_1(X)^{ab} = H_1(X,\mathbb{Z}) & \underset{\text{lattice}}{\hookrightarrow} & H^0(X,\Omega^1_X)^* \\ c & \longmapsto & \left[\omega \mapsto \int_c \omega\right] \end{array}$$

- Jac_X = abelian variety of dim. g

Abelianized π_1 : the case of Riemann surfaces

Modular interpretation :

- $$\mathrm{Div}^0(X) = \left\{ \sum_{x \in X} m_x[x] \mid m_x \in \mathbb{Z},\ \sum_x m_x = 0 \right\}$$

  degree 0 divisors.

- For $f \in \mathcal{M}(X)$,

  $$\mathrm{div}(f) = \sum_{x \in X} \mathrm{ord}_x(f)[x] \in \mathrm{Div}^0(X)$$

  where if z_x is a local coordinate at x and $f = \sum_{n \geq k} a_n z_x^n$ near x, $a_k \neq 0$, $\mathrm{ord}_x(f) := k$

- $$\mathrm{Div}^0(X)/\sim\, = \mathrm{Pic}^0(X)$$

  where $D \simeq D'$ if $D - D' = \mathrm{div}(f)$, f meromorphic

Abelianized π_1 : the case of Riemann surfaces

Modular interpretation of Jac_X :

Theorem (Abel, Jacobi)

$$\begin{array}{rcl} \mathrm{Div}^0(X) & \longrightarrow & H^0(X,\Omega^1_X)^*/H_1(X,\mathbb{Z}) \\ [x]-[y] & \longmapsto & \left[\omega \mapsto \int_y^x \omega\right] \textit{ mod } H_1(X,\mathbb{Z}) \end{array}$$

induces

$$\mathrm{Pic}^0(X) = \mathrm{Div}^0(X)/\sim \xrightarrow{\sim} \mathrm{Jac}_X$$

Abelianized π_1 : the case of Riemann surfaces

- Pic_X = Picard scheme of line bundles on X,

$$0 \longrightarrow \underbrace{\mathrm{Pic}^0_X}_{\mathrm{Jac}_X} \longrightarrow \underbrace{\mathrm{Pic}_X}_{\coprod_{d\in\mathbb{Z}} \mathrm{Pic}^d_X} \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0$$

- Splitting given by the choice of $\infty \in X$, identifies

$$\begin{array}{ccc} \mathrm{Pic}^0_X & \xrightarrow{\sim} & \mathrm{Pic}^d_X \\ \mathcal{L} & \longmapsto & \mathcal{L}(d[\infty]). \end{array}$$

- Canonical identification independent of the choice of ∞

$$\pi_1(\mathrm{Pic}^0_X) \xrightarrow{\sim} \pi_1(\mathrm{Pic}^d_X)$$

Abelianized π_1 : the case of Riemann surfaces

- $$\begin{aligned} \mathrm{AJ}^1 : X &\longrightarrow \mathrm{Pic}^1_X \\ x &\longmapsto \mathcal{O}([x]) \end{aligned}$$

  induces

  $$\pi_1(X)^{ab} \xrightarrow{\sim} \pi_1(\mathrm{Pic}^1(X)) = \pi_1(\mathrm{Jac}_X).$$

- $\rightarrow$ any abelian Galois cover of X come by pullback via AJ^1 from a cover of Jac_X

The geometric Langlands point of view

- X smooth projective algebraic curve over k alg. closed
- $\mathrm{Jac}_X = \mathrm{Pic}^0_X$ abelian variety$/k$
- construct canonical isomorphism (Groth. π_1 =profinite)

$$\pi_1(X)^{ab} \xrightarrow{\sim} \pi_1(\mathrm{Jac}_X).$$

- Reduced to :

Théorème

Any rank 1 étale $\overline{\mathbb{Q}}_\ell$-local system $\mathscr{E}$ on X descends along $\mathrm{AJ}^1 : X \longrightarrow \mathrm{Pic}^1_X$ to a rang 1 étale $\overline{\mathbb{Q}}_\ell$-local system

$$\mathrm{Aut}_{\mathscr{E}} = \textit{autmorphic loc. syst. associated to } \mathscr{E}$$

on Pic^1_X.

The geometric Langlands point of view

Sketch of proof :

- $d \geq 1$,

$$\mathrm{Div}_X^d = \text{Hilbert scheme of deg. } d \text{ effective divisors on } X$$

- One has

$$\begin{array}{rcl} X^d/\mathfrak{S}_d & \xrightarrow{\sim} & \mathrm{Div}_X^d \\ (x_1, \ldots, x_d) \bmod \mathfrak{S}_d & \longmapsto & \sum_{i=1}^{d} [x_i] \end{array}$$

- Abel-Jacobi morphism

$$\begin{array}{rcl} \mathrm{AJ}^d : \mathrm{Div}_X^d & \longrightarrow & \mathrm{Pic}_X^d \\ D & \longmapsto & \mathcal{O}(D) \end{array}$$

The geometric Langlands point of view

- $\pi_d : X^d \to X^d/\mathfrak{S}_d = \mathrm{Div}_X^d$,

$$(x_1, \ldots, x_d) \longmapsto \sum_{i=1}^{d} [x_i]$$

- $\mathscr{E}$ = rank 1 étale $\overline{\mathbb{Q}}_\ell$-local system on X

$$\mathscr{E} \longmapsto \underbrace{\left[\pi_{d*}\mathscr{E}^{\boxtimes d}\right]^{\mathfrak{S}_d}}_{\mathscr{F}_d} = \text{rank 1 étale loc. sys. on } \mathrm{Div}_X^d$$

$\to$ for $d > 1$, $\pi_1(X^d/\mathfrak{S}_d) = \pi_1(X)^{ab}$

- R.R. : for $d > 2g-2$, AJ^d = locally trivial fibration with fiber $\mathbb{P}^{d-g}$

- $\mathbb{P}^{d-g}$ simply connected

  $\Rightarrow$ for $d > 2g-2$,

$$\mathscr{F}_d = (\mathrm{AJ}^d)^* \underbrace{\mathscr{G}_d}_{\substack{\text{rk. 1 loc. sys.} \\ \text{on } \mathrm{Pic}^d_X}}$$

  $\rightarrow$ $\mathscr{F}_d$ descends along AJ^d in high degree d

The geometric Langlands point of view

- Collection $(\mathscr{G}_d)_{d>2g-2}$ of rk. 1 loc. sys.
- For $d, d' > 2g-2$, via $m : \mathrm{Pic}_X \times \mathrm{Pic}_X \to \mathrm{Pic}_X$

$$m^*\mathscr{G}_{d+d'} = \mathscr{G}_d \boxtimes \mathscr{G}_{d'}$$

- using the group structure on Pic one deduces that $(\mathscr{G}_d)_{d>2g-2}$ extends canonically to an equivariant loc. sys. on Pic_X,
- on Pic^1_X

$$\mathrm{Aut}_{\mathscr{E}} := \mathscr{G}_1$$

  easy to verify

$$(\mathrm{AJ}^1)^* \, \mathrm{Aut}_{\mathscr{E}} = \mathscr{E}.$$

Geometric class field in equal char.

- X smooth proj. curve $/\mathbb{F}_q$
- $W_X \subset \pi_1(X)$ pullback of $\mathrm{Frob}_q^{\mathbb{Z}}$ via

$$1 \longrightarrow \pi_1^{geo}(X) \longrightarrow \pi_1(X) \longrightarrow \overbrace{\mathrm{Gal}(\overline{\mathbb{F}}_q|\mathbb{F}_q)}^{\mathrm{Frob}_q^{\widehat{\mathbb{Z}}}} \longrightarrow 1$$

- Preceding implies

$$W_X^{ab} = \mathrm{Pic}_X(\mathbb{F}_q) = F^\times \backslash \mathbb{A}_F^\times / \prod_v \mathcal{O}_{F_v}^\times$$

where $F = \mathbb{F}_q(X)$

Abelian π_1's in number theory

- Kronecker-Weber : $\mathbb{Q}^{ab} = \bigcup_{n\geq 1} \mathbb{Q}(\zeta_n)$
- local K.-W. : $\mathbb{Q}_p^{ab} = \bigcup_{n\geq 1} \mathbb{Q}_p(\zeta_n)$
- more generally, $[E : \mathbb{Q}_p] < +\infty$,

$$E^{un} = \bigcup_{(n,p)=1} E(\zeta_n)$$

LT_π = Lubin-Tate one dimensional formal group law with logarithm

$$f = \sum_{n\geq 0} \frac{T^{q^n}}{\pi^n}$$

i.e $X \underset{LT_\pi}{+} Y = f^{-1}(f(X) + f(Y)) \in \mathcal{O}_E[\![X, Y]\!]$

Local reciprocity

$T_\pi(\mathrm{LT}_\pi) = $ rk. 1 free $\mathcal{O}_E$-module $\to$ character

$$\chi_0 : \mathrm{Gal}(\overline{E}|E) \longrightarrow \mathcal{O}_E^\times$$

Théorème

Let $w : W_E \to \mathbb{Z}$ *be s.t.* $\sigma \equiv \mathrm{Frob}_q^{w(\sigma)}$*. Then*

$$\chi = \chi_0.\pi^w : W_E \longrightarrow E^\times$$

induces

$$W_E^{ab} \xrightarrow{\sim} E^\times$$

i.e.

$$E^{ab} = E^{un}(\textit{torsion points of } LT_\pi).$$

Local reciprocity via the curve

$* = \mathsf{Spd}(\overline{\mathbb{F}}_q)$,

$$\mathrm{Div}^d \to *$$

moduli of degree d divisors on the curve

$$\mathcal{P}ic \to *$$

Picard stack of line bundles on the curve

Théorème (F.)

For $d \geq 2$,

$$\mathsf{AJ}^d : \mathrm{Div}^d \longrightarrow \mathcal{P}ic^d$$

is a pro-étale locally trivial fibration in simply connected spatial diamonds

Local reciprocity via the curve

One has

$$\begin{array}{rcl} \mathrm{Div}^1 & = & \mathsf{Spa}(\breve{E})^\diamond/\varphi^{\mathbb{Z}} \\ \mathcal{P}ic^1 & = & [*/\underline{E}^\times] \end{array}$$

and the morphism

$$W_E = \pi_1(\mathrm{Div}^1) \longrightarrow \pi_1(\mathcal{P}ic^1) = E^\times$$

is given by χ. Preceding result $\Rightarrow$ local reciprocity :

$$\chi : W_E^{ab} \xrightarrow{\sim} E^\times.$$

Local reciprocity via the curve

Théorème (F.)

1. *The morphism* $(\mathrm{Div}^1)^d \to \mathrm{Div}^d$ *is quasi-pro-étale surjective and induces*

$$\underbrace{(\mathrm{Div}^1)^d/\mathfrak{S}_d}_{\substack{\text{pro-étale}\\ \text{quotient}}} \xrightarrow{\sim} \mathrm{Div}^d$$

2. AJ^d *is a pro-étale fibration in*

$$\underbrace{\mathbb{B}^{\varphi=\pi^d} \setminus \{0\}}_{\substack{\text{punctured absolute}\\ \text{BC space}}}$$

that is a simply connected spatial diamond if $d \geq 2$