Some new geometric structures in the Langlands program

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More than 20 years of work on the (local) Langlands program / p-adic Hodge theory :

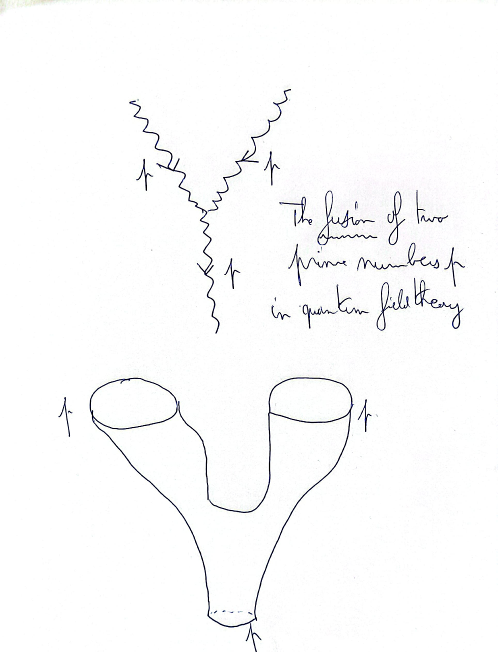
- from Harris-Taylor work for GL_n, my PHD work on the geometric realization of the local Langlands correspondence,
- going through my joint work with Fontaine on "the curve" where we gave a meaning to the notion of "holomorphic function of the variable p",
- through my geometrization conjecture I formulated at the MSRI in 2014,
- until my recent joint work with Scholze on the geometric realization of the local Langlands correspondence.

- The (local) Langlands program has been completely reformulated.
- This has allowed us to construct jointly with Scholze recently the semi-simple local Langlands correspondence

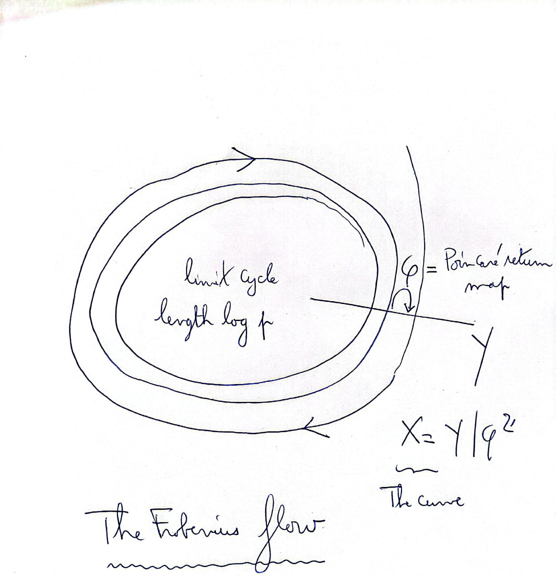
$$\underbrace{\pi}_{\substack{\text{smooth rep.} \\ \text{of } G(\mathbb{Q}_p)}} \longmapsto \underbrace{\varphi_\pi^{ss}}_{\substack{\text{Langlands parameter} \\ W_{\mathbb{Q}_p} \to {}^L G}} .$$

Here typically $\pi =$ local component at p of an automorphic representation of a reductive group over $\mathbb{Q}$ and ${}^L G =$ Langlands dual.

Fusion



The Frobenius flow



Holomorphic functions of the variable p

- $A =$ perfect $\mathbb{F}_p$-algebra i.e. $\mathrm{Frob}_A : A \xrightarrow{\sim} A$
- $$\underbrace{W(A)}_{\text{Witt vectors}} \underbrace{=}_{\substack{\text{unique}\\ \text{writing}}} \left\{ \sum_{n\geq 0} [a_n] p^n \;\middle|\; a_n \in A \right\}$$

  unique p-torsion free p-adically complete lift of A
- $[a] = \underbrace{\lim_{n\to+\infty} \left(\widetilde{a^{1/p^n}}\right)^{p^n}}_{\text{renormalization process}}$, $\widetilde{b} :=$ any lift of b

Holomorphic functions of the variable p

- Addition / multiplication given by universal generalized polynomials in
$$\mathbb{F}_p\big[X_i^{1/p^\infty}, Y_j^{1/p^\infty}\big]_{i,j\geqslant 0}$$
- Example :
$$[a]+[b]=[a+b]+[P(a,b)]p+p^2\cdots$$
where
$$P(X,Y)=\frac{(X^{1/p}+Y^{1/p})^p-X-Y}{p}\in\mathbb{Z}[X^{1/p},Y^{1/p}]$$

Holomorphic functions of the variable p

- $A = \mathbb{F}_p$-perfectoid algebra i.e. $A =$ perfect Banach ring
- $A^+ \subset A$ ring of bounded by 1 holomorphic functions
- $W(A^+)$ equipped with topology=mix of p-adic and Banach ring top of A^+, $f \in W(A^+)$,

$$f = \sum_{n \geq 0} [a_n] p^n = \textit{hol. fct. variable } p.$$

- $\varpi \in A^{\circ\circ} \cap A^\times$ pseudo-uniformizer, $S = \mathrm{Spa}(R, R^+)$

$$Y_S = \mathrm{Spa}\big(W(A^+), W(A^+)\big) \setminus V(p\,[\varpi])$$

adic space, "open punctured disk variable p"

Holomorphic fct. variable p

- Y_S = *Stein* adic space
- $\mathcal{O}(Y_S)$ = a completion of

$$W(A^+)[\tfrac{1}{p},\tfrac{1}{[\varpi]}] = \Big\{ \underbrace{\sum_{n \gg -\infty}}_{\substack{\text{meromorphic}\\ \text{at } p=0}} [a_n]p^n \mid \underbrace{\sup_n \|a_n\| < +\infty}_{\substack{\text{meromorphic at}\\ [\varpi]=0}} \Big\}$$

$\to$ non explicit Frechet ring, *Fontaine's type ring*

- Equipped with Frobenius

$$Y_S \circlearrowleft^{\varphi}$$

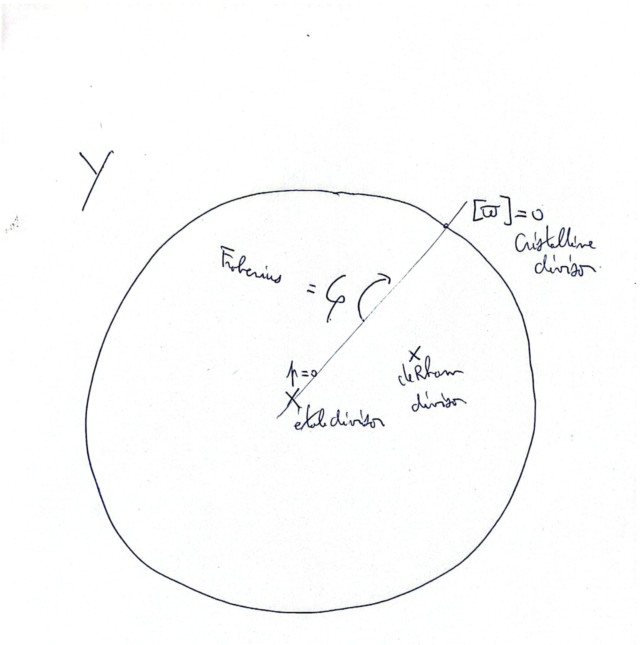
induced by Frob of the Witt vectors $\sum_n [a_n]p^n \mapsto \sum_n [a_n^p]p^n$

- Acts properly discontinuously without fixed points, continuous map

$$|Y_S| \longrightarrow]0,1[$$

if $y \mapsto t$ then $\varphi(y) \mapsto t^{1/p}$.

The space Y



Y
[ϖ]=0
Cristalline
divisor
Frobenius
= φ
p=0
étale divisor
deRham
divisor

Holomorphic functions variable p

$S = \mathrm{Spa}(F)$, F = perfectoid field.

- With Fontaine : develop a theory of hol. fct. for $f \in \mathcal{O}(Y_S)$.
- Example

Théorème (F.-Fontaine, Weierstrass factorization)

If F is algebraically closed and

$$f = \sum_{n \geq 0} [a_n] p^n \in W(\mathcal{O}_F) = \mathcal{O}(Y_S)^+,$$

$|a_0| < 1, \ldots, |a_{d-1}| < 1$, $a_0 \neq 0$ *and* $|a_d| = 1$, *can write*

$$f = \mathit{unit} \times (p - [z_1]) \times \cdots \times (p - [z_d])$$

The curve

Définition (F.-Fontaine)

For $S = \mathbb{F}_p$-perfectoid space

$$X_S = Y_S/\varphi^{\mathbb{Z}}$$

the relative curve associated to S

- Functorial in S
- Can think of X_S as the family associated to the collection of curves
$$(X_{K(s),K(s)^+})_{s\in S}$$
- For $S = \text{Spa}(F)$, F perfectoid field, one has (F.-Fontaine) $X_F =$ *Noetherian analytic adic space of dimension* 1
$$\underbrace{H^1(\mathcal{O}) = 0}_{\text{like } \mathbb{P}^1} \text{ but } \underbrace{H^1(\mathcal{O}(-1)) \neq 0}_{\text{unlike } \mathbb{P}^1}$$

The moduli of bundles on the curve

G = reductive group over $\mathbb{Q}_p$

Définition

Bun_G is the moduli on $\overline{\mathbb{F}}_p$-perfectoid spaces

$$S \longmapsto \underbrace{\{G\text{-bundles on } X_S\}}_{\text{groupoid}}$$

$\to$ v-stack on $\mathrm{Perf}_{\overline{\mathbb{F}}_p}$, v-top= analog of fpqc topology

The moduli of bundles on the curve

Théorème (F.-Scholze)

The v-stack Bun_G is an Artin v-stack ℓ-coho. smooth of dimension 0.

$\rightarrow$ nice charts made of locally spatial diamonds for the ℓ-coho. smooth topology, $\ell \neq p$

$\rightarrow$ nice geometric structure, not an abstract v-stack

The moduli of bundles on the curve

$S = \mathrm{Spa}(F)$, F alg. closed $\to$ geometric point

Théorème (F.-Fontaine, F.)

1. *For each $\lambda \in \mathbb{Q}$, $\lambda = \frac{d}{h}$, $\mathcal{O}(\lambda)$ = stable vector bundle of rank h degree d on the curve.*

$$\begin{aligned} \{\lambda_1 \geq \cdots \geq \lambda_r \mid r \in \mathbb{N}, \lambda_i \in \mathbb{Q}\} &\xrightarrow{\sim} \mathrm{Bun}(F)/\sim \\ (\lambda_1, \ldots, \lambda_r) &\longmapsto \Big[\bigoplus_{i=1}^{r} \mathcal{O}(\lambda_i)\Big]. \end{aligned}$$

2. *$B(G) = G(\breve{\mathbb{Q}}_p)/\sigma$-conjugacy, $b \sim gbg^{-\sigma}$,*

$$\begin{aligned} B(G) &\xrightarrow{\sim} \mathrm{Bun}_G(F)/\sim \\ [b] &\longmapsto [\mathscr{E}_b] \end{aligned}$$

The space $|\mathrm{Bun}_G|$

- $|\mathrm{Bun}_G| \underbrace{=}_{\text{as sets}} B(G)$
- First Chern class

$$c_1 : |\mathrm{Bun}_G| \xrightarrow{\sim} \pi_1(G)_\Gamma$$

- Harder-Narasimhan stratification : semi-continuous

$$|\mathrm{Bun}_G| \longrightarrow \underbrace{X_*(A)^+_{\mathbb{Q}}}_{\substack{\text{positive} \\ \text{Weyl chamber}}}$$

- Usual order on $X_*(A)^+_{\mathbb{Q}}$ defines the topology of $|\mathrm{Bun}_G|$,

$$\underbrace{\mathcal{O}^2}_{\substack{\text{semi-stable locus} \\ =\text{one point}}} \geqslant \mathcal{O}(1) \oplus \mathcal{O}(-1) \geqslant \mathcal{O}(2) \oplus \mathcal{O}(-2) \geqslant \dots$$

for Bun_2^0. For Bun_2^1 one has

$$\underbrace{\mathcal{O}(\tfrac{1}{2})}_{\substack{\text{semi-stable locus} \\ =\text{one point}}} \geqslant \mathcal{O}(1) \oplus \mathcal{O} \geqslant \mathcal{O}(2) \oplus \mathcal{O}(-1) \geqslant \dots$$

Étale sheaves

$\Lambda =$ any $\mathbb{Z}_\ell$-algebra, $\ell \neq p$

- We define with Scholze

$$D_{lis}(\mathrm{Bun}_G, \Lambda)$$

  as a stable ∞-category

- This is $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ when Λ is torsion and in general

$$D_{lis}(\mathrm{Bun}_G, \Lambda) \underbrace{\subset}_{\substack{\text{explicitely}\\ \text{defined as}\\ \text{a sub-cat}}} D_{\text{pro-ét},\blacksquare}(\mathrm{Bun}_G, \Lambda)$$

- Semi-orthogonal decomposition by

$$\underbrace{D(G_b(E), \Lambda)}_{\text{smooth rep. of } G_b(E)}, \ [b] \in B(G).$$

  via the HN-stratification of Bun_G

The Hecke action

- Div^1 = sheaf of degree 1 effective divisors on the curve
- $\mathrm{Spa}(\breve{\mathbb{Q}}_p)^\diamond/\varphi^{\mathbb{Z}} \xrightarrow{\sim} \mathrm{Div}^1$ via : if $S^\sharp$ = an untilt of S over $\mathbb{Q}_p$,

$$S^\sharp \underbrace{\hookrightarrow}_{\substack{\text{deg. 1}\\ \text{Cartier}\\ \text{divisor}}} X_S.$$

- For any finite set I, moduli of modifications

$$\begin{array}{ccc} & \mathrm{Hecke}_I & \\ \swarrow & & \searrow \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times (\mathrm{Div}^1)^I \end{array}$$

along $\sum_{i \in I} D_i$, $(D_i)_{i \in I}$ collection of deg. 1 Cartier divisors

The Hecke action : from global to local

- Local Hecke stack

$$\mathscr{H}\mathsf{ecke}_I \longrightarrow (\mathrm{Div}^1)^I$$

obtained by replacing the curve by its formal completion along $\sum_{i \in I} D_i$,

- Loop group interpretation

$$\mathscr{H}\mathsf{ecke}_I = \left[L_I^+ G \backslash L_I G / L_I^+ G \right]$$

where $L_I^+ G = G(\mathbb{B}_{dR,I})$, $\mathbb{B}_{dR,I}$ = generalized Fontaine's v-sheaf of rings of formal functions on the formal completion

- Global to local map

$$\begin{array}{ccc} \mathsf{Hecke}_I & \xrightarrow{\ loc\ } & \mathscr{H}\mathsf{ecke}_I \\ \big\downarrow & & \big\downarrow \\ \mathrm{Bun}_G \times (\mathrm{Div}^1)^I & \longrightarrow & (\mathrm{Div}^1)^I \end{array}$$

The Hecke action : geometric Satake

Théorème (F.-Scholze)

Monoidal equivalence

$$\Big(\mathrm{Rep}_\Lambda\big(({}^LG)^I\big),\otimes\Big)\xrightarrow{\sim}\Big(\mathrm{Perv}^{ULA}(\mathscr{H}ecke_I,\Lambda),\underbrace{*}_{\text{convolution product}}\Big)$$

here ULA : relative to $\mathscr{H}ecke_I\to(\mathrm{Div}^1)^I$.

The Hecke action

- Via global to local map, via pullback : can upgrade the global Hecke correspondence to *a cohomological one*
- The action of those Hecke correspondences defines monoidal ∞-functors between stable ∞-categories

$$\mathsf{Rep}_\Lambda\left(({}^LG)^I), \otimes\right) \to \mathsf{Hom}\left(D_{lis}(\mathrm{Bun}_G, \Lambda), D_{lis}(\mathrm{Bun}_G \times (\mathrm{Div}^1)^I, \Lambda)\right)$$

- Drinfeld lemma :

$$\underbrace{D_{lis}(\mathrm{Bun}_G, \Lambda)}_{\substack{\text{condensed}\\\text{stable}\\\infty\text{-cat}}}{}^{BW_E^I} \xrightarrow{\sim} D_{lis}(\mathrm{Bun}_G \times (\mathrm{Div}^1)^I, \Lambda)$$

The Hecke action

At the end, a sequence of monoidal functors between monoidal stable ∞-categories

$$F_I : \mathrm{Rep}_\Lambda \left(({}^L G)^I \right) \longrightarrow \mathrm{End}(D_{lis}(\mathrm{Bun}_G, \Lambda))^{BW_E^I}$$

that is

1. Linear over $\mathrm{Rep}_\Lambda(W_E^I)$ via $({}^L G)^I \to W_E^I$
2. Functorial in the finite set I (see, factorization objects à la Beilinson-Drinfeld/Kapranov)

For example (*fusion property*) :
$\mathrm{Res}^{W_E^2}_{W_E} F_{1,2}(W_1 \boxtimes W_2) = F_1(W_1 \otimes W_2)$.

The moduli of Langlands parameters

Théorème

There is a natural moduli of condensed Langlands parameters from $\underbrace{W_E}_{\text{condensed group}}$ *to* ${}^L G$*, over Spec*(Λ)*,*

$$\mathrm{LocSys}_{\widehat{G}} = [\ \underbrace{Z^1(W_E, \widehat{G})}_{\coprod \text{finite type affine schemes}}\ / \widehat{G}\ ]$$

that is an algebraic stack locally complete intersection of relative dimension 0*.*

The spectral action

Théorème (F.-Scholze)

The family of functors F_I, I a finite set, together with the factorization property define a monoidal action of the stable ∞-category

$$\mathrm{Perf}(\mathrm{LocSys}_{\widehat{G}})$$

on

$$D_{lis}(\mathrm{Bun}_G, \Lambda)$$

$\rightarrow$ write $\underbrace{BW_E}_{\infty\text{-groupoid}}$ as a homotopy colimit of finite sets and link to the finite sets I

The local Langlands correspondence

- $\pi =$ smooth irreducible $\overline{\mathbb{Q}}_\ell$-rep. of $G(E)$
- $\pi \mapsto \mathscr{F}_\pi$ local system on $\mathrm{Bun}_G^{ss,0} \simeq [*/\underline{G(E)}]$
- $j : \mathrm{Bun}_G^{ss,0} \hookrightarrow \mathrm{Bun}_G$ open immersion
- look at the spectral action on $j_! \mathscr{F}_\pi$
- this defines φ_π^{ss}

The geometrization conjecture

G quasi-split. Fix $\psi : U(E) \to \overline{\mathbb{Q}}_\ell^\times$ non-degenerate. Let

$$\underbrace{\mathcal{W}_\psi}_{\text{Whittaker sheaf}} = j_! \, \text{c-Ind}_{U(E)}^{G(E)} \psi \in D_{lis}(\mathrm{Bun}_G, \Lambda)$$

Conjecture

The functor

$$\mathrm{Perf}(\mathrm{LocSys}_{\widehat{G}}) \longrightarrow D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$$

$$\mathcal{E} \longmapsto \underbrace{\mathcal{E} \overbrace{\ast}^{\text{spectral action}} \mathcal{W}_\psi}_{\text{non-abelian Fourier transform}}$$

extends to an equivalence

$$D^b_{coh}(\mathrm{LocSys}_{\widehat{G}}) \xrightarrow{\sim} D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega.$$

Rethinking the Langlands program

Natural objects are not smooth rep. of $G(E)$ but complexes in $D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$. Let $A \in D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$:

Théorème

1. *A is compact $\Leftrightarrow$ its support is finite and $\forall [b]$, $i_b^* A \in D^b_{ft}(G_b(E), \overline{\mathbb{Q}}_\ell)$.*
2. *A is ULA $\Leftrightarrow \forall [b],\ \forall K,\ (i_b^* A)^K \in D^b_{ft}(\overline{\mathbb{Q}}_\ell)$*
3. *$\exists \mathbb{D}_{BZ}$ involution of $D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$ generalizing Berstein-Zelevinsky involution.*