The curve and the Langlands program

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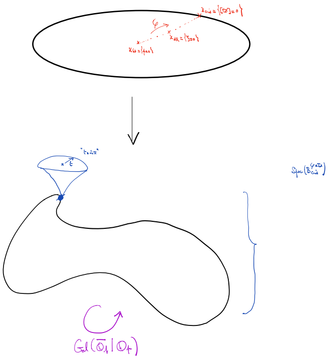

The curve

- Defined and studied in our joint work with Fontaine
- Two aspects : compact p-adic Riemann surface / algebraic curve

- Starting datum : $F|\mathbb{F}_p$ perfectoid field
- p=variable, $Y = \{0 < |p| < 1\}$ open punctured disk, adic space

$$W(\mathcal{O}_F)[\tfrac{1}{p}, \tfrac{1}{[\varpi]}] = \Big\{ \sum_{n \gg -\infty} [x_n]p^n \mid x_n \in F, \ \sup_n |x_n| < +\infty \Big\}$$

- $\mathcal{O}(Y)$ completion of the preceding. Fontaine's ring. No explicit formula for elements of $\mathcal{O}(Y)$

The curve

- $Y \circlearrowleft \varphi$ Frobenius
$$X = Y/\varphi^{\mathbb{Z}}$$
compact adic space $/\mathbb{Q}_p$
- $\mathcal{O}(1)$ line bundle $/X \leftrightarrow$ automorphic factor $j(\varphi, p) = p$
- Schematical curve
$$\mathfrak{X} = \mathrm{Proj}\big(\oplus_{d\geq 0} \mathcal{O}(Y)^{\varphi = p^d} \big)$$
- GAGA type morphism
$$X \longrightarrow \mathfrak{X}$$

The curve

Theorem (F.-Fontaine)

- *The curve is a curve : $\mathfrak{X}$ is a Dedekind scheme, X noetherian regular dimension 1 adic space*
- *Perfectoid residue fields :*

$$\forall x \in |\mathfrak{X}|, \quad k(x)|\mathbb{Q}_p \textit{ perfectoid}, \quad [k(x)^\flat : F] < +\infty$$

- $\widehat{\mathcal{O}}_{\mathfrak{X},x} = B_{dR}^+(k(x))$
- *$\mathfrak{X}$ is "complete" :* $\deg(x) := [k(x)^\flat : F]$

$$\forall f \in \mathbb{Q}_p(\mathfrak{X})^\times, \quad \deg(\mathrm{div} f) = 0$$

The curve

- $\mathfrak{X} \setminus \{x\} = \mathrm{Spec}(B_x)$, B_x Dedekind, P.I.D. if F alg. closed. $(B_x, -ord_x)$ not euclidean i.e.

$$H^1(\mathcal{O}(-1)) \neq 0$$

- F alg. closed. $\Rightarrow$ $|\mathfrak{X}| =$ untilts of F up to Frob.

Vector bundles on the curve

- GAGA verified : $\mathrm{Bun}_{\mathfrak{X}} \xrightarrow{\sim} \mathrm{Bun}_X$
- $\mathfrak{X}$ complete $\Rightarrow$ good Harder-Narasimhan reduction theory

$$\mu = \frac{\deg}{\mathrm{rk}}$$

- $\lambda \in \mathbb{Q} \rightsquigarrow \mathcal{O}(\lambda)$ stable slope λ vector bundle, $\lambda = \frac{d}{h}$, pushforward of $\mathcal{O}(d)$ via cyclic cover $Y/\varphi^{h\mathbb{Z}} \to Y/\varphi^{\mathbb{Z}}$

Theorem (F.-Fontaine)

F alg. closed

$$\begin{array}{rcl} \{\lambda_1 \geq \cdots \geq \lambda_n \mid \lambda_i \in \mathbb{Q}\} & \xrightarrow{\sim} & \mathrm{Bun}_{\mathfrak{X}}/\sim \\ (\lambda_1, \ldots, \lambda_n) & \longmapsto & [\oplus_i \mathcal{O}(\lambda_i)]. \end{array}$$

Vector bundles on the curve : applications

- Quick simple proofs of the two fundamentals theorems of p-adic Hodge theory :
  - weakly admissible $\Rightarrow$ admissible (Colmez-Fontaine)
  - de Rham $\Rightarrow$ pot. semi-stable (Berger)
- $F = \mathbb{C}_p^\flat$,

$$\mathfrak{X} \circlearrowleft \mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$$

  $\infty \in |\mathfrak{X}|$ fixed. Study Galois equivariant vector bundles and their modifications at ∞
- Look at modifications of vector bundles
  - φ-modules over A_{inf}
  - Scholze-Weinstein

G-bundles

- G reductive group $/\mathbb{Q}_p$, $\breve{\mathbb{Q}}_p := \widehat{\mathbb{Q}_p^{un}}$, $\sigma = Frob$,

$$B(G) = G(\breve{\mathbb{Q}}_p)/\sigma\mathrm{-conj}, \quad b \sim gbg^{-\sigma}$$

- $b \in G(\breve{\mathbb{Q}}_p)$

$$\mathcal{E}_b = Y \underset{\varphi}{\times} G$$

G-bundle $/X$, φ acts on G via conjugation by $b\sigma$

Theorem (F)

F alg. closed

$$\begin{array}{ccc} B(G) & \xrightarrow{\sim} & H^1_{\acute{e}t}(\mathfrak{X}, G) \\ [b] & \mapsto & [\mathcal{E}_b] \end{array}$$

- Dictionary : reduction theory (Atiyah-Bott) for G-bundles / Kottwitz description of $B(G)$.
- *Example* : $\mathcal{E}_b$ semi-stable $\Leftrightarrow$ b is basic (isoclinic)

G-bundles : applications

- (Chen, F, Shen) : Proof of F.-Rapoport conjecture on period spaces (p-adic analogs of Griffith's period spaces).
  Example : computation of p-adic period spaces for $SO(2, n-2)$
- Construction of local Shimura varieties (Scholze) $\mathrm{Sht}(G, b, \mu)$ as moduli of modifications $\mathcal{E}_1 \rightsquigarrow \mathcal{E}_b$
- Newton stratification of Hodge-Tate flag manifold (Caraiani-Scholze), modification $\mathcal{E}_{1,x}$, $x \in$ flag manifold, $\mathcal{E}_{1,x} \simeq \mathcal{E}_b$ for some $b \rightsquigarrow$ stratification by the set of such $[b]$

The stack Bun_G of G-bundles/curve

- Introduced to formulate a geometrization conjecture of the local Langlands correspondence

$$\underbrace{\varphi}_{\text{Langlands parameter}} \longmapsto \underbrace{\mathscr{F}_\varphi}_{\text{perverse sheaf on } \mathrm{Bun}_G}$$

- $S = \overline{\mathbb{F}}_p$-perfectoid space $\rightsquigarrow X_S$ adic space/$\mathbb{Q}_p$ = "family of curves $(X_{k(s)})_{s\in S}$"
- Bun_G = v-stack on $\mathrm{Perf}_{\overline{\mathbb{F}}_p}$ of G-bundles/curve

$$B(G) \xrightarrow{\sim} |\mathrm{Bun}_G|$$

$$\mathrm{Bun}_G = \coprod_{\alpha\in\pi_1(G)_\Gamma} \mathrm{Bun}_G^{c_1=\alpha}$$

The stack Bun_G

- $$\mathrm{Bun}_G^{c_1=0,ss} = [\bullet / \underline{G(\mathbb{Q}_p)}] \xhookrightarrow{\text{open}} \mathrm{Bun}_G$$
- $\underline{\mathrm{Aut}}$(trivial G-bundle) $= \underline{G(\mathbb{Q}_p)}$ topological group $\neq$ classical case $\rightsquigarrow$ G algebraic group
- More generally $\forall \alpha \in \pi_1(G)_\Gamma$
$$\mathrm{Bun}_G^{c_1=\alpha,ss} = [\bullet / \underline{J_b(\mathbb{Q}_p)}]$$
b basic, J_b inner form of G
- In general each H.N. stratum is a classifying stack
$$[\bullet / \mathcal{J}_b]$$
$\mathcal{J}_b = \mathcal{J}_b^0 \rtimes \underline{J_b(\mathbb{Q}_p)}$, $\mathcal{J}_b^0$ =unipotent diamond

The geometrization conjecture

- φ = discrete Langlands parameter

$$\varphi : W_{\mathbb{Q}_p} \longrightarrow {}^L G$$

- Conjecture

$$\varphi \longmapsto \mathscr{F}_\varphi$$

S_φ-equivariant perverse Hecke eigensheaf on Bun_G

- s.t. the stalks of $\mathscr{F}_\varphi$ at semi-stable points gives local Langlands + internal structure of L-packets for all extended pure inner forms of G

The geometrization conjecture

For this :

- Need to give a meaning to "perverse sheaf on Bun_G"
- Need to give a meaning to the Hecke eigensheaf property : establish geometric Satake in this context

$\rightsquigarrow$ joint work with Scholze : give a precise statement of the conjecture + construction of the local Langlands correspondence

$$\pi \mapsto \varphi_\pi$$

à la V. Lafforgue using Bun_G

Constructible and perverse étale sheaves on Bun_G

Joint with Scholze.

$\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Q}}_\ell\}$ étale coefficients, $\ell \neq p$

- Bun_G is a (cohomologically) smooth stack of dimension 0
- Good notion of constructible sheaf on Bun_G = reflexive sheaves w.r.t. Verdier duality
- Fiberwise criterion of constructibility in terms of representation theory : $\forall b \in G(\breve{\mathbb{Q}}_p)$, the stalk at the point given by b is an admissible representation of $J_b(\mathbb{Q}_p)$

Geometric Satake

- $\mathrm{Div}^1 = \mathsf{Spa}(\mathbb{Q}_p)^\diamond/\varphi^{\mathbb{Z}}$ sheaf of deg. 1 effective div. on the curve
- $$\mathrm{Gr}_G^{B_{dR}} \longrightarrow \mathsf{Spa}(\mathbb{Q}_p)^\diamond$$

  Scholze's B_{dR} affine grassmanian,

  $$\mathrm{Gr}_G^{B_{dR}}/\varphi^{\mathbb{Z}} \to \mathrm{Div}^1$$

  Beilinson-Drinfeld type affine Grassmanian

Theorem (F.-Scholze)

Geometric Satake holds for $\mathrm{Gr}_G^{B_{dR}}$, Satake category $\simeq \mathrm{Rep}({}^L G, \Lambda)$.

Construction of Langlands parameters

- Factorization enhancement :

$$I \text{ finite set } \rightsquigarrow \mathrm{Gr}_{G,I}^{B_{dR}} \to (\mathrm{Spa}(\mathbb{Q}_p)^\diamond)^I$$

  $+$ factorization property when I varies.

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$$\begin{array}{ccc} & \mathrm{Hecke}_I & \\ \swarrow & & \searrow \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times (\mathrm{Spa}(\mathbb{Q}_p)^\diamond)^I \end{array}$$

- $W \in \mathrm{Rep}({}^LG^I, \Lambda) \rightsquigarrow IC_W$ kernel on Hecke_I via geo Satake

Construction of Langlands parameters

- Coupled with V. Lafforgue strategy (global function field) we construct local Langlands

$$\underbrace{\pi}_{\substack{\text{irred. rep.}\\ \text{of } G(\mathbb{Q}_p)}} \longmapsto \underbrace{\varphi_\pi}_{\substack{\text{semi-simple Langlands}\\ \text{parameter}}}$$

- Very little known about this correspondence : surjectivity, finiteness of fibers ?
- Geometrization conjecture goes in the other direction

$$\text{Langlands parameter} \longmapsto \text{representation}$$

and would give this + internal structure of L-packets

Back to the geometrization conjecture

- The GL_1-case. Classically : Abel-Jacobi morphism locally trivial fibration in simply connected alg. var. (projective spaces) in high degree
- Reduced to the following theorem

Theorem (F)

For $d \geq 3$, the Abel-Jacobi morphism

$$AJ^d : \mathrm{Div}^d \longrightarrow \mathscr{P}\mathrm{ic}^d$$

is a pro-étale locally trivial fibration in simply connected diamonds.

Here

$$\mathrm{Div}^d = \text{Hilbert diamond} = (\mathrm{Div}^1)^d/\mathfrak{S}_d$$

with $\mathrm{Div}^1 = \mathrm{Spa}(\mathbb{Q}_p)^\diamond/\varphi^\mathbb{Z}$.

That's only the beginning of the story!