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EILENBERG/HAUSDORFF LECTURES ON THE GEOMETRIZATION OF THE LOCAL LANGLANDS CORRESPONDENCE

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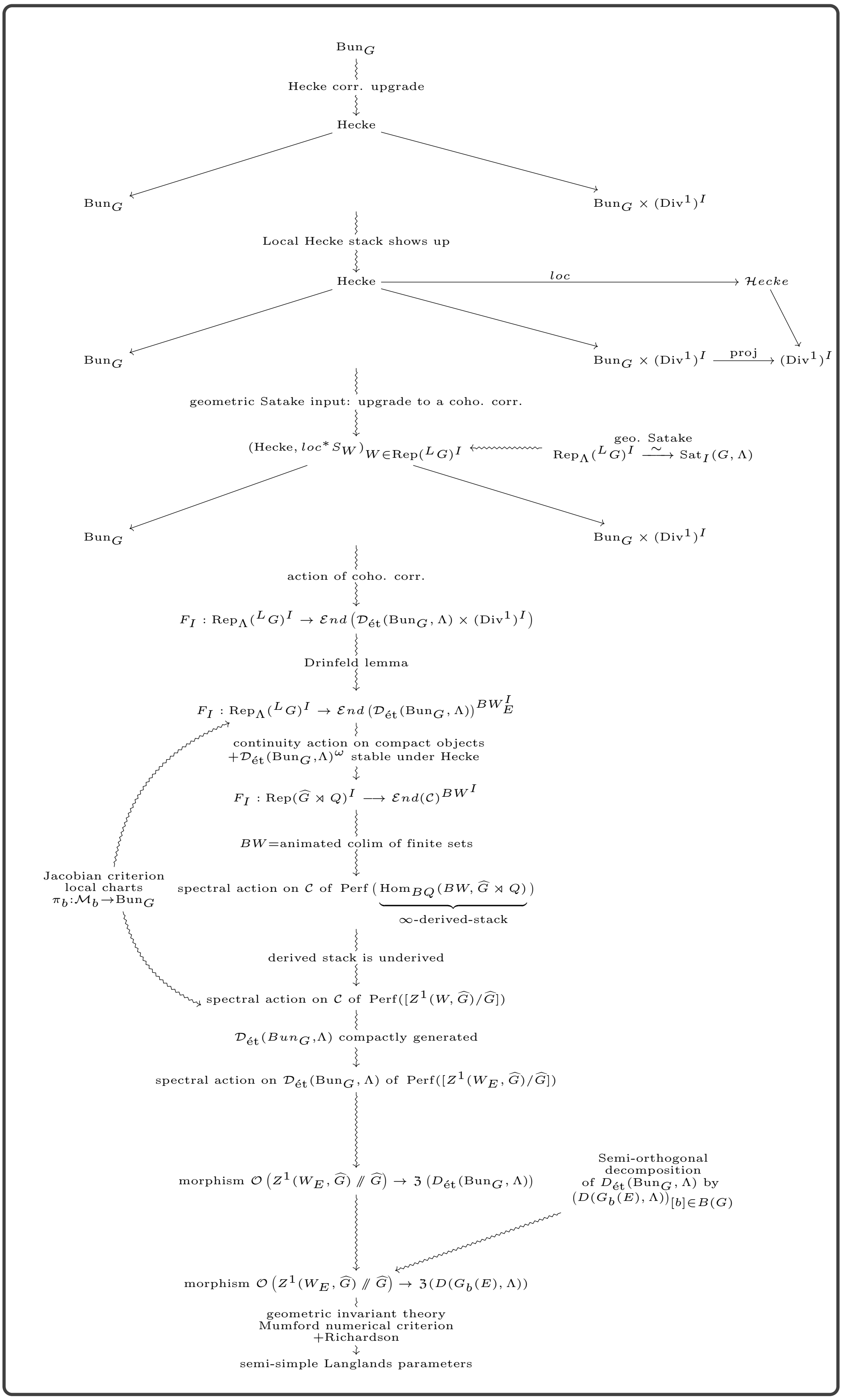
Bun_G
Hecke corr. upgrade
Hecke
Bun_G
$\mathrm{Bun}_G \times (\mathrm{Div}^1)^I$
Local Hecke stack shows up
Hecke
loc
$\mathcal{H}ecke$
Bun_G
$\mathrm{Bun}_G \times (\mathrm{Div}^1)^I$
proj
$(\mathrm{Div}^1)^I$
geometric Satake input: upgrade to a coho. corr.
$(\mathrm{Hecke}, loc^* S_W)_{W \in \mathrm{Rep}({}^L G)^I}$
geo. Satake
$\mathrm{Rep}_\Lambda({}^L G)^I \xrightarrow{\sim} \mathrm{Sat}_I(G, \Lambda)$
Bun_G
$\mathrm{Bun}_G \times (\mathrm{Div}^1)^I$
action of coho. corr.
$F_I : \mathrm{Rep}_\Lambda({}^L G)^I \to \mathcal{E}nd\left(\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda) \times (\mathrm{Div}^1)^I\right)$
Drinfeld lemma
$F_I : \mathrm{Rep}_\Lambda({}^L G)^I \to \mathcal{E}nd\left(\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda)\right)^{BW_E^I}$
continuity action on compact objects
$+\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G,\Lambda)^\omega$ stable under Hecke
$F_I : \mathrm{Rep}(\widehat{G} \rtimes Q)^I \longrightarrow \mathcal{E}nd(\mathcal{C})^{BW^I}$
BW=animated colim of finite sets
Jacobian criterion
local charts
$\pi_b : \mathcal{M}_b \to \mathrm{Bun}_G$
spectral action on $\mathcal{C}$ of Perf $\big(\underbrace{\mathrm{Hom}_{BQ}(BW, \widehat{G} \rtimes Q)}_{\infty\text{-derived-stack}}\big)$
derived stack is underived
spectral action on $\mathcal{C}$ of $\mathrm{Perf}([Z^1(W, \widehat{G})/\widehat{G}])$
$\mathcal{D}_{\text{ét}}(Bun_G,\Lambda)$ compactly generated
spectral action on $\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ of $\mathrm{Perf}([Z^1(W_E, \widehat{G})/\widehat{G}])$
morphism $\mathcal{O}\left(Z^1(W_E, \widehat{G}) /\!\!/ \widehat{G}\right) \to \mathfrak{Z}\left(D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)\right)$
Semi-orthogonal
decomposition
of $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ by
$(D(G_b(E), \Lambda))_{[b] \in B(G)}$
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geometric invariant theory
Mumford numerical criterion
+Richardson
semi-simple Langlands parameters

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PART I

EILENBERG LECTURES: SOME NEW GEOMETRIC STRUCTURES IN THE LANGLANDS PROGRAM

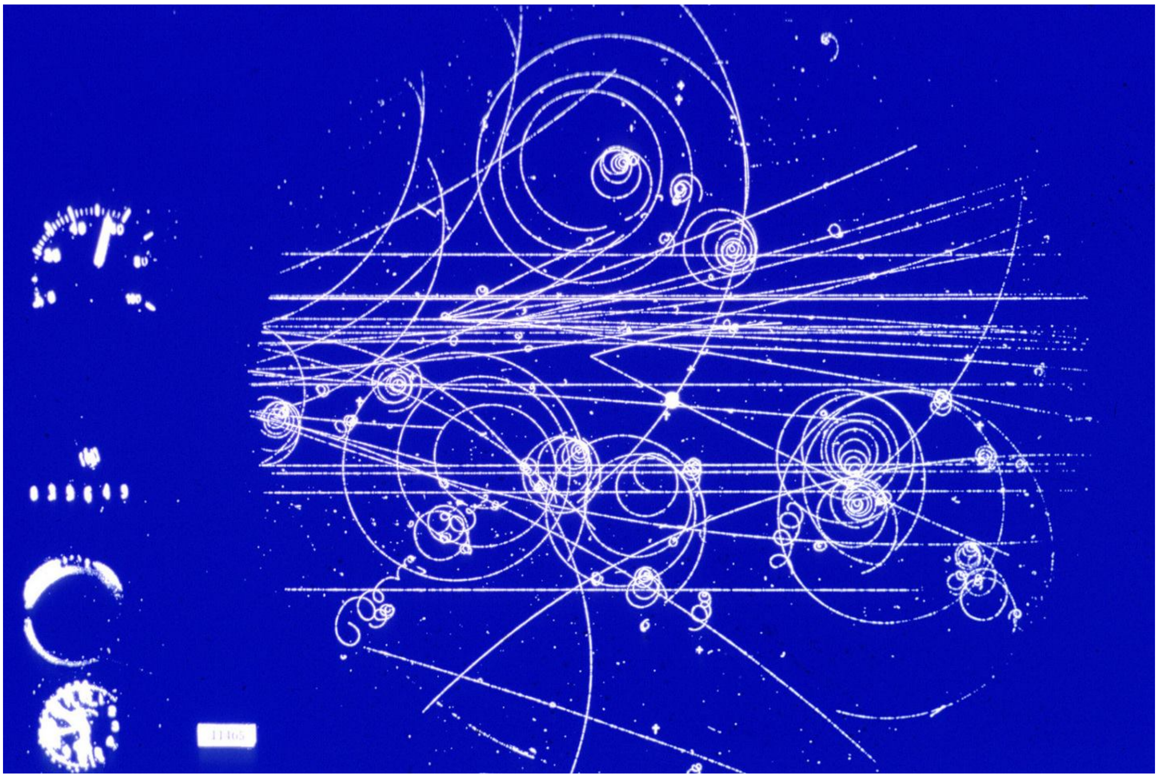


Figure 1. The experimental observation at CERN's LHC of the collision of two primes numbers p and p producing as a sub-product of the fusion some p, ℓ and ∞'s.

Préface

Those are the notes of the Eilenberg lectures given at Columbia university during fall 2024. The author would like to thank Johan de Jong, Michael Harris and Eric Urban for the invitation and attending the lectures. This was a great opportunity to expose this work that spans over 20 years.

LECTURE 1

THE LOCAL LANGLANDS CORRESPONDENCE

In this chapter we present the local Langlands correspondence as stated in the "classical case". We explain at the end its reformulation as it appears in [**58**].

1.1. Notations

We fix a prime number p. We need the following datum:

— E is a finite degree extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$ and uniformizer π.

— We fix an algebraic closure $\overline{E}$ of E and let

$$\Gamma_E = \mathrm{Gal}(\overline{E}|E)$$

and

$$W_E \subset \Gamma_E$$

be the associated Weil group of elements of Γ_E acting as Frob_q^n for some $n \in \mathbb{Z} \subset \widehat{\mathbb{Z}}$ on the residue field.

— G is a reductive group over E.

— We fix some $\ell \neq p$ and consider $\overline{\mathbb{Q}}_\ell$ an algebraic closure of $\mathbb{Q}_\ell$.

We let

$${}^L G = \widehat{G} \rtimes \Gamma_E$$

be *the associated L-group* over $\mathbb{Z}$ (seen as a pro-algebraic group). Here $\widehat{G}$ is a split reductive group over $\mathbb{Z}$ equipped with an action of Γ_E factorizing through the quotient by an open subgroup of Γ_E. We refer to [**18**] for L-groups.

Example 1.1.1. — *1. If $G = T$ is a torus then $\widehat{T} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{G}_m$ with the Γ_E action deduced from the one on $X^*(T)$.*

2. If $G = \mathrm{GL}_{n/E}$ then $\widehat{G} = \mathrm{GL}_n$ with trivial Γ_E action.

3. If $G = \mathrm{SL}_{n/E}$ then $\widehat{G} = \mathrm{PGL}_n$ with trivial Γ_E action.

4. *If $K|E$ is a quadratic extension with Galois group $\{\mathrm{Id}, *\}$, $A \in M_n(K)$ is hermitian non-degenerate, i.e. satisfies ${}^tA^* = A$ and $\det(A) \neq 0$, the associated unitary group G such that*

$$G(E) = \{B \in \mathrm{GL}_n(K) \mid BA\,{}^tB^* = A\}$$

satisfies $\widehat{G} = \mathrm{GL}_n$ with the action of Γ_E factorizing through Gal$(K|E)$, and where the non-trivial element of the Galois group acts as $g \mapsto w\,{}^tg^{-1}w$ where $w = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ \iddots & & & \end{pmatrix}$.

1.2. The local Langlands correspondence: expectations

1.2.1. Smooth representations. — Let Λ be a $\mathbb{Z}\left[\frac{1}{p}\right]$-algebra. Recall the following definition.

Definition 1.2.1. — *A smooth representation of $G(E)$ with coefficients in Λ is a Λ-module M equipped with a linear action of $G(E)$ such that the stabilizer of any vector is open in $G(E)$. We note*

$$\mathrm{Rep}_\Lambda(G(E))$$

for the category of smooth representations with coefficients in Λ.

Let

$$\mathscr{C}(G(E), \Lambda)$$

be the Λ-module of locally constant with compact support functions on $G(E)$ with coefficients in Λ. Let

$$\mathscr{H}_\Lambda(G(E)) = \mathrm{Hom}_\Lambda(\mathscr{C}(G(E), \Lambda), \Lambda)$$

be *the Hecke convolution algebra* of distributions on $G(E)$ with coefficients in Λ that are smooth with compact support. The choice of a Haar measure μ on $G(E)$ with values in $\mathbb{Z}\left[\frac{1}{p}\right]$ defines an isomorphism

$$\begin{array}{ccc} \mathscr{C}(G(E), \Lambda) & \xrightarrow{\sim} & \mathscr{H}_\Lambda(G(E)) \\ f & \longmapsto & f\mu \end{array}$$

where the ring structure on $\mathscr{C}(G(E))$ is now given by

$$(f * g)(x) = \int_{G(E)} f(xy^{-1})g(y)d\mu(y).$$

For each $K \subset G(E)$ an open pro-p subgroup there is associated an *idempotent*

$$e_K \in \mathscr{H}_\Lambda(G(E))$$

given by $\langle e_K, \varphi\rangle = \int_K \varphi$ where, in this formula, the integration on K is with respect to the Haar measure with volume 1. In other words, $e_K = \frac{1}{\mu(K)}\mathbf{1}_K \in \mathscr{C}(G(E), \Lambda)$ via the preceding identification. Then, one has $e_K * e_{K'} = e_K$ if $K \subset K'$ and

$$\mathscr{H}_\Lambda(G(E)) = \bigcup_K \underbrace{e_K * \mathscr{H}_\Lambda(G(E)) * e_K}_{\mathscr{H}_\Lambda(K\backslash G(E)/K)}$$

where $\mathscr{H}_\Lambda(K\backslash G(E)/K)$ is the Hecke algebra of K-bi-invariant distributions on $G(E)$ with compact support.

To any $\pi \in \mathrm{Rep}_\Lambda(G(E))$ with associated Λ-module M_π, one can associate a module over $\mathscr{H}_\Lambda(G(E))$ by setting for $m \in M_\pi$ and $T \in \mathscr{H}_\Lambda(G(E))$,

$$T.m = \int_{G(E)} \pi(g).m \; dT(g).$$

One then has

$$e_K.M_\pi = M_\pi^K$$

as an $\mathscr{H}(K\backslash G(E)/K, \Lambda)$-module. This induces an equivalence

$$\boxed{\{\text{smooth rep. of } G(E) \text{ wt. coeff. in } \Lambda\} \xrightarrow{\sim} \left\{\begin{matrix}\mathscr{H}_\Lambda(G(E))\text{-modules } M \\ \text{s.t. } M = \cup_K e_K.M\end{matrix}\right\}.}$$

One verifies that if Λ is a field and K is compact open with order invertible in Λ this induces a bijection

$$\boxed{\{\pi \in \mathrm{Rep}_\Lambda(G(E)) \text{ irred. s.t. } \pi^K \neq 0\}/\sim \xrightarrow{\sim} \left\{\begin{matrix}\text{irreducible} \\ \mathscr{H}_\Lambda(K\backslash G(E)/K)\text{ -modules}\end{matrix}\right\}/\sim}$$

We refer to [**119**], [**12**] and [**11**] for the basics of smooth representations of p-adic groups.

Later in this text we will consider

$$D(G(E), \Lambda)$$

the derived category of smooth representations of $G(E)$ with coefficients in Λ. The category $\mathrm{Rep}_\Lambda(G(E))$ has enough injective and projective objects. For projective objects it suffices to consider the collection

$$\Big(\underbrace{\text{c-ind}_K^{G(E)} \Lambda}_{\text{compact induction}} \Big)_K$$

where K goes through the set of compact open pro-p subgroups of $G(E)$. For the injective objects it suffices to consider the collection

$$\Big(\underbrace{\mathrm{Ind}_{\{1\}}^{G(E)} M}_{\text{smooth induction}}\Big)_M$$

where M goes through the set of injective Λ-modules. When Λ is a characteristic zero field the category $\mathrm{Rep}_\Lambda(G(E))$ has finite cohomological dimension, see [**127**] where this is deduced from the contractibility of the Bruhat-Tits building.

1.2.2. Langlands parameters. — The local Langlands correspondence seeks to attach to any *irreducible* $\pi \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G(E))$ a *Langlands parameter*

$$\boxed{\varphi_\pi : W_E \longrightarrow {}^L G(\overline{\mathbb{Q}}_\ell).}$$

Here the terminology "Langlands parameter" means

- that the composite of φ_π with the projection to Γ_E is the canonical inclusion $W_E \subset \Gamma_E$ i.e. φ_π is given by a 1-cocycle $W_E \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$,
- that moreover this cocycle takes values in $\widehat{G}(L)$ where L is a finite degree extension of $\mathbb{Q}_\ell$,
- that this cocycle with values in $\widehat{G}(L)$ is continuous.

Remark 1.2.2. — *There's a way to make this notion of Langlands parameter independent of the choice of the ℓ-adic topology. In fact, Grothendieck's ℓ-adic monodromy theorem ("any ℓ-adic representation is potentially semi-stable", see* [**35**, Theorem 8.2]*) applies in this context and a Langlands parameter*

$$\varphi : W_E \to {}^L G(\overline{\mathbb{Q}}_\ell)$$

as before is in fact the same as a couple (ρ, N) where

- *$\rho : W_E \to {}^L G(\overline{\mathbb{Q}}_\ell)$ is a Langlands parameter that is trivial on an open sub-group of W_E,*
- *$N \in \mathfrak{g}_{\overline{\mathbb{Q}}_\ell}(-1)$ is nilpotent and satisfies: $\forall \tau \in W_E,\ \mathrm{Ad}\,\rho(\tau).N = q^{v(\tau)}N$ where τ acts as $\mathrm{Frob}_q^{v(\tau)}$ on the residue field.*

The couples

$$(\rho, N)$$

are the so-called **Weil-Deligne parameters**. *There is a 1-cocycle*

$$t_\ell : W_E \to \mathbb{Z}_\ell(1)$$

sending τ to $\big(\tau(\pi^{1/\ell^n})/\pi^{1/\ell^n}\big)_{n\geq 1}$. The correspondence sends (ρ, N) to the parameter φ such that for $\tau \in W_E$,

$$\varphi(\tau) = \rho(\tau)\exp(t_\ell(\tau)N) \rtimes \tau.$$

Nevertheless, since we fix a prime number ℓ in our work with Scholze we prefer to give a formulation using the ℓ-adic topology. This is justified by the fact that we construct such parameters over $\overline{\mathbb{F}}_\ell$ too and *our correspondence is compatible with mod ℓ reduction.*

Remark 1.2.3. — *There is a p-adic local Langlands program too for which peoples look at the case $\ell = p$, see* [**22**] *for example. We only look at the case $\ell \neq p$ here which is the case of the "classical" local Langlands correspondence.*

One last remark: φ_π *is only defined up to $\widehat{G}(\mathbb{Q}_\ell)$-conjugation* i.e. we see it as an element of $H^1(W_E, \widehat{G}(\overline{\mathbb{Q}}_\ell))$. Up to now the local Langlands correspondence is a map

$$\boxed{\mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G(E))/\sim \;\longrightarrow\; \left\{\varphi : W_E \to {}^L G(\overline{\mathbb{Q}}_\ell)\right\}/\widehat{G}(\overline{\mathbb{Q}}_\ell)}$$

i.e. a map between isomorphism classes of object. We will later see *this correspondence has some categorical flavors* (and this is quite important since at the end we formulate a real categorical local Langlands correspondence with Scholze) but up to now we deal with *objects up to isomorphisms.*

1.2.3. What to expect from the local Langlands correspondence.

— Here is what we expect from the local Langlands correspondence.

1. **Frobenius semi-simplicity** First, there is one condition on φ_π: this has to be *Frobenius semi-simple* in the sense that the associated couple (ρ, N) has to be such that for all τ, $\rho(\tau)$ is semi-simple (i.e. $\rho(\tau)$ is semi-simple for a τ satisfying $v(\tau) = 1$).

2. **Finiteness of the L-packets** The fibers of $\{\pi\} \mapsto \{\varphi_\pi\}$ are finite: those are the so-called *L-packets.*

3. **Description of the image** When G is quasi-split the correspondence
$$\{\pi\} \mapsto \{\varphi_\pi\}$$
should be surjective. For other groups G, there is the so-called relevance condition: a (Frobenius semi-simple) parameter
$$\varphi : W_E \to {}^L G(\overline{\mathbb{Q}}_\ell)$$
is isomorphic to some φ_π if and only if as soon as φ factorizes (up to $\widehat{G}(\overline{\mathbb{Q}}_\ell)$-conjugacy) through some parabolic subgroup ${}^L P(\overline{\mathbb{Q}}_\ell)$ where P is a parabolic subgroup of G^* then P transfers to G. We refer to [**18**, section 3] for the notion of a relevant parabolic subgroup.

   For example: if $G = D^\times$ where D is a central division algebra over E with $[D : E] = n^2$ then a Langlands parameter
$$\varphi : W_E \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell) = \widehat{G}(\overline{\mathbb{Q}}_\ell)$$
is relevant if and only if φ, as a linear representation of W_E, is indecomposable.

4. **Compatibility with local class field theory** ([**18**, Section 9]) If $G = T$ is a torus, class field theory gives an isomorphism of groups
$$\mathrm{Hom}(T(E), \overline{\mathbb{Q}}_\ell^\times) \xrightarrow{\sim} H^1(W_E, {}^L T(\overline{\mathbb{Q}}_\ell))$$
this has to be the local Langlands correspondence for tori. Typically, when T is a spli torus, there is an Artin reciprocity isomorphism
$$T(E) \xrightarrow{\sim} W_E^{ab} \otimes_{\mathbb{Z}} X_*(T)$$
deduced from
$$\mathrm{Art}_E : E^\times \xrightarrow{\sim} W_E^{ab},$$
and this isomorphism induces the local Langlands correspondence for T.

5. **Compatibility with the unramified local Langlands correspondence (Satake isomorphism)** A good reference for the Satake isomorphism is [**70**]. If G is unramified, K is hyperspecial, after the choice of a square root of q in $\overline{\mathbb{Q}}_\ell$, there is a *Satake isomorphism* given by a *constant term map*
$$\mathscr{H}_{\overline{\mathbb{Q}}_\ell}(K\backslash G(E)/K) \xrightarrow{\sim} \mathscr{H}_{\overline{\mathbb{Q}}_\ell}(T(\mathcal{O}_E)\backslash T(E)/T(\mathcal{O}_E))^W$$
where T is an unramified torus coming from an integral model associated to the choice of K. If $A \subset T$ is the maximal split torus inside T then
$$\mathscr{H}_{\overline{\mathbb{Q}}_\ell}(T(\mathcal{O}_E)\backslash T(E)/T(\mathcal{O}_E))^W = \mathscr{H}_{\overline{\mathbb{Q}}_\ell}(A(\mathcal{O}_E)\backslash A(E)/A(\mathcal{O}_E))^W$$
that is identified with
$$\overline{\mathbb{Q}}_\ell[X_*(A)]^W = \overline{\mathbb{Q}}_\ell\big[X^*(\widehat{A})\big]^W.$$
If π is such that $\pi^K \neq 0$ then the irreducible module π^K over the spherical Hecke algebra thus defines a character
$$\overline{\mathbb{Q}}_\ell\big[X^*(\widehat{A})\big]^W \longrightarrow \overline{\mathbb{Q}}_\ell$$
that is to say an element of $\widehat{A}(\overline{\mathbb{Q}}_\ell)/W$. One can prove that this is the same as an element of
$$\big\{\text{unramified (semi-simple) } \varphi : W_E/I_E \to {}^L G(\overline{\mathbb{Q}}_\ell)\big\} \,/\, \widehat{G}(\overline{\mathbb{Q}}_\ell),$$
see [**18**, Section 7]. We ask that this this correspondence is our local Langlands correspondence for unramified representations.

6. **Compatibility with Kazhdan-Lusztig depth 0 local Langlands**

   If G is split and I is an Iwahori subgroup of $G(E)$ then the category
$$\mathrm{Rep}^I_{\overline{\mathbb{Q}}_\ell}(G(E))$$
of $\pi \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G(E))$ generated by π^I form a block in $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G(E))$ in the sense that there is an indecomposable idempotent e in the Bernstein center of $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G(E))$ such that
$$e.\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G(E)) = \mathrm{Rep}^I_{\overline{\mathbb{Q}}_\ell}(G(E)).$$

This is the so-called central block, see [**17**] for the beginning of this story. This category is then identified with the category of modules over the Iwahori-Hecke algebra

$$\mathscr{H}(I\backslash G(E)/I).$$

The identification of this Iwahori-Hecke algebra with the equivariant K-theory of the Steinberg variety has allowed Kazhdan and Lusztig to give a parametrization of irreducible $\mathscr{H}(I\backslash G(E)/I)$-modules as couples

$$(s, N)$$

where $s \in \widehat{G}(\overline{\mathbb{Q}}_\ell)$ is semi-simple and $N \in \widehat{\mathfrak{g}}_{\overline{\mathbb{Q}}_\ell}$ is nilpotent and satisfies $\mathrm{Ad}(s).N = qN$, see [**83**]. We ask that this is the local Langlands correspondence in this case.

7. **Compatibility up to semi-simplification with parabolic induction**

   We say a parameter φ is semi-simple if the associated Weil-Deligne Langlands parameter (ρ, N) is such that $N = 0$. Equivalently, $\varphi_{|I_E}$ is trivial on an open subgroup. For a parameter φ we can define

$$\varphi^{ss}$$

its semi-simplification: if φ corresponds to (ρ, N) then φ^{ss} corresponds to $(\rho, 0)$. Then, if P is a parabolic subgroup with Levi quotient M we ask the following: for π an irreducible smooth representation of $M(E)$, if π' is an irreducible subquotient of the finite length representation

$$\mathrm{Ind}_{P(E)}^{G(E)} \pi$$

(parabolic induction), then

$$\varphi_{\pi'}^{ss}$$

is the composite of φ_π^{ss} with the inclusion ${}^L M(\overline{\mathbb{Q}}_\ell) \hookrightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$.

Let us remark that, of course, this is false without the semi-simplification since the Steinberg representation of $\mathrm{GL}_n(E)$ and the trivial one do not have the same Langlands parameters.

8. **Categorical flavor: description of supercuspdial L-packets**

   We are now introducing some categorical flavor inside the Langlands parameters: we are not looking at the set quotient

$$\{\varphi : W_E \to {}^L G(\overline{\mathbb{Q}}_\ell)\}/\widehat{G}(\overline{\mathbb{Q}}_\ell)$$

but *the quotient as a groupoid*

$$\boxed{\Big[\big\{\varphi : W_E \to {}^L G(\overline{\mathbb{Q}}_\ell)\big\} \,/\, \widehat{G}(\overline{\mathbb{Q}}_\ell)\Big],}$$

and thus

$$\big\{\varphi : W_E \to {}^L G(\overline{\mathbb{Q}}_\ell)\big\} / \widehat{G}(\overline{\mathbb{Q}}_\ell) = \pi_0 \Big[\big\{\varphi : W_E \to {}^L G(\overline{\mathbb{Q}}_\ell)\big\} / \widehat{G}(\overline{\mathbb{Q}}_\ell)\Big].$$

Suppose G is quasi-split (we will see later, following the work of Vogan, Kottwitz and Kaletha what to do in the non-quasi-split case). For a parameter φ we define

$$S_\varphi = \{g \in \widehat{G}(\overline{\mathbb{Q}}_\ell) \mid g\varphi g^{-1} = \varphi\}.$$

This is the automorphism group of φ in the preceding groupoid. There is always an inclusion

$$Z(\widehat{G})(\overline{\mathbb{Q}}_\ell)^{\Gamma_E} \subset S_\varphi.$$

We say that φ *is cuspidal* if it is semi-simple and $S_\varphi/Z(\widehat{G})(\overline{\mathbb{Q}}_\ell)^{\Gamma_E}$ is finite. We say a packet is supercuspidal if all of its elements are supercuspidal. Then

$$\boxed{\{\text{supercuspidal L-packets}\} \xrightarrow{\sim} \{\varphi : W_E \to {}^L G(\overline{\mathbb{Q}}_\ell) \text{ cuspidal }\}/\widehat{G}(\overline{\mathbb{Q}}_\ell).}$$

Moreover, the choice of a Whittaker datum defines a bijection for φ a cuspidal parameter

$$\boxed{\mathrm{Irr}(\underbrace{S_\varphi/Z(\widehat{G})(\overline{\mathbb{Q}}_\ell)^{\Gamma_E}}_{\text{finite group}})/\sim \xrightarrow{\sim} \text{L-packet associated to } \varphi}$$

where the trivial representation should correspond to the unique generic (with respect to the choice of the Whittaker datum) representation of the L-packet.

This phenomenon of L-packets already shows up in the unramified case. The choice of a Whittaker datum fixes a conjugacy class of hyperspecial subgroup. One this hyperspecial subgroup is fixed the characters of the finite abelian group $\pi_0(S_\varphi/Z(\widehat{G})^{\Gamma_E})$ are in natural bijection with the set of conjugacy classes of compact hyperspecial subgroups, see [**110**] for example.

9. Local global compatibility

Let K be a number field and Π be an algebraic automorphic representation of G where now G is a reductive group over K, see [**30**] and [**23**]. Conjecturally, Π_f is defined over a number field as a smooth representation of $G(\mathbb{A}_f)$. Let us fix an embedding of this number field inside $\overline{\mathbb{Q}}_\ell$. Then one should be able to attach to Π an ℓ-adic Langlands parameter

$$\varphi_\Pi : \mathrm{Gal}(\overline{K}|K) \longrightarrow {}^L G(\overline{\mathbb{Q}}_\ell).$$

For a place v of K dividing $p \neq \ell$,

$$\varphi_{\Pi \,|\, W_{K_v}}$$

depends only on Π_v and is given up to conjugation by

$$\varphi_{\Pi_v}.$$

1.3. Background on the global Langlands correspondence and global Langlands parameters

Let G be a reductive group over a number field K. Let Π be an automorphic representation of G i.e. an irreducible sub-quotient of the space of automorphic forms on G. As an abstract representation

$$\Pi \simeq \bigotimes_v \Pi_v$$

where v goes through the places of K. If $v|\infty$, the local Langlands correspondence is known for Π_v ([**94**]), and we can define

$$\varphi_{\Pi_v} : W_{K_v} \longrightarrow {}^L G_{\mathbb{C}}.$$

There is a natural morphism

$$\mathbb{C}^\times \longrightarrow W_{K_v}$$

that is an isomorphism if $K_v \simeq \mathbb{C}$ and fits into a non-split exact sequence

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow W_{K_v} \longrightarrow \mathrm{Gal}(\mathbb{C}|\mathbb{R}) \longrightarrow 1$$

if $K_v \simeq \mathbb{R}$.

Definition 1.3.1. — *An automorphic representation* Π *of* G *is algebraic if for all* $v|\infty$, $\varphi_{\Pi_v|\mathbb{C}^\times} : \mathbb{C}^\times \longrightarrow \widehat{G}(\mathbb{C})$ *is algebraic i.e. is given by an algebraic morphism* $\mathbb{S}_{\mathbb{C}} \to \widehat{G}_{\mathbb{C}}$ *where* $\mathbb{S}$ *is Deligne's torus* $\mathrm{Res}_{\mathbb{C}|\mathbb{R}}\,\mathbb{G}_m$ *via the inclusion* $\mathbb{C}^\times = \mathbb{S}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{C})$.

It is the same as to ask that for all $v|\infty$, Π_v has the same infinitesimal character as the one of an algebraic irreducible finite dimension representation of the algebraic group $G_{\overline{K_v}}$ with coefficients in $\mathbb{C}$.

Conjecturally, there exists a global Langlands group

$$\mathscr{L}_K$$

that is a locally compact topological group sitting in an exact sequence

$$1 \longrightarrow \mathscr{L}_K^\circ \longrightarrow \mathscr{L}_K \longrightarrow \mathrm{Gal}(\overline{K}|K) \longrightarrow 1$$

and with an identification

$$\boxed{\mathscr{L}_K/(\mathscr{L}_K^\circ)' = W_K}$$

the global Weil group ([**135**], [**3**]). Let us note that a candidate for $\mathscr{L}_K$ has been proposed in [**2**]. Moreover, on expects the following. In the following conjecture a continuous representation ρ of $\mathscr{L}_K$ on a finite dimensional $\mathbb{C}$-vector space is said to be algebraic if for each $v|\infty$, the composite of ρ with $\mathbb{C}^\times \to W_{K_v} \to \mathscr{L}_K$ is an algebraic representation of Deligne's torus $\mathbb{S}$.

Conjecture 1.3.2. — *The following is expected:*

1. *To each automorphic representation* Π *of* G *one can associate a Langlands parameter*
$$\varphi_\Pi : \mathscr{L}_K \longrightarrow {}^L G_{\mathbb{C}}$$
compatibly with the local Langlands correspondence at archimedean places and the unramified one at almost all finite places
2. *If* Π *is algebraic then* Π_f *is defined over a number field inside* $\mathbb{C}$ *and to the choice of an embedding of such a number field inside* $\overline{\mathbb{Q}}_\ell$ *is associated an* ℓ*-adic Langlands parameter*
$$\varphi_{\Pi,\ell} : Gal(\overline{K}|K) \longrightarrow {}^L G_{\overline{\mathbb{Q}}_\ell}$$
3. *The Tannakian category of continuous representations of* $\mathscr{L}_K$ *on finite dimensional* $\mathbb{C}$*-vector spaces that are algebraic is identified with the category of Grothendieck motives for numerical equivalence with* $\mathbb{C}$*-coefficients.*

This is known for tori when we consider the category of CM-motives for absolute Hodge cycles. In fact, this is a consequence of the identification of the Taniyama group with the motivic Galois group of the category of CM motives equipped with absolute Hodge cycles, see [**36**].

The construction of the ℓ-adic Langlands parameters is know for regular algebraic automorphic representations of GL_n over a totally real or CM field, see [**71**] and [**130**]. Other cases are known using the cohomology of Shimura varieties, see for example [**89**].

For example, if $f = \sum_{n\geq 1} a_n q^n$ is a normalized weight $k \geq 1$ holomorphic modular form for $\Gamma_0(N)$ that is new and an Hecke eigenvector of the Hecke operators $(T_p)_{p\not| N}$ then one can associate (Shimura, Deligne ([**34**]), Deligne-Serre ([**37**]) a Galois representation

$$\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$$

such that for $p \not| N$, that characteristic polynomial of $\rho_f(\mathrm{Frob}_p)$ is $X^2 - a_p X + p^{k-1}$.

1.4. What we do

We prove the following theorem in [**58**].

***Theorem 1.4.1* (F.-Scholze).** — *For ℓ a good prime with respect to G (any ℓ if $G = \mathrm{GL}_n$, $\ell \neq 2$ for classical groups) there exists a monoidal action of the category of perfect complexes*

$$\mathrm{Perf}(\mathrm{LocSys}_{\widehat{G}/\overline{\mathbb{Z}}_\ell})$$

on

$$D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$$

where $\mathrm{LocSys}_{\widehat{G}} \to Spec(\mathbb{Z}\big[\frac{1}{p}\big])$ *is the moduli space of Langlands parameter, an algebraic stack locally complete intersection of dimension* 0 *over* $Spec(\mathbb{Z}\big[\frac{1}{p}\big])$.

As a consequence of the preceding theorem we can construct the semi-simple local Langlands correspondence

$$\pi \mapsto \varphi_\pi^{ss}$$

for any reductive group over E, over $\overline{\mathbb{F}}_\ell$ and $\overline{\mathbb{Q}}_\ell$ (and compatibly with mod ℓ reduction). We will explain later the

As for now the statement of the local Langlands conjecture is the following.

***Conjecture 1.4.2* (Categorical local Langlands)**
Suppose G is quasi-split and fix a Whittaker datum (B, ψ). Suppose ℓ is a good prime. There exists an equivalence of stable ∞-categories

$$\mathcal{D}^b_{coh}(\mathrm{LocSys}_{\widehat{G}/\overline{\mathbb{Z}}_\ell})_{nilp.ss.supp} \xrightarrow{\sim} \mathcal{D}_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)^\omega$$

compatible with the preceding spectral action and sending the structural sheaf $\mathcal{O}$ to the Whittaker sheaf.

The goal of this text is to explain how after 20 years of work, starting from the classical local Langlands correspondence in terms of parameters of smooth irreducible representations as in the work of Harris-Taylor, we arrived at such a statement and what are those geometric objects showing up in the preceding statement, starting with the so-called Lubin-Tate spaces continuing with Rapoport-Zink spaces, Hodge-Tate periods, the curve and so on.

LECTURE 2

SHIMURA VARIETIES, GALOIS REPRESENTATIONS, AND THE WORK OF HARRIS-TAYLOR

2.1. Introduction

The problem of the following chapter is the following: construct the local Langlands correspondence for a given group using local-global compatibility coupled with some known cases of the global construction of ℓ-adic parameters via the cohomology of Shimura varieties.

More precisely, if $\Pi \simeq \bigotimes_v \Pi_v$ is a cohomological automorphic representation of the reductive group G defined over a number field $\mathbb{Q}$ and

$$\Pi \longmapsto r_\mu \circ \varphi_{\Pi|\mathrm{Gal}(\overline{\mathbb{Q}}|L)}$$

via the cohomology of Shimura varieties where

— $\varphi_\Pi : \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \to {}^L G(\overline{\mathbb{Q}}_\ell)$ is the expected global ℓ-adic parameter,

— L is the reflex field associated to the Shimura variety, a number field inside $\mathbb{C}$,

— $r_\mu \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\widehat{G} \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}|L))$ is an algebraic representation associated to our Shimura datum,

one expects that for $p \neq \ell$,

$$\varphi_{\Pi|W_{\mathbb{Q}_p}} = \varphi_{\Pi_p}$$

and thus, if $v|p$ is a place of L associated to the choice of an embedding f L inside $\overline{\mathbb{Q}}_p$,

$$\boxed{r_\mu \circ \varphi_{\Pi|W_{L_v}} = r_\mu \circ \varphi_{\Pi_p|W_{L_v}}}$$

Remark 2.1.1. — *1. By definition, a cohomological automorphic representation is a particular type of algebraic automorphic representation that shows up in the cohomology of locally symmetric spaces. For example, for* GL_2*, the automorphic representation associated to an holomorphic modular form of weight* $k \geq 1$ *is algebraic but cohomological only when* $k \geq 2$*. The* ℓ*-adic Langlands parameter associated to a weight* ≥ 2 *holomorphic modular forms is obtained inside the* *intersection cohomology cohomology of modular curves with coefficients in some local systems (Shimura, Deligne).*

For weight 1 holomorphic modular forms this ℓ-adic Langlands parameter is obtained by ℓ-adic interpolation from the weight ≥ 2 case (Deligne-Serre).

There is another class of automorphic representation of $\mathrm{GL}_{2/\mathbb{Q}}$ that are algebraic but not cohomological: the one associated to non-holomorphic Maass forms f that satisfy $\Delta f = \frac{1}{4} f$ where $\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ is the hyperbolic Laplacian. We do not know how to construct their ℓ-adic Langlands parameter.

2. *Suppose that $G_{\mathbb{R}}$ has discrete series, that is to say $G_{\mathbb{R}}$ is an inner form of its compact form. This is for example the case if G can be enhanced to a Shimura datum. One can prove that one can globalize any supercuspdial representation of $G(\mathbb{Q}_p)$ to an automorphic representation Π such that Π_∞ is a discrete series representation (*[**29**, Theorem 1B]*). Those are cohomological and show up in middle degree in the cohomology of locally symmetric spaces.*
3. *One of the difficulties of the preceding approach is that we can not construct φ_Π but its composition with r_μ where r_μ is a very particular type representation of the Laglands dual since μ is minuscule. This difficulty is removed over function fields over $\mathbb{F}_q$ using general Shtuka moduli spaces but we don't know, even for* GL_2*, how to define Shimura varieties for non-minuscule μ. We will see later how to remove this difficulty for local Shimura varieties at p as diamonds.*

We would like to use this type of formula to define φ_{Π_v} after choosing suitable Shimura data giving rise to different representations r_μ, This leads to the question: why, after composing with r_μ, would $\varphi_{\Pi|W_{K_v}}$ depend only on Π_v? This is the problem of *local-global compatibility.* The answer is that there are *local Shimura varieties* linked to the global one via a process of p-adic uniformization.

Remark 2.1.2. — *The use of the local-global compatibility is common in the domain. Let us cite for example the proof of the fundamental lemma by Ngô (*[**112**]*) that uses a globalization to a smooth projective curve over a finite field or the proof of the arithmetic fundamental lemma (*[**143**]*).*

2.2. Shimura varieties

2.2.1. Hermitian symmetric spaces. — Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}|\mathbb{R}}\,\mathbb{G}_m$ be Deligne's torus. Recall the the Tannakian category of real Hodge structures is equivalent to $\mathrm{Rep}_{\mathbb{R}}(\mathbb{S})$.

Let

$$(G, \{h\})$$

be a couple where

1. G reductive group over $\mathbb{R}$,
2. $h : \mathbb{S} \to G$ with $G(\mathbb{R})$-conjugacy class $\{h\}$.

This is the same as the datum of G together with a faithful $\otimes$-functor

$$\mathrm{Rep}(G) \longrightarrow \mathbb{R}\text{-Hodge structures},$$

i.e. *a* G-$\mathbb{R}$-*Hodge structure*, such that the composite

$$\mathrm{Rep}(G) \longrightarrow \mathbb{R}\text{-Hodge structures} \xrightarrow{can} \mathrm{Vect}_{\mathbb{R}}$$

is isomorphic to the canonical fiber functor on $\mathrm{Rep}\, G$.

We note $\mu_h : \mathbb{G}_{m/\mathbb{C}} \to G_{\mathbb{C}}$ for the composite of $h_{\mathbb{C}}$ with $z \mapsto (z,1)$ from $\mathbb{G}_{m/\mathbb{C}} \to \mathbb{S}_{\mathbb{C}} = \mathbb{G}_{m/\mathbb{C}} \times \mathbb{G}_{m/\mathbb{C}}$. This defines the Hodge filtration.

Hypothesis:

1. (**Weight 0 adjoint Hodge structure**) $w_h : \mathbb{G}_m \to G$, obtained by composing h with the morphism $\mathbb{G}_m \to \mathbb{S}$ inducing $\mathbb{R}^\times \hookrightarrow \mathbb{C}^\times$ on the $\mathbb{R}$-points, is central that is to say the Hodge structure $(\mathfrak{g}, \mathrm{Ad} \circ h)$ is pure of weight 0.
2. (**Polarization**) Conjugation by $h(i)$ is a Cartan involution on G_{ad} that is to say the Killing form on $\mathfrak{g}_{ad}$ defines a polarization of the weight 0 Hodge structure $(\mathfrak{g}_{ad}, \mathrm{Ad} \circ h)$.
3. (**Griffiths transversality**) $\mu_h : \mathbb{G}_{m/\mathbb{C}} \to G_{\mathbb{C}}$ is minuscule that is to say the weights of $\mathrm{Ad} \circ \mu_h$ on $\mathfrak{g}_{\mathbb{C}}$ are in $\{-1, 0, 1\}$ that is to say the Hodge structure $(\mathfrak{g}_{\mathbb{R}}, \mathrm{Ad} \circ h)$ is of type $(-1,1), (1,-1), (0,0)$.

Under those hypothesis, if K_∞ is the centralizer of $h(i)$ in $G(\mathbb{R})$, a sub-group of $G(\mathbb{R})$ that is compact modulo the center,

$$X = G(\mathbb{R})/K_\infty$$

is an hermitian symmetric space. More precisely, if $\mathcal{F}$ is the complex analytic flag manifold defined by μ_h, the map

$$X \longrightarrow \mathcal{F}$$

that sends some h',$G(\mathbb{R})$-conjugate to h, to the class of $\mu_{h'}$ is an open embedding,

$$X \underset{\text{open}}{\subset} \mathcal{F}.$$

Furthermore, X is a moduli space of rigidified variations of Hodge structures equipped with an additional G-structure.

More precisely, if S is a smooth complex analytic space then $X(S)$ is the set of equivalence classes of $(\mathscr{F}, \mathrm{Fil}^\bullet \mathscr{F} \otimes_{\mathbb{R}} \mathcal{O}_S, \eta)$ where

— $\mathscr{F} : \mathrm{Rep}\ G \to \{\mathbb{R}$ local systems on S$\}$ is a $\otimes$-functor,

— $\mathrm{Fil}^\bullet \mathscr{F} \otimes_{\mathbb{R}} \mathcal{O}_S$ is a finite decreasing filtration of the $\otimes$-functor

$$\mathscr{F} \otimes_{\mathbb{R}} \mathcal{O}_S : \mathrm{Rep}\ G \longrightarrow \{\text{vector bundles on } S\}$$

satisfying Griffiths transversality: if $\nabla = \mathrm{Id} \otimes d$ then $\nabla \mathrm{Fil}^k \subset \mathrm{Fil}^{k-1} \otimes \Omega^1_S$

— for each $\mathbb{R}$-linear representation (V,ρ) of G and $s \in S$, the complex conjugate of the associated filtration of $V_{\mathbb{C}}$ is $\rho \circ w_h$-opposite to the filtration of $V_{\mathbb{C}}$ and thus defines a weight $\rho \circ w_h$ Hodge structure,

— η is an isomorphism between tensor functors between $\mathscr{F}$ and the canonical functor $(V,\rho) \mapsto \underline{V}$,

— We ask that for each $s \in S$, the associated morphism $\mathbb{S} \to G$ defined by taking the stalk at s of the preceding variation is $G(\mathbb{R})$-conjugated to h.

Thus, X *is a moduli space of Hodge structures.* We will see later that we can define moduli spaces of p-adic Hodge structures using the curve. But we are first going to treat a particular case: *Lubin-Tate spaces.*

2.2.2. Shimura varieties. — Let us begin by recalling the definition of a Shimura datum ([**38**], [**39**], [**104**], [**107**], [**108**], [**64**]).

Shimura datum:

1. G is a reductive group over $\mathbb{Q}$.
2. $h : \mathbb{S} \to G_{\mathbb{R}}$.

Hypothesis:

1. (**Weight** 0 **adjoint Hodge structure**) $w_h : \mathbb{G}_m \to G_{\mathbb{R}}$, obtained by composing h with the morphism $\mathbb{G}_m \to \mathbb{S}$ inducing $\mathbb{R}^\times \hookrightarrow \mathbb{C}^\times$ on the $\mathbb{R}$-points, is central that is to say the Hodge structure $(\mathfrak{g}_{\mathbb{R}}, \mathrm{Ad} \circ h)$ is pure of weight 0.
2. (**Polarization**) Conjugation by $h(i)$ is a Cartan involution on $G_{\mathbb{R},ad}$ that is to say the Killing form on $\mathfrak{g}_{ad}$ defines a polarization of the weight 0 Hodge structure $(\mathfrak{g}_{\mathbb{R},ad}, \mathrm{Ad} \circ h)$.
3. (**Griffiths transversality**) μ_h is minuscule that is to say the Hodge structure $(\mathfrak{g}_{\mathbb{R}}, \mathrm{Ad} \circ h)$ is of type $(-1,1),(1,-1),(0,0)$.
4. (**Density of CM points**) For any simple $\mathbb{Q}$-factor H of G_{ad}, $H(\mathbb{R})$ is not compact.

Example 2.2.1. — *1.* $G = \mathrm{GL}_2$ *and* $h(a+ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. *This is the modular curves case.*

2. Same as before but $G = D^\times$ *with* D *a quaternion division algebra over* $\mathbb{Q}$. *This is the case of Shimura curves.*

3. $G = \mathrm{Gsp}_{2n}$ *associated with the symplectic form* $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. *Set* $h(a+ib) = \begin{pmatrix} aI_n & bI_n \\ -bI_n & aI_n \end{pmatrix}$. *This is the case of Siegel modular varieties (modular curves for* $n = 1$*).*

4. *Let K be a CM field and B be a central simple algebra over K equipped with an involution $*$ inducing complex conjugation on K. Let $G = GU(D, *)$ be the associated similitude unitary group. Let's fix an isomorphism*

$$G_{\mathbb{R}} \simeq G\left(\prod_{\tau\in\Phi} U(p_\tau, q_\tau)\right)$$

where $(p_\tau, q_\tau)_{\tau\in\Phi}$ is a set of signatures index by a CM type Φ of K. Then if $h(z) = (h_\tau(z))_{\tau\in\Pi}$ with $h_\tau(z) = \mathrm{diag}(\underbrace{z, \dots, z}_{p_\tau}, \underbrace{\overline{z}, \dots, \overline{z}}_{q_\tau})$ this defines a unitary type Shimura variety.

5. *All the preceding cases are particular cases of PEL type Shimura varieties, see* [**88**].

As a complex analytic space, the Shimura variety associated to the preceding datum is

$$\mathrm{Sh}_K = G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)/K)$$

for $K \subset G(\mathbb{A}_f)$ compact open "sufficiently small". Writing $G(\mathbf{A}_f) = \coprod_{i\in I} G(\mathbb{Q})g_iK$ with I finite (finiteness of the class number), one has

$$\mathrm{Sh}_K = \coprod_{i\in I} \Gamma_i\backslash X$$

where $\Gamma_i = G(\mathbb{Q}) \cap g_iKg_i^{-1}$ is an arithmetic subgroup of $G(\mathbb{R})$.

The smooth complex analytic space Sh_K has an interpretation as a moduli of variations of $\mathbb{Q}$-Hodge structures equipped with a G-structure. To be more precise, the natural moduli space is not Sh_K but

$$\coprod_{\ker^1(\mathbb{Q},G)} \mathrm{Sh}_K$$

a finite disjoint union of copies Sh_K where $\ker^1(\mathbb{Q}, G)$ is a finite group measuring the obstruction to the Hasse principle for G (see [**88**] for the PEL case). More precisely, if S is a smooth complex analytic space then $\coprod_{\ker^1(\mathbb{Q},G)} \mathrm{Sh}_K(S)$ is the set of equivalence classes of $(\mathscr{F}, \mathrm{Fil}^\bullet \otimes_{\mathbb{Q}} \mathcal{O}_S, \overline{\eta})$ where

— $\mathscr{F} : \mathrm{Rep}\ G \to \{\mathbb{Q} - \text{local systems on } S\}$ is a $\otimes$-functor,

— $\mathrm{Fil}^\bullet\mathscr{F} \otimes_{\mathbb{Q}} \mathcal{O}_S$ is a finite filtration of the $\otimes$-functor

$$\mathscr{F} \otimes_{\mathbb{R}} \mathcal{O}_S : \mathrm{Rep}\ G \longrightarrow \{\text{vector bundles on } S\}$$

satisfying Griffiths transversality: if $\nabla = \mathrm{Id} \otimes d$ then $\nabla \mathrm{Fil}^k \subset \mathrm{Fil}^{k-1} \otimes \Omega^1_S$,

— for each $\mathbb{R}$-linear representation (V, ρ) of G and $s \in S$, the complex conjugate of the associated filtration of $V_{\mathbb{C}}$ is $\rho \circ w_h$-opposite to the filtration of $V_{\mathbb{C}}$ and thus defines a weight $\rho \circ w_h$ Hodge structure,

— for each $s \in S$, the associated functor $\mathrm{Rep}\, G_{\mathbb{R}} \to \mathrm{Vect}_{\mathbb{R}}$ obtained by taking the stalk at s is trivial and the associated $G_{\mathbb{R}}$-Hodge structure is in the $G(\mathbb{R})$-conjugacy class of h,

— $\bar{\eta}$ is a K^p-orbit of trivialization $\eta : \mathrm{can} \otimes_{\mathbb{Q}} \mathbb{A}_f \xrightarrow{\sim} \mathscr{F} \otimes_{\mathbb{Q}} \mathbb{A}_f$.

Recall the following. We note L for the reflex field of the Shimura datum (G, X). This is the field of definition of the conjugacy class of μ_h.

Theorem 2.2.2. — *The tower of complex analytic spaces* $(\mathrm{Sh}_K)_K$ *is a tower of smooth quasi-projective algebraic varieties defined over L. When G is anisotropic modulo its center those are projective smooth algebraic varieties over L.*

Algebraicity as a $\mathbb{C}$-analytic space is due due Baily and Borel ([**4**]) where they prove that if one adds a boundary to X by forming X^*, a generalization of $\mathbb{H}^* = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$, whose boundary components are parametrized by conjugacy classes of maximal parabolic subgroups of G over $\mathbb{Q}$, equipped with the so-called Satake topology, then $\Gamma_i \backslash X^*$ *is a compact normal* $\mathbb{C}$*-analytic space*. The quasi-projectivity assertion is then done by proving that the dualizing sheaf ω on those spaces is ample. This is donne via the construction of Eisentein-Poincaré series that are automorphic forms sections of $\omega^{\otimes n}$ for $n \gg 0$. The co-compact case, i.e. when G is anisotropic modulo its center, was done before by Cartan and is much more simple via the construction of Poincaré series and the realization of X as a bounded domain ([**26**]).

The descent datum from $\mathbb{C}$ to L is first constructed on CM-points via the theory of Shimura and Taniyama, and the proof that it extends to an effective descent datum to the entire Shimura variety is “easy” in the Hodge type and more generally abelian type case and delicate, essentially due to Deligne, in the general case. We refer to [**104**] and [**106**]. The case of mixed Shimura varieties and their compactifications, that contains for example the case of the universal abelian scheme over Siegel modular varieties, is handled in [**113**].

This is equipped with an action of $G(\mathbb{A}_f)$ when K varies, for $g \in G(\mathbb{A}_f)$

$$\mathrm{Sh}_K \xrightarrow{g} \mathrm{Sh}_{g^{-1}Kg} .$$

This induces correspondences for g and K as before

$$\begin{tikzcd} & \mathrm{Sh}_{K \cap gKg^{-1}} \arrow[r, "\sim"] \arrow[dl] & \mathrm{Sh}_{g^{-1}Kg \cap K} \arrow[dr] & \\ \mathrm{Sh}_K & & & \mathrm{Sh}_K . \end{tikzcd}$$

Furthermore, if ρ is an algebraic representation of G with values in a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space, it induces an equivariant (with respect to the $G(\mathbb{A}_f)$-action or the preceding correspondences that are upgraded to cohomological one) étale $\overline{\mathbb{Q}}_\ell$-local system $\mathscr{L}_\rho$ on $(\mathrm{Sh}_K)_K$. We can look at

$$\varinjlim_K H^\bullet_{\text{ét}}(\mathrm{Sh}_K \otimes_L \overline{L}, \mathscr{L}_\rho)$$

as a smooth representation of $G(\mathbb{A}_f)$ equipped with a continuous commuting action of $\mathrm{Gal}(\overline{L}|L)$. The action of the Hecke algebra on the K-invariants of those $G(\mathbb{A}_f)$-smooth representations is given by the action of the preceding cohomological correspondences.

Let us now recall the following (see [**19**] and [**142**]).

Theorem 2.2.3 **(Mastushima, Borel, Franke).** — *For G a reductive group over $\mathbb{Q}$, $K_\infty \subset G(\mathbb{R})$ compact whose neutral connected component is the neutral connected component of a maximal compact subgroup, and $K \subset G(\mathbf{A}_f)$ compact open "sufficiently small", if*

$$X_K = G(\mathbb{Q})\backslash(G(\mathbb{R})/K_\infty A_G(\mathbb{R})^+ \times G(\mathbf{A}_f)/K)$$

as a locally symmetric space, where A_G is the maximal split torus in Z_G, then for any finite dimensional complex representation ρ of $G_{\mathbb{C}}$,

1. *If G is anisotropic modulo its center then, as a module over the Hecke algebra $\mathscr{H}(K\backslash G(\mathbb{A}_f/K)$,*

$$H^\bullet(X_K, \mathscr{L}_\rho) = \bigoplus_\Pi m_\Pi . \dim_{\mathbb{C}} H^\bullet(\mathfrak{g}_\infty, K_\infty; \Pi_\infty \otimes \rho).\Pi_f^K$$

   where
   - Π *goes through the set of automorphic representations of G with trivial central character when restricted to $A_G(\mathbb{R})^+$,*
   - m_Π *is the multiplicity of Π in the space of automorphic forms,*
   - $H^\bullet(\mathfrak{g}_\infty, K_\infty, \Pi_\infty)$ *is a finite dimensional cohomology $\mathbb{C}$-vector space associated to Π_∞.*

   In particular this cohomology space is semi-simple as a module over the Hecke algebra $\mathscr{H}(K\backslash G(\mathbb{A}_f)/K)$.

2. *For any G, any constituent of $H^\bullet(X_K, \mathscr{L}_\rho)$ as a module over the Hecke algebra is automorphic in the sense that it is isomorphic to Π_f^K where Π is a cohomological automorphic representation of G.*

This result is in fact deeper: $H^\bullet(X_K, \mathscr{L}_\rho)$ is isomorphic to the $(\mathfrak{g}_\infty, K_\infty)$-cohomology of the space of automorphic forms with level K, see [**142**]. This result has variants, for example the cohomology of the discrete part (the so-called discrete spectrum that is orthogonal to the Eisentein part) of the space of L^2 automorphic forms is identified with the L^2-cohomology of the locally symmetric space. This is itself identified with the intersection cohomology of the minimal compactifications.

2.3. Harris-Taylor Shimura varieties ([72, Chapter III])

2.3.1. Generic fiber. — Let E be a given p-adic field. We are looking to define the local Langlands correspondence for $G = \mathrm{GL}_{n/E}$.

Harris and Taylor have exhibited some PEL-type Shimura datum (G, X) such that

$$G_{\mathbb{R}} \simeq G(U(1,n-1) \times U(n) \times \cdots \times U(n))$$

and

$$G_{\mathbb{Q}_p} \simeq \mathrm{GL}_{n/E} \times \mathbb{G}_m.$$

Moreover, one has

$$\widehat{G} = \mathrm{GL}_n \times \mathrm{GL}_n \times \cdots \times \mathrm{GL}_n \times \mathbb{G}_m$$

with r_μ the standard representation of dimension n on the first GL_n-factor, trivial on the other GL_n factors and all of this is twisted by the standard representation of $\mathbb{G}_m$. We can moreover suppose that G is anisotropic modulo its center.

In fact, G is a similitude unitary group attached to to a division algebra over a CM field equipped with an involution inducing complex conjugation on the CM field. Those are particular cases of Shimura varieties that were already studied by Kottwitz ([**88**], [**87**]) and Clozel ([**31**]).

We get

$$\begin{tikzcd}\mathrm{Sh}_K \arrow[d] \\ \mathrm{Spec}(L)\end{tikzcd}$$

a *proper smooth algebraic variety that is in fact a moduli of abelian varieties equipped with additional structures* like a polarization and an action of an order in a division algebra. Set $L_v = E$ our p-adic field where v is a place of L dividing p.

We are going to analyze the cohomology of $(\mathrm{Sh}_K)_K \otimes_L L_v$ by making a degeneration from $p \neq 0$ to $p = 0$.

2.3.2. Integral models. — If $K_p \subset G(E)$ is compact hyperspecial, $K_p = \mathrm{GL}_n(\mathcal{O}_E) \times \mathbb{Z}_p^\times$ ("minimal level at p"), then $\mathrm{Sh}_{K_pK^p}$ degenerates smoothly for any K^p compact open inside $G(\mathbb{A}_f^p)$ there exists a smooth projective model

$$\begin{tikzcd}S_{K^p} \arrow[r, hook] \arrow[d] & \mathrm{Sh}_{K_pK^p} \otimes_L L_v \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_E) \arrow[r, hook] & \mathrm{Spec}(E)\end{tikzcd}$$

with $S_{K^p} \otimes_{\mathcal{O}_E} E = \mathrm{Sh}_{K_pK^p} \otimes_L L_v$. This is a moduli space of abelian schemes with additional structures.

The main point is the following. Let

$$\begin{tikzcd}\mathcal{A} \arrow[d] \\ S_{K^p}\end{tikzcd}$$

be the universal abelian scheme. The fact is that the p-divisible group $\mathcal{A}[p^\infty]$ splits as

$$\mathcal{A}[p^\infty] = \mathcal{G} \oplus \mathcal{G}^D$$

where $\mathcal{G}$ is equipped, as an extra additional structure, with an action of $M_n(\mathcal{O}_E)$. The additional structure that is the polarization on $\mathcal{A}[p^\infty]$ is the canonical polarization on $\mathcal{G} \oplus \mathcal{G}^D$. Let $e = \begin{pmatrix} 1 & \\ & \end{pmatrix}$ as an idempotent of $M_n(\mathcal{O}_E)$. Then (Morita equivalence), a p-divisible group such as $\mathcal{G}$ equipped with an action of $M_n(\mathcal{O}_E)$ is the same as a p-divisible group equipped with an action of $\mathcal{O}_E$,

$$H := e.\mathcal{G}$$

in our case. The fact now is that the signature at ∞ of our unitary group

$$(1, n-1) \times (0, n) \times \cdots \times (0, n)$$

transfers at p as the condition that

1. H is a 1-dimensional p-divisible group with an action of $\mathcal{O}_E$
2. The action of $\mathcal{O}_E$ on $\operatorname{Lie} H$ is the canonical one via $S_{K^p} \to \mathcal{O}_E$.

We call such an object a 1-*dimensional* π-*divisible* $\mathcal{O}_E$-*module.*

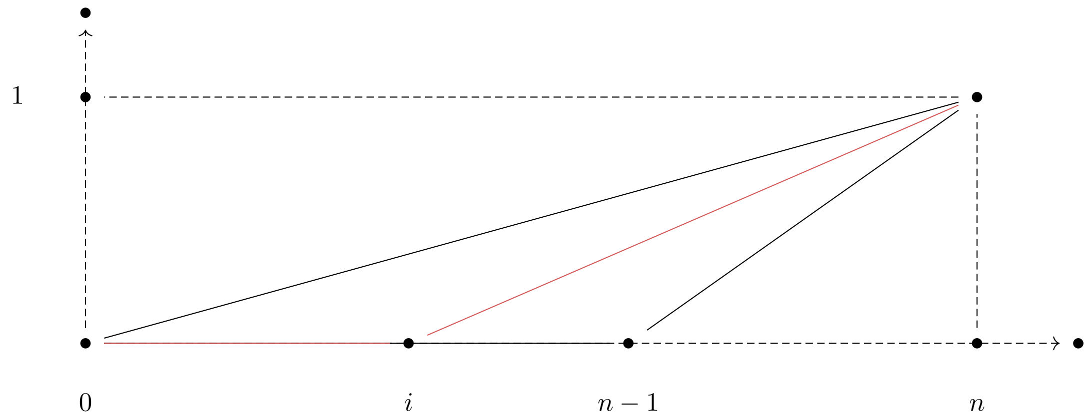
2.3.3. Newton stratification. — Let

$$\overline{S}_{K^p} = S_{K^p} \otimes_{\mathcal{O}_E} \mathbb{F}_q$$

be the reduction modulo π of our Shimura variety. This again forms a tower of étale coverings equipped with an action of $G(\mathbb{A}_f^p)$ when K^p varies. Let

$$\begin{tikzcd}\overline{H} \arrow[d] \\ \overline{S}_{K^p}\end{tikzcd}$$

be our 1-dimensional π-divisible $\mathcal{O}_E$-modules. Geometrically fiberwise on $\overline{S}_{K^p}$ this has a Newton polygon that is of the following shape in red:



for an integer $i \in \{0, \dots, n-1\}$. In the preceding picture the Hodge polygon has slope 0 with multiplicity $n-1$ and 1 with multiplicity 1. The basic (i.e. isoclinic in Kottwitz terminology) polygon has slope $1/n$. The integer i is the $\mathcal{O}_E$-height of the étale part. More precisely, there is a stratification by locally closed subsets

$$\overline{S}^{(i)}_{K^p}, \quad 0 \leq i \leq n-1$$

where a geometric point x of $\overline{S}_{K^p}$ lies in $\overline{S}^{(i)}_{K^p}$ if an only if

$$0 \longrightarrow \underbrace{\overline{H}_x^{\circ}}_{\substack{\text{1-dim.formal} \\ \text{of } \mathcal{O}_E\text{-height } n-i}} \longrightarrow \overline{H}_x \longrightarrow \underbrace{\overline{H}_x^{\text{ét}}}_{\mathcal{O}_E\text{-height } i} \longrightarrow 0.$$

1. The closed stratum is $\overline{S}^{(0)}_{K^p}$ that is a finite set of closed points, the so-called basic locus,
2. The open stratum is $\overline{S}^{(n-1)}_{K^p}$ that is the so-called μ-ordinary locus.

Remark 2.3.1. — *The Newton strata of Shimura varieties are in general parametrized by Kottwitz set $B(G,\mu)$ (*[**88**],[**86**],[**121**]*, and* [**98**] *who proves* [**50**, Conjecture 3.1.1]*). The appearance of the set $B(G)$ in the geometry of Shimura varieties, as in* [**50**]*, has been a great motivation over the years for finding a more geometric interpretation of it.*

2.3.4. Level structures at p. — We worked before with a level structure at p for which $K_p = \mathrm{GL}_n(\mathcal{O}_E) \times \mathbb{Z}_p^\times$. In this case the integral models are smooth. Drinfeld ([**40**], [**72**, Section II.2]) defined a "good" notion of level structures at p for the principal congruence subgroups $K_p = \mathrm{Id} + \pi^m M_n(\mathcal{O}_E) \times \mathbb{Z}_p^\times$ when $m \geq 1$. This is very particular to 1-dimensional p-divisible groups. By "good" we mean that the associated integral models

$$S_{m,K^p}$$

are *regular* and the change of level morphism

$$\begin{tikzcd} S_{m,K^p} \arrow[d, "\text{finite flat}"] & \text{regular} \arrow[l, squiggly] \arrow[d] \\ S_{K^p} & \text{smooth}/\mathcal{O}_E \arrow[l, squiggly] \end{tikzcd}$$

is *finite flat* (see [**72**, Lemma III.4.1]). Moreover those morphisms are totally ramified over the points of the basic locus. We obtain a tower

$$(S_{m,K^p})_{m\geq 1}$$

that is equipped at the limit when $m \to +\infty$ with an action of $G(\mathbb{Q}_p)$ and commuting Hecke correspondences associated to elements of $K^p\backslash G(\mathbf{A}_f^p)/K^p$.

2.3.5. Analysis of the ℓ-adic cohomology at p via nearby cycles. —

2.3.5.1. *Background on nearby cycles. — Nearby cycles are a construction that allows us to analyze the cohomology of an algebraic variety via the cohomology of the special fiber of a "*1*-parameter degeneration" of this algebraic variety i.e. a degeneration parametrized by what we call a trait (the spectrum of a rank* 1 *valuation ring).* We refer to [**79**] for an historical introduction to the subject.

Let

$$\begin{tikzcd} X \arrow[d] \\ \mathrm{Spec}(V) \end{tikzcd}$$

be finite presentation morphism of schemes where V is an Henselian rank 1 valuation ring. Let $K = \mathrm{Frac}(V)$ and k be the residual field of V. Fix an algebraic closure $\overline{K}$ of K and let $\overline{k}$ be the associated algebraic closure of k. We note $\mathrm{Spec}(k)$, $\overline{s} = \mathrm{Spec}(\overline{k})$, $\eta = \mathrm{Spec}(K)$ and $\overline{\eta} = \mathrm{Spec}(\overline{K})$.

There is a diagram

$$\begin{tikzcd} X_s \arrow[r, hook] \arrow[d] & X \arrow[d] & X_\eta \arrow[l, hook'] \arrow[d] \\ s \arrow[r, hook] & \mathrm{Spec}(V) & \eta \arrow[l, hook'] \end{tikzcd}$$

Let $\mathscr{F} \in D_c^b(X_\eta, \overline{\mathbb{Q}}_\ell)$ with ℓ invertible in V. We want to understand

$$H^\bullet(X_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)$$

with its $\mathrm{Gal}(\overline{K}|K)$ action in terms of the special fiber X_s of our degeneration. There is a "nearby cycle fiber functor"

$$D^b_c(X_\eta, \overline{\mathbb{Q}}_\ell) \xrightarrow{R\Psi_{\bar\eta}} \left\{\begin{matrix}\text{objects in } D^b_c(X_{\bar s}, \overline{\mathbb{Q}}_\ell) + \text{ action of } \mathrm{Gal}(\overline{K}|K) \\ \text{compatible with the one of } \mathrm{Gal}(\bar k|k)\end{matrix}\right\}$$

such that for any geometric point $\bar x$ of $X_{\bar s}$,

$$R\Psi_{\bar\eta}(\mathscr{F})_{\bar x} = R\Gamma\Big(\underbrace{\mathrm{Spec}(\mathcal{O}^{sh}_{\overline{X},\bar x}[\tfrac{1}{\varpi}])}_{\substack{\text{schematical Milnor fiber} \\ \text{over } \bar x}}, \mathscr{F}\Big)$$

where ϖ is a pseudo-uniformizer in V and $\overline{X} = X \otimes_V \overline{V}$ with $\overline{V}$ the integral closure of V in $\overline{K}$.

Remark 2.3.2. — *The fiber at geometric points of $R\Psi_{\bar\eta}(\mathscr{F})$ is thus identified with the cohomology complex of those schematical Milnor fibers. Grothendieck's construction of the functor $R\Psi_{\bar\eta}$ is a way to take all those cohomology complexes of the different "classical" Milnor fibers and build a sheaf out of it. Deligne's theorem says that this complex has constructible cohomology and thus the cohomology of those Milnor fibers "varies constructibly".*

Proper base change then says that if $X \to \mathrm{Spec}(V)$ is proper then

$$R\Gamma(X_{\bar s}, R\Psi_{\bar\eta}(\mathscr{F})) \xrightarrow{\sim} R\Gamma(X_{\bar\eta}, \mathscr{F}).$$

We will now use the following very important result that says that the nearby cycles depend only on the formal completion and not the henselelinzation. Suppose that the residue field k is perfect.

Theorem 2.3.3. — *(Berkovich ([**9**], [**10**]), Huber ([**78**, Corollary 3.5.16, Theorem 3.5.8])) Let x be a closed point of $X_{\bar s}$ and $\mathfrak{X}_x$ be the formal completion of $X \otimes_V V^{un}$ at x where V^{un} is the integral closure of V in the maximal unramified extension K^{un} of K. This is a formal scheme over Spf$(\widehat{V^{un}})$. Let $\mathfrak{X}^{ad}_x$ be its generic fiber as an adic space over* Spa$(\widehat{K^{un}})$*. There is then an isomorphism*

$$\underbrace{R\Psi_{\bar\eta}(\mathscr{F})_x}_{\substack{\textit{cohomology of the} \\ \textit{schematical Milnor fiber}}} \xrightarrow{\sim} \underbrace{R\Gamma_{\text{ét}}\big(\mathfrak{X}_{x,\eta} \hat\otimes_{\widehat{K^{un}}} \widehat{\overline{K}}, \mathscr{F}^{ad}\big)}_{\substack{\textit{cohomology of the} \\ \textit{rigid analytic Milnor fiber}}}$$

2.3.6. A localization phenomenon. — The geometry of non-basic Newton strata implies the following result. For $m \geq 1$ we note

$$R\Psi_{\bar\eta}(\overline{\mathbb{Q}}_\ell)_{m,K^p} \in D^b_c(\overline{S}_{m,K^p} \otimes \overline{\mathbb{F}}_q, \overline{\mathbb{Q}}_\ell).$$

If $m' \geq m$ and $\Pi_{m',m} : \overline{S}_{m',K^p} \to \overline{S}_{m,K^p}$ then

$$R\Psi_{\bar{\eta}}(\overline{\mathbb{Q}}_\ell)_{m,K^p} = \Pi_{m',m*} R\Psi_{\bar{\eta}}(\overline{\mathbb{Q}}_\ell)_{m',K^p}.$$

Moreover if $\mathscr{H}_m = \mathscr{H}(G(E) /\!\!/ \mathrm{Id} + \pi^m M_n(\mathcal{O}_E))$,

$$R\Psi_{\bar{\eta}}(\overline{\mathbb{Q}}_\ell)_{m,K^p}$$

is equipped with an action of $\mathscr{H}_m \otimes \mathscr{H}(K^p \backslash G(\mathbf{A}_f^p)/K^p)$.

The following result is essentially a consequence of [**72**, Lemma II.2.1], see point 4 and 5 of this lemma.

Theorem 2.3.4. — *For any $m \geq 1$,*

$$\big[R\Psi_{\bar{\eta}}(\overline{\mathbb{Q}}_\ell)_{m,K^p}\big]_{supercusp.\,at\ p} \xrightarrow{\sim} \bigoplus_{x \in \overline{S}^{(0)}_{m,K^p}(\overline{\mathbb{F}}_q)} i_{x*}\big[R\Psi_{\bar{\eta}}(\overline{\mathbb{Q}}_\ell)_{m,K^p,x}\big]_{supercusp.\,at\ p}$$

that is to say the supercuspidal at p part of the complex of nearby cycles localizes on supersingular points.

Remark 2.3.5. — *This localization phenomenon is called Boyer's trick: the cohomology of non-basic Newton strata is parabolically induced at p. This phenomenon generalizes to Newton strata of PEL type Shimura varieties for which the Newton and Hodge polygon (in a generalized sense as elements of a positive Weyl chamber) touch at a breakpoint of the Newton polygon. This is the Hodge Newton decomposability condition (*[**103**]*,* [**133**]*,* [**68**]*,* [**27**] *for an application of this to the geometry of p-adic period domains).*

2.4. Lubin-Tate spaces ([49], [69])

Definition 2.4.1. — *Let $\mathbb{H}$ be a one dimensional formal p-divisible group over $\overline{\mathbb{F}}_q$ equipped with an action of $\mathcal{O}_E$ such that the action of $\mathcal{O}_E$ on Lie $\mathbb{H}$ is the canonical one. We note*

$$\mathcal{LT}$$

for the deformation space of $\mathbb{H}$ as a Spf$(\mathcal{O}_{\breve{E}})$-formal scheme.

This is a formal scheme (non-canonically) isomorphic to

$$\mathrm{Spf}(\mathcal{O}_{\breve{E}}[\![x_1, \dots, x_{n-1}]\!]).$$

We note

$$\mathcal{LT}_\eta \simeq \mathring{\mathbb{B}}^{n-1}_{\breve{E}}$$

for its generic fiber as a locally of finite type adic space over Spa$(\breve{E})$.

On this open ball the Tate module of the universal deformation is an $\mathcal{O}_E$-étale local system of rank n. The moduli of its trivializations defines a tower of rigid analytic spaces with finite étale transition morphisms

$$\boxed{(\mathcal{LT}_{\eta,K})_{K\subset \mathrm{GL}_n(\mathcal{O}_E)} \longrightarrow \mathcal{LT}_\eta}$$

equipped with an action of $\mathrm{GL}_n(E)^1 = \{g \in \mathrm{GL}_n(E) \mid \det(g) \in \mathcal{O}_E^\times\}$ at the limit. There is another group that shows up: the group of automorphisms by quasi-isogenies of $\mathbb{H}$, $\mathrm{End}(\mathbb{H})_{\mathbb{Q}}^\times$, that is identified with

$$D^\times$$

where D is a division algebra with invariant $\frac{1}{n}$ over E. At the end the tower $(\mathcal{LT}_{\eta,K})_K$ has a commuting action of $(D^\times \times \mathrm{GL}_n(E))^1$, the subgroup of $D \times \mathrm{GL}_n(E)$ formed by elements (d,g) such that $v(\mathrm{Nrd}(d)) + v(\det(g)) = 0$ where Nrd is the reduced norm.

In fact we prefer to work with

$$\mathcal{M}_K = \mathcal{LT}_{\eta,K} \overset{\mathcal{O}_D^\times}{\times} D^\times$$

that is a $\coprod_{\mathbb{Z}}$ of copies of the Lubin-Tate space. This is a particular case of Rapoport-Zink space. The tower $(\mathcal{M}_K)_K$ has an action of $D^\times \times \mathrm{GL}_n(E)$ and a (non-effective since this shifts everything by $+1$ in the components $\coprod_{\mathbb{Z}}$) descent datum

$$\mathcal{M}_K^{(\sigma)} \xrightarrow{\sim} \mathcal{M}_K$$

from $\check{E}$ to E ([**117**, Section 3.48] in general). We now define

$$R\Gamma(\mathcal{M}_K \hat{\otimes}_{\check{E}} \widehat{\overline{E}}, \overline{\mathbb{Q}}_\ell) := \bigoplus_{\alpha \in \pi_0(\mathcal{M}_K)} R\Gamma(\mathcal{M}_K^\alpha \hat{\otimes}_{\check{E}} \widehat{\overline{E}}, \overline{\mathbb{Q}}_\ell).$$

This has an action of $D^\times \times W_E$ where the action of $D^\times$ is smooth (using [**10**, Theorem 4.1], see [**72**, Lemma II.2.8]) and a commuting action of the Hecke algebra $\mathscr{H}_{\overline{\mathbb{Q}}_\ell}(K\backslash \mathrm{GL}_n(E)/K)$.

Remark 2.4.2. — *As for Harris-Taylor Shimura varieties, the notion of Drinfeld level structure allows us to define some regular integral models of* $\mathcal{LT}_{\eta,K}$ *when* $K = \mathrm{Id} + \pi^m M_n(\mathcal{O}_E)$, *a principal congruence subgroup. Those are formal spectrum of complete regular Noetherian rings that are finite free over* $\mathcal{O}_{\check{E}}[\![x_1, \ldots, x_{n-1}]\!]$, *see* [**40**]. *Nevertheless, we don't need them to define the cohomology of the Lubin-Tate tower. This is one of the main ideas of* [**50**]*: there's no need of any integral models anywhere, we can do everything "in generic fiber" directly. At the end this has been reflected in* [**58**] *where we look at vector bundles on the curve and its moduli as an "analytic stack" instead of the moduli of* F*-isocrystals on perfect schemes. The first one is a "nice" Artin* v*-stack, the second one is not a classical Artin stack in any sense we can imagine.*

2.4.1. The basic locus as a zero dimensional locally symmetric space. — Let I be the algebraic reductive group over $\mathbb{Q}$ that is the endomorphism by quasi-isogenies of an abelian variety over $\overline{\mathbb{F}}_q$ equipped with its additional structures defining an $\overline{\mathbb{F}}_q$-point of $\overline{S}^{(0)}_{K^p}$. This satisfies

1. $I(\mathbb{R})$ is compact modulo its center,
2. $I(\mathbb{Q}_p) = D^\times \times \mathbb{Z}_p^\times$ via the action of an automorphism on the Dieudonné module,
3. $I(\mathbf{A}_f^p) = G(\mathbf{A}_f^p)$ via the action of an automorphism on the étale cohomology outside p.

In fact I is an inner form of G that is isomorphic to G outside $p\infty$. We refer to [**117**, Chapter 6] for this and more generally to [**88**] and even more generally to [**93**] and [**105**].

The fact is, like for modular curves, that all basic points are in an unique isogeny class. From this we deduce that, after fixing a base point,

$$I(\mathbb{Q})\backslash(I(\mathbb{Q}_p)/\mathcal{O}_D^\times \times I(\mathbf{A}_f^p)/K^p) \xrightarrow{\sim} \overline{S}^{(0)}_{K^p}(\overline{\mathbb{F}}_q)$$

(to be correct we should add in fact a $\coprod_{\ker^1(\mathbb{Q},G)}$ to the left hand term).

Remark 2.4.3. — *This last formula is a particular case of Rapoport-Zink uniformization of the basic locus (*[**117**, Chapter 6]*),*

$$I^\varphi(\mathbb{Q})\Big\backslash\left(\underbrace{\mathcal{M}}_{\textit{R.Z. space}} \times G(\mathbb{A}_f^{(p)})/K^p\right) \xrightarrow{\sim} \begin{array}{c}\textit{formal completon of Shimura variety}\\ \textit{with hyperspecial level at } p\\ \textit{along the basic locus of the special fiber}\end{array}$$

with I an inner form of G satisfying

- $I_{\mathbb{A}_f^{(p)}} \simeq G_{\mathbb{A}_f^{(p)}}$,
- $I_{\mathbb{Q}_b} \simeq G_b$, $[b] \in B(G,\mu)$ *basic,*
- $I_\mathbb{R}$ *is the compact mod center inner form of* $G_\mathbb{R}$.

We can also see this as a particular case of the fact that basic Igusa varieties ([**72**, Chapter IV], [**102**]*) are zero dimensional locally symmetric spaces attached to I. All of this is the starting point of* [**50**].

2.4.2. Harris-Taylor theorem. — From the preceding we obtain that

$$\underbrace{\varinjlim_K R\Gamma(\mathrm{Sh}_K \otimes_L \overline{L}, \overline{\mathbb{Q}}_\ell)_{|W_E,\text{cusp at } p}}_{\substack{\text{expressed in terms of automorphic}\\ \text{representations of } G}} \xrightarrow{\sim} \underbrace{\mathcal{A}(I)}_{\substack{\text{expressed in}\\ \text{terms of}\\ \text{automorphic}\\ \text{representations}\\ \text{of } I}} \otimes^{\mathbb{L}}_{\mathscr{H}_{\overline{\mathbb{Q}}_\ell}(D^\times)} \varinjlim_K R\Gamma(\mathcal{M}_K \hat{\otimes}_{\breve{E}} \widehat{\overline{E}}, \overline{\mathbb{Q}}_\ell)_{\text{cusp at } p}$$

Via a comparison between automorphic representations on the two inner forms I and G (global Jacquet-Langlands) obtained via a comparison of Arthur trace formulas Harris and Taylor prove the following result. This result is obtained via global methods using the fact that any supercuspdial representation globalizes to an automorphic representation that is a discrete series at ∞.

***Theorem 2.4.4* (Harris-Taylor ([72])).** — *The cuspidal part of the middle degree cohomology*

$$\varinjlim_{K} H^{n-1}(\mathcal{M}_K \hat{\otimes}_{\breve{E}} \widehat{\overline{E}}, \overline{\mathbb{Q}}_\ell)$$

is, up to a Tate twist, of the form

$$\bigoplus_{\substack{\pi \\ supercuspidal}} \mathrm{JL}^{-1}(\pi) \otimes \pi \otimes \varphi_\pi$$

where φ_π is an n-dimensional $\overline{\mathbb{Q}}_\ell$-representation of W_E. The correspondence $\pi \mapsto \varphi_\pi$ defines a local Langlands correspondence for $\mathrm{GL}_{n/E}$.

Here the term ”local Langlands correspondence“ refers to the formulation due to Henniart that characterize it in terms of ε-factors of pairs ([**75**]), see [**25**] for more details about Harris-Taylor work.

Remark 2.4.5. — *Henniart gave another proof of the local Langlands correspondence in* [**76**]*. Moreover, Scholze gave another proof in* [**129**]*. This last proof is relevant to* [**58**] *in terms of some kind of philosophy of “character sheaves” on the moduli of p-divisible groups.*

2.5. Final thoughts

One of the main ideas of [**50**] is to “do everything in generic fiber” after remarking that in the work of Harris-Taylor the use of integral models is a tool to prove results but at the end we can define everything in generic fiber and integral models are just a tool for the proofs. Since we did not speak so much about them: the Igusa varieties ([**72**, Chapter IV], [**102**]) played an important role for [**58**] via [**24**] and some expected local/global compatibility properties.

Finally, [**129**] has been an important motivation via the “character sheaf” property that appears there linked to the stack of p-divisible groups.

LECTURE 3

p-ADIC PERIOD MORPHISMS

In this chapter we discuss *period morphisms for p-divisible groups.* Historically, the first appearance of de Rham period morphisms goes back to Katz ([**82**]) for deformation spaces of ordinary p-divisible groups like $\mathbb{Q}_p/\mathbb{Z}_p \oplus \mu_{p^\infty}$ where the (de Rham) period morphism is then identified with the p-adic logarithm. This was later defined and studied by Gross and Hopkins in [**69**] for Lubin-Tate spaces. Rapoport and Zink extended this to all Rapoport-Zink spaces in [**117**].

The Hodge-Tate period morphism first appeared in [**47**] and [**51**]. It later appeared for global Shimura varieties in [**52**] at the level of Berkovich topological spaces. It finally appeared at the level of perfectoid spaces for Shimura varieties with infinite level in [**130**].

3.1. Some general thoughts on period morphisms

For $p = \infty$ there is only on period morphism and this is a $G(\mathbb{R})$-equivariant embedding

$$G(\mathbb{R}) \circlearrowright X \xhookrightarrow{\text{open}} \mathscr{F}_{\mu_h} \circlearrowleft G(\mathbb{C})$$

where G is a reductive group over $\mathbb{R}$, X is an hermitian symmetric space defined by the $G(\mathbb{R})$-conjugacy class of $h : \mathbb{S} \to G$, and $\mathscr{F}_{\mu_h}$ is the complex analytic flag manifold defined by μ_h. This embedding is nothing else than the map that sends a Hodge structure to the Hodge filtration.

Moreover, the image of this embedding is easy to describe. In fact, the complex conjugate of μ_h is $\mu_h^c = w_h.\mu_h^{-1}$ with $w_h : \mathbb{G}_m \to G$ central and thus complex conjugation defines $\overline{(-)} : \mathscr{F}_{\mu_h} \xrightarrow{\sim} \mathscr{F}_{\mu_h^{-1}}$, and

$$\boxed{\begin{array}{rcl} X & \underset{\text{open/closed}}{\subset} & \{z \in \mathscr{F}_{\mu_h} \mid z \text{ and } \overline{z} \text{ are opposite parabolic subgroups}\} \\ & = & \underbrace{\{z \in \mathscr{F}_{\mu_h} \mid \operatorname{inv}(z, \overline{z}) = 1\}}_{\text{Deligne-Luztig variety at } \infty} \end{array}}$$

where here $P_{\mu_h^{-1}}$ is opposite to P_{μ_h} and

$$\begin{array}{rcl} \operatorname{inv} : G_{\mathbb{C}}/P_{\mu_h} \times G_{\mathbb{C}}/P_{\mu_h^{-1}} & \longrightarrow & P_{\mu_h^{-1}} \backslash G_{\mathbb{C}}/P_{\mu_h} \\ (gP_{\mu_h}, g'P_{\mu_h^{-1}}) & \longmapsto & P_{\mu_h^{-1}} g'^{-1} g P_{\mu_h} \end{array}$$

Here the open/closed condition defining X is that for z satisfying $\operatorname{inv}(x, \overline{z}) = 1$, one has an associated $h_z : \mathbb{G}_m \to G$ and we ask this is $G(\mathbb{R})$-conjugated to h.

Example 3.1.1. — 1. *Consider $G = \mathrm{Gsp}_{2n}$. Then, $\mathscr{F}_{\mu_h}$ is the variety of Lagrangians in $\mathbb{C}^{2n}$ equipped with the standard symplectic structure. Moreover, for a Lagrangian subspace $L \subset \mathbb{C}^{2n}$, the condition defining our open subset is that $L \cap \overline{L} = (0)$. It is clear that if $L \cap \big(\mathbb{C}^n \oplus (0)\big) \neq (0)$ then L is not in our open subset. The subset of $\mathscr{F}_{\mu_h}$ formed by Lagrangian subspace L satisfying $L \cap \big(\mathbb{C}^n \oplus (0)\big) = \emptyset$ is identified with the affine space of symmetric matrices $A \in M_n(\mathbb{C})$, ${}^tA = A$. To such a matrix A one associated the image of $\mathbb{C}^n \oplus (0)$ by $\begin{pmatrix} I & 0 \\ A & I \end{pmatrix}$. Now, the associated Lagrangian subspace L satisfies $L \cap \overline{L} = (0)$ iff $\mathrm{Im}(A)$ (imaginary part) is invertible. Our open subset has thus n connected components given by the signature of the symmetric non-singular matrix $\mathrm{Im}(A)$.*

The open/closed subspace X is the union of the two connected components that correspond to the signatures $(n, 0)$ and $(0, n)$ that is to say $\mathrm{Im}(A)$ or $-\mathrm{Im}(A)$ is positive definite. This is $\pm\mathcal{H}_n$ where $\mathcal{H}_n$ is Siegel upper half space.

2. *Let $G = GU(1, n-1)$ with $h(z) = \operatorname{diag}(z, \overline{z}, \dots, \overline{z})$. One has $\mathscr{F}_{\mu_h} = \mathbb{P}^{n-1}(\mathbb{C})$ and our open subset is*

$$\big\{[z_1 : \dots : z_n] \mid |z_1|^2 - \sum_{i=2}^{n} |z_i|^2 \neq 0\big\}.$$

This has two connected components: the first one is an open ball

$$\{[1 : z_2 : \dots : z_n] \mid \sum_{i=2}^{n} |z_i|^2 < 1\} \subset \mathbb{C}^{n-1}$$

and the other one is $\{[1 : z_2 : \dots : z_n] \mid \sum_{i=2}^{n} |z_i|^2 > 1\} \cup \{[0 : z_2 : \dots : z_n] \in \mathbb{P}^{n-2}(\mathbb{C})\}$. The space X is the first connected component identified with an open ball.

For $p \neq \infty$ the story is different:

1. There are *two period maps* and *two groups acting*
2. Those are linked to the *two cohomology theories*: crystalline cohomology and p-adic étale cohomology. For $p = \infty$ we only have Betti cohomology.
3. Those two period maps correspond to the *two spectral sequences*: the Hodge to de Rham spectral sequence and the Hodge-Tate spectral sequence (see [**13**, Theorem 1.7])
4. The period maps aren't embeddings in general.

3.2. The case of Lubin-Tate spaces

3.2.1. The Lubin-Tate tower (see section 2.4). — Take $E = \mathbb{Q}_p$ to simplify. Let

$$\mathbb{H}$$

be a one dimensional 1-dimensional formal p-divisible group over $\overline{\mathbb{F}}_p$ (such an $\mathbb{H}$ is unique up to a non-unique isomorphism). This can be seen, after fixing a coordinate $\mathrm{Spf}(\overline{\mathbb{F}}_p[\![T]\!]) \xrightarrow{\sim} \mathbb{H}$ as a one dimensional formal group law $\mathfrak{F} \in \overline{\mathbb{F}}_p[\![X, Y]\!]$ that gives the addition: $X \underset{\mathfrak{F}}{+} Y = \mathfrak{F}(X, Y)$.

Let n be the height of $\mathbb{H}$ that is to say $[p]_{\mathfrak{F}} = aT^{p^n} + \dots$ with $a \neq 0$.

Definition 3.2.1. — *The moduli space of deformations of $\mathbb{H}$ over complete local $W(\overline{\mathbb{F}}_p)$-algebras is the Lubin-Tate space*

$$\begin{tikzcd} \mathcal{LT} \arrow[d] \\ \mathit{Spf}(W(\overline{\mathbb{F}}_p)). \end{tikzcd}$$

This is non-canonically isomorphic to

$$\mathrm{Spf}\big(W(\overline{\mathbb{F}}_p)[\![x_1, \dots, x_{n-1}]\!]\big).$$

Let D be a division algebra with invariant $\frac{1}{n}$ over $\mathbb{Q}_p$, $D = \mathbb{Q}_{p^n}[\Pi]$ where $\mathbb{Q}_{p^n}$ is the degree n unramified extension of $\mathbb{Q}_p$, $\Pi^n = p$ and if σ is the Frobenius of $\mathbb{Q}_{p^n}|\mathbb{Q}_p$ then $\Pi x \Pi^{-1} = x^\sigma$. One has an identification

$$\mathcal{O}_D = \mathrm{End}(\mathbb{H})$$

where $\mathcal{O}_D$ is the maximal order in D, $\mathcal{O}_D = \mathbb{Z}_{p^n}[\Pi]$.

There is an evident action of $\mathcal{O}_D^\times$ on $\mathcal{LT}$

$$\begin{tikzcd} \mathcal{LT} \arrow[loop below, "\mathcal{O}_D^\times"] \end{tikzcd}$$

Definition 3.2.2. — *Let* $\mathcal{LT}_\eta$ *be the generic fiber of* $\mathcal{LT}$ *as a locally of finite type adic space over* $\mathrm{Spa}(\breve{\mathbb{Q}}_p)$.

After fixing some formal coordinates

$$\mathcal{LT}_\eta \simeq \mathring{\mathbb{B}}^{n-1}_{\breve{\mathbb{Q}}_p}$$

that is again equipped with an action of $\mathcal{O}_D^\times$. The Tate module of the universal deformation defines an étale $\mathbb{Z}_p$-local system T of rank n on $\mathcal{LT}_\eta$.

Definition 3.2.3. — *For* $K \subset \mathrm{GL}_n(\mathbb{Z}_p)$ *we note*

$$\mathcal{LT}_{\eta,K}$$

the moduli space of trivializations mod K *of the* $\mathbb{Z}_p$*-local system* T.

This means

$$\mathcal{LT}_{\eta,K} = \underline{(K/\operatorname{Id}+p^m M_n(\mathbb{Z}_p))}\backslash \underline{\mathrm{Isom}}\Big(\underline{(Z/p^m\mathbb{Z}})^n, T/p^mT\Big)$$

for $m \gg 0$.

We obtain a tower of rigid analytic spaces

$$\begin{tikzcd} \big(\mathcal{LT}_{\eta,K}\big)_K \arrow[loop right, "\mathrm{GL}_n(\mathbb{Q}_p)^1"] \arrow[loop below, "\mathcal{O}_D^\times"] \end{tikzcd}$$

where
— the action of $\mathcal{O}_D^\times$ is horizontal,
— the action of $\mathrm{GL}_n(\mathbb{Q}_p)^1$ is vertical: for $g \in \mathrm{GL}_n(\mathbb{Q}_p)^1$, $g : \mathcal{LT}_{\eta,K} \xrightarrow{\sim} \mathcal{LT}_{\eta,g^{-1}Kg}$,
— both actions commute.

Here the action of $\mathrm{GL}_n(\mathbb{Z}_p)$ is the evident one. To extend it to an action of $\mathrm{GL}_n(\mathbb{Q}_p)^1$ we have to go back to some integral models of $\mathcal{LT}_{\eta,K}$ for K a principal congruence subgroup, $K = \operatorname{Id}+p^m M_n(\mathbb{Z}_p)$, $m \geq 1$. This is given by the notion of Drinfeld level structure ([**72**, Chapter II.2]) that defines an integral model $\mathrm{Spf}(R_m)$

where R_m is a complete regular $W(\overline{\mathbb{F}}_p)$-algebra. We then use the following two elementary results:

1. if S is a formal scheme over $\mathrm{Spf}(\mathbb{Z}_p)$ and H a one dimensional height n formal p-divisible group over S equipped with a level m Drinfeld structure
$$\eta : \underline{(\mathbb{Z}/p^m)^n} \longrightarrow H[p^m]$$
then any subgroup M of $(\mathbb{Z}/p^m\mathbb{Z})^n$ defines a finite flat closed subgroup scheme $G \subset H[p^m]$ such that $\eta_{|M} : \underline{M} \to G$.
2. if S is a reduced $\mathbb{F}_p$-scheme and $f : H \to H'$ is a height 0 quasi-isogeny between one dimensional formal p-divisible groups then f is an isomorphism.

At the end we obtain an action of $(\mathrm{GL}_n(\mathbb{Q}_p) \times D^\times)^1$ on our tower.

3.2.2. The de Rham period morphism. — Let
$$\boldsymbol{D} = \mathbb{D}(\mathbb{H})$$
be the covariant rational Dieudonné module of $\mathbb{H}$. This is an n-dimensional $\breve{\mathbb{Q}}_p$-vector space equipped with a crystalline Frobenius φ,
$$\boldsymbol{D} \circlearrowleft \varphi .$$
The matrix of the associated Verschiebung $p\varphi^{-1}$ is given in a suitable basis by
$$p\varphi^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \sigma^{-1}$$

We now use the following property. Recall that a quasi-isogeny between p-divisible groups H and H' over a quasi-compact scheme S is an element of $f \in \mathrm{Hom}(H, H')\left[\frac{1}{p}\right]$ such that there exists $g \in \mathrm{Hom}(H', H)\left[\frac{1}{p}\right]$ satisfying $g \circ f = \mathrm{Id}$ and $f \circ g = \mathrm{Id}$.

***Lemma 3.2.4* (rigidity of quasi-isogenies).** — *Let $S_0 \hookrightarrow S$ be a nilpotent closed immersion of schemes and H, H' be p-divisible groups over S. Then, reduction to S_0 induces an isomorphism*
$$\mathrm{Qisog}(H, H') \xrightarrow{\sim} \mathrm{Qisog}(H \times_S S_0, H' \times_S S_0).$$

We now use *the crystalline nature of the Dieudonné crystal of a p-divisible group.* Let R be a p-adic ring, H a p-divisible group over $\mathrm{Spf}(R)$ and H_0 be a p-divisible group over $\mathrm{Spec}(R/pR)$. Suppose given a quasi-isogeny
$$\rho : H_0 \to H \otimes_R R/pR.$$
Let $\mathscr{E}$ be the covariant Dieudonné crystal of H on $(\mathrm{Spec}(R)/\mathrm{Spec}(\mathbb{Z}_p))_{crys}$ and $\mathscr{E}_0$ be the one of H_0 on $(\mathrm{Spec}(R/pR)/\mathrm{Spec}(\mathbb{Z}_p))_{crys}$. This gives rise to an isomorphism
$$\rho_* : \mathscr{E}_{0,R \twoheadrightarrow R/pR}\left[\tfrac{1}{p}\right] \xrightarrow{\sim} \mathscr{E}_{R \twoheadrightarrow R}\left[\tfrac{1}{p}\right].$$
From this and the rigidity of quasi-isogenies we deduce the following result.

Proposition 3.2.5. — *Let $(\mathscr{E},\nabla)$ be the convergent isorcystal on $\mathcal{LT}_\eta$ associated to the universal deformation as an $\mathcal{O}_D^\times$-equivariant vector bundle equipped with an integrable connection. There is a canonical $\mathcal{O}_D^\times$-equivariant isomorphism*

$$\big(\boldsymbol{D}\otimes_{\breve{\mathbb{Q}}_p}\mathcal{O}_{\mathcal{LT}_\eta},\mathrm{Id}\otimes d\big)\xrightarrow{\sim}\big(\mathscr{E},\nabla\big)$$

and thus $(\mathscr{E},\nabla)$ is generated by its horizontal sections that are identified with D,

$$\boldsymbol{D}\xrightarrow{\sim}\mathscr{E}^{\nabla=0}.$$

The rank n vector bundle $\mathscr{E}$ can be though of as being the $(\mathcal{H}^1_{dR})^\vee$ of the universal deformation. There is an *Hodge filtration*

$$\mathrm{Fil}\,\mathscr{E}\subset\mathscr{E}$$

that is identified with $\omega_{H^D}\left[\frac{1}{p}\right]$ where H is the universal deformation and fits into the Hodge exact sequence

$$0\longrightarrow\underbrace{\omega_{H^D}\left[\tfrac{1}{p}\right]}_{\text{rk. }n-1}\longrightarrow\mathscr{E}\longrightarrow\underbrace{\omega_H\left[\tfrac{1}{p}\right]}_{\text{rk. }1}\longrightarrow 0$$

***Definition 3.2.6* (de Rham period morphism for Lubin-Tate spaces)** *We note*

$$\pi_{dR}:\mathcal{LT}_\eta\longrightarrow\mathbb{P}(\boldsymbol{D})\simeq\mathbb{P}^{n-1}$$

for the $\mathcal{O}_D^\times$-equivariant morphism defined by the Hodge filtration and Proposition 3.2.5.

Grothendieck-Messing theory says that to deform a p-divisible group is the same as to deform its Hodge filtration. From this the following basic result is elementary.

Proposition 3.2.7. — *The de Rham period morphism π_{dR} satisfies the following:*

1. *It is (partially proper) étale,*
2. *Its geometric fibers are the Hecke orbits*

The following result is quite deep and will be later reinterpreted in terms of the curve.

***Theorem 3.2.8* (Gross-Hopkins ([69])).** — *The de Rham period morphism*

$$\pi_{dR} : \mathscr{LT}_\eta \longrightarrow \mathbb{P}^{n-1}_{\breve{\mathbb{Q}}_p}$$

is surjective.

At the end we thus have an étale cover

$$\mathring{\mathbb{B}}^{n-1}_{\breve{\mathbb{Q}}_p} \longrightarrow \mathbb{P}^{n-1}_{\breve{\mathbb{Q}}_p}$$

with infinite discrete fibers.

The following result can be verified in an elementary way. We note $\mathbb{Q}_p^{cyc} := \widehat{\cup_{n\geq 1}\mathbb{Q}_p(\zeta_n)}$.

Proposition 3.2.9. — *The projective limit*

$$\mathscr{LT}_{\eta,\infty} := \varprojlim_K \mathscr{LT}_{\eta,K}$$

makes sense as a $\mathbb{Q}_p^{cyc}$*-perfectoid space.*

At the end we obtain the following picture.

$$\begin{tikzcd} \mathscr{LT}_{\eta,\infty} \arrow[d] \arrow[d, bend left, dash, "\underline{\mathrm{GL}_n(\mathbb{Z}_p)}"] \arrow[dd, bend right=60, dash, "\underline{\mathrm{GL}_n(\mathbb{Q}_p)^1}"'] \\ \mathscr{LT}_\eta \arrow[d, "\pi_{dR}"] \\ \mathbb{P}^{n-1}_{\breve{\mathbb{Q}}_p} \end{tikzcd}$$

where the torsors are pro-étale torsors.

3.2.3. The Hodge-Tate period morphism. — We now come to the other period morphism in the game. This first appeared in [**47**] and [**51**] where this is defined using integral models and flatification by blowups ([**118**]). This later appeared at the level of the Berkovich topological space for infinite level Shimura varieties in [**52**]. The next main step was its construction for infinite level perfectoid Shimura varieties in [**130**].

Recall that if G is a (commutative) finite locally free group scheme over a scheme there is a morphism of fppf sheaves

$$\begin{array}{rcl} G = \mathscr{H}om(G^D, \mathbb{G}_m) & \longrightarrow & \omega_{G^D} \\ f & \longmapsto & f^*\dfrac{dT}{T}. \end{array}$$

from G toward the fppf sheaf associated to the coherent sheaf ω_{G^D}.

Let now H be a p-divisible group over $\mathrm{Spec}(R)$ where R is a p-torsion free p-adic ring. Suppose moreover that R is integrally closed in $R\big[\frac{1}{p}\big]$. The preceding construction applied to the collection $(H[p^n])_{h\geq 1}$ defines a $\mathbb{Z}_p$-linear morphism

$$\boxed{\alpha_H : \underbrace{\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, H_\eta)}_{\mathbb{Z}_p\text{-module}} \underbrace{=}_{\substack{R\text{ int. closed}\\ \text{in } R\left[\frac{1}{p}\right]}} \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, H) \longrightarrow \underbrace{\omega_{H^D}}_{\substack{\text{projective of}\\ \text{finite type}\\ R\text{-module}\\ \text{of rk. ht}(H)-\dim(H)}}}$$

where H_η is the étale p-divisible group $H \otimes_R R\big[\frac{1}{p}\big]$. We note $\alpha_H \otimes 1$ for its linearization

$$\alpha_H \otimes 1 : \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, H_\eta) \otimes_{\mathbb{Z}_p} R \longrightarrow \omega_{H^D}.$$

The key result is now the following.

***Proposition 3.2.10* (Faltings, F. ([51])).** — *If $R = \mathcal{O}_C$ with $C|\mathbb{Q}_p$ a complete algebraically closed extension of $\mathbb{Q}_p$ then the preceding induces a complex*

$$0 \longrightarrow \omega_H^\vee(1) \xrightarrow{(\alpha_{H^D}\otimes 1)^\vee(1)} T_p(H) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \xrightarrow{\alpha_H \otimes 1} \omega_{H^D} \longrightarrow 0$$

whose cohomology is killed by $p^{\frac{1}{p-1}}$ if $p \neq 2$ and 4 if $p = 2$.

In particular one has an Hodge-Tate exact sequence

$$0 \longrightarrow \omega_H^\vee(1)\big[\tfrac{1}{p}\big] \xrightarrow{(\alpha_{H^D}\otimes 1)^\vee(1)} V_p(H) \otimes_{\mathbb{Q}_p} C \xrightarrow{\alpha_H \otimes 1} \omega_{H^D}\big[\tfrac{1}{p}\big] \longrightarrow 0.$$

Let us remark that $\frac{1}{p-1} = v_p(2i\pi)$ in the preceding proposition (the appearance of this is well explained in [**13**] via the functor $L\eta$). Using this result we can construct a morphism

$$\pi_{HT} : \mathcal{LT}_{\eta,\infty} \longrightarrow \check{\mathbb{P}}^{n-1}_{\breve{\mathbb{Q}}_p}$$

that is $\mathrm{GL}_n(\mathbb{Q}_p)^1$-equivariant and $\mathcal{O}_D^\times$-invariant. Here $\check{\mathbb{P}}^{n-1}_{\breve{\mathbb{Q}}_p}$ is the dual projective space classifying rank $n-1$ quotients of $\mathcal{O}^n$. Let us fix the isomorphism $\check{\mathbb{P}}^{n-1}_{\breve{\mathbb{Q}}_p} \xrightarrow{\sim} \mathbb{P}^{n-1}_{\breve{\mathbb{Q}}_p}$ given by the identification of $(\mathcal{O}^n)^\vee$ and $\mathcal{O}^n$ deduced from the dual of the canonical basis. This commutes with the action of $\mathrm{GL}_n(\mathbb{Q}_p)$ twisted by $g \mapsto {}^t g^{-1}$.

***Theorem 3.2.11* (Faltings, F.).** — *The image of*

$$\pi_{HT} : \mathcal{LT}_{\eta,\infty} \longrightarrow \check{\mathbb{P}}^{n-1}_{\breve{\mathbb{Q}}_p} \xrightarrow{\sim} \mathbb{P}^{n-1}_{\breve{\mathbb{Q}}_p}$$

is Drinfeld's space Ω*. Moreover,* $\mathcal{LT}_{\eta,\infty} \to \Omega$ *is a pro-étale* $\underline{\mathcal{O}_D}^\times$*-torsor that is identified with Drinfeld-tower.*

At the end we obtain the following diagram.

$$\begin{tikzcd}
 & \mathcal{LT}_{\eta,\infty} \arrow[loop above, "\underline{[\mathrm{GL}_n(\mathbb{Q}_p)\times D^\times]^1}"] \arrow[dl, "\pi_{dR}"] \arrow[dr, "\pi_{HT}"'] \arrow[dl, bend right=40, dash, "\underline{\mathrm{GL}_n(\mathbb{Q}_p)^1}"'] \arrow[dr, bend left=40, dash, "\underline{\mathcal{O}_D}^\times"] & \\
\mathbb{P}^{n-1}_{\breve{\mathbb{Q}}_p} \arrow[loop below, "\underline{\mathcal{O}_D^\times}"] & & \Omega_{\breve{\mathbb{Q}}_p} \arrow[loop below, "\underline{\mathrm{GL}_n(\mathbb{Q}_p)^1}"]
\end{tikzcd}$$

3.3. Rapoport-Zink spaces ([117])

3.3.1. Integral models in hyperspecial level. — Rapoport-Zink spaces are generalizations of Lubin-Tate and Drinfeld spaces. We only explain the $G = \mathrm{GL}_n$-case.

Let $\mathbb{H}$ be a p-divisible group over $\overline{\mathbb{F}}_p$ of dimension n and dimension d. Let (D, φ) be its covariant rational Dieudonné isocrystal. We note:

1. $G = \mathrm{GL}_n$,
2. G_b the reductive algebraic group over $\mathbb{Q}_p$ whose R-points are $\mathrm{Aut}(D \otimes_{\mathbb{Q}_p} R, \varphi \otimes \mathrm{Id})$.

Here the $b \in G(\breve{\mathbb{Q}}_p)$ refers to the matrix of Frobenius in a basis of D, in which case φ can be identified with $b\sigma \in G(\breve{\mathbb{Q}}_p) \rtimes \sigma$, see [**86**] where G_b is notes J_b. Then, G_b is identified with the twisted centralizer of b,

$$G_b(R) = \{g \in G(R \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p) \mid gb\sigma = b\sigma g\}$$

that is to say

$$gbg^{-\sigma} = b.$$

If $(\lambda_1, \ldots, \lambda_r)$ are the slopes of (D, φ) with respective multiplicities $(m_1, \ldots, m_r)$, then

$$G_b \simeq \prod_{i=1}^{r} \mathrm{GL}_{m_i}(D_{-\lambda_i})$$

where D_λ is the division algebra with invariant λ over $\mathbb{Q}_p$.

Definition 3.3.1. — *We note $\mathcal{M}$ for the functor on $W(\overline{\mathbb{F}}_p)$-schemes on which p is locally nilpotent such that*

$$\mathcal{M}(S) = \{(H, \rho)\}/\sim$$

where

— *H is a p-divisible group over $\overline{\mathbb{F}}_p$,*

— *$\rho : \mathbb{H} \times_{\overline{\mathbb{F}}_p} (S \bmod p) \longrightarrow H \times_S (S \bmod p)$ is quasi-isogeny.*

The $\overline{\mathbb{F}}_p$-points of this moduli are identified via Dieudonné theory with

$$\mathcal{M}(\overline{\mathbb{F}}_p) = \{M \subset D \text{ a lattice s.t. } pM \subset \varphi(M) \subset M\}.$$

This can be rewritten in the following way. Let $\mu : \mathbb{G}_m \to G$ be the Hodge cocharacter

$$\mu(z) = (\underbrace{z, \ldots, z}_{d \text{ times}}, \underbrace{1, \ldots, 1}_{n-d \text{ times}}).$$

Then, we have

$$\boxed{\mathcal{M}(\overline{\mathbb{F}}_p) = \left\{ g \in G\left(W(\overline{\mathbb{F}}_p)\left[\tfrac{1}{p}\right]\right)/G\left(W(\overline{\mathbb{F}}_p)\right) \;\middle|\; \mathrm{inv}(bg^\sigma, g) = \{\mu\} \right\}}$$

where

$$\begin{tikzcd}
G\left(W(\overline{\mathbb{F}}_p)\left[\frac{1}{p}\right]\right)/G\left(W(\overline{\mathbb{F}}_p)\right) \times G\left(W(\overline{\mathbb{F}}_p)\left[\frac{1}{p}\right]\right)/G\left(W(\overline{\mathbb{F}}_p)\right) \arrow[d, "\mathrm{inv}"] & ([g_1], [g_2]) \arrow[d, mapsto] \\
G\left(W(\overline{\mathbb{F}}_p)\right)\backslash G\left(W(\overline{\mathbb{F}}_p)\left[\frac{1}{p}\right]\right)/G\left(W(\overline{\mathbb{F}}_p)\right) & {[g_1^{-1} g_2]}
\end{tikzcd}$$

that is identified with $\mathrm{Hom}(\mathbb{G}_m, G)/G$-conjugacy via $\mu \mapsto [\mu(p)]$.

Thus, *the $\overline{\mathbb{F}}_p$-points of $\mathcal{M}$ can be identified with an affine Deligne-Lusztig set.* We refer to [**144**] and [**15**] for a more precise point of view on this where one can prove that such a set has a natural structure of perfect scheme locally of perfect finite type.

Further more, for any $x \in \mathcal{M}(\overline{\mathbb{F}}_p)$ if H_x is the associated p-divisible group, there is an identification

$$\widehat{\mathcal{M}}_{/x} = \mathrm{Def}(H_x)$$

that is representable by a formal scheme isomorphic to

$$\mathrm{Spf}(W(\overline{\mathbb{F}}_p)[\![x_1, \dots, x_{d(n-d)}]\!]).$$

The moduli space $\mathcal{M}$ is a much subtler version on the naive formal scheme

$$\coprod_{x \in \mathcal{M}(\overline{\mathbb{F}}_p)} \mathrm{Def}(H_x).$$

We have in fact the following theorem.

***Theorem 3.3.2* (Rapoport-Zink).** — *The functor $\mathcal{M}$ is a representable by a Spf($W(\overline{\mathbb{F}}_p)$)-formal scheme locally formally of finite type that is to say locally isomorphic to*

$$\mathit{Spf}\big(W(\overline{\mathbb{F}}_p)[\![X_1, \dots, X_s]\!]\langle Y_1, \dots, Y_t\rangle / \,\mathrm{Ideal}\big).$$

Moreover the irreducible components of $\mathcal{M}_{red}$ are projective algebraic varieties over $\overline{\mathbb{F}}_p$.

The action of $G_b(\mathbb{Q}_p)$ on the quasi-isogeny ρ defines a continuous action of $G_b(\mathbb{Q}_p)$ on $\mathcal{M}$,

$$\mathcal{M} \circlearrowleft G_b(\mathbb{Q}_p)$$

Example 3.3.3. — *From the fact that any degree* 0 *quasi-isogeny between* 1*-dimensional formal p-divisible groups over $\overline{\mathbb{F}}_p$ we deduce that in the Lubin-Tate case*

$$\mathcal{M} = \mathcal{LT} \overset{\mathcal{O}_D^\times}{\times} D^\times$$

that is (non-canonically) isomorphic to $\coprod_{\mathbb{Z}} \mathcal{LT}$ where the action of $\mathcal{O}_D^\times$ on the factor $\mathcal{LT}$ associated to $k \in \mathbb{Z}$ is the canonical one twisted by $d \mapsto \Pi^k d \Pi^{-k}$.

Remark 3.3.4. — *Although $\mathcal{M}$ is formally smooth, in general $\mathcal{M}_{red}$ is not smooth. The study of the geometry of $\mathcal{M}_{red}$ is an ongoing subject of research, see for example* [**60**] *following* [**141**]*. Their geometry when one adds ramified additional structures at parahoric levels, the search for a "nice" integral model has been thoroughly studied via the theory of local models (see for example* [**46**] *for some recent work) that began in* [**117**]*. They have been involved in many subjects related to the geometry of Shimura varieties.*

3.3.2. The tower. — Let

$$\begin{tikzcd}\mathcal{M}_\eta \arrow[loop right, "G_b(\mathbb{Q}_p)"]\end{tikzcd}$$

be the generic fiber of $\mathcal{M}$ as a locally of finite type adic space over $\mathrm{Spa}(\breve{\mathbb{Q}}_p)$. As before with the Lubin-Tate tower one obtains a tower

$$\begin{tikzcd}\left(\mathcal{M}_{\eta,K}\right)_K \arrow[loop right, "G(\mathbb{Q}_p)"] \arrow[loop below, "G_b(\mathbb{Q}_p)"]\end{tikzcd}$$

where K goes through the set of compact open subgroups of $G(\mathbb{Q}_p)$ and both actions commute. The definition of the action of $G(\mathbb{Q}_p)$ is more subtle than in the Lubin-Tate case since there is no "good notion" of integral level structures like this is the case for one dimensional p-divisible groups according to Drinfeld.

This relies on *Raynaud's flatification by blow-ups* ([**118**]): if S is a quasi-compact quasi-separated scheme, $G \to S$ is a finite locally free group scheme, $U \subset S$ is an open subset and $H \subset G \times_S U$ is a closed finite locally free sub-group scheme then after a blow-up supported on $S \smallsetminus U$ we can suppose that H extends to a closed subgroup scheme of G finite locally free over S. We refer fo this to [**117**, Section 5.34].

Example 3.3.5. — *For the Lubin-Tate tower, the associated RZ tower is*

$$(\mathcal{M}_{\eta,K})_K = (\mathcal{LT}_{\eta,K})_K \overset{\left(\mathrm{GL}_n(\mathbb{Q}_p)\times D^\times\right)^1}{\times} \mathrm{GL}_n(\mathbb{Q}_p) \times D^\times .$$

3.3.3. Period morphisms. — As for Lubin-Tate spaces, if $(\mathscr{E}, \nabla)$ is the convergent isocrystal associated to the universal deformation H on $\mathcal{M}$, the universal quasi-isogeny ρ induces an isomorphism

$$(D \otimes_{\breve{\mathbb{Q}}_p} \mathcal{O}_{\mathcal{M}_\eta}, \mathrm{Id} \otimes d) \xrightarrow{\sim} (\mathscr{E}, \nabla).$$

The Hodge filtration then defines a $G_b(\mathbb{Q}_p)$-equivariant morphism

$$\pi_{dR} : \mathcal{M}_\eta \longrightarrow \mathscr{F}_\mu$$

where $\mathscr{F}_\mu$ is the rigid analytic flag manifold associated to μ. This satisfies:

— This is étale and thus in particular its image is open,

— Its geometric fibers are the Hecke orbits.

The image of the étale morphism π_{dR},

$$\mathscr{F}_\mu^a := \mathrm{Im}(\pi_{dR}),$$

is the so-called *admissible open subset* of $\mathscr{F}_\mu$. This is a partially proper open subset inside the flag manifold $\mathscr{F}_\mu$. Little is known in general about it outside of the fact that

- there is an inclusion
$$\mathscr{F}_\mu^a \subset \mathscr{F}_\mu^{wa}$$
where $\mathscr{F}_\mu^{wa}$ is the so-called *weakly admissible open subset*, a very concrete open subset that is of the form
$$\mathscr{F}_\mu \setminus \bigcup_{\text{profinite}} \text{Schubert varities},$$
see [**117**].
- For $[K : \breve{\mathbb{Q}}_p] < +\infty$,
$$\mathscr{F}_\mu^a(K) = \mathscr{F}_\mu^{wa}(K)$$
that is to say $\mathscr{F}_\mu^a$ and $\mathscr{F}_\mu^{wa}$ have the same Tate classical points.
- There is a complete characterization of when $\mathscr{F}_\mu^a = \mathscr{F}_\mu^{wa}$, see [**73**] and [**28**].

The picture at this point for the Hodge-Tate period morphism is more difficult to describe since we first need to give a meaning to
$$\mathcal{M}_{\eta,\infty} := \varprojlim_K \mathcal{M}_{\eta,K}.$$
The fact is that this is a perfectoid space (if $\mathbb{H}$ is not étale) but we can make a sense out of it using integral models and blow-ups as in the Lubin-Tate case. At the end there is picture

$$\begin{tikzcd}
 & \mathcal{M}_{\eta,\infty} \arrow[loop above, "\underline{G(\mathbb{Q}_p)\times G_b(\mathbb{Q}_p)}"] \arrow[dl, "\pi_{dR}"'] \arrow[dr, "\pi_{HT}"] \arrow[dl, bend right=40, dash, "\underline{G(\mathbb{Q}_p)}"'] \arrow[dr, bend left=40, dash, "\underline{G_b(\mathbb{Q}_p)}"] & & \\
\mathscr{F}_\mu^a \arrow[loop below, "\underline{G_b(\mathbb{Q}_p)}"] & & \mathrm{Im}(\pi_{HT}) \arrow[loop below, "\underline{G(\mathbb{Q}_p)}"] \arrow[r, hook, "\substack{\text{locally}\\ \text{closed}}", "\substack{\text{open if}\\ [b] \text{ basic}}"'] & \mathscr{F}_{\mu^{-1}}
\end{tikzcd}$$

where $\mathrm{Im}(\varphi_{HT}) \subset \mathscr{F}_{\mu^{-1}}$ is not an open subset in general and is well defined in general only as a locally spatial diamond. When b is basic i.e. the isocrystal (D, φ) is isoclinic then $\mathrm{Im}(\pi_{HT})$ is open inside the dual flag manifold $\mathscr{F}_{\mu^{-1}}$ and this is a classical rigid analytic open subset.

3.3.4. Cohomology. — As for Lubin-Tate spaces one can use the cohomology spaces
$$H_c^\bullet(\mathcal{M}_K \hat{\otimes}_{\breve{\mathbb{Q}}_p} \mathbb{C}_p, \overline{\mathbb{Q}}_\ell)$$

as representations of $\mathscr{H}_{\overline{\mathbb{Q}}_\ell}(K\backslash G(\mathbb{Q}_p)/K)$ and $G_b(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$ to define *a kernel for the local Langlands correspondence.* More precisely, we look at the correspondence

$$\underbrace{\pi}_{\text{smooth rep.of } G_b(\mathbb{Q}_p)} \longmapsto \underbrace{\varinjlim_{K} \operatorname{Ext}^{\bullet}_{G_b(\mathbb{Q}_p)}(H^{\bullet}_c(\mathcal{M}_K \hat{\otimes}_{\breve{\mathbb{Q}}_p} \mathbb{C}_p, \overline{\mathbb{Q}}_\ell), \pi)}_{\text{smooth}\times\text{continuous representation of } G(\mathbb{Q}_p)\times W_{\mathbb{Q}_p}} \quad . \tag{1}$$

This was first studied in [**50**] and [**102**] and later in [**134**].

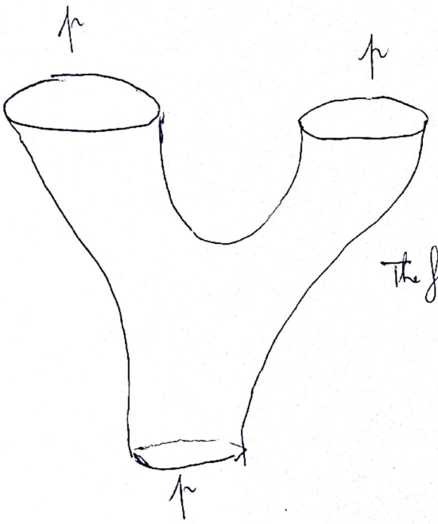
3.4. Final thoughts

Diagram (1) has been a great motivation for the geometrization conjecture of the local Langlands correspondence with relation with the correspondence given by the Hecke stack

$$\begin{tikzcd}
 & \text{Hecke} \arrow[dl] \arrow[dr] & \\
\text{Bun}_G & & \text{Bun}_G
\end{tikzcd}$$

The "cohomological kernel" of equation (1) given by the cohomology of Rapoport-Zink spaces is even a reminder that the preceding correspondence should be upgraded to a cohomological one.

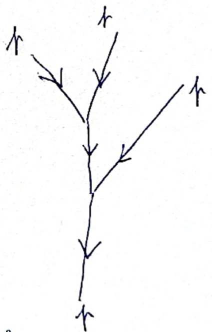
Little is known right now in general about the image of period morphisms for Rapoport-Zink spaces and more generally local Shimura varieties, the so-called admissible locus. The most general result is [**28**] that characterizes when the weakly admissible and admissible loci coincide. This gives for example explicit formulas for the admissible locus in the $U(1, n-1)$ or $SO(2, n-2)$ cases. The works [**128**] and [**58**] can be used to compute the connected components of the admissible locus ([**65**]).



The fusion of two copies of p

$$\mathrm{Div}^1 \xhookrightarrow{\mathrm{diag}} \mathrm{Div}^1 \times \mathrm{Div}^1$$

$$\mathrm{Div}^1 = \mathrm{Spa}(\mathbb{Q}_p)^{\diamond} / \varphi^{\mathbb{Z}}$$



The fusion of 3 copies of p

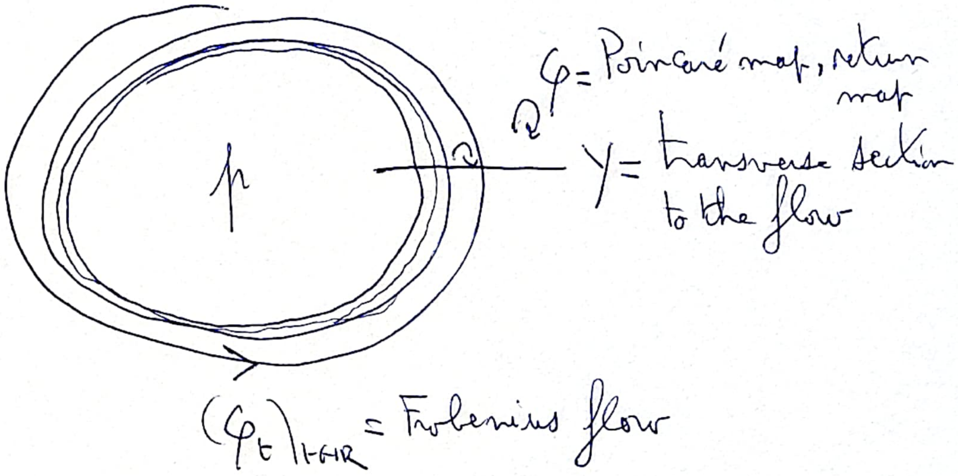
$$\mathrm{Div}^1 \hookrightarrow \mathrm{Div}^1 \times \mathrm{Div}^1 \hookrightarrow \mathrm{Div}^1 \times \mathrm{Div}^1 \times \mathrm{Div}^1$$

$$x \longmapsto (x, x)$$

$$(y, z) \longmapsto (y, y, z)$$

LECTURE 4

THE CURVE



Frobenius flow: limit cycle of length log p for each prime number p.

In this chapter we expose some of the main results of [**57**]. The appearance of the curve in p-adic Hodge theory has changed the domain as it is exposed in [**59**].

4.1. Holomorphic functions of the variable p ([**57**, Chapter 1])

Let E be a finite degree extension of $\mathbb{Q}$ with residue field $\mathbb{F}_q$. *Contrary to the "classical case", the curve "X" does not exists absolutely over* $\mathbb{F}_q$, it exists only after pull-back to an $\mathbb{F}_q$-perfectoid field F i.e. "X" makes no sense but X_F makes sense for each such F. Let us thus fix an $\mathbb{F}_q$-perfectoid field F. This is nothing else than a perfect, complete with respect to a non-trivial rank 1 valuation, non-archimedean field. One may, for example, want to consider $F = \mathbb{F}_q(\!(T^{1/p^\infty})\!)$ or $F = \widehat{\overline{\mathbb{F}_q(\!(T)\!)}}$.

Definition 4.1.1. — *We note $A_{\text{inf}} = W_{\mathcal{O}_E}(\mathcal{O}_F)$ equipped with its Frobenius φ lifting* Frob_q *modulo π.*

One has

$$A_{\text{inf}} \underbrace{=}_{\substack{\text{unique}\\ \text{writting}}} \Big\{ \sum_{n\geq 0} [a_n]\pi^n \mid a_n \in \mathcal{O}_F \Big\}$$

and

$$\varphi\Big(\sum_{n\geq 0}[a_n]\pi^n\Big) = \sum_{n\geq 0}[a_n^q]\pi^n.$$

We think of A_{inf} as being a ring of holomorphic functions where π is the variable and the coefficients are in $\mathcal{O}_F$. In fact, we want to define an open punctured disk of the variable π over F. This is the space Y_F that will come. For this space Y_F, the ring A_{inf} is the subring of $\mathcal{O}(Y_F)$ formed by holomorphic functions that are holomorphic at $\pi = 0$ and bounded by 1. We fix a pseudo-uniformizer ϖ of F.

Definition 4.1.2. — *We note $Y_F = \text{Spa}(A_{\text{inf}}, A_{\text{inf}}) \setminus V(\pi.[\varpi])$ equipped with its Frobenius φ.*

Let us begin by saying the following to remove any doubt.

Theorem 4.1.3. — *The following is satisfied:*

1. *Y_F is sous-perfectoid in the sense that for any $K|E$ perfectoid, $Y_F\hat{\otimes}_E K$ is a K-perfectoid space with tilting $\mathrm{Spa}(F)\times_{\mathrm{Spa}(\mathbb{F}_q)}\mathrm{Spa}(K^\flat)$ where φ is identified with $\mathrm{Frob}_q\times\mathrm{Id}$ ([**53**]).*
2. *Y_F is strongly Noetherian ([**85**]).*

In particular, via point (1) or (2), Huber's presheaf of holomorphic functions on $|Y_F|$ is a sheaf.

Remark 4.1.4. — *We will define later Y_S for any $\mathbb{F}_q$-perfectoid space S. Property (1) is still valid in this context but property (2) does not hold anymore in general.*

There is a radius continuous function

$$\begin{array}{rcl}\rho:|Y_F| & \longrightarrow &]0,1[\\ y & \longmapsto & q^{-\frac{v(\pi(y^{max}))}{v([\varpi](y^{max}))}}\end{array}$$

where y^{max} is the maximal generalization of y seen as a Berkovich point that is to say a valuation with values in $\mathbb{R}$. This extends to a continuous function

$$|\mathrm{Spa}(A_{\mathrm{inf}},A_{\mathrm{inf}})|\longrightarrow[0,1].$$

where $\rho=0$ corresponds to the Cartier divisor $\pi=0$ and $\rho=1$ to $[\varpi]=0$. Those two divisors are fixed by φ and one has the formula

$$\rho(\varphi(y))=\rho(y)^{1/q}.$$

In particular, φ acts properly discontinuously without fixed points on $|Y_F|$.

For any compact interval $I\subset]0,1[$ of the form $[a,b]$ with $a,b\in p^{\mathbb{Q}}$, the annulus

$$Y_{F,I}=\{y\mid\rho(y)\in I\}$$

is a rational domain and in particular affinoid (even affinoid sous-perfectoid). One has

$$Y_F=\bigcup_{\substack{0<a\leq b<1\\ a,b\in q^{\mathbb{Q}}}}\underbrace{Y_{F,[a,b]}}_{\substack{\text{affinoid}\\ \text{annulus}}}.$$

The main difficulty (and this is one of the main reasons why "p-adic Hodge theory is difficult") is that $\mathcal{O}(Y_F)$ is defined as a (Frechet) completion of $A_{\mathrm{inf}}[\frac{1}{\pi},\frac{1}{[\varpi]}]$ and *there is no explicit formula, typically as a power series expansion, for elements in this ring.*

Nevertheless, functions that are holomorphic at $\pi=0$ have an explicit description. One can in fact introduce

$$\mathcal{Y}_F=\mathrm{Spa}\left(W_{\mathcal{O}_E}(\mathcal{O}_F),W_{\mathcal{O}_E}(\mathcal{O}_F)\right)\smallsetminus V([\varpi]).$$

This is an adic space equipped with a radius function

$$\rho:|Y_F|\longrightarrow[0,1[$$

with $Y_F = \{\rho \neq 0\} = \mathcal{Y}_F \setminus V(\pi)$. For any $a \in p^{\mathbb{Q}} \cap]0,1[$,

$$\mathcal{Y}_{F,[0,a]} = \{y \in \mathcal{Y}_F \mid \rho(y) \in [0,a]\}$$

is affinoid and

$$\mathcal{O}(\mathcal{Y}_{F,[0,a]}) \subset W_{\mathcal{O}_E}(F)$$

explicit.

Remark 4.1.5. — *Although we are mainly interested in the unequal characteristic case, we can consider the so-called equal characteristic case too, $E = \mathbb{F}_q((\pi))$. In this case, one has $W_{\mathcal{O}_E}(R^+) = R^+[\![\pi]\!]$ and $Y_F = \mathbb{D}^*_F$, an open punctured disk over* Spa(F) *where the variable is π, and $\mathcal{Y}_F = \mathbb{D}_F$ the non-punctured disk. This is the case studied in* [**74**] *"before the curve".*

4.2. Newton polygons and Weierstrass factorization

A Key definition is the following.

Definition 4.2.1. — *An element $\xi = \sum_{n\geq 0}[a_n]\pi^n \in A_{\text{inf}}$ is distinguished of degree $d \geq 1$ if*

— $a_0, \dots, a_{d-1} \in \mathfrak{m}_F$,
— $a_0 \neq 0$,
— $a_d \in \mathcal{O}_F^\times$.

The product of a degree d and degree d' distinguished elements is a degree $d + d'$ distinguished element. If ξ is distinguished of degree d and $u \in A_{\text{inf}}^\times$ then $u\xi$ is distinguished of degree d.

Another key property if the following. Let us normalize the valuation v on F such that $v(\varpi) = 1$. For any $r > 0$ and $f = \sum_{n\geq 0}[a_n]\pi^n \in A_{\text{inf}}$, the formula

$$v_r(f) = \inf_{n\geq 0} v(a_n) + rn$$

defines a Gauss valuation $Gauss_r \in |Y_F|$ with $\rho(Gauss_r) = q^{-r}$. The function $r \mapsto v_r(f)$ is a concave polygon and using a process of (inverse) Legendre transform we can deduce from it a Newton polygon. More precisely:

For any interval $I \subset]0,1[$ with extremities in $q^{\mathbb{Q}}$ and any $f \in \mathcal{O}(Y_{F,I}) \setminus \{0\}$, one can define naturally a *Newton polygon* $\text{Newt}_I(f)$ with breakpoints at integral x-coordinates and whose slopes are in $-\log_q I$ in such a way that

1. For $f = \sum_{n\gg -\infty}[a_n]\pi^n \in A_{\text{inf}}[\frac{1}{\pi}, \frac{1}{[\varpi]}]$, $\text{Newt}_{]0,1[}(f)$ is the convex envelope of $(v(a_n), n)_{n\in\mathbb{Z}}$,
2. $\text{Newt}_I(fg)$ is obtained by concatenation from $\text{Newt}_I(f)$ and $\text{Newt}_I(g)$.

Here is the main factorization result we obtained with Fontaine.

Theorem 4.2.2 **([57, Chapter 2 and 3]).** — *The following is satisfied:*

1. *For $\xi \in A_{\mathrm{inf}}$ distinguished irreducible of degree d, $K_\xi = A_{\mathrm{inf}}[\frac{1}{\pi}]/\xi$ is a perfectoid field and the map $x \mapsto \big([x^{1/p^n}] \bmod \xi\big)_{n\geq 0}$ induces an embedding $F \hookrightarrow K_\xi^\flat$ such that*
$$[K_\xi^\flat : F] = d.$$
2. *If F is algebraically closed then any irreducible distinguished element ξ is of degree 1. We thus has*
$$K_\xi^\flat = F.$$
Moreover $\xi = u.(\pi - [a])$ with $a \in \mathfrak{m}_F \smallsetminus \{0\}$ and $u \in A_{\mathrm{inf}}^\times$.
3. *For any $I \subset]0,1[$ with extremities in $q^{\mathbb{Q}}$, for any $f \in \mathcal{O}(Y_{F,I}) \smallsetminus \{0\}$, and any slope λ of* $\mathrm{Newt}_I(f)$*, there exists a factorization*
$$f = g.\xi$$
where $g \in \mathcal{O}(Y_{F,I})$, ξ is distinguished irreducible with $\mathrm{Newt}_{]0,1[}(\xi)$ *a line with slope λ between 0 and* $\deg(\xi)$*.*

Example 4.2.3 **(Weierstrass factorization).** — *If F is algebraically closed and ξ is distinguished of degree d one can write*
$$\xi = u(\pi - [a_1]) \times \cdots \times (\pi - [a_d])$$
where u is a unit and $v(a_1), \ldots, v(a_d)$ are the slopes of $\mathrm{Newt}_{]0,1[}(\xi)$.

Definition 4.2.4. — *A point $y \in |Y_F|$ of the form $V(\xi)$ with ξ distinguished irreducible is called a classical point of Y_F. By definition,* $\deg(y) := \deg(\xi)$.

Thus, for $y \in |Y_F|^{cl}$, $K(y)$ is a perfectoid field with
$$[K(y)^\flat : F] = \deg(y).$$
This is a form of the point of view that one may think of Y_F as a moduli of untilts of the perfectoid field F.

4.3. The adic curve

We finally arrive to the curve.

Definition 4.3.1. — *We note*

$$X_F = Y_F/\varphi^{\mathbb{Z}}$$

as a quasi-compact quasi-separated E-adic space.

This is thus strongly Noetherian sous-perfectoid with

$$(X_F\hat{\otimes}_F K)^\flat = (\mathrm{Spa}(F) \times_{\mathrm{Spa}(\mathbb{F}_q)} \mathrm{Spa}(K^\flat))/\varphi^{\mathbb{Z}} \times \mathrm{Id}\,.$$

This is a curve because of the following. This uses heavily the preceding factorization results.

Theorem 4.3.2. — *For any compact interval $I \subset]0,1[$ with extremities in $p^{\mathbb{Q}}$, the Banach E-algebra $\mathcal{O}(Y_{F,I})$ is a P.I.D. with an identification*

$$\mathrm{Spm}(\mathcal{O}(Y_{F,I})) = |Y_{F,I}|^{cl}.$$

One deduces from this result that for any $U \subset Y_F$ an affinoid open subset, $\mathcal{O}(U)$ is a P.I.D. and thus *X_F is a curve*. In particular one has the following: for any $x \in |X_F|^{cl}$,

- $\mathcal{O}_{X,x}$ is an Henselian D.V.R. such that if $y \mapsto x$ with $y \in |Y_F|^{cl}$, $y = V(\xi)$, $\mathcal{O}_{X_F,x} \xrightarrow{\sim} \mathcal{O}_{Y_F,y}$,
- in particular *the residue field at x, $K(x)$, is perfectoid*,
- and one has

$$\widehat{\mathcal{O}}_{X_F,x} \xrightarrow{\sim} B^+_{dR}(K(x))$$

as complete D.V.R..

4.3.1. The schematical curve. — The adic curve X_F does not come alone. It is in fact equipped with an "ample" line bundle.

Definition 4.3.3. — *We note $\mathcal{O}_{X_F}(1)$ for the line bundle on X_F associated to the automorphy factor $\varphi \mapsto \pi^{-1}$ on Y_F equipped with its action of $\varphi^{\mathbb{Z}}$.*

This means that the pullback of $\mathcal{O}_{X_F}(1)$ to Y_F is trivialized and the descent datum along the cover $Y_F \to X_F$ is given by $\varphi \mapsto \pi^{-1}$.

Let us define

$$\mathbb{B}(F) := \mathcal{O}(Y_F)$$

as a Frechet E-algebra equipped with the continuous automorphism φ. One has for any $d \in \mathbb{Z}$,

$$H^0(X_F, \mathcal{O}(d)) = \underbrace{\mathbb{B}(F)^{\varphi=\pi^d}}_{\{f \in \mathbb{B}(F) \mid \varphi(f)=\pi^d f\}}$$

that is

- 0 if $d < 0$,
- E if $d = 0$,
- an infinite dimension E-Banach space if $d > 0$.

Remark 4.3.4. — *Suppose $E = \mathbb{Q}_p$. If $y \in |Y_F|^{cl}$ there is an inclusion*

$$\bigcap_{n\geq 0} \varphi^n\big(B^+_{cris}(\mathcal{O}_{K(y)}/p)\big) \subset \mathbb{B}(F)$$

that induces for all $d \in \mathbb{Z}$ an identification

$$B^+_{cris}(\mathcal{O}_{K(y)}/p)^{\varphi=p^d} \xrightarrow{\sim} \mathbb{B}(F)^{\varphi=p^d}.$$

*This makes the link between the "classical" Fontaine's period rings ([**59**]) and the ring $\mathbb{B}(F)$.*

We now declare that $\mathcal{O}(1)$ is ample.

Definition 4.3.5. — *We define*

$$P_F = \bigoplus_{d\geq 0} H^0(X_F, \mathcal{O}_{X_F}(d))$$

as a graded E-algebra and

$$\mathfrak{X}_F = \mathrm{Proj}(P_F)$$

as an E-scheme.

One of the main structure results for the graded algebra P_F is the following.

Theorem 4.3.6. — *Suppose that F is algebraically closed. The graded E-algebra P_F is graded factorial in the sense that the commutative monoid*

$$\coprod_{n\geq 0} P_{F,n} \setminus \{0\}/E^\times$$

is commutative free on degree 1 non-zero elements up to $E^\times$.

In other terms, for any $f \in P_{F,d} \setminus \{0\}$, one can write

$$f = t_1 \dots t_d$$

where $t_1, \dots t_d \in P_{F,1} \setminus \{0\}$ are uniquely determined up to multiplication by an element of $E^\times$. The proof of this theorem relies on two facts:

1. Using the preceding results on the factorization of elements and Newton polygons one defines
$$\mathrm{Div}^+(Y_F) = \Big\{ \sum_{y\in |Y_F|^{cl}} a_y[y] \mid a_y \in \mathbb{N}, \{y \mid a_y \neq 0\} \text{ is locally finite} \Big\}$$
and an injection of monoids
$$\mathrm{div} : \mathcal{O}(Y_F) \smallsetminus \{0\}/E^\times \hookrightarrow \mathrm{Div}^+(Y_F)$$
given by "the divisor of an holomorphic function". In particular, this defines an injection
$$\coprod_{n\geq 0} P_{F,n} \smallsetminus \{0\}/E^\times \hookrightarrow \mathrm{Div}^+(Y_F)^{\varphi=\mathrm{Id}}$$
where the right hand side is the free commutative monoid on $\{\sum_{n\in\mathbb{Z}}[\varphi^n(y)] \in \mathrm{Div}^+(Y_F)\ y \in |Y_F|^{cl} \bmod \varphi^{\mathbb{Z}}\}$.
2. For any $y \in |Y_F|^{cl}$ one can construct (this is where the hypothesis F alg. closed shows up) some $t \in P_{F,1} \smallsetminus \{0\}$ such that $\mathrm{div}(t) = \sum_{n\in\mathbb{Z}}[\varphi^n(y)]$. In fact, when $E = \mathbb{Q}_p$, it suffices to take $t =$ Fontaine's $2i\pi$ associated to the algebraically closed field $K(y)|\mathbb{Q}_p$.

Theorem 4.3.7. — *The scheme $\mathfrak{X}_F$ is a Dedekind scheme.*

One can go further into the structure of $\mathfrak{X}_F$ using GAGA. More precisely, for any $t \in P_{F,1} \smallsetminus \{0\}$, one has $D^+(t) = \mathrm{Spec}(B_{e,t})$ with
$$B_{e,t} = \mathbb{B}(F)[\tfrac{1}{t}]^{\varphi=\mathrm{Id}}$$
that is identified with $P_F[\frac{1}{t}]_0$. The morphism
$$B_{e,t} \hookrightarrow \mathbb{B}(F)[\tfrac{1}{t}] \to \mathcal{O}(Y_F \smallsetminus V(t))$$
induces a morphism of ringed spaces $(Y_F \smallsetminus V(t))/\varphi^{\mathbb{Z}} \to D^+(t)$. When t varies this defines a GAGA morphism of ringed spaces
$$X_F \longrightarrow \mathfrak{X}_F.$$
One then has the following result.

Theorem 4.3.8. — *Consider the GAGA morphism $X_F \to \mathfrak{X}_F$.*

1. *It induces a bijection $|X_F|^{cl} \xrightarrow{\sim} |\mathfrak{X}_F|^{closed}$ (closed points).*
2. *For any $x \in |X_F|^{cl}$, if $x \mapsto x' \in |\mathfrak{X}_F|$, the morphism of D.V.R. $\mathcal{O}_{\mathfrak{X}_F,x'} \to \mathcal{O}_{X_F,x}$ induces an isomorphism*
$$\widehat{\mathcal{O}}_{\mathfrak{X}_F,x'} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X_F,x} = B^+_{dR}(K(x)).$$

In particular the residue fields at closed points of $\mathfrak{X}_F$ are perfectoid fields.

Let us note for x a closed point of $\mathfrak{X}_F$

$$\deg(x) = [K(x)^\flat : F].$$

We can now dig a little bit deeper into the structure of $\mathfrak{X}_F$.

Theorem 4.3.9. — *1. The curve is complete: for any $f \in E(\mathfrak{X}_F)^\times$,*

$$\deg(\mathrm{div}(f)) = 0.$$

2. If F is algebraically closed then for any $t \in P_{F,1} \smallsetminus \{0\}$, $V^+(t)$ is one closed point ∞_t and $\mathfrak{X}_F \smallsetminus \{\infty_t\} = \mathrm{Spec}(B_{e,t})$ with $B_{e,t}$ a P.I.D.. In other words,

$$\mathrm{Pic}^0(\mathfrak{X}_F) = 0.$$

3. If F is algebraically closed one has

$$H^1(\mathfrak{X}_F, \mathcal{O}) = 0$$

and

$$H^1(\mathfrak{X}_F, \mathcal{O}(-1)) \neq 0.$$

Said in another way, for the stathme $\deg_t := -\mathrm{ord}_{\infty_t} : B_{e,t} \to \mathbb{N} \cup \{-\infty\}$, the couple $(B_{e,t}, \deg_t)$ is not euclidean but almost euclidean: for any $a, b \in B_{e,t}$ with $b \neq 0$ we can write $a = bx + y$ with $\deg_t(y) \leq \deg_t(b)$ but not $\deg_t(y) < \deg_t(b)$ in general.

4.4. GAGA

The following GAGA result is satisfied.

Theorem 4.4.1 **([58, Chapter II]).** — *The GAGA morphism $X_F \to \mathfrak{X}_F$ induces an equivalence of categories*

$$\big\{\textit{vector bundles on } \mathfrak{X}_F\big\} \xrightarrow{\sim} \big\{\textit{vector bundles on } X_F\big\}.$$

At the heart of the preceding theorem is the following result due to Kedlaya: for any vector bundle $\mathscr{E}$ on X_F, for $n \gg 0$ one has

— $H^1(X_F, \mathscr{E}(n)) = 0$,

— $\mathscr{E}(n)$ is generated by its global sections.

We refer to [**58**, Chapter II] for this. Let us remark that the preceding result extends when we replace F by any $\mathbb{F}_p$-perfectoïd ring. More precisely, if (R, R^+) is an $\mathbb{F}_p$-affinoid perfectoid ring one can define a schematical curve $\mathfrak{X}_{R,R^+}$ as before by declaring $\mathcal{O}(1)$ ample. Although this is not a Noetherian scheme in general, *GAGA theorem still holds in this context*:

$$\boxed{\{\text{vector bundles on } \mathfrak{X}_{R,R^+}\} \xrightarrow{\sim} \{\text{vector bundles on } X_{R,R^+}\}.}$$

4.5. Recovering classical p-adic Hodge theoretic objects geometrically

Let $\mathcal{R}^{cl}$ be the "classical non-perfectoid" Robba ring, if $\mathbb{B}^1_{\mathbb{Q}_p} = \{|T| \leq 1\}$ is the adic closed ball with radius 1 over $\mathbb{Q}_p$ with coordinate T, $S = \{|T| = 1\} \subset \mathbb{B}^1_{\mathbb{Q}_p}$, one has

$$\mathring{\mathbb{B}}^1_{\mathbb{Q}_p} = \mathbb{B}^1_{\mathbb{Q}_p} \setminus \overline{S}$$

and

$$\mathcal{R}^{cl} = \varinjlim_{S \subset\subset U} \mathcal{O}\left(U \setminus \overline{S}\right)$$

with U open and where $S \subset\subset U$ means $\overline{S} \subset U$ in the adic space. Let $F = \mathbb{Q}_p^{cyc,\flat}$ with pseudo-uniformizer $\varpi = \epsilon - 1$ where ε is a generator of $\mathbb{Z}_p(1)$. We note $\mathcal{R}_F$ for the "perfectoid Robba ring"

$$\mathcal{R}_F = \varinjlim_{\substack{U \subset \mathcal{Y}_F \\ V(\pi) \subset U}} \mathcal{O}\left(U \setminus V(\pi)\right).$$

The ring $W(\mathcal{O}_F)$ is $(p, [\varpi])$-adically complete and there is a morphism

$$\begin{array}{ccc} \mathbb{Q}_p\langle T\rangle & \longrightarrow & W(\mathcal{O}_F)\left[\frac{1}{p}\right] \\ T & \longmapsto & [\varpi] \end{array}$$

This induces a morphism of Robba rings

$$\mathcal{R}^{cl} \longrightarrow \mathcal{R}_F.$$

One then one has the following result.

Theorem 4.5.1 **([8]).** — *Scalar extension from $\mathcal{R}^{cl}$ to $\mathcal{R}_F$ induces an equivalence between*

1. *the category of (φ, Γ)-modules over $\mathcal{R}^{cl}$,*
2. *the category of Γ-equivariant vector bundles on X_F,*
3. *the category of Gal$(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$-equivariant vector bundles on $X_{\mathbb{C}_p^\flat}$.*

We won't go further in this arithmetic direction. Let us just say that this result gives an arithmetic description of the groupoid of morphisms

$$\mathrm{Spa}(\mathbb{Q}_p)^\diamond \longrightarrow \mathrm{Bun}_n.$$

4.6. Étale covers and the starting point of a geometric Langlands program on the curve

We have the following theorem that is worth stating since this was a motivation for developing a geometric Langlands programm on the curve.

***Theorem 4.6.1* ([57]).** — *Suppose the perfectoid field $F|\mathbb{F}_q$ is algebraically closed. Then,*

$$\pi_1(\mathfrak{X}_F) = Gal(\overline{E}|E)$$

in the sense that there is an equivalence of categories via pullback by $\mathfrak{X}_F \to Spec(E)$

$$\{\textit{finite étale covers of } Spec(E)\} \xrightarrow{\sim} \{\textit{finite étale covers of } \mathfrak{X}_F\}.$$

The starting point of the geometric Langlands program of Drinfeld ([**41**]) and Laumon ([**95**]) is, in fact, a geometrically irreducible rank n local system $\mathscr{E}$ on a curve over a finite field X. Starting from this datum the purpose is to construct a perverse sheaf $\mathscr{F}$ on Bun_n the Artin stack of rank n vector bundles on X satisfying:

1. This is an Hecke eigensheaf with eigenvalue $\mathscr{E}$ in the following sense. Let

$$\begin{tikzcd}
 & \mathrm{Hecke}_{n,i} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathrm{Bun}_n & & \mathrm{Bun}_n \times X
\end{tikzcd}$$

be the stack of quadruples $(\mathscr{E}_1, \mathscr{E}_2, u, f)$ where $f : S \to X$ with associated degree 1 relative Cartier divisor $i = f \times \mathrm{Id} : S \hookrightarrow X \times S$, $\mathscr{E}_1$ and $\mathscr{E}_2$ are rank n vector bundles on $X \times S$ and

$$u : \mathscr{E}_1 \hookrightarrow \mathscr{E}_2$$

is a monomorphism of vector bundles with cokernel isomorphic to $i_*\mathcal{G}$ with $\mathcal{G}$ a rank i vector bundle on S. Then one asks that

$$Rp_{2*}p^{1*}\mathscr{F} \simeq \mathscr{F} \boxtimes \wedge^i \mathscr{E}$$

up to a shift and a Tate twist.

2. The trace of Frobenius function of $\mathscr{F}$ is, up to a scalar, the everywhere unramified automorphic function associated to $\mathscr{E}$ via the "classical" everywhere unramified Langlands correspondence for GL_n ([**90**]).

This result has been established for GL_n ([**61**], [**63**], [**96**]).

One of the starting points of the geometrization conjecture for "the curve" was the preceding result: a $\overline{\mathbb{Q}}_\ell$ étale local system on the curve is nothing else than an irreducible continuous representation of $\mathrm{Gal}(\overline{E}|E)$ with values in a rank n $\overline{\mathbb{Q}}_\ell$-vector space; the classical local Langlands program for GL_n ([**72**]) seeks to attach to this type of datum an irreducible supercuspidal representation of $\mathrm{GL}_n(E)$ with values in a $\overline{\mathbb{Q}}_\ell$-vector space. The temptation to upgrade this to a "perverse sheaf on the stack of rank n vector bundles is then quite tempting".

At the end, this is not the Galois group $\mathrm{Gal}(\overline{E}|E)$ and the curve that shows up but rather the Weil group W_E and the object Div^1 (see section 9.1) that looks like the curve but isn't the curve (see remark 9.1.3).

4.7. Final thoughts

The curve has been used in [**57**] to give new simpler and more conceptual proofs of "weakly admissible implies admissible" and "the p-adic monodromy theorem" by upgrading a galois representation V of $\mathrm{Gal}(\overline{K}|K)$ to a Galois equivariant vector bundle

$$V \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathfrak{X}_{\widehat{\overline{K}}}}$$

and looking at its Galois equivariant modifications at $\infty \in |\mathfrak{X}_{\widehat{\overline{K}}^\flat}|$ corresponding to the untilt $\widehat{\overline{K}}$ of $\widehat{\overline{K}}^\flat$. The theorem ([**57**]) that says that slope 0 semi-stable vector bundles on $\mathfrak{X}_F$ are the same as Galois representations of $\mathrm{Gal}(\overline{F}|F)$ is very reminiscent of Narasimhan-Seshadri'w work ([**111**]).

This is now a standard object in p-adic Hodge theory, see for example [**114**] for a use in Iwasawa theory. Its similarities with "classical curves" has been a great motivation for the development of a geometric Langlands program on it. Some of the results of [**57**] are still at the heart of the geometrization of the local Langlands correspondence. For example, the fact that, when F is algebraically closed, the factorization result that says that any $f \in \mathbb{B}(F)^{\varphi=\pi^d}$ non zero one can write

$$f = t_1 \dots t_d$$

with $t_1, \dots, t_d \in \mathbb{B}(F)^{\varphi=\pi} \setminus \{0\}$ well defined up to multiplication by a scalar, is at the heart of [**56**].

LECTURE 5

G-BUNDLES ON THE CURVE

Vector bundles on the curve were first introduced and classified in [**57**] were they are used to give a new more conceptual proof of "weakly admissible implies admissible" and the p-adic monodromy theorem, see [**57**, Chapter 10]. Here the main tool is to see a Galois representation V of $\mathrm{Gal}(\overline{K}|K)$, $K|\mathbb{Q}_p$ discrete with perfect residue field, as a Galois equivariant vector bundle $V \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathfrak{X}_{\widehat{\overline{K}}^\flat}}$ and look at its Galois equivariant modifications.

It later appeared in [**55**] that the classification of G-bundles for any reductive p-adic group G was very rich and interesting, forgetting any arithmetic structure like a Galois action. The results of [**55**], in particular the appearance of Kottwitz set $B(G)$ has been a key point for the author. Harisch-Chandra/Langlands philosophy says that we have to work with any reductive group G, not just only GL_n, in the Langlands program, and this is an application of this mindset.

5.1. Vector bundles on the curve

5.1.1. Isocrystals. — Let $\overline{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$. We note $\breve{E} = \widehat{E^{un}}$ with its Frobenius σ lifting Frob_q. Recall the following definition.

Definition 5.1.1. — *An isocrystal is a pair (D, φ) where D is a finite dimensional $\breve{E}$-vector space and φ a σ-linear automorphism of D.*

Those are classified by Dieudonné-Manin in terms of slopes: the category of isocrystals is semi-simple with a unique isoclinic of slope λ object for each $\lambda \in \mathbb{Q}$. If $\lambda = \frac{d}{h}$ with $h \geq 1$ and $(d, h) = 1$ then the associated simple object has dimension h over $\breve{E}$

and in a suitable basis φ is given by

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & \pi^d \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \sigma.$$

More precisely, one has an orthogonal decomposition

$$\mathrm{Isoc} = \overset{\perp}{\bigoplus_{\lambda \in \mathbb{Q}}} \underbrace{\mathrm{Isoc}^{\lambda}}_{\substack{\text{Slope } \lambda \\ \text{isoclinic isocrystals}}}$$

where Isoc^{λ} has a unique simple object as described before. We refer to [**148**] for a proof of the Dieudonné-Manin decomposition.

5.1.2. A simple construction. — Let $F|\overline{\mathbb{F}}_q$ be a perfectoid field. We have a morphism

$$\begin{tikzcd} Y_F \arrow[loop right, "\varphi"] \arrow[d] \\ \mathrm{Spa}(\breve{E}) \arrow[loop right, "\sigma"] \end{tikzcd}$$

By pullback this induces a functor from σ-equivariant vector bundles on $\mathrm{Spa}(\breve{E})$, i.e. isocrystals, to φ-equivariant vector bundles on Y_F, i.e. vector bundles on X_F.

Definition 5.1.2. — *1. We note $\mathscr{E}(D,\varphi)$ the vector bundle*

$$Y_F \overset{\varphi^{\mathbb{Z}}}{\times} D$$

on X_F associated to the isocrystal (D,φ).

2. For $\lambda \in \mathbb{Q}$ we note

$$\mathcal{O}_{X_F}(\lambda)$$

for $(D,\varphi) = (\breve{E}^h, \varphi)$ where $\lambda = \frac{d}{h}$ with $h \geq 1$, $(d,h) = 1$, and

$$\varphi = \begin{pmatrix} 0 & 0 & \cdots & 0 & \pi^{-d} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \sigma.$$

The global sections of $(\mathscr{E},\varphi)$ are given by

$$\boxed{H^0(X_F,\mathscr{E}(D,\varphi)) = \big(D\otimes_{\breve{E}}\mathbb{B}(F)\big)^{\varphi=\mathrm{Id}}}$$

where "φ" here means $\varphi\otimes\varphi$ acting on $D\otimes_{\breve{E}}\mathbb{B}(F)$. In particular

$$H^0(X_F,\mathcal{O}(\lambda)) = \mathbb{B}(F)^{\varphi^h=\pi^d}.$$

We use the same notations for the associated vector bundle on $\mathfrak{X}_F$ via GAGA (Theorem 4.4.1). In fact we have the following formula: if

$$M(D,\varphi) = \bigoplus_{d\geq 0}\big(D\otimes_{\breve{E}}\mathbb{B}(F)\big)^{\varphi=\pi^d}$$

as a graded P_F-module then

$$\mathscr{E}(D,\varphi) = \widetilde{M(D,\varphi)}$$

on $\mathfrak{X}_F = \mathrm{Proj}(P_F)$.

At the end there is a $\otimes$-*exact functor* between monoidal categories

$$\boxed{(2) \qquad \mathscr{E}(-) : \mathrm{Isoc} \xrightarrow{\otimes} \{\text{vector bundles on } \mathfrak{X}_F\}.}$$

Remark 5.1.3. — *The upgrade of an isocrystal to a vector bundle on the curve is a key point, see remark 2.4.2. In fact, "*$\mathrm{Spa}(\breve{E})/\sigma^{\mathbb{Z}}$*" has no "nice" geometric structure contrary to* $Y_F/\varphi^{\mathbb{Z}}$*.*

5.1.3. Cohomology. — For $n\geq 1$ let $E_n|E$ be the degree n unramified extension of E inside $\breve{E}$. There is an identification

$$\begin{aligned} X_{F,E}\otimes_E E_n &= X_{F,E_n}\\ \mathfrak{X}_{F,E}\otimes_E E_n &= \mathfrak{X}_{F,E_n}\end{aligned}$$

where X_{F,E_n} and $\mathfrak{X}_{F,E_n}$ are defined using $Y_{F,E_n} = Y_{F,E}$ but where the Frobenius has been replace by φ^n. At the end the finite Galois cover

$$\begin{tikzcd} Y_F/\varphi^{n\mathbb{Z}} \arrow[d] \arrow[d, phantom, bend right=60, "\mathbb{Z}/n\mathbb{Z}"'] \\ Y_F/\varphi^{\mathbb{Z}} \end{tikzcd}$$

is identified with

$$\begin{tikzcd} X_{F,E_n} \arrow[r, equal] \arrow[d] \arrow[d, phantom, bend right=60, "\mathrm{Gal}(E_n|E)"'] & X_{F,E} \otimes_E E_n \\ X_{F,E}. & \end{tikzcd}$$

Let us note π_n this finite étale morphism. One easily verifies that if $\lambda = \frac{d}{h}$ as before then

$$\mathcal{O}_{X_{F,E}}(\lambda) = \pi_{h*}\mathcal{O}_{X_{F,E_h}}(d).$$

Using this, up to replacing E by a finite unramified extension, one deduces using Theorem 4.3.9 the following for $\lambda \in \mathbb{Q}$ and F algebraically closed:

- $H^0(X_F, \mathcal{O}(\lambda)) = \begin{cases} 0 \text{ if } \lambda < 0 \\ E \text{ if } \lambda = 0 \\ \text{an infinite dim. } E\text{-Banach space if } \lambda > 0 \end{cases}$
- $H^1(X_F, \mathcal{O}(\lambda)) = \begin{cases} \text{an infinite dim. } E\text{-Banach space if } \lambda < 0 \\ 0 \text{ if } \lambda \geq 0 \end{cases}$

5.1.4. Upgrade of the construction. — Let G be a reductive group over E. By definition, an isocrystal with a G-structure is a $\otimes$-functor

$$\mathrm{Rep}(G) \xrightarrow{\otimes} \mathrm{Isoc}\,.$$

Another way to phrase it is to consider the Dieudonné gerbe

$$\begin{tikzcd} \mathfrak{D} \arrow[d] \\ \mathrm{Spec}(E) \end{tikzcd}$$

of fiber functors on Isoc seen as a stack over Spec(E) that is a cofiltered limit of algebraic stacks. More precisely, if $\mathbb{D}$ is the slope pro-torus with $X^*(\mathbb{D}) = \mathbb{Q}$ then Isoc is banded by $\mathbb{D}$ via the equivalence

$$\mathrm{Isoc} \otimes_E E^{un} \xrightarrow{\sim} \{\mathbb{Q}\text{-graded } E^{un}\text{-vector spaces}\}$$

given by the functor

$$(D, \varphi) \longmapsto \bigoplus_{\lambda \in \mathbb{Q}} \bigcup_{n \gg 1} D^{\varphi^n = \pi^{n\lambda}}.$$

One has $\mathrm{Isoc} = \cup_{n\geq 1} \mathrm{Isoc}_n$ where Isoc_n is the Tannakian category of isocrystals with slopes in $\frac{1}{n}\mathbb{Z}$. Then,

$$\mathfrak{D} = 2 - \varprojlim_{n\geq 1} \underbrace{\mathfrak{D}_n}_{\substack{\text{algebraic stack,}\\ \text{gerbe banded by } \mathbb{G}_m\\ \text{neutral over } E_n}}$$

There is then an identification

$$\{\text{Isocrystals with a } G\text{-structure}\} \xrightarrow{\sim} \{\text{étale } G\text{-torsors on } \mathfrak{D}\}.$$

***Definition 5.1.4* (Kottwitz [88]).** — *We note $B(G)$ for the set of isomorphism classes of isocrystals equipped with a G-structure.*

One thus has

$$B(G) = H^1_{\text{ét}}(\mathfrak{D}, G).$$

According to Steinberg, $H^1(\breve{E}, G)$ is trivial. From this one deduces that

$$B(G) = G(\breve{E})/\sim$$

where $\sim$ is the σ-conjugacy relation,

$$b \sim gbg^{-\sigma}.$$

To $b \in G(\breve{E})$ one associates the G-isocrystal that sends $(V, \rho) \in \mathrm{Rep}(G)$ to the isocrystal $(V \otimes_E \breve{E}, \rho(b)\sigma)$.

The functor (2) from isocrystals to vector bundles on the curve $\mathscr{E}(-)$ defines a morphism of stacks

$$\begin{tikzcd}
\mathfrak{X}_F \arrow[dd, "\text{structural morphism}"'] \arrow[dr, "\text{induced by } \mathscr{E}(-)"] & \\
& \mathfrak{D} \arrow[dl] \\
\mathrm{Spec}(E) &
\end{tikzcd}$$

and thus by pullback a map

$$\{G\text{-isocrystals}\} \longrightarrow \{\text{étale } G\text{-torsors on } \mathfrak{X}_F\}$$

inducing

$$\boxed{B(G) \longrightarrow H^1_{\text{ét}}(\mathfrak{X}_F, G).}$$

The following definition is introduced in [**55**].

Definition 5.1.5. — *For $b \in G(\breve{E})$ we note $\mathscr{E}_b$ the associated G-bundle on $\mathfrak{X}_F$.*

5.2. Semi-stability

5.2.1. Vector bundles. — Since "$\mathfrak{X}_F$ is complete", there is a "nice" degree function

$$\deg : \mathrm{Pic}(\mathfrak{X}_F) \longrightarrow \mathbb{Z}$$

simply defined by the formula $\deg(\mathscr{L}) = \deg(\mathrm{div}(s))$ where s is any rational section of $\mathscr{L}$, $s : E(\mathfrak{X}_F) \xrightarrow{\sim} \mathscr{L}_\eta$. This allows us to define the degree of a vector bundle $\mathscr{E}$ via the formula

$$\deg(\mathscr{E}) := \deg(\det(\mathscr{E})).$$

Its main property is that if $u : \mathscr{E} \to \mathscr{E}'$ is a morphism between vector bundles that is generically an isomorphism then $\deg(\mathscr{E}) \leq \deg(\mathscr{E}')$ with equality if and only if u is an isomorphism. This property implies the existence and uniqueness of Harder-Narasimhan filtrations for the slope function

$$\mu = \frac{\deg}{\mathrm{rk}}.$$

Example 5.2.1. — *For any $\lambda \in \mathbb{Q}$ the vector bundle $\mathcal{O}(\lambda)$ is semi-stable with slope λ. In fact $\mathcal{O}(\lambda)$ is the pushforward via a finite étale morphism of a semi-stable vector bundle: the direct sum of line bundles of the same degree (the direct sum of two semi-stable vector bundles with same slope is semi-stable).*

5.2.2. Principal G-bundles. — Here we suppose G is quasi-split to simplify (in fact $\mathfrak{X}_F \times G$ is a quasi-split reductive group scheme over $\mathfrak{X}_F$ for any G). If $\mathscr{E}$ is an étale G-torsor recall that $\mathscr{E}$ is *semi-stable* if for any parabolic subgroup P of G, for any reduction $\mathscr{E}_P$ of $\mathscr{E}$ to P,

$$\deg(s^* T_{P\backslash\mathscr{E}}) \geq 0$$

(tangent bundle of $P\backslash\mathscr{E} \to \mathfrak{X}_F$) where s is the section

$$\begin{tikzcd} P\backslash\mathscr{E} \arrow[d] \\ \mathfrak{X}_F \arrow[u, bend left, "s"] \end{tikzcd}$$

corresponding to the reduction $\mathscr{E}_P$ i.e. the P-torsor $\mathscr{E}_P$ is the pullback by s of the étale P-torsor $\mathscr{E} \to P\backslash\mathscr{E}$.

One can then prove that for any $\mathscr{E}$ there exists (up to $G(E)$-conjugacy) a unique parabolic subgroup P and a reduction $\mathscr{E}_P$ of $\mathscr{E}$ to E satisfying:

1. $\mathscr{E}_P \overset{P}{\times} \underbrace{P/R_uP}_{\text{Levi quotient}}$ is semi-stable,
2. for any $\chi \in X^*(P/Z_G) \smallsetminus \{0\} \cap \mathbb{N}.\Delta$ we have $\deg \chi_*\mathscr{E} > 0$ where Δ is the set of simple roots.

This is the so-called *canonical reduction of* $\mathscr{E}$.

5.3. Vector and numerical invariants

5.3.1. The case of tori. — Let T be a torus over E. Recall the following key elementary result in the theory ([**88**]).

Proposition 5.3.1. — *There is a canonical in T identification*

$$B(T) \xrightarrow{\sim} X_*(T)_{\Gamma_E}$$

such that for $\mathbb{G}_m$ this is given by the π-adic valuation of an element of $\breve{E}^\times$.

The associated element via $X_*(T)_{\Gamma_E} \otimes \mathbb{Q} = [X_*(T)_{\mathbb{Q}}]^{\Gamma_E}$ is called the generalized Newton polygon of an element of $B(T)$.

5.3.2. *B(G)* ([88]). — There is an exact sequence of pointed sets

$$1 \to \underbrace{H^1(E,G)}_{\substack{\text{unit root} \\ G\text{-isocrystal}}} \longrightarrow B(G) \longrightarrow \big[\mathrm{Hom}(\mathbb{D}_{\overline{E}}, G_{\overline{E}}) \underbrace{\;/\;}_{\substack{\text{action by} \\ \text{conjugation}}} G(\overline{E})\big]^{\Gamma_E}$$

that is identified with a low degree Hochschild-Serre spectral sequence

$$1 \to H^1(E,G) \longrightarrow H^1_{\text{ét}}(\mathfrak{D}, G) \longrightarrow H^0\big(\Gamma_E, H^1_{\text{ét}}(\mathfrak{D}_{\overline{E}}, G)\big).$$

Here by "unit root" we mean slope 0. For $[b] \in B(G)$ we note $[\nu_b]$ for the class of the associated morphism $\mathbb{D}_{\overline{E}} \xrightarrow{\nu_b} G_{\overline{E}}$.

Let us suppose, to simplify, that G is quasi-split. Let $A \subset T \subset B$ be the inclusion of a maximal split torus inside a maximal torus inside a Borel subgroup. Then,

$$[\nu_b] \in X_*(A)^+_{\mathbb{Q}}$$

is the *generalized Newton polygon of* $[b]$.

There is a second invariant associated to $[b]$,

$$\boxed{\kappa(b) \in \underbrace{\pi_1(G)}_{\substack{\text{Borovoi}\\ \text{fund. group}}}{}_{\Gamma}.}$$

that is the generalization of the endpoint of the Newton polygon of an isocrystal. In fact, the images of $[\nu_b]$ and $\kappa(b)$ in $\pi_1(G)_\Gamma \otimes \mathbb{Q}$ are equal.

The abelian group $\pi_1(G)$ is Borovoi's fundamental group,

$$\pi_1(G) = X_*(T)/\langle \check{\Phi} \rangle$$

where $\check{\Phi}$ is the set of coroots ([**20**], [**21**]). Its profinite completion is identified with Grothendieck's étale fundamental group:

$$\widehat{\pi_1(G)} \quad \underset{\underset{\Gamma_E}{\circlearrowleft}}{=} \quad \pi_1^{\text{ét}}(G_{\overline{E}}).$$

When G_{der} is simply connected one has

$$\pi_1(G) = X_*(G/G_{der})$$

and the map κ is given by the projection

$$B(G) \longrightarrow B(G/G_{der})$$

coupled with proposition 5.3.1.

In general, this is defined via an *abelianization map*

$$B(G) = H^1(\sigma^{\mathbb{Z}}, G(\breve{E})) \longrightarrow \underbrace{H^1\big(\sigma^{\mathbb{Z}}, \underbrace{[G_{sc}(\breve{E}) \to G(\breve{E})]}_{\text{crossed module}}\big)}_{\substack{\text{group coho. with coeff.}\\ \text{in a crossed module}}}.$$

Here G_{sc} is the universal cover of the derived subgroup G_{der} and G acts on G_{sc} via the morphism

$$G \longrightarrow G_{ad} \xrightarrow{\text{conj. action}} \mathrm{Aut}(G_{sc}).$$

If T_{sc} is the pullback of $T \cap G_{der}$ to G_{sc} the morphism of crossed modules

$$[T_{sc} \to T] \longrightarrow [G_{sc} \to G]$$

is a quasi-isomorphism and thus induces a bijection

$$H^1(\sigma^{\mathbb{Z}}, T_{sc}(\breve{E}) \to T(\breve{E})) \xrightarrow{\sim} H^1\big(\sigma^{\mathbb{Z}}, [G_{sc}(\breve{E}) \to G(\breve{E})]\big).$$

We deduce an exact sequence

$$B(T_{sc}) \longrightarrow B(T) \longrightarrow H^1(\sigma^{\mathbb{Z}}, T_{sc}(\breve{E}) \to T(\breve{E})) \longrightarrow 0.$$

We deduce our κ map from proposition 5.3.1,

$$\kappa : B(G) \longrightarrow \pi_1(G)_\Gamma.$$

5.3.3. Principal G-bundles. — Suppose again that G is quasi-split. Let $\mathscr{E}$ be an étale G-torsor on $\mathfrak{X}_F$ and let $\mathscr{E}_P$ be its canonical reduction where P is a standard parabolic subgroup with respect to the choice of B as before. The morphism

$$\begin{aligned} X^*(P) &\longrightarrow \mathbb{Z} \\ \chi &\longmapsto \deg(\chi_*\mathscr{E}_P) \end{aligned}$$

can be seen as an element of $X_*(A)_{\mathbb{Q}}$. Moreover the second condition in the definition of the canonical reduction of $\mathscr{E}$ implies this is an element of $X_*(A)^+_{\mathbb{Q}}$. We note it

$$\boxed{[\nu_{\mathscr{E}}] \in X_*(A)^+_{\mathbb{Q}}}$$

and we think about it as a *generalized Harder-Narasimhan polygon.*

As before there is an abelianization map

$$H^1_{\text{ét}}(\mathfrak{X}_F, G) \longrightarrow H^1_{\text{ét,ab}}(\mathfrak{X}_F, G) := H^1_{\text{ét}}(\mathfrak{X}_F,\ \underbrace{[G_{sc} \to G]}_{\substack{\text{crossed module} \\ \text{on } \mathfrak{X}_{F,\text{ét}}}}\ ).$$

One can prove that when F is algebraically closed then for a torus S over E

$$B(S) = H^1_{\text{ét}}(\mathfrak{D}, S) \xrightarrow{\sim} H^1_{\text{ét}}(\mathfrak{X}_F, S)$$

(the proof is reduced to the $\mathbb{G}_m$-case where one of the key ingredients is to prove that $\text{Br}(\mathfrak{X}_F) = 0$; we will later see that this isomorphism is true for any reductive group G but one can give a simpler proof for a torus) and thus

$$H^1_{\text{ét,ab}}(\mathfrak{D}, G) \xrightarrow{\sim} H^1_{\text{ét,ab}}(\mathfrak{X}_F, G).$$

At the end this allows us to define

$$\boxed{c_1(\mathscr{E}) \in \pi_1(G)_\Gamma}$$

the first Chern class of $\mathscr{E}$.

5.4. Classification of G-isocrystals

Recall the following definition due to Kottwitz that generalizes the definition of an isoclinic isocrystal.

> ***Definition 5.4.1.*** — *The element $[b] \in B(G)$ is basic if ν_b is central.*

One of the first basic results in the domain is the following.

Proposition 5.4.2. — *Kottwitz κ map induces a bijection*

$$\kappa_{|B(G)_{bsc}} : B(G)_{bsc} \xrightarrow{\sim} \pi_1(G)_\Gamma.$$

Remark 5.4.3. — *This result that seems mysterious at first will be fully understood later: any connected component of* Bun_G *contains a unique semi-stable point. For any* $[b]$*, the basic element associated to* $\kappa(b)$ *in* $B(G)$ *will correspond to the maximal generalization of the point of* Bun_G *associated to* $[b]$.

We still suppose that G is quasi-split and we fix $A \subset T \subset B$. Then, if M_b is is the Standard Levi subgroup that is the centralizer of the slope morphism $[\nu_b] \in X_*(A)^+_{\mathbb{Q}}$, $[b]$ *has a canonical basic reduction* $[b_{M_b}] \in B(M_b)_{\text{basic}}$.

Finally, one can prove that the map $[b] \mapsto ([\nu_b], \kappa(b))$ is an injection

$$B(G) \hookrightarrow \pi_1(G)_\Gamma \times X_*(A)^+_{\mathbb{Q}}.$$

One can describe its image but this is not useful for what we do. Let us just remark that the injectivity of this map will later be reinterpreted as saying that on any connected component $\mathcal{C}$ of Bun_G, the map given by $[b] \mapsto [\nu_b]$ induces an injection $|\mathcal{C}| \hookrightarrow X_*(A)^+_{\mathbb{Q}}$.

5.5. Automorphisms of G-isocrystals

Recall the following definition due to Kottwitz.

Definition 5.5.1. — *For* $[b] \in B(G)$ *we note* G_b *for the algebraic group of automorphisms of the associated* G*-isocrystals.*

Concretely, for R an E-algebra, one has

$$G_b(R) = \left\{ g \in G(R \otimes_E \breve{E} \mid gb\sigma = b\sigma g \right\}.$$

Less concretely, if $\mathcal{T}_b \to \mathfrak{D}$ is the associated étale G-torsor over the Dieudonné gerbe,

$$G_b(R) = \mathrm{Aut}_{\mathfrak{D} \times_{\mathrm{Spec}(E)} \mathrm{Spec}(R)} \left(\mathcal{T}_b \times_{\mathrm{Spec}(E)} \mathrm{Spec}(R) \right)$$

(automorphisms of torsors). The Dieudonné gerbe splits over E^{un} and we obtain an isomorphism

$$G_b \otimes_E E^{un} \simeq C_G(\nu_b).$$

In particular, G_b is an inner form of the centralizer of ν_b (which is a Levi subgroup of the quasi-split inner form of G), and

$$[b] \text{ is basic } \Leftrightarrow G_b \text{ is an inner form of } G.$$

Those are the so-called extended pure inner forms of Kottwitz that generalize Vogan's pure inner forms ([**140**]) via the embedding

$$\underbrace{H^1(E,G)}_{\substack{\text{unit root}\\ G\text{-isocrystals}}} = \{[b] \mid \nu_b = 0\} \subset B(G).$$

5.6. Classification of principal G-bundles

5.6.1. Vector bundles. — The following classification result is a difficult very important result in the domain, see [**57**].

Theorem 5.6.1. — *Suppose F is algebraically closed. There is a bijection*

$$\begin{array}{ccc} \big\{\lambda_1 \geq \cdots \geq \lambda_r \mid r\in\mathbb{N},\ \lambda_i\in\mathbb{Q}\big\} & \xrightarrow{\sim} & \big\{\text{v.b. on } \mathfrak{X}_F\big\}/\sim \\ (\lambda_1,\dots,\lambda_r) & \longmapsto & \Big[\bigoplus_{i=1}^{r} \mathcal{O}(\lambda_i)\Big]. \end{array}$$

In terms of reduction theory, this can be split in two parts:

1. Slope λ semi-stable vector bundles are isomorphic to directs sums of $\mathcal{O}(\lambda)$,
2. The Harder-Narasimhan filtration of a vector bundle is (non-canonically) split.

Point (2) is an immediate consequence of point (1) since

$$\mathrm{Ext}^1(\mathcal{O}(\lambda),\mathcal{O}(\mu)) = H^1(\mathfrak{X}_F, \underbrace{\mathcal{O}(-\lambda)\otimes\mathcal{O}(\mu)}_{\substack{\text{finite direct}\\ \text{sum of } \mathcal{O}(\mu-\lambda)}})$$

is zero if $\lambda \leq \mu$ when F is algebraically closed.

5.6.2. Principal G-bundles. — Here is the main result.

Theorem 5.6.2. — *When F is algebraically closed there is a bijection of pointed sets*

$$\begin{array}{ccc} B(G) & \xrightarrow{\sim} & H^1_{\text{ét}}(\mathfrak{X}_F, G) \\ [b] & \longmapsto & [\mathscr{E}_b]. \end{array}$$

Via this theorem we have the following *dictionary between arithmetic and geometry*:

1. $\underbrace{[b] \text{ is basic}}_{\substack{\text{arithmetic condition}\\ \text{on the } p\text{-adic}\\ \text{valuations of the eigenvalues}\\ \text{of Frob}}} \Leftrightarrow \underbrace{\mathscr{E}_b \text{ is semi-stable}}_{\substack{\text{geometric semi-stability}\\ \text{condition}}}$
2. $\underbrace{[\nu_b]}_{\text{Newton polygon}} = w.(-\underbrace{[\nu_{\mathscr{E}_b}]}_{\text{HN polygon}})$ where w is the longest element in the Weyl group.
3. $\underbrace{\kappa(b)}_{\substack{\text{terminal point}\\ \text{of Newton}}} = -\underbrace{c_1(\mathscr{E}_b)}_{\text{first Chern class}}$.

5.7. On the proof of the classification theorem

5.7.1. Background on Beauville-Laszlo. — Let $\infty \in |\mathfrak{X}_F|$ be a degree 1 closed point with residue field K. We suppose that F is algebraically closed. We note $B_{dR}^+ := B_{dR}^+(K) = \widehat{\mathcal{O}}_{\mathfrak{X}_F,\infty}$ with uniformizer t.

A modification of a G-bundle $\mathscr{E}$ at ∞ is the data given by a a G-bundle $\mathscr{E}'$ together with an isomorphism

$$\mathscr{E}_{|\mathfrak{X}_F \smallsetminus \{\infty\}} \xrightarrow{\sim} \mathscr{E}'_{|\mathfrak{X}_F \smallsetminus \{\infty\}}.$$

When $G = \mathrm{GL}_n$, Beauville-Laszlo ([**6**]) tells us that such a modification is the same as the datum of a B_{dR}^+-lattice in $\widehat{\mathscr{E}}_\infty[\frac{1}{t}]$ where here we see $\mathscr{E}$ as a vector bundle. In general, this is the same as an étale G-torsor $\mathscr{F}$ on $\mathrm{Spec}(B_{dR}^+)$ together with an isomorphism

$$\mathscr{E} \times_{\mathfrak{X}_F} \mathrm{Spec}(B_{dR}) \xrightarrow{\sim} \mathscr{F} \times_{\mathrm{Spec}(B_{dR}^+)} \mathrm{Spec}(B_{dR}).$$

Since B_{dR}^+ is complete with algebraically closed residue field any étale G-torsor on $\mathrm{Spec}(B_{dR}^+)$ is trivial. We deduce that this is the same as an element of

$$\mathscr{E}(B_{dR})/G(B_{dR}^+).$$

where $\mathscr{E}(B_{dR})$ is the set of sections

$$\begin{tikzcd} & \mathscr{E} \arrow[d] \\ \mathrm{Spec}(B_{dR}) \arrow[ur, bend left, "\text{section}"] \arrow[r] & \mathfrak{X}_F \end{tikzcd}$$

5.7.2. Some piece of the Hecke groupoid. —

Let us now remark that *for any $b \in G(\breve{E})$ there is a canonical trivialization of*

$$\mathscr{E}_b \times_{\mathfrak{X}_F} \mathrm{Spec}(B_{dR}^+).$$

Suppose now that $\mathscr{E} = \mathscr{E}_b$ and $\mathscr{E}' = \mathscr{E}_{b'}$. Then, modifications

$$\mathfrak{M} : \mathscr{E}_b \dashrightarrow \mathscr{E}_{b'}$$

at ∞ are given by an element of

$$p_1(\mathfrak{M}) \in G(B_{dR})/G(B_{dR}^+)$$

and its inverse

$$p_2(\mathfrak{M}) := p_1(\mathfrak{M}^{-1}) \in G(B_{dR})/G(B_{dR}^+).$$

The images of $p_1(\mathfrak{M})$ and of $p_2(\mathfrak{M})^{-1} \in G(B_{dR}^+)\backslash G(B_{dR})$ in

$$G(B_{dR}^+)\backslash G(B_{dR})/G(B_{dR}^+)$$

are equal. We call this the type of the modification.

Suppose that G is split to simplify. There is then a bijection

$$\begin{array}{ccc} X_*(T)^+ & \xrightarrow{\sim} & G(B_{dR}^+)\backslash G(B_{dR})/G(B_{dR}^+) \\ \mu & \longmapsto & G(B_{dR}^+)\mu(t)G(B_{dR}^+). \end{array}$$

We equip $X_*(T)^+$ with the order $\mu \le \mu'$ if $\mu' - \mu \in \mathbb{N}.\Delta^\vee$, $\Delta^\vee$ being the simple coroots.

Definition 5.7.1. — *For G split over E and $\{\mu\}$ a conjugacy class of cocharaters of G we define for $F|\overline{\mathbb{F}}_q$ a perfectoid field*

1. $$\mathrm{Sh}(G, b, b', \mu)(F)$$
   as the set of a degree 1 point $\infty \in |\mathfrak{X}_F|$ and a modification at ∞
   $$\mathscr{E}_b \dashrightarrow \mathscr{E}_{b'}$$
   of type $\leqslant \mu$.
2. $$\mathrm{Gr}_{G,\leqslant\mu}^{B_{dR}}(F)$$
   as the set of a degree 1 closed point ∞ on $\mathfrak{X}_F$ with residue field K and an element of $G(B_{dR}(K))/G(B_{dR}^+(K))$ whose image in
   $$G(B_{dR}^+(K))\backslash G(B_{dR}(K))/G(B_{dR}^+(K))$$
   is $\leqslant \mu$.

We thus have two maps

$$\begin{tikzcd}
 & \mathrm{Sh}(G,b,b',\mu)(F) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathrm{Gr}^{B_{dR}}_{G,\leqslant\mu}(F) & & \mathrm{Gr}^{B_{dR}}_{G,\leqslant\mu^{-1}}(F)
\end{tikzcd}$$

Those local Shimura varieties are generalizations of Rapoport-Zink spaces.

5.7.3. Modifications of vector bundles associated to p-divisible groups ([57], [131]).

— The following is the starting point of the link between Rapoport-Zink spaces and modification of vector bundles on the curve.

Proposition 5.7.2. — *Let $\mathcal{M}$ be the deformation space by quasi-isogenies of the p-divisible group $\mathbb{H}$ over $\overline{\mathbb{F}}_p$ as defined by Rapoport-Zink. Let $C|\breve{\mathbb{Q}}_p$ be algebraically closed and consider an element $x \in \mathcal{M}(\mathcal{O}_C)$. Let*

- *V be the rational Tate module of the universal deformation specialized at x,*
- *(D, φ) be covariant isocrystal of $\mathbb{H}$,*
- *$\mathrm{Fil}\, D_C$ be the Hodge filtration.*

There is a canonical exact sequence of coherent sheaves on $\mathfrak{X}_{C^\flat}$

$$0 \longrightarrow V \otimes \mathcal{O}_{\mathfrak{X}_{C^\flat}} \longrightarrow \mathscr{E}(D, p^{-1}\varphi) \longrightarrow i_{\infty *} D_C / \mathrm{Fil}\, D_C \longrightarrow 0$$

where $\infty \in |\mathfrak{X}_{C^\flat}|$ is the closed point associated to the untilt C of $C^\flat$.

This is a rewriting in terms of the curve of Fontaine/Faltings comparison theorems:

$$V \otimes_{\mathbb{Q}_p} \mathbb{B}(C^\flat)[\tfrac{1}{t}] \xrightarrow{\sim} D \otimes_{\breve{\mathbb{Q}}_p} \mathbb{B}(C^\flat)[\tfrac{1}{t}].$$

where

$$\mathrm{Id} \otimes \varphi \leftrightarrow p^{-1}\varphi \otimes \varphi.$$

Define now for $F|\overline{\mathbb{F}}_p$ algebraically closed

$$\mathcal{M}^\diamond_{\eta,\infty}/\varphi^{\mathbb{Z}}(F)$$

as the set of

- an untilt C of F over E up to a power of Frobenius (i.e. the identification between F and $C^\flat$ is taken up to a power of Frobenius),
- an object of $\mathcal{M}(\mathcal{O}_C)$,
- an infinite level structure on this object i.e. a base of the associated rational Tate module.

Let d be the dimension of $\mathbb{H}$ and set $\mu(z) = \mathrm{diag}(\underbrace{z, \dots, z}_{d\text{-times}}, 1, \dots, 1)$ for $G = \mathrm{GL}_n$ over $\mathbb{Q}_p$. Set $G = \mathrm{GL}_n$ where n is the height of $\mathbb{H}$. The preceding proposition defines a map for F algebraically closed

$$\begin{tikzcd}
& \mathcal{M}^{\diamond}_{\eta,\infty}/\varphi^{\mathbb{Z}}(F) \arrow[ddl, bend right, "\pi_{dR}"'] \arrow[d] \arrow[ddr, bend left, "\pi_{HT}"] & \\
& \mathrm{Sh}(G,b,1,\mu)(F) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathrm{Gr}^{B_{dR}}_{G,\leqslant\mu}(F) & & \mathrm{Gr}^{B_{dR}}_{G,\leqslant\mu^{-1}}(F)
\end{tikzcd}$$

where here there is an identification between $\mathrm{Gr}^{B_{dR}}_{G,\leqslant\mu}(F)$ and $\mathscr{F}^{\diamond}_{\mu}/\varphi^{\mathbb{Z}}(F)$ since μ is minuscule.

5.7.4. Application to the classification. — As a consequence of the study of de Rham and Hodge-Tate periods of Lubin-Tate spaces one deduces the following from the preceding construction

$$\big\{p\text{-divisible groups}/\mathcal{O}_C\big\} \longrightarrow \big\{\text{modifications of vector bundles }/\mathfrak{X}_{C^\flat}\big\}$$

For F algebraically closed:

- (*Surjectivity of the de Rham period morphism for L.T. spaces*) For any exact sequence of coherent sheaves on $\mathfrak{X}_F$

  $$0 \longrightarrow \mathscr{E} \longrightarrow \mathcal{O}(\tfrac{1}{n}) \longrightarrow \mathscr{F} \longrightarrow 0,$$

  where $\mathscr{F}$ is a torsion coherent sheaf of degree 1, one has

  $$\mathscr{E} \simeq \mathcal{O}^n.$$

- (*Computation of the image of π_{HT} for L.T. spaces*) For any exact sequence of coherent sheaves on $\mathfrak{X}_F$

  $$0 \longrightarrow \mathcal{O}^n \longrightarrow \mathscr{E} \longrightarrow \mathscr{F} \longrightarrow 0$$

  where $\mathscr{F}$ is a torsion coherent sheaf of degree 1, one has

  $$\mathscr{E} \simeq \mathcal{O}^{n-r} \oplus \mathcal{O}(\tfrac{1}{r})$$

  for some integer $1 \leq r \leq n$.

Using those two results about degree 1 modifications of vector bundles on the $\mathfrak{X}_F$ one can obtain by elementary manipulations the classification theorem 5.6.1.

5.7.5. Equivalence. — Reciprocally one can prove the following result (using the classification of vector bundles on the curve).

***Theorem 5.7.3* ([54], [131]).** — *The following is satisfied:*

1. *For F algebraically closed the map*
$$\mathcal{M}_{\eta,\infty}^{\diamond}/\varphi^{\mathbb{Z}}(F) \longrightarrow \mathrm{Sh}(G,b,1,\mu)(F)$$
is a bijection.
2. *The functor that sends a p-divisible group over $\mathcal{O}_C$ to the corresponding modification of vector bundles on $\mathfrak{X}_{C^\flat}$*
$$\mathscr{E} \dashrightarrow \mathscr{E}',$$
with $\mathscr{E}$ a trivial vector bundle, together with a lattice in $H^0(\mathfrak{X}_{C^\flat},\mathscr{E})$, is an equivalence of categories.

Using this result together with some arguments about Banach-Colmez spaces and [**57**, Section 11.1] one can prove that in infinite level Rapoport-Zink spaces are perfectoid and are moduli of modifications of vector bundles on the curve.

Remark 5.7.4. — *Using this point of view one can define local Shimura varieties for any triple (G,b,μ) as diamonds and even rigid analytic spaces if μ is minuscule, see* [**132**, Section 23]*. Those are generalizations of Rapoport-Zink spaces. Contrary to Rapoport-Zink spaces, that exist only for classical groups G, they are constructed directly in generic fiber without the use of an integral model. Reciprocally, the question to construct "natural" integral models of those general local Shimura varieties is still unsolved. Nevertheless, the work of Zhu (*[**144**]*) and Bhatt-Scholze (*[**15**]*) allows us to define the perfection of their special fiber as a perfect scheme for G unramified.*

5.8. p-adic Hodge structures and local Shtukas

The following result says that a (geometric) p-adic Hodge structure is the same as a local Shtukas.

Theorem 5.8.1 **([53], [132,** Theorem 11.4.5] **).** — *Let* $F|\mathbb{F}_q$ *be algebraically closed. Let* $(F_i^\sharp)_{i\in I}$ *be a finite collection of untilts of* F *over* E *associated to* $(\xi_i)_{i\in I}$ *a finite collection of degree* 1 *primitive elements in* $W_{\mathcal{O}_E}(\mathcal{O}_F)$*. There is an equivalence of categories between*

- *local Shtukas i.e. couples* (M,φ) *where* M *is a free* $A_{\inf}$*-module of finite type and* φ *an isomorphism*
$$\varphi : M\left[\frac{1}{\prod_{i\in I}\varphi^{-1}(\xi_i)}\right] \xrightarrow{\sim} M\left[\frac{1}{\prod_{i\in I}\xi_i}\right]$$
- *modifications of vector bundles*
$$\mathscr{E} \dashrightarrow \mathscr{E}'$$
at $\sum_{i\in I}\infty_i$ *where* $\mathscr{E}$ *is a trivial vector bundle and* $\infty_i \in |X_F|^{cl}$ *corresponds to the untilt* $F_i^\sharp$*.*

Remark 5.8.2. — *This last result is the starting point of* $A_{\inf}$*-cohomology (*[**13**]*). In fact if* X *is a proper smooth algebraic variety over* K *where* $[K:\mathbb{Q}_p] < +\infty$ *and* $C = \widehat{\overline{K}}$*, the "classical" comparison theorems (Fontaine, Fontaine-Messing, Tsuji, Faltings) associates to any cohomologycal degree* $i \in \mathbb{N}$ *a modification of vector bundles* $\mathscr{E} \dashrightarrow \mathscr{E}'$ *on* $\mathfrak{X}_{C^\flat}$ *where*

- $\mathscr{E} = H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathcal{O}$
- $\mathscr{E}' = \mathscr{E}(D,\varphi)$ *where* $D = H^i_{\text{cris}}(X_{\overline{k}_K}, W)\left[\frac{1}{p}\right]$ *with its crystalline Frobenius.*

The lattice
$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)/torsion$$
then gives rise by application of the preceding theorem to a φ*-module over* $A_{\inf}$*. One of the starting point of* $A_{\inf}$*-cohomology is to refine this construction by construction a cohomology complex in the derived category of* $A_{\inf}$*-modules to take into account torsion.*

5.9. Understanding the twin towers isomorphism

The isomorphism of [**47**] and [**51**] was mysterious during some time. This is now understood in the following way via the following remark.

Lemma 5.9.1. — *If* $[b] \in B(G)$ *is basic then the group-scheme* $G_b \times_{Spec(E)} \mathfrak{X}_F$ *is the inner twisting of* $G \times_{Spec(E)} \mathfrak{X}_F$ *by the étale torsor* $\mathscr{E}_b$*,*
$$G_b \times_{Spec(E)} \mathfrak{X}_F = \underline{Aut}\,(\mathscr{E}_b/\mathfrak{X}_F)\,.$$

As a corollary one deduces an identification between G-torsors on $\mathfrak{X}_F$ and G_b-torsors. This fact extends to the relative curve $\mathfrak{X}_{R,R^+}$ for any $\overline{\mathbb{F}}_q$-affinoid perfectoid ring (R,R^+) and induces an isomorphism of Artin v-stacks

$$\begin{array}{ccc} \mathrm{Bun}_G & \xrightarrow{\sim} & \mathrm{Bun}_{G_b} \\ \mathscr{E} & \longrightarrow & \mathscr{E} \overset{G}{\times} \mathscr{E}_b. \end{array}$$

This induces Kottwitz's identification $B(G) \xrightarrow{\sim} B(G_b)$ (see [**86**]). This isomorphism is compatible with modifications and this induces and identification of the associated Hecke stacks. This allows us to recover [**47**], [**51**], and [**48**] in a very elegant way, see [**28**].

5.10. Some final thoughts

Theorem 5.7.3 has been a great motivation for the introduction of the geometrization conjecture. Already in [**57**], in the proof we gave of weakly admissible implies admissible, modification of vector bundles and B_{dR}^+-lattices showed up in an essential way. The appearance of Kottwitz set already, strongly linked to the mod p geometry of Shimura varieties, is a great sign.

Theorem 5.8.1 has been very important in relation with Drinfeld's work on Shtukas and more recently V. Lafforgue's work ([**91**]). The appearance of Shutkas, as defined by Drinfeld, is quite stunning.

Finally let us note that the upgrade of isocrystal to vector bundles on the curve is part of the philosophy to upgrade everything analytically as in [**50**] where we considered the tube over the Newton strata as analytic spaces. As a matter of fact, the analytic world is more natural than the algebraic one in this situation. For example, one can do this in family and consider the fpqc stack of G-isocrystals

$$\mathrm{Isoc}_G$$

on perfect schemes that associates to a perfect ring R the groupoid of étale G-torsors on Spec $\big(W_{\mathcal{O}_E}(R)[\frac{1}{\pi}]\big)$. This stack is not algebraic in any sense. In the equal characteristic case, the case $E=\mathbb{F}_q(\!(\pi)\!)$, this is

$$LG/_{\varphi}\, LG$$

where LG is the loop group of G over $\mathrm{Spec}(\overline{\mathbb{F}}_q)$ and the quotient is for the twisted φ-conjugation. This is not a "good" algebraic object although (Dieudonné-Manin)

$$B(G) = |\,\mathrm{Isoc}_G\,|.$$

The stack Bun_G is much better: this is an Artin v-stack. This fact was an important motivation for the work [**58**]. We refer to remark 8.1.4 for more details.

LECTURE 6

DIAMONDS

Diamonds are the basic geometric objects that show up in the geometry of Bun_G. They appeared in [**132**] and are everywhere in the domain.

6.1. What is a diamond ?

- Schemes are obtained by gluing affine schemes for the Zariski topology.
- Algebraic spaces are obtained by gluing affine schemes for the étale topology.
- Usually one adds a coherence condition in the definition of an algebraic space, one typically assumes that they are quasi-separated to remove pathological objects like $\mathbb{G}_{a,\mathbb{C}}/\mathbb{Z}$ (action by translations) that is not quasi-separated since the étale topology is not coarse enough contrary to the analytic topology where $\mathbb{C}/\mathbb{Z}$ is a nice good object as a complex analytic space.
- There is an Artin criterion for algebraic spaces.

The theory of diamonds follows the same path by replacing schemes by $\mathbb{F}_p$-perfectoid spaces and the étale topology by the pro-étale topology. The nice coherence condition that one adds to make them look like analytic adic spaces is called *the spatialness condition.* There is even an analog of *Artin's criterion.*

Historically there has been different sources of diamonds:

1. The first one comes from he theory of *finite dimensional Banach spaces in the sense of Colmez* ([**32**], [**97**] for the interpretation in terms of the curve). The first, from the historical point of view, typical question being to describe geometrically

$$\mathbb{B}^{\varphi=p^2}$$

   that, contrary to $\mathbb{B}^{\varphi=p}$ that is a 1-dimensional open ball, is a quotient of a 2-dimensional ball by a pro-étale pro-p equivalence relation but is not represented by a perfectoid space.

2. The second one is the remark that points of the curve correspond to *untilts* up to Frobenius, this has lead to the introduction of
$$\mathrm{Spa}(\mathbb{Q}_p)^\diamond.$$
3. The third one is the remark due to Faltings and Colmez that *pro-étale locally analytic adic spaces are perfectoid*; this is typically the remark due to Faltings that "the Frobenius of $\overline{R}/p\overline{R}$ is surjective". This has lead later to the construction of
$$X^\diamond$$
where X is an analytic adic space.
4. The fourth one come from the desire to put *a geometric structure on the set of* B_{dR}^+*-lattices in* B_{dR}^n, a construction that already showed up in the curve proof of weakly admissible implies admissible in [**57**] and has been put in form in [**132**] as the so-called B_{dR}-affine Grassmanian.

6.2. Background on the pro-étale and the v-topology ([132], [128])

Let
$$\mathrm{Perf}$$
be the category of all perfectoid spaces.

Here, and this is essential for our work, see Remark 6.2.2, we consider affinoid perfectoid algebras that may not contain a field. They are classified as triples $((R, R^+, I)$ where:

1. (R, R^+) is an $\mathbb{F}_p$-affinoid perfectoid algebra,
2. $I \subset W(R^+)$ is an ideal generated by a degree 1 distinguished element i.e. an element of the form $\sum\limits_{n\geq 0}[a_n]p^n$ where $a_0 \in R^{\circ\circ}$ and $a_1 \in (R^+)^\times$ (such elements are regular and such an ideal I is a Cartier divisor).

This correspondence is given by the following rules:

1. To (A, A^+) affinoid perfectoid we associate
$$\big(A^\flat, A^{\flat,+}, \ker\theta\big)$$
where $\theta : W(A^{\flat,+}) \to A^+$.
2. In the other direction, to $((R, R^+, I)$ we associate
$$\big(W(R^+)/I[\tfrac{1}{[\varpi]}], W(R^+)/I\big).$$

If (A, A^+) contains a field and corresponds to $((R, R^+, (\xi))$ with $\xi = \sum_{n\geq 0}[a_n]p^n$ then either $a_0 = 0$ i.e. A contains $\mathbb{F}_p$, either $a_0 \in R^\times$ i.e. A contains $\mathbb{Q}_p$.

Example 6.2.1. — *If we take*

$$(R, R^+) = (K\langle T^{1/p^\infty}\rangle, \mathcal{O}_K\langle T^{1/p^\infty}\rangle)$$

with K a characteristic p perfectoid field and

$$I = ([T] + p)$$

then the corresponding perfectoid space $S = \mathrm{Spa}(A, A^+)$ satisfies $|S| = |\mathbb{B}^1_K|$ that is connected. The open subset $|\mathbb{B}^1_K \setminus \{0\}|$ is a $\mathbb{Q}_p$-perfectoid space and the origin $\{0\} \subset |\mathbb{B}^1_K|$ is $\mathrm{Spa}(K)$ that is an $\mathbb{F}_p$-perfectoid space.

Remark 6.2.2. — *From this example we deduce a quotient map*

$$|\mathbb{B}^1_K| \to \underbrace{|\mathrm{Spa}(\mathbb{Z}_p)^\diamond|}_{\substack{\textit{top. space}\\ \textit{associated}\\ \textit{to a small}\\ \textit{v-sheaf}\\ \textit{(see later)}}} \underbrace{=}_{\substack{\textit{as a}\\ \textit{set}}} \{s, \eta\}$$

where the image of $|\mathbb{B}^1_K \setminus \{0\}|$ is η and the one of $\{0\}$ is s. This implies that $\eta \geqslant s$ and thus $|\mathrm{Spa}(\mathbb{Z}_p)^\diamond| = \{s, \eta\}$ with $\eta \geqslant s$ as a topological space. This fact is crucial for the proof of the geometric Satake correspondence where we use a degeneration of the B_{dR}-affine Grassmanian from $\mathrm{Spa}(\mathbb{Q}_p)^\diamond$ to $\mathrm{Spa}(\mathbb{F}_p)^\diamond$ via $\mathrm{Spa}(\mathbb{Z}_p)^\diamond$ to the usual Witt vector affine Grassmanian where we can apply some classical arguments using the decomposition theorem.

The category Perf is equipped with three natural Grothendieck topologies.

6.2.1. The étale topology ([128, Section 6]. —

This is the usual étale topology on perfectoid spaces. One of its main properties is that it is compatible with the tilting equivalence: if S is a perfectoid space, via the equivalence

$$(-)^\flat : \mathrm{Perf}_S \xrightarrow{\sim} \mathrm{Perf}_{S^\flat},$$

$T \to S$ is étale if and only if $T^\flat \to S^\flat$ is étale. This is part of the so-called *purity theorem.* Among its elementary properties is the fact that any étale morphism is open.

In general the étale site of a perfectoid space is considered as a small site.

6.2.2. The pro-étale topology ([128, Section 8]). —

6.2.2.1. *Definition.* — One of the great features of perfectoid spaces, compared to "classical Noetherian analytic adic spaces" is that some operations that do not exist in the Noetherian world make a sense for perfectoid spaces. Typically, if $(S_i)_i$ is a cofiltered projective system of affinoid perfectoid spaces, $S_i = \mathrm{Spa}(R_i, R_i^+)$, then

$$\varprojlim_i S_i$$

is well defined, and affinoid perfectoid, as $\mathrm{Spa}(R_\infty^+[\frac{1}{\varpi}], R_\infty^+)$ where R_∞^+ is the ϖ-adic completion of $\varinjlim_i R_i^+$ and ϖ is the image of some pseudo-uniformizer in R_i for some index i.

Recall the following definition.

Definition 6.2.3. — *A morphism $T \to S$ of perfectoid spaces is pro-étale if it can be written locally on T and S as*

$$T = \varprojlim_{i \geq i_0} S_i \longrightarrow S_{i_0} = S$$

where $(S_i)_i$ is a cofiltered projective system of affinoid perfectoid spaces with étale transition morphisms.

The pro-étale topology has to be manipulated carefully for the following reason: contrary to étale morphisms of perfectoid spaces, *in general pro-étale morphisms are not open.* This is for example the case for any $s \in S$ where

$$\mathrm{Spa}(K(s), K(s)^+) = \varprojlim_{U \ni s} U \hookrightarrow S$$

is pro-étale not open. This may still be the case for surjective morphisms of affinoid perfectoid spaces, typically $S \coprod \mathrm{Spa}(K(s), K(s)^+) \to S$.

One thus has to add the following condition in the definition of a pro-étale cover:

Definition 6.2.4. — *A family of morphisms of perfectoid spaces $(T_i \to S)_{i \in I}$ is a pro-étale cover if for any quasi-compact open subset U in S there exists $I' \subset I$ finite and for each $i \in I'$ a quasi-compact open subset $V_i \subset T_i$ such that*

$$U = \bigcup_{i \in I'} \mathrm{Im}(V_i \to T).$$

If for all indices $i \in I$, $T_i \to S$ is open this "strong surjectivity condition" is equivalent to "the weak one" saying that $\coprod_{i \in I} |T_i| \to |S|$ is surjective. But as we said before this is not true in general. *The pro-étale site is seen as a big site.*

6.2.2.2. *Pro-étale local structure of perfectoid spaces.* — One of the most important results is the following structure of perfectoid spaces pro-étale locally. In fact, recall the following definition. We use the fact that for any qc qs perfectoid space X there is a morphism

$$X \longrightarrow \underline{\pi_0(X)}$$

whose fibers are the connected components of X (that are perfectoid spaces). Here

$$\pi_0(X) = \overbrace{\pi_0(\underbrace{|X|}_{\substack{\text{spectral}\\ \text{space}}})}^{\text{profinite}}.$$

Remark 6.2.5. — *Here we use the following construction. If T is a topological space then we define $\underline{T}$ as a functor on* Perf *via the formula*

$$\underline{T}(S) = \mathscr{C}(|S|, T).$$

This defines a pro-étale (and even a v)-sheaf on Perf.

Definition 6.2.6. — *A perfectoid qc qs space X is strictly totally discontinuous if it satisfies the following equivalent properties:*

1. *Every connected components of X contains a unique closed point i.e. is of the form* $\mathrm{Spa}(K, K^+)$ *with (K, K^+) an affinoid perfectoid field. We moreover ask that all residue fields are algebraically closed i.e. any connected component is of the form* $\mathrm{Spa}(C, C^+)$ *with C algebraically closed.*
2. *Any étale cover of X splits i.e. admits a section.*

Strictly totally disconnected perfectoid spaces can be though of as a "amalgamations" of collections Spec$(C(x), C(x)^+)$ with $C(x)$ algebraically closed when x goes along a profinite set.

The following says that pro-étale locally any perfectoid space if a disjoint union of strictly totally disconnected perfectoid spaces, see [**128**].

Proposition 6.2.7 **(Pro-étale local structure of perfectoid spaces)**
For any qc qs perfectoid space X there exists an open pro-étale surjective morphism

$$\widetilde{X} \longrightarrow X$$

with $\widetilde{X}$ strictly totally discontinuous.

Example 6.2.8. — *For any perfectoid space X, if $X_\bullet \to X$ is an hypercover by $\coprod$ strictly totally disconnected perfectoid spaces then*

$$\{\textit{étale sheaves on } X\} \xrightarrow{\sim} \{\textit{cartesian sheaves on } |X_\bullet|\}.$$

From this point of view étale cohomology of perfectoid spaces is simpler than étale cohomology of schemes: everything is reduced to cartesian sheaves on simplicial topological spaces.

6.2.2.3. *A geometric fiberwise criterion to be pro-étale pro-étale locally.* — *Pro-étale morphisms do not satisfy descent for the pro-étale topology.* This problems has lead to the following.

Proposition 6.2.9. — *A morphism of perfectoid spaces $X \to S$ is pro-étale pro-étale locally on S if and only if for all its geometric fibers, $X \times_S \mathrm{Spa}(C, C^+)$ for $\mathrm{Spa}(C, C^+) \to S$, are locally profinite, i.e. locally of the form $\underline{P} \times \mathrm{Spa}(C, C^+)$ for a profinite set P.*

This has lead to the definition of *quasi-pro-étale morphisms* and his a very useful criterion for application to morphisms of moduli spaces for which computing the geometric fibers is usually easy.

Example 6.2.10. — *Let $T \to S$ be a morphism of qc qs perfectoid spaces such that $|T| \to |S|$ is surjective (i.e. this is a v-cover) and such that for all $s : \mathrm{Spa}(C, C^+) \to S$, $T_s \simeq \underline{P} \times \mathrm{Spa}(C, C^+)$ with P a profinite set. Then, up to replacing S by a pro-étale cover, $T \to S$ is a pro-étale cover. From this we deduce that $T \to S$ is a surjective morphism of pro-étale sheaves. This is for example the case for the Kummer map $\underbrace{\mathbb{B}_K^{1,1/p^\infty}}_{\substack{\textit{perfectoid}\\ \textit{ball}}} \xrightarrow{z \mapsto z^n} \mathbb{B}_K^{1,1/p^\infty}$ when K is a perfectoid field.*

6.2.3. The v-topology ([132, Chapter 17], [128, Section 8]. — The v-topology is an analog of the fpqc topology for schemes. This is a big site on perfectoid spaces where we take the same definition for covers as for the pro-étale topology but by taking any morphism of perfectoid spaces instead of the pro-étale one. This is the most general topology we use. *It is subcanonical*: the functor defined by a perfectoid space is a v-sheaf. It moreover satisfies some nice descent properties. For example:

1. Vector bundles satisfy descent for the v-topology ([**132**, Lemma 17.1.8]).
2. Separated étale morphisms satisfy descent for the v-topology ([**128**, Proposition 9.7]).

This last (difficult) result is used all the times.

Example 6.2.11. — *Let G be a locally profinite group and $T \to S$ be a $\underline{G}$ torsor for the v-topology where S is a perfectoid space. One has, as v-sheaves,*

$$T \xrightarrow{\sim} \varprojlim_{K} \underline{K}\backslash T$$

where K goes through the set of compact open subgroups of G. Since v-locally $\underline{K}\backslash T \to S$ is separated étale, one deduces that $\underline{K}\backslash T \to S$ is representable by a separated étale morphism of perfectoid spaces. In particular, $T \to S$ is a pro-étale morphism of perfectoid spaces,

$$T = \varprojlim_{K' \subset K} \underline{K'}\backslash T \underbrace{\longrightarrow}_{\substack{\textit{pro-}\\ \textit{étale finite}}} \underline{K}\backslash T \underbrace{\longrightarrow}_{\substack{\textit{étale}\\ \textit{separated}}} S,$$

and thus a pro-étale torsor and we have

$$H^1_{\textit{pro-ét}}(S, \underline{G}) \xrightarrow{\sim} H^1_v(S, \underline{G}).$$

Example 6.2.12. — *Let $\mathbb{Q}_p^{cyc} = \cup_{n \geq 1} \mathbb{Q}_p(\zeta_n)$. Then, $\widehat{\mathbb{Q}_p^{cyc}}$ is a perfectoid field. The morphism*

$$\mathrm{Spa}(\mathbb{C}_p) \longrightarrow \mathrm{Spa}(\widehat{\mathbb{Q}_p^{cyc}})$$

is a v (and even pro-étale) cover. Let $H = Gal(\overline{\mathbb{Q}}_p|\mathbb{Q}_p^{cyc})$. The preceding morphism is an $\underline{H}$-torsor. The fact that vector bundles descend along this morphisms is then equivalent to (Sen)

$$\underbrace{\mathrm{Vect}_{\widehat{\mathbb{Q}_p^{cyc}}}}_{\substack{\textit{finite dim.}\\ \widehat{\mathbb{Q}_p^{cyc}}\textit{-v.s.}}} \xrightarrow{\sim} \underbrace{\mathrm{Rep}_{\mathbb{C}_p}(H)}_{\substack{\textit{semi-linear}\\ \textit{rep. of } H\\ \textit{on finite dim.}\\ \mathbb{C}_p\textit{-v.s.}}}$$

Faltings' Simpson correspondence has been retaken in this context, peoples looking at vector bundles on Perf_X *equipped wit the pro-étale topology when X is a $\mathbb{Q}_p$-rigid analytic space, see for example* [**77**]. *This has allowed peoples to rethink the theory of de Rham $\mathbb{Q}_p$-local systems on a rigid analytic space X, using the functor*

$$\begin{array}{rcl} \textit{Pro-étale } \mathbb{Q}_p\textit{-local systems on } X & \longrightarrow & \textit{Vector bundles on } X_{\textit{proét}} \\ \mathscr{F} & \longmapsto & \mathscr{F} \otimes_{\underline{\mathbb{Q}_p}} \mathcal{O}_{X_{\textit{proét}}}. \end{array}$$

6.3. Diamonds ([**128**, Section 11])

6.3.1. Definition and elementary results. — As we said, diamonds are algebraic spaces for the pro-étale topology. One of the ides of the theory is to push everything in characteristic p.

Definition 6.3.1. — *A diamond is a pro-étale sheaf X on $\mathrm{Perf}_{\mathbb{F}_p}$ such that there exists an $\mathbb{F}_p$-perfectoid space $\widetilde{X}$ and an equivalence relation $R \subset \widetilde{X} \times \widetilde{X}$*

- *that is representable by a perfectoid space,*
- *such that both maps $R \rightrightarrows \widetilde{X}$ are pro-étale,*
- *and we have*
$$X \simeq \widetilde{X}/R$$
(quotient as pro-étale sheaves).

As is well known (Gabber), any algebraic space is an fppf sheaf. The same holds for diamonds: one can prove that *any diamond is a v-sheaf* ([**128**, Proposition 11.9]).

The category of diamonds is very behaved: it has fibered products and finite products.

6.3.2. Spatial diamonds. — Let X be a v-sheaf on $\mathrm{Perf}_{\mathbb{F}_p}$. Suppose it is small in the sense that there exists a perfectoid space S and a surjection $S \to X$. One can then define

$$|X| = \big\{\mathrm{Spa}(K, K^+) \to X \mid (K, K^+) \text{ affinoid perf. field}\big\}/\sim$$

where two morphisms $\mathrm{Spa}(K_1, K_1^+) \to X$ and $\mathrm{Spa}(K_2, K_2^+) \to X$ are equivalent if there exists a diagram

$$\begin{tikzcd}
 & \mathrm{Spa}(K_1, K_1^+) \arrow[dr] & \\
\mathrm{Spa}(K_3, K_3^+) \arrow[ur] \arrow[dr] & & X \\
 & \mathrm{Spa}(K_2, K_2^+) \arrow[ur] &
\end{tikzcd}$$

where $\mathrm{Spa}(K_3, K_3^+) \to \mathrm{Spa}(K_1, K_1^+)$ and $\mathrm{Spa}(K_3, K_3^+) \to \mathrm{Spa}(K_2, K_2^+)$ sends the closed point to the closed point. This is equipped with the structure of a topological space where the open subsets are the subsets

$$|U| \subset |X|$$

where $U \hookrightarrow X$ is a morphisms of v-sheaves representable by an open immersion. One can verify that if

$$S_1 \rightrightarrows S_0 \longrightarrow X$$

is a 1-v-hypercover by perfectoid spaces then

$$\underbrace{\mathrm{coeq}}_{\substack{\text{coeq in}\\ \text{the cat. of}\\ \text{top. spaces}}} \big(|S_1| \rightrightarrows |S_0| \big) \underbrace{\xrightarrow{\sim}}_{\text{homeomorphism}} |X|.$$

Here is the “good notion” of diamonds we use.

Definition 6.3.2. — *A diamond X is spatial if X is qc qs and each point of $|X|$ has a basis of neighborhoods formed of qc open subsets.*

One can verify that in fact the topological condition is equivalent to saying that $|X|$ *is a spectral space.* This makes spatial diamonds look like qc qs analytic adic spaces. This throws out some pathological objects that are not related to rigid analytic geometry like

$$\underline{T} \times \mathrm{Spa}(K)$$

where T is a compact Hausdorff space and K is a perfectoid field. In fact, this last object is a diamond with topological space T. More precisely, if βT_{disc} is the Stone-Chech compactification of T_{disc} there is a surjective quotient map

$$\underbrace{\beta T_{disc}}_{\text{profinite}} \longrightarrow T$$

sending an ultrafilter to its limit. This shows that $\underline{T} \times \mathrm{Spa}(K)$ is a diamond with topological space T.

This spatialness notion is extremely flexible, giving rise to a new geometry. For example, if $(X_i)_i$ is a cofiltered projective system of spatial diamonds then $\varprojlim_i X_i$ is a spatial diamond with $|\varprojlim_i X_i| = \varprojlim_i |X_i|$ as spectral spaces.

Maybe one of the greatest features of the geometry of spatial diamonds is the following. If X is a v-sheaf and $Z \subset |X|$ a subset then Z defines a sub-v-sheaf of X via the formula

$$Z(S) = \{S \to X \mid \mathrm{Im}(|S| \to |X|) \subset Z\}.$$

We have the following result that is a consequence of the fact that *if X is a strictly totally disconnected perfectoid space then any pro-constructible generalizing subset of $|X|$ is representable by a perfectoid space.*

Proposition 6.3.3. — *Let X be a spatial diamond and let $Z \subset |X|$ be pro-constructible generalizing subset. Then Z defines a spatial diamond with topological space Z equipped with a qc injection inside X.*

The geometry of (locally) spatial diamonds is much more flexible than the geometry of classical rigid spaces à la Tate.

6.3.3. Some abstract construction: tilting anything. — Let X be a v-sheaf on Perf.

***Definition 6.3.4* (Tilting anything).** — *We note $X^\diamond$ for the v-sheaf on* $\mathrm{Perf}_{\mathbb{F}_p}$ *whose value on S is given by the datum $(S^\sharp, \iota, s)$ where*
- *$S^\sharp$ is a perfectoid space,*
- *$\iota : S \xrightarrow{\sim} (S^\sharp)^\flat$,*
- *s is an element of $X(S^\sharp)$.*

Of course, if X is a perfectoid space then

$$X^\diamond = X^\flat.$$

This abstract construction allows us to tilt anything in characteristic p. If S is any v-sheaf then

$$\boxed{(-)^\diamond : \mathrm{Perf}\,/S \xrightarrow{\sim} \mathrm{Perf}_{\mathbb{F}_p}\,/S^\diamond}$$

that is a generalized form of the tilting equivalence. This extends to equivalences of topoï (étale, pro-étale or v)

$$\widetilde{\mathrm{Perf}/S} \xrightarrow{\sim} \widetilde{\mathrm{Perf}_{\mathbb{F}_p}}/S^\diamond$$

For example, $\mathbb{Q}_p$-perfectoid spaces are the same as $\mathbb{F}_p$-perfectoid spaces sitting over $\mathrm{Spa}(\mathbb{Q}_p)^\diamond$:

$$\boxed{\mathrm{Perf}_{\mathbb{Q}_p} \xrightarrow{\sim} \mathrm{Perf}_{\mathbb{F}_p}/\mathrm{Spa}(\mathbb{Q}_p)^\diamond.}$$

6.3.4. First example: $\mathrm{Spa}(\mathbb{Q}_p)^\diamond$**.** — The v-sheaf $\mathrm{Spa}(\mathbb{Q}_p)^\diamond$ is a spatial diamond. This is the moduli of untilts of a characteristic p perfectoid space in characteristic 0. In fact,

$$\mathrm{Spa}(\mathbb{Q}_p)^\diamond = \mathrm{Spa}(\mathbb{C}_p^\flat)/\underline{\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)}.$$

We refer to [**132**, Section 8.4] for a detailed discussion. If $E|\mathbb{Q}_p$ is a finite degree extension and E_∞ the completion of the extension generated by the π-torsion points of a Lubin-Tate group then

$$\begin{aligned} \mathrm{Spa}(E)^\diamond &= \mathrm{Spa}(\mathbb{C}_p^\flat)/\underline{\mathrm{Gal}(\overline{E}|E)} \\ &= \mathrm{Spa}(E_\infty^\flat)/\underline{\mathcal{O}_E^\times}. \end{aligned}$$

Remark 6.3.5. — *One has to be careful that* $\mathrm{Spa}(\mathbb{Z}_p)^\diamond$ *is not a diamond. In fact, one can prove that any sub-v-sheaf of a diamond is a diamond, but* $\mathrm{Spa}(\mathbb{F}_p)^\diamond$ *is not a diamond. Nevertheless, one can wrok with such type of objects, this is typically needed in* [**58**, Chapter VI] *for the geometric Satake correspondence.*

6.3.5. $X^\diamond$ for X an analytic adic space. — The starting point of this is the following due to Colmez: *If R is a uniform complete Tate Huber ring then there exists a filtered inductive system $(R_i)_{i\geq i_0}$ of complete uniform Tate Huber rings with $R_{i_0}=R$ such that all transition morphisms are finite étale and*

$$\widehat{\varinjlim_i R_i}$$

is perfectoid. From this one deduces the following.

Proposition 6.3.6. — *If X is an analytic adic space then $X^\diamond$ is a locally spatial diamond with $|X^\diamond|=|X|$.*

Example 6.3.7. — *If X is characteristic p then $X^\diamond$ is a perfectoid space equal to X^{1/p^∞}. In fact, a complete $\mathbb{F}_p$-Tate affinoid ring (A,A^+) is perfectoid if and only if A is a perfect $\mathbb{F}_p$-algebra. As a consequence, if (A,A^+) is any $\mathbb{F}_p$-Tate affinoid ring with pseudo-uniformizer ϖ then $\widehat{A^{+,1/p^\infty}}[\frac{1}{\varpi}]$ is perfectoid.*

***Example 6.3.8* (Faltings).** — *Let R be a p-torsion free p-adic integral normal domain. Let $K=\mathrm{Frac}(R)$ and $\overline{K}$ an algebraic closure of K. Let $\overline{R}$ be the integral closure of R in the maximal extension of K inside $\overline{K}$ that is étale over $R\big[\frac{1}{p}\big]$ i.e. $\mathrm{Aut}_R(\overline{R})=\pi_1(Spec(R\big[\frac{1}{p}\big],\overline{x})$ with $\overline{x}$ given by the choice of $\overline{K}$. Then $\widehat{\overline{\boldsymbol{R}}}$ is perfectoid. In fact, if $x\in\overline{R}$ then the polynomial $P(T)=T^p+pT-x$ is separable over $\overline{R}\big[\frac{1}{p}\big]$. A zero of this polynomial in $\overline{K}$ is then an element of $\overline{R}$ whose p-power is congruent to x modulo p.*

Here is how to explicitly construct some perfectoid charts on $X^\diamond$. Let

$$\widetilde{X}\to X$$

be a pro-étale cover with $\widetilde{X}$ perfectoid. Let

$$\widetilde{X}\times_X\widetilde{X}$$

be the categorical product in the category of X-perfectoid spaces. This product exists since (locally on X and $\widetilde{X}$ that we can suppose affinoid) if $\widetilde{X}=\varprojlim_{i\geq i_0}X_i$ with $X_{i_0}=X$ and finite étale transition morphisms then the purity theorem says that for all indices i,

$$\widetilde{X}\times_X X_i$$

is perfectoid. On can then take

$$\widetilde{X}\times_X\widetilde{X}=\varprojlim_i\widetilde{X}\times_X X_i.$$

Then one has

$$\boxed{X^\diamond = \mathrm{coeq}\left(\ (\widetilde{X} \times_X \widetilde{X})^\flat \rightrightarrows \widetilde{X}^\flat \ \right).}$$

Remark 6.3.9. — *One has to be careful that the product $\widetilde{X} \times_X \widetilde{X}$ should not be taken in the category of adic spaces but in the category of perfectoid spaces sitting over X. For example, $\mathbb{C}_p \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p$ is not perfectoid contrary to $\mathscr{C}(Gal(\overline{\mathbb{Q}}_p|\mathbb{Q}_p), \mathbb{C}_p)$. In fact, $\overline{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \overline{\mathbb{Z}_p} \subset \overline{\overline{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \overline{\mathbb{Z}_p}}^{\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p}$ are not commensurable lattices inside $\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$.*

Example 6.3.10. — *One has*

$$\begin{aligned} & \mathrm{Spa}\big(\mathbb{Q}_p\langle T, T^{-1}\rangle, \mathbb{Z}_p\langle T, T^{-1}\rangle\big)^\diamond \\ = \ & \mathrm{Spa}\big(\mathbb{C}_p^\flat\langle T^{\pm 1/p^\infty}\rangle, \mathcal{O}_{\mathbb{C}_p^\flat}\langle T^{\pm 1/p^\infty}\rangle\big)/\underline{\mathbb{Z}_p(1) \rtimes Gal(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)}. \end{aligned}$$

Taking a product of this situation and using the purity theorem one deduces that for any smooth $\mathbb{Q}_p$-adic space locally of finite type X, locally on X, there is a diagram

$$\begin{tikzcd}
 & \widetilde{X} \arrow[dl, "\text{\it pro-étale-finite}"'] \arrow[dl, bend right, dash, "\mathbb{Z}_p(d)\rtimes Gal(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)"'] \arrow[dr, "\text{\it étale}"] & \\
X \arrow[dr, "\text{\it étale}"'] & \square & \widetilde{\mathbb{T}}^d \arrow[dl, "\text{\it pro-étale-finite}"'] \arrow[dl, bend left, dash, "\mathbb{Z}_p(d)\rtimes Gal(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)"] \\
 & \mathbb{T}^d &
\end{tikzcd}$$

where $\mathbb{T}^d = \mathrm{Spa}(\mathbb{Q}_p\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle, \mathbb{Z}_p\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle)$ and $\widetilde{X}$ is perfectoid since $\widetilde{\mathbb{T}}^d$ is perfectoid thanks to the purity theorem.

In the next example we use quasi-pro-étale morphisms. More precisely, if a morphism of small v-sheaves

$$f : X \to Y$$

is quasicompact, satisfies $|f| : |X| \to |Y|$ is surjective and has profinite geometric fibers then f is an epimorphism of pro-étale sheaves; see section 6.2.2.3.

Example 6.3.11. — *Let's compute $(\mathbb{B}^d_{\mathbb{Q}_p})^\diamond$. One has a quasi-pro-étale cover*

$$\mathbb{B}^{d,1/p^\infty}_{\mathbb{C}_p^\flat} = \left(\mathbb{B}^{d,1/p^\infty}_{\mathbb{C}_p}\right)^\flat \longrightarrow (\mathbb{B}^d_{\mathbb{Q}_p})^\diamond.$$

One has moreover a quasi-pro-étale surjection

$$\left(\mathbb{B}^{d,1/p^\infty}_{\mathbb{C}_p}\right)^\flat \times_{\mathrm{Spa}(\mathbb{C}_p^\flat)} \underline{\mathbb{Z}_p(1)^d \rtimes Gal(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)} \longrightarrow \left(\mathbb{B}^{d,1/p^\infty}_{\mathbb{C}_p}\right)^\flat \times_{(\mathbb{B}^d_{\mathbb{Q}_p})^\diamond} \left(\mathbb{B}^{d,1/p^\infty}_{\mathbb{C}_p}\right)^\flat .$$

One deduces that

$$\left(\mathbb{B}^d_{\mathbb{Q}_p}\right)^\diamond \simeq \mathbb{B}^{d,1/p^\infty}_{\mathbb{C}_p^\flat}/\underline{\mathbb{Z}_p(1)^d \rtimes Gal(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)}$$

(pro-étale quotient) where the action of $\underline{\mathbb{Z}_p(1)^d \rtimes Gal(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)}$ *is free outside the origin but not on the entire object.*

Let us conclude with the following (see [**128**, Section 15]).

Theorem 6.3.12. — *The functor $X \mapsto X^\diamond$ satisfies the following:*

1. *It is fully faithful from the category of Noetherian* **normal** *analytic adic spaces to the category of locally spatial diamonds,*
2. *For X a Noetherian analytic adic space one has an equivalence of topoï*

$$\widetilde{X^\diamond}_{\text{ét}} \xrightarrow{\sim} \widetilde{X}_{\text{ét}}$$

and in particular one can compute the étale cohomology of X in terms of the one of $X^\diamond$.

6.4. An example: the twin towers isomorphism

After collapsing the towers on their base, the isomorphism of [**51**] can be rewritten as an isomorphism of pro-étale stacks over $\mathrm{Spa}(\breve{E})^\diamond$

$$\left[\ \mathbb{P}^{n-1,\diamond}_{\breve{E}}/\underline{D}^\times\ \right] \simeq \left[\ \Omega^\diamond_{n-1,\breve{E}}/\underline{\mathrm{GL}_n(E)}\ \right]$$

where D is the division algebra with invariant $\frac{1}{n}$ and Ω_{n-1} the $n-1$-dimensional Drinfeld's space over E.

This type of isomorphism has been generalized to any reductive group and any basic $[b] \in B(G)$, see section 5.9 and [**28**].

6.5. Some final thoughts

The theory of diamonds gives access to some new geometry. For example, if X is an analytic adic space and $Z \subset |X|$ a subset that is locally on X pro-constructible generalizing then Z defines a sub-locally spatial diamond of $X^\diamond$ that is not attached to a classical analytic adic space in general. For example, the étale cohomology of diamonds allows us to define the étale cohomology of such a Z.

The key points that make the theory work are the associated descent results; typically the fact that separated étale morphismes descend for the v-topology. The theory is very flexible, typically of $(X_i)_i$ is a filtered projective system of spatial diamonds then $\varprojlim_i X_i$ is a spatial diamonds. This makes the theory of locally spatial diamonds the natural one for the study of étale cohmology in a non-archimedean setting.

The first appearance of the use of diamonds with an arithmetic application is [**24**] where the authors define a stratification of a p-adic flag manifold whose strata are not classical rigid spaces in general using [**55**]; see remark 8.6.6. An example of the use of diamonds outside of the work [**58**] is [**66**].

Finally let us note that the theory of diamonds is the one that lead to the theory of condensed sets, a condensed set being nothing else than a pro-étale sheaf on $\mathrm{Spa}(C)$, C an algebraically closed perfectoid field. If X is a perfectoid space there is a continuous morphism of sites

$$\lambda : X_v \longrightarrow X_{\text{proét}}$$

where X_v is the big v site of X and $X_{\text{proét}}$ its small pro-étale site. If $\mathscr{F}$ is a pro-étale sheaf of abelian groups on X that comes from an étale sheaf of abelian groups on $X_{\text{ét}}$ by pullback via $X_{\text{proét}} \to X_{\text{ét}}$ then

$$\mathscr{F} \xrightarrow{\sim} R\lambda_* \lambda^* \mathscr{F},$$

but in general this may be false for any pro-étale sheaf of abelian groups. The functor

$$\lambda^* : D(X_{\text{proét}}, \Lambda) \longrightarrow D(X_v, \Lambda),$$

where Λ is any ring, may not be fully faithful. Nevertheless, the functor λ^* commutes with all small limits ([**128**, Lemma 14.4]) and thus if $U = \varprojlim_i \underbrace{U_i}_{\text{finite set}}$ is a cofiltered limit of affinoid perfectoid spaces étale over X, with $U = \varprojlim_i U_i$ that is pro-étale over X,

$$\Lambda[U]^{\blacksquare} := \varprojlim_i \Lambda[U_i]$$

as a pro-étale sheaf on X, one has

$$\lambda^* \Lambda[U]^{\blacksquare} = \varprojlim_i \underbrace{\Lambda[U_i]}_{\substack{v\text{-sheaf that} \\ \text{comes from} \\ \text{an étale sheaf}}}$$

and thus, since $R\lambda_*$ commutes with limits,

$$\Lambda[U]^{\blacksquare} \xrightarrow{\sim} R\lambda_* \lambda^* \Lambda[U]^{\blacksquare}$$

which may be false if we replace $\Lambda[U]^{\blacksquare}$ by the standard generator $\Lambda[U]$ of the category of pro-étale sheaves of Λ-modules on X. This remark may be one of the starting points of solid condensed Λ-modules; we have fully faithful embeddings

$$D_{\text{ét}}(X, \Lambda) \xhookrightarrow{\nu^*} D_{\text{proét},\blacksquare}(X, \Lambda) \xhookrightarrow{\lambda^*} D_v(X, \Lambda).$$

We refer to [**58**, Chapter VII] for the development of solid quasi-pro-étale sheaves on small v-stacks.

LECTURE 7

EXAMPLES OF DIAMONDS

7.1. The linear objects of the category of diamonds: BC spaces

7.1.1. The relative curve ([58, Section II.1]). — For S an $\overline{\mathbb{F}}_q$-perfectoid space we can define

$$\boxed{X_S = Y_S/\varphi^{\mathbb{Z}}}$$

the *relative curve associated to S* as an E-adic space. This is defined the same way as when S is the spectrum of a perfectoid field. More precisely, if $S = \mathrm{Spa}(R, R^+)$ is affinoid perfectoid then we can define

$$\boxed{Y_S = \mathrm{Spa}(W_{\mathcal{O}_E}(R^+), W_{\mathcal{O}_E}(R^+)) \smallsetminus V(\pi[\varpi])}$$

with its Frobenius φ. This construction glues and lead to the definition of Y_S for any S. The preceding constructions when $S = \mathrm{Spa}(F)$ extends:

1. For S affinoid perfectoid Y_S is sous-perfectoid and thus *Huber's structural presheaf of holomorphic functions is in fact a sheaf.* That being said, *in general this is not a Noetherian adic space.*
2. If $S = \mathrm{Spa}(R, R^+)$ is affinoid perfectoid we can define the associated schematical curve $\mathfrak{X}_{R,R^+}$ as before and GAGA theorem extends:

$$\{\text{vector bundles on } \mathfrak{X}_{R,R^+}\} \xrightarrow{\sim} \{\text{vector bundles on } X_{R,R^+}\}.$$

Remark 7.1.1. — *In fact, Y_S extends naturally to an analytic adic space $\mathcal{Y}_S$ over* $\mathrm{Spa}(\mathcal{O}_E)$ *where $Y_S = \mathcal{Y}_S \smallsetminus V(\pi)$. When $S = \mathrm{Spa}(R, R^+)$ is affinoid perfectoid,*

$$\mathcal{Y}_S = \mathrm{Spa}\left(W_{\mathcal{O}_E}(R^+), W_{\mathcal{O}_E}(R^+)\right) \smallsetminus V([\varpi]).$$

The construction $S \mapsto X_S$ is functorial in S and *one can thought of X_S as being "$X \times S$" although "X" does not exist.*

Here is a computation.

Proposition 7.1.2. — *One has*

$$X_S^\diamond = \left(S \times \mathrm{Spa}(E)^\diamond\right)/\varphi^{\mathbb{Z}} \times \mathrm{Id}$$

as a locally spatial diamond over $\mathrm{Spa}(E)^\diamond$.

This result is reduced to the computation of $Y_S^\diamond$ together with its Frobenius action. One has to compute morphisms $\mathrm{Spa}(A, A^+) \to Y_S$ for (A, A^+) an affinoid perfectoid E-algebra. Suppose that $S = \mathrm{Spa}(R, R^+)$. Such a morphism is given by a morphism

$$W_{\mathcal{O}_E}(R^+) \longrightarrow A^+ \tag{3}$$

such that the image of $[\varpi]$ is a pseudo-uniformizer of A.
We now use the adjunction

$$\text{Perfect } \mathbb{F}_q\text{-algebras} \underset{(-)^\flat}{\overset{W_{\mathcal{O}_E}(-)}{\rightleftarrows}} p\text{-adically separated complete } \mathcal{O}_E\text{-algebras}$$

where the adjunction maps are given by $x \mapsto ([x^{1/p^n}])_{n \geq 0}$ and Fontaine's θ map. From this adjunction we deduce that to give oneself a morphism as in Equation (3) is the same as a morphism

$$R^+ \longrightarrow A^{\flat,+}$$

sending ϖ to a pseudo-uniformizer in A. The result is easily deduced.

Remark 7.1.3. — *In the equal characteristic case, the case $E = \mathbb{F}_q(\!(\pi)\!)$, one has $Y_S = \mathbb{D}_S^*$ and $X_S = \mathbb{D}_S^*/\varphi^{\mathbb{Z}}$ where φ is the Frobenius of S that act trivially on the coordinate π of the punctured disk.*

7.1.2. BC spaces and their families ([132, Section 15.2], [58, Section II.2 and II.3]).

— Affine spaces and their twisted versions (vector bundles) are the natural linear objects showing up in the "classical case" as the relative cohomology of vector bundles. The linear objects of the category of diamonds are the Banach-Colmez spaces.

Before beginning let us recall that *there is a good notion of vector bundles on analytic adic spaces like X_S*. More precisely we have the following. Let (A, A^+) be stably uniform complete Tate Huber ring (for example (A, A^+) is sous-perfectoid). Then

$$(-)^{\text{adification}} : \Big\{ \underbrace{\text{vector bundles on } \mathrm{Spec}(A)}_{\substack{\text{projective finite type} \\ A\text{-modules}}} \Big\} \xrightarrow{\sim} \Big\{ \underbrace{\text{vector bundles on } \mathrm{Spa}(A, A^+)}_{\substack{\text{locally free} \\ \mathcal{O}\text{-modules}}} \Big\}$$

This sends a projective finite type A-module P to $P \otimes_A \mathcal{O}_{\mathrm{Spa}(A,A^+)}$. Moreover for any such vector bundle $\mathscr{E}$ on $\mathrm{Spa}(A, A^+)$ one has

$$H^i(\mathrm{Spa}(A, A^+), \mathscr{E}) = 0 \text{ for } i > 0.$$

***Proposition 7.1.4* (Relative cohomology of vector bundles)**
If S is an $\overline{\mathbb{F}}_q$-perfectoid space and $\mathscr{E}$ a vector bundle on X_S then the functors on S-perfectoid spaces

1. *$T \mapsto H^0(X_T, \mathscr{E}_{|X_T})$,*
2. *the pro-étale sheaf associated to the presheaf $T \mapsto H^1(X_T, \mathscr{E}_{|X_T})$*

are locally spatial diamonds.

If $S = \mathrm{Spa}(R, R^+)$ is affinoid perfectoid and $\mathscr{E}$ is a vector bundle on X_S with pull-back $\mathscr{F}$ to Y_S, since "Y_S is Stein",

$$R\Gamma(X_S, \mathscr{E}) = \Big[\ \Gamma(Y_S, \mathscr{F}) \xrightarrow{\mathrm{Id} - \varphi} \Gamma(Y_S, \mathscr{F}) \ \Big]$$

and there is thus only H^0 and H^1 like any "classical curve".

Example 7.1.5. — *Here are some examples of Banach-Colmez spaces*

1. *The relative cohomology of the structural sheaf $\mathcal{O}$ is $\underline{E}$,*
2. *Let us note $\mathbb{B}$ for the v-sheaf of E-algebras $S \mapsto \mathcal{O}(Y_S)$. Then if $\lambda = \frac{d}{h} \geq 0$ with $(d, h) = 1$ then $H^1(X_S, \mathcal{O}(\lambda)) = 0$ if S is affinoid perfectoid, and the H^0 is the v-sheaf*

$$\mathbb{B}^{\varphi^h = \pi^d}.$$

Remark 7.1.6. — *We use the "period ring" $\mathbb{B}$ over $\mathrm{Spa}(\mathbb{F}_p)^\diamond$ but we could use some "more classical one". As a matter of fact, if $E = \mathbb{Q}_p$, there is a v-sheaf in rings*

$$\begin{tikzcd}
\mathbb{B} \times \mathrm{Spa}(\mathbb{Q}_p)^\diamond \arrow[rr, hook] \arrow[dr] & & \mathbb{B}^+_{cris} \arrow[dl] \\
& \mathrm{Spa}(\mathbb{Q}_p)^\diamond &
\end{tikzcd}$$

where for (R, R^+) affinoid perfectoid with untilt $(R^\sharp, R^{\sharp,+})$ over $\mathbb{Q}_p$

$$\mathbb{B}^+_{cris}(R^\sharp, R^{\sharp,+}) = H^0_{cris}(\mathit{Spec}(R^+/pR^+)/\mathit{Spec}(\mathbb{Z}_p), \mathcal{O}).$$

This induces equalities

$$\mathbb{B} \times \mathrm{Spa}(\mathbb{Q}_p)^\diamond = \bigcap_{n \geq 0} \varphi^n \left(\mathbb{B}^+_{cris}\right),$$

and thus

$$(\mathbb{B}\times \mathrm{Spa}(\mathbb{Q}_p)^\diamond)^{\varphi^h=p^d} \xrightarrow{\sim} (\mathbb{B}^+_{cris})^{\varphi^h=p^d}.$$

Example 7.1.7. — *Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be two vector bundles on X_S. Then the v-sheaf on* Perf_S

$$T\longmapsto \mathrm{Isom}(\mathscr{E}_{1|X_T},\mathscr{E}_{2|X_T})$$

is representable by a locally spatial diamond as an open sub-diamond of $T\mapsto H^0(X_T,\mathscr{E}^\vee_{1|X_T}\otimes\mathscr{E}_{2|X_T})$. This means that the diagonal of the stack Bun *of vector bundles on the curve (see later) is representable in locally spatial diamonds.*

Remark 7.1.8 **(Kedlaya Liu's point of view ([84]))**

Let $S=\mathrm{Spa}(R,R^+)$ be affinoid perfectoid. We define the associated Robba ring as a ring of germs of holomorphic functions around $\pi=0$,

$$\mathcal{R}_{R,R^+}=\varinjlim_{\substack{U\subset\mathcal{Y}_{R,R^+}\\ open\\ V(\pi)\subset U}} \mathcal{O}\left(U\smallsetminus V(\pi)\right).$$

According to Kelaya and Liu ([**84**]*), the restriction functor from φ-equivariant vector bundles on Y_{Y,Y^+} to "germs of φ-equivariant vector bundles around $\pi=0$" induces an equivalence*

$$\{vector\ bundles\ on\ X_S\}\xrightarrow{\sim}\left\{(M,\varphi)\ \middle|\ \begin{array}{c} M\ is\ a\ projective\ of\ finite\ type\\ module\ over\ \mathcal{R}_{R,R^+}\\ and\ \varphi: M\xrightarrow{\sim} M\ semi\text{-}linear\end{array}\right\}.$$

The content of this equivalence is the fact that for a φ-equivariant vector bundle $\mathscr{F}$ on Y_{R,R^+} and $U\subset\mathcal{Y}_{R,R^+}$ a quasicompact open subset, $\Gamma(U\smallsetminus V(\pi),\mathscr{F})$ is a projective module of finite type. One this is verified the equivalence follows easily since for such a U, $Y_{Y,Y^+}=\bigcup_{n\geq 0}\varphi^n\left(U\smallsetminus V(\pi)\right)$.

From this point of view, the cohomology complex of the vector bundle associated to (M,φ) is

$$\left[\,M\xrightarrow{\varphi-\mathrm{Id}} M\right].$$

***Theorem 7.1.9* ([97]).** — *When $S = \mathrm{Spa}(C^\flat)$ with $C|E$ algebraically closed the relative cohomology construction gives an equivalence between*

1. *objects $\mathscr{E} \in D^{[-1,0]}(\mathcal{O}_{X_{C^\flat}})$ satisfying the perversity conditions*
   - *$\mathcal{H}^{-1}(\mathscr{E})$ is a vector bundle with < 0 H.N. slopes,*
   - *$\mathcal{H}^{0}(\mathscr{E})$ is a coherent sheaf with $\geqslant 0$ H.N slopes,*
2. *the sub-abelian category of the category of pro-étale sheaves on $\mathrm{Spa}(C)$ of $\underline{E}$-vector spaces that is the smallest one that*
   - *contains $\underline{E}$*
   - *contains $\mathbb{G}_a$,*
   - *is stable under extensions.*

Let us remark that the only case when those Banach-Colmez spaces are representable by perfectoid spaces is when $\mathscr{E}$ is a vector bundle with slopes in $[0,1]$, in which case it representable by the universal cover of a p-divisible group ([**131**]) . Typically, if the slopes are in $]0,1]$ this is representable by an open perfectoid ball over $\mathrm{Spa}(C^\flat)$. More precisely, we have the following result.

Proposition 7.1.10. — *Let (R, R^+) be an $\mathbb{F}_q$-affinoid perfectoid algebra and $S = \mathrm{Spa}(R, R^+)$. Let $\mathcal{G}$ be a formal π-divisible $\mathcal{O}_E$-module over Spf(R^+) with associated covariant crystal (M, φ) where M is projective of finite type $W_{\mathcal{O}_E}(R^+)$-module and $\varphi : M \to M$ is semi-linear with $\pi M \subset \varphi M \subset M$. There is then a period isomorphism*

$$\underbrace{\left(\underbrace{\varprojlim_{\times \pi} \mathcal{G}}_{\substack{\textit{universal cover of } \mathcal{G} \\ \simeq \textit{Spf}(R^+[\![x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}]\!]) \\ \textit{loc. on Spf}(R^+)}} \right)^{\diamond}}_{\substack{\textit{loc. isom. to} \\ \mathring{\mathbb{B}}_S^{d,1/p^\infty}}} \xrightarrow{\sim} \underbrace{\mathrm{BC}\left(M[\tfrac{1}{\pi}], \pi^{-1}\varphi\right)}_{\substack{\textit{sheaf of global sections} \\ \textit{of the v.b. associated to} \\ \left(M \otimes_{W_{\mathcal{O}_E}(R^+)} \mathcal{O}_{Y_{R,R^+}}, \pi^{-1}\varphi\right)}} \quad .$$

Example 7.1.11. — *Suppose F is algebraically closed. Let $t_1, t_2 \in \mathbb{B}(F)^{\varphi=\pi}$ be linearly independant. Note $\infty_1, \infty_2 \in |X_F|^{cl}$ the associated vanishing loci. There is associated an exact sequence of vector bundles on X_F*

$$0 \longrightarrow \mathcal{O}_{X_F} \xrightarrow{a\mapsto(at_1,at_2)} \mathcal{O}_{X_F}(1) \oplus \mathcal{O}_{X_F}(1) \xrightarrow{(a,b)\mapsto at_2-bt_1} \mathcal{O}_{X_F}(2) \longrightarrow 0.$$

Applying the relative cohomology functor in the curve one obtains an exact sequence of Banach-Colmez spaces

$$0 \longrightarrow \underline{E} \longrightarrow \mathbb{B}^{\varphi=\pi} \oplus \mathbb{B}^{\varphi=\pi} \longrightarrow \mathbb{B}^{\varphi=\pi^2} \longrightarrow 0$$

that proves that $\mathbb{B}^{\varphi=\pi^2}$, the Banach-Colmez spaces associated to $\mathcal{O}(2)$, is a pro-étale quotient of a 2-dimensional ball by a free action of $\underline{E}$,

$$\mathbb{B}_F^{\varphi=\pi^2} \simeq \mathbb{B}_F^{2,1/p^\infty}/\underline{E}.$$

Example 7.1.12. — *Suppose F is algebraically closed as before. Chose $t \in \mathbb{B}(F)^{\varphi=\pi}$ non-zero. Note $\{\infty\} = V^+(t)$ with residue field C, $C^\flat \simeq F$. There is an exact sequence of vector bundles on X_F*

$$0 \longrightarrow \mathcal{O}_{X_F}(-1) \xrightarrow{\times t} \mathcal{O}_{X_F} \longrightarrow i_{\infty *}C \longrightarrow 0.$$

Taking relative cohomology we deduce an isomorphism of Banach-Colmez spaces

$$\underbrace{\mathrm{BC}(\mathcal{O}(-1)[1])}_{\substack{\textit{relative } H^1\\ \textit{of } \mathcal{O}(-1)}} \simeq (\mathbb{G}_{a/C})^\diamond/\underline{E}.$$

7.2. Artin criterion

To go further and give new examples of diamonds we will need the following.

Theorem 7.2.1 **(Artin criterion for spatial diamonds,** [**128**, Proposition 12.20]**)**
Let X be v-sheaf on $\mathrm{Perf}_{\mathbb{F}_p}$. This is a spatial diamond if and only if

1. *it is small,*
2. *it is spatial i.e. X is qc qs and $|X|$ is spectral,*
3. *for any $x \in |X|$, $X_x := \varprojlim_{U\ni x} U$ is a diamond that is to say isomorphic to $\mathrm{Spa}(C, C^+)/\underline{G}$ where $G \subset \mathrm{Aut}(C, C^+)$ is a profinite subgroup.*

Like the classical Artin criterion:

	Diamonds	Algebraic spaces
Global hypothesis	spatial v-sheaf	finite presentation fppf sheaf
Local hypothesis	$\forall x \in \lvert X\rvert, X_x$ is a diamond	the formal completion at each point is representable by a formal scheme

The way we are going to apply the preceding result is the following. We will first prove that X is spatial using the following elementary result.

Lemma 7.2.2 **([128, Lemma 2.9]).** — *Let X be a spectral space and $R \subset X \times X$ be a pro-constructible equivalence relation such that both maps $R \rightrightarrows X$ are open. Then, X/R is a spectral space.*

Here is the corollary we will use. We use the notion of ℓ-cohomological smoothness (to be seen later): the only thing to know is that

$$\ell\text{-cohomologically smooth} \implies \text{open}.$$

Proposition 7.2.3. — *Let X be a spatial diamond and $R \subset X \times X$ be an equivalence relation such that R is a spatial diamond. Suppose both maps $R \rightrightarrows X$ are ℓ-cohomologically smooth for some $\ell \neq p$. Then, X/R is a spatial v-sheaf.*

Example 7.2.4. — *Let $X \to S$ be a morphism of spatial diamonds and G a spatial diamond that is group over S action on X. Suppose that $G \to S$ is ℓ-cohomologically smooth for some $\ell \neq p$. Then X/G is a spatial v-sheaf.*

The second step is the following one we know that X is a spatial v-sheaf. We will exhibit a finite stratification

$$|X| = \bigcup_i Z_i,$$

where Z_i is locally closed generalizing, such that for all indices i, Z_i is a diamond. This will prove that X is a spatial diamond.

Let's put this in a corollary.

Corollary 7.2.5. — *Let X be a qc qs v-sheaf that is ℓ-cohomologically smooth locally a spatial diamond and such that $|X| = \bigcup_i Z_i$ with $Z_i \subset |X|$ locally closed generalizing that is a diamond. Then X is a spatial diamond.*

7.3. Schubert cells in the B_{dR}-affine Grassmanian ([**132**, Section 19])

7.3.1. The B_{dR}-affine Grassmanian. — Let G be our reductive group over E. We can consider the v-sheaf or filtered E-algebras

$$\begin{tikzcd} \mathbb{B}_{dR}^{+} \arrow[d] \\ \mathrm{Spa}(E)^{\diamond}. \end{tikzcd}$$

This sends (R, R^+) an $\mathbb{F}_p$-perfectoid algebra to

- an untilt $(R^\sharp, R^{\sharp,+})$ over E,
- an element of the completion of $W_{\mathcal{O}_E}(R^+)\left[\frac{1}{p}\right]$ for the $\ker\theta$-adic topology where

$$\theta : W_{\mathcal{O}_E}(R^+)\left[\tfrac{1}{p}\right] \longrightarrow R^{\sharp,+}\left[\tfrac{1}{p}\right].$$

We note $\mathbb{B}_{dR}$ for the localization of $\mathbb{B}_{dR}^{+}$ obtained after inverting a generator of $\ker\theta$.

Definition 7.3.1. — *We note*

$$\begin{tikzcd} \mathrm{Gr}_G^{B_{dR}} \arrow[d] \\ \mathrm{Spa}(E)^{\diamond} \end{tikzcd}$$

for $G(\mathbb{B}_{dR})/G(\mathbb{B}_{dR}^{+})$ (étale quotient).

Remark 7.3.2. — *This B_{dR}-affine Grassmanian can be thought of as a Beilinson-Drinfeld type affine Grassmanian i.e. a relative one (see* [**146**] *for basic facts about affine Grassmanian in diverse contexts). In fact, for any $S \to \mathrm{Spa}(E)^{\diamond}$ with S a characteristic p perfectoid space, the associated untilt $S^\sharp$ of S over E defines a "relative degree 1 Cartier divisor" (one can give a precise meaning to this)*

$$S^\sharp \hookrightarrow X_S.$$

If $\mathscr{E}$ is a G-bundle on X_S, its pullback to $S^\sharp \hookrightarrow X_S$ is étale locally on S trivial. It is thus still the case for the pullback of $\mathscr{E}$ to the formal completion along this Cartier divisor. Then, Beauville-Laszlo gluing ([**6**]*) implies that $\mathrm{Gr}_G^{B_{dR}}(S)$ is the set of modifications supported on the Cartier divisor $S^\sharp$ between the trivial G-bundle and another G-bundle.*

7.3.2. Schubert cells. — Suppose G is split to simplify. Fix $T \subset B$ a maximal torus inside a Borel subgroup. For each $\mu \in X_*(T)^+$ there is defined an open Schubert cell inside a closed one

$$\mathrm{Gr}^{B_{dR}}_{G,\mu} \underbrace{\subset}_{\text{open}} \mathrm{Gr}^{B_{dR}}_{G,\leqslant\mu}\,.$$

This is defined via a pointwise condition for each morphism $\mathrm{Spa}(C,C^+) \to \mathrm{Gr}^{B_{dR}}_{G}$. The fact that the inclusion is an open immersion is not completely evident. Nevertheless, as soon as one can write μ as a sum of minuscule cocharacters there is a Bialynicki-Birula morphism

$$\mathrm{BL}_\mu : \mathrm{Gr}^{B_{dR}}_{G,\mu} \longrightarrow \mathscr{F}^\diamond_\mu$$

where $\mathscr{F}_\mu$ is the flag variety associated to μ. This morphism is an iterated étale fibration in $(\mathbb{A}^1)^\diamond$ and we deduce that the open Schubert cell is a locally spatial diamond. This proves that the open Schubert cell is an ℓ-cohomologically smooth locally spatial diamond over $\mathrm{Spa}(E)^\diamond$.

In general, it happens that one can not write such a μ as a sum of minuscule cocharacters (for example for E_8 since there does not exist any minuscule cocharacter in this situation). In this situation one can always write μ as a sum of minuscule cohcarcters plus, eventually, a quasi-minuscule one. Nevertheless, one can look at the associated affine flag dR manifold

$$G(\mathbb{B}_{dR})/I_{dR} = \mathscr{F}\ell^{B_{dR}}_{G} \longrightarrow \mathrm{Gr}^{B_{dR}}_{G}$$

where $I_{dR} \subset G(\mathbb{B}^+_{dR})$ is the reciprocal image of a Borel subgroup via $\theta : G(\mathbb{B}^+_{dR}) \longrightarrow G^\diamond$,

$$I_{dR} = \theta^{-1}B^\diamond.$$

This affine version of $\mathrm{Gr}^{B_{dR}}_{G}$ is stratified by the affine Weyl group

$$\widetilde{W} = X_*(T) \rtimes W.$$

Any element of $\widetilde{W}$ can be written as a product of minimal elements in the affine Weyl groups, giving rise to a Bialynicki-Birula morphism in this context. Now, if $w \in \widetilde{W}$ maps to μ in $X_*(T)^+$ and is of longest length among those we have a diagram

$$\begin{tikzcd}
\mathscr{F}\ell^{B_{dR}}_{G,w} \arrow[r, "\ell\text{-coho. sm.}"] \arrow[d, "\ell\text{-coho. sm.}"', "\mathrm{BL}_w"] & \mathrm{Gr}^{B_{dR}}_{G,\mu} \\
\mathscr{F}\ell^\diamond_{G,\mu} &
\end{tikzcd}$$

As an application of Artin's criterion, using the stratification by open Schubert cells, one can prove the following.

Theorem 7.3.3 **([132**, 19.2.4]**).** — *For all μ, the closed Schubert cell* $\mathrm{Gr}^{B_{dR}}_{G,\leqslant\mu}$ *is a spatial diamond proper over* $\mathrm{Spa}(E)^\diamond$*. The open Schubert cell* $\mathrm{Gr}^{B_{dR}}_{G,\mu}$ *is* ℓ*-cohomologically smooth over* $\mathrm{Spa}(E)^\diamond$*.*

The same goes on with the factorization B_{dR}-affine Grassmanians of section 10: if G is split, for any $(\mu_i)_{i\in I} \in (X_*(T)^+)^I$,

$$\mathrm{Gr}^{B_{dR}}_{G,I,\leqslant(\mu_i)_i} \longrightarrow (\mathrm{Spa}(E)^\diamond)^I$$

is a spatial diamond proper over $(\mathrm{Spa}(E)^\diamond)^I$.

7.4. Punctured absolute Banach-Colmez spaces

This case is new compared to [**132**]. It first appeared in [**56**]. Let

$$* = \mathrm{Spa}(\overline{\mathbb{F}}_q)$$

be the final object of the v-topos on $\overline{\mathbb{F}}_q$-perfectoid spaces (that is not representable since $\mathrm{Spa}(\overline{\mathbb{F}}_q)$ is not perfectoid). For each $\lambda \in \mathbb{Q}_{>0}$ let us note

$$\mathrm{BC}(\mathcal{O}(\lambda)) \longrightarrow *$$

for the v-sheaf

$$S \mapsto H^0(X_S, \mathcal{O}(\lambda)).$$

We call this *an absolute Banach-Colmez space.* Here the terminology “absolute” refers to the fact that we don’t fix a perfectoid base as in Theorem 7.1.9.

When $\lambda \in]0,1]$, this is represented by the adic space associated to a formal scheme isomorphic to

$$\mathrm{Spf}(\overline{\mathbb{F}}_q[\![x_1^{1/p^\infty},\dots,x_d^{1/p^\infty}]\!])$$

where $\lambda = \frac{d}{h}$ with $(d,h)=1$. More precisely, this is the universal cover of a dimension d and height h π-divisible $\mathcal{O}_E$-module. More precisely, if $\mathcal{G}$ is a formal π-divisible $\mathcal{O}_E$-module over $\overline{\mathbb{F}}_q$ with covariant relative isocrystal (D,φ) then

$$BC(\mathscr{E}(D,\pi^{-1}\varphi)) \simeq \varprojlim_{F} \mathcal{G} = \varprojlim_{\times\pi} \mathcal{G}.$$

This is clearly not represented by a perfectoid space or even a diamond (this is represented by a perfect adic space that is not analytic and thus not perfectoid). Nevertheless,

$$BC(\mathcal{O}(\lambda)) \setminus \{0\} = \mathrm{Spa}\big(\overline{\mathbb{F}}_q[\![x_1^{1/p^\infty},\dots,x_d^{1/p^\infty}]\!], \overline{\mathbb{F}}_q[\![x_1^{1/p^\infty},\dots,x_d^{1/p^\infty}]\!]\big) \setminus V(x_1,\dots,x_d)$$

that is a qc qs perfectoid space.

For $\lambda < 0$ we can consider similarly

$$\mathrm{BC}(\mathcal{O}(\lambda)[1]) \longrightarrow *$$

that is the sheaf whose value on S affinoid perfectoid is

$$H^1(X_S, \mathcal{O}(\lambda)).$$

On then has the following result.

Theorem 7.4.1 **([58, Section II.3.3]).** — *For all $\lambda \in \mathbb{Q}_{>0}$, the punctured absolute Banach-Colmez spaces*

$$\mathrm{BC}(\mathcal{O}(\lambda)) \smallsetminus \{0\}$$

and

$$\mathrm{BC}(\mathcal{O}(-\lambda)[1]) \smallsetminus \{0\}$$

are spatial diamonds.

Those last spatial diamonds are not associated to any classical usual objects like Noetherian analytic adic spaces or formal schemes. They are among the most "original" and new objects showing up in [**58**] and are completely unrelated to any usual classical object like formal schemes.

7.5. Some final thoughts

Those last objects, the negative punctured absolute Banach-Colmez spaces, are a key ingredient in our joint work with Scholze. They allow use to construct some very particular charts of the stack of G-bundles on the curve, the so-called "$\mathcal{M}_b$" ([**58**, Section 5.3]). The spatialness of $\mathrm{BC}(\mathcal{O}(-\lambda)[1]) \smallsetminus \{0\}$ is one of the reason why we consider

$$\begin{tikzcd}\mathrm{Bun}_G \arrow[d] \\ *\end{tikzcd}$$

absolutely and not by replacing $*$ by $\mathrm{Spa}(C)$ where C is an algebraically closed $\mathbb{F}_p$-perfectoid field. Working absolutely over $\mathrm{Spa}(\overline{\mathbb{F}}_q)$ is an essential point.

The geometry of locally spatial diamonds is much more flexible than the usual one of Noetherian analytic adic spaces; typically the fact that any locally pro-constructible generalizing subset of $|X|$, X locally spatial, defines a sub-locally spatial diamond is extremely useful.

As a final remark: for any small v-sheaf (and even any small v-stack) X and Λ a torsion ring we can define

$$D_{\text{ét}}(X, \Lambda)$$

via a descent procedure: this is

$$\underbrace{\mathrm{Ho}}_{\substack{\text{homotopy}\\ \text{category}}} \ \varprojlim_{S \to X} \ \underbrace{\mathcal{D}(|S|, \Lambda)}_{\text{stable } \infty\text{-category}}$$

where S is a strictly totally disconnected perfectoid space. More concretely, if

$$S_\bullet \longrightarrow X$$

is a v-hypercover by $\coprod$ of strictly totally disconnected perfectoid spaces then $D_{\text{ét}}(X, \Lambda)$ is identified with the derived category of cartesian sheaves of Λ-modules on $|S_\bullet|$. This makes the category $D_{\text{ét}}(X, \Lambda)$ quite abstract, in particular the functor Rf_*, when f is a morphism of small v-sheaves, is not explicit: if we have a diagram

$$\begin{tikzcd} S'_\bullet \arrow[r, "g"] \arrow[d] & S_\bullet \arrow[d] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

then Rf_* is computed as

$$D_{\text{cart}}(|S'_\bullet|, \Lambda) \xrightarrow{R|g|_*} D(|S_\bullet|, \Lambda) \xrightarrow[\text{functor}]{\text{cartesianification}} D_{\text{cart}}(|S_\bullet|, \Lambda)$$

where the cartesianification functor is not explicit.

Nevertheless, *if X is a locally spatial diamond, it has a nice étale site $X_{\text{ét}}$ of locally separated étale morphisms*; this is a consequence of the fact that separated étale morphisms descend for the v-topology and one has

$$\underbrace{D^+(X_{\text{ét}}, \Lambda)}_{\substack{\text{concrete usual} \\ \text{derived category of} \\ \text{sheaves on } X_{\text{ét}}}} \xrightarrow{\sim} \underbrace{D^+_{\text{ét}}(X, \Lambda)}_{\text{abstractly defined via descent}}$$

with

$$\underbrace{Rf_{\text{ét}*}}_{\substack{\text{concrete} \\ \text{usual derived functor} \\ \text{of étale pushfoward}}} = \underbrace{Rf_*}_{\substack{\text{not explicit} \\ \text{in general}}}$$

when f is a morphism of locally spatial diamonds. We refer to [**128**, Section 17] for the cohomological formalism.

LECTURE 8

Bun_G

We discuss here one of the main objects of [**58**], the moduli space of G-bundles on the curve, see [**58**, Chapter III]. This object is completely new compared to [**131**].

8.1. The moduli stack of G-bundles on the curve

Recall the following. We have $E|\mathbb{Q}_p$ with $\mathcal{O}_E/\pi = \mathbb{F}_q$. We let

$$* = \mathrm{Spa}(\overline{\mathbb{F}}_q)^\diamond$$

be the final object of $(\mathrm{Perf}_{\overline{\mathbb{F}}_q})_v^\sim$, the v-topos. For each $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ we have, functorially in S,

$$X_S$$

an E-adic sous-perfectoid space that can be thought of as "$X \times S$" allthought X does not exist. Being sous-perfectoid, there is a good notion of vector bundles on it.

Definition 8.1.1. — *For any S as before, a G-bundle on X_S is a faithful tensor functor*

$$\mathrm{Rep}\, G \xrightarrow{\otimes} \{\textit{vector bundles on } X_S\}.$$

When $S = \mathrm{Spa}(R, R^+)$ is affinoid perfectoid, if $\mathfrak{X}_{R,R^+}$ is the schematical curve as an E-scheme, there is a GAGA equivalence

$$\big\{G\text{-bundles on } X_{R,R^+}\big\} \xrightarrow{\sim} \big\{\text{étale } G\text{-torsors on } \mathfrak{X}_{R,R^+}\big\}.$$

It is sometimes easier to work with the schematical curve, typically if $\mathscr{E}$ is an étale G-torsor on $\mathfrak{X}_{R,R^+}$ and P a parabolic closed subgroup of G one can form $P\backslash\mathscr{E} \to \mathfrak{X}_{R,R^+}$ that is representable by a proper scheme over $\mathfrak{X}_{R,R^+}$.

Here is the main object of our study.

Definition 8.1.2. — *We note* Bun_G *the fibered groupoid over* $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$

$$S \longmapsto \underbrace{\{G\text{-bundles on } X_S\ \}}_{groupoid}.$$

The first basic result is the following.

Proposition 8.1.3. — *The fibered groupoid* Bun_G *is a stack for the v-topology on* $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$.

This is easily derived from the fact that the fibered groupoid on Perf

$$T \longmapsto \{\text{ vector bundles on } T\ \}$$

is a v-stack i.e. vector bundles satisfy descent for the v-topology, see [**131**, Lemma 17.1.8].

Remark 8.1.4. — *With the notations of section 5.10, there is a morphism of v-stacks*

$$\mathrm{Isoc}_G^{\diamond} \longrightarrow \mathrm{Bun}_G$$

where $\mathrm{Isoc}_G^{\diamond}$ *is the v-stack associated to the pre-stack* $(R, R^+) \mapsto \mathrm{Isoc}_G(R^+)$. *This is given by sending a couple* (D, φ), *where*

- D *is a* $W_{\mathcal{O}_E}(R^+)[\frac{1}{\pi}]$*-module that is projective of finite type,*
- $\varphi : D \xrightarrow{\sim} D$ *is a semi-linear isomorphism.*

to $\Big(D \otimes_{W(R^+)[\frac{1}{\pi}]} \mathcal{O}_{Y_{R,R^+}}, \varphi \otimes \varphi\Big)$.

Nevertheless the Artin v-stack Bun_G *is more suited to our needs, its geometry being more natural. This is an occurrence of the principle that says that the analytic world is more suited to our needs than the algebraic one. For example, in* [**58**, Section II.2.4] *we use the local constancy of* $c_1 : |\mathrm{Bun}_G| \to \pi_1(G)_{\Gamma_E}$ *to give a new and simpler proof of one of the results in* [**121**] *about families of G-isocrystals, using the preceding morphism of v-stacks.*

8.2. Six operations ([128])

We discuss here the formalism of 6 operations for small v-stacks as developed in [**128**].

8.2.1. Small v-stacks. — We need a definition to start with.

Definition 8.2.1. — *1. A small v-stack is a v-stack X on $\mathrm{Perf}_{\mathbb{F}_p}$ such that there exists a v-surjective morphism*

$$\underbrace{S}_{\textit{perf. space}} \longrightarrow X$$

and

$$\underbrace{T}_{\textit{perf. space}} \longrightarrow \underbrace{S \times_X S}_{v\textit{-sheaf}}.$$

2. A morphism $X \to Y$ of small v-stacks is 0-truncated if for any

$$\underbrace{S}_{\textit{perf. space}} \longrightarrow Y$$

the stacky fibered product

$$X \times_Y S$$

is a v-sheaf i.e. $X \times_Y S \xrightarrow{\sim} \underbrace{\pi_0(X \times_Y S)}_{\textit{coarse moduli}}$.

The point is that for any small v-stack X there is a v-hypercovering

$$\underbrace{S_\bullet}_{\substack{\text{simplicial} \\ \text{perf. space}}} \longrightarrow X.$$

8.2.2. $D_{\text{ét}}(X, \Lambda)$ for X a small v-stack. — Let Λ be a *prime to p torsion ring.* We now would like to define

$$D_{\text{ét}}(X, \Lambda)$$

for any small v-stack X. The way we define it is via descent: we want, functorially in X,

$$D_{\text{ét}}(X, \Lambda) = \underbrace{\mathrm{Ho}}_{\substack{\text{homotopy} \\ \text{category}}} \quad \underbrace{\mathcal{D}_{\text{ét}}(X, \Lambda)}_{\substack{v\text{-hypersheaf} \\ \text{of presentable stable } \infty\text{-cat.}}}$$

where the v-hypersheaf condition means that if

$$S_\bullet \longrightarrow X$$

is a v-hypercover of X by perfectoid spaces then

$$\mathcal{D}_{\text{ét}}(X, \Lambda) \xrightarrow{\sim} \varprojlim_{[n] \in \Delta} \mathcal{D}_{\text{ét}}(S_n, \Lambda)$$

where the limit is taken in the ∞-category of presentable stable ∞-categories. The key remark is now the following.

Lemma 8.2.2. — *The correspondence $S \mapsto \mathcal{D}(S, \Lambda)$ from the category of spectral spaces equipped with qc generalizing morphisms is an hypersheaf.*

This is a consequence of the fact that if $T \to S$ is a qc generalizing map between spectral spaces then it is a quotient map. Let us now recall that if S is a strictly totally disconnected perfectoid space then étale sheaves on S are the same as sheaves on $|S|$. Coupled with the preceding lemma we can thus define the following.

Definition 8.2.3. — *For X a small v-stack we set*

1.

$$\mathcal{D}_{\text{ét}}(X,\Lambda) = \varprojlim_{S\to X} \mathcal{D}(|S|,\Lambda)$$

where S is a strictly totally disconnected perfectoid space.

2.

$$D_{\text{ét}}(X,\Lambda) = \operatorname{Ho}\mathcal{D}_{\text{ét}}(X,\Lambda).$$

One can compute the following that does not require any ∞-categories: there are morphisms of sites for X a locally spatial diamond

$$\mathrm{Perf}_{X,v} \xrightarrow{\lambda} \mathrm{Perf}_{X,\text{pro-ét}} \xrightarrow{\nu} \mathrm{Perf}_{X,\text{ét}}\,.$$

Then on can prove that for X a locally spatial diamond

$$D_{\text{ét}}(X,\Lambda) \overset{\nu^*}{\hookrightarrow} \underbrace{D_{\text{pro-ét},\blacksquare}(X,\Lambda)}_{\substack{\text{solid pro-étale}\\ \text{sheaves}}} \overset{\lambda^*}{\hookrightarrow} D_v(X,\Lambda).$$

As a consequence one can check that for X a small v-stack

1. $D_{\text{ét}}(X,\Lambda) = \{A \in D_v(X,\Lambda) \mid \forall S \to X, S \text{ s.t.d. perf. space}, A_{|S} \in D(|S|,\Lambda)\}$.
2. If $S_\bullet \to X$ is a v-hypercover by s.t.d. perfectoid spaces then
$$D_{\text{ét}}(X,\Lambda) \xrightarrow{\sim} D_{\text{cart}}(|S_\bullet|,\Lambda).$$

8.2.3. 4 operations.

— It is now easy to define a formalism of 4 operations for $D_{\text{ét}}(-,\Lambda)$

$$(f^*, Rf_*, R\mathscr{H}om_\Lambda(-,-), \otimes^{\mathbb{L}}_\Lambda)$$

where here f is a 0-truncated morphism of small v-stacks. Here f^* and $\otimes^{\mathbb{L}}_\Lambda$ are explicit but Rf_* and $R\mathscr{H}om(-,-)$ are not explicit in general, they are constructed as adjoints of explicit functors.

Since separated étale morphisms descend for the v-topology and thus the pro-étale one, for any locally spatial diamond X there is a "good" small étale site $X_{\text{ét}}$ whose objects are locally separated étale morphisms of locally spatial diamonds $X' \to X$. One then has

$$\underbrace{\widehat{D}(X_{\text{ét}}, \Lambda)}_{\substack{\text{left completion of } D(X_{\text{ét}},\Lambda) \\ = \text{Ho} \varprojlim_{n \geq 0} \mathcal{D}^{\geq -n}(X_{\text{ét}},\Lambda)}} \xrightarrow{\sim} D_{\text{ét}}(X, \Lambda)$$

where he the process of left completion corresponds to the fact that, in general without any finite cohomological assumption, Postnikov towers of an object may not converge to the original object (see [**101**, Section 5.5.6] for convergence of Postnikov towers). Then, for $f : X \to Y$ a morphism of locally spatial diamonds the preceding operations are explicit and are the usual one, for example

$$Rf_* = \underbrace{Rf_{\text{ét}*}}_{\substack{\text{usual derived} \\ \text{functor extended} \\ \text{via left completion}}} .$$

8.2.4. 6 **operations.** — It remains to define the two other operations $(Rf_!, Rf^!)$. For this we need to upgrade our assumptions and add some technical hypothesis: f is not only 0-truncated but *representable in locally spatial diamonds, compactifiable of finite geometric transcendance degree.* We don't enter into the details, let's just say that those hypothesis are satisfied for all the morphisms we consider. One the obtains a formalism of 6 operations

$$\boxed{(f^*, Rf_*, Rf_!, Rf^!, R\mathscr{H}om(-,-), \otimes^{\mathbb{L}}_{\Lambda})}$$

where $Rf^!$ is formally defined as a right adjoint to $Rf_!$.

8.3. Cohomological smoothness ([**128**, Section 23])

By definition, a morphism is cohomologically smooth if a version of relative Poincaré duality, in the sense of Verdier, is satisfied. More precisely, let

$$f : X \longrightarrow Y$$

be a morphism of small v-stacks that is representable in locally spatial diamonds, compactifiable of finite dim. trg.. By formal adjunction properties there is always a natural transformation

$$Rf^!(\Lambda) \otimes^{\mathbb{L}}_{\Lambda} f^*(-) \longrightarrow Rf^!(-)$$

between functors $D_{\text{ét}}(Y, \Lambda) \to D_{\text{ét}}(X, \Lambda)$.

Definition 8.3.1. — *The morphism f is cohomologically smooth if for any $\ell \neq p$, for any $S \to Y$ with S a strictly totally disconnected perfectoid space, if*

$$f_S : X \times_Y S \longrightarrow S$$

then

1. *$Rf_S^!(\mathbb{F}_\ell) \otimes_\Lambda^{\mathbb{L}} f_S^*(-) \xrightarrow{\sim} Rf_S^!(-)$ as a natural transformation between functors from $D_{\text{ét}}(Y, \mathbb{F}_\ell)$ to $D_{\text{ét}}(X, \mathbb{F}_\ell)$,*
2. *$Rf_S^!(\mathbb{F}_\ell)$ is invertible i.e. étale locally isomorphic to $\mathbb{F}_\ell[2d]$ for some $d \in \frac{1}{2}\mathbb{Z}$.*

One has to be careful that this has to be checked after any base change i.e. we force the cohomological smoothness property to be stable under base change. Reciprocally, if f is cohomologically smooth then

$$Rf^!(\Lambda) \otimes_\Lambda^{\mathbb{L}} f^*(-) \xrightarrow{\sim} Rf^!(-)$$

and $Rf^!(\Lambda)$ is invertible.

8.4. Smooth charts on Bun$_G$

We now have a setup in which we can speak about smooth charts on Bun$_G$.

Theorem 8.4.1. — *The v-stack* Bun$_G$ *is an Artin v-stack in the sense that*

1. *Its diagonal is representable in locally spatial diamonds,*
2. *There exists a locally spatial diamond U together with an ℓ-cohomologically smooth surjective morphism $U \longrightarrow \mathrm{Bun}_G$.*

It is moreover cohomologically smooth of dimension 0 *in the sense that one can choose such a $U \to \mathrm{Bun}_G$ satisfying: $U \to *$ is cohomologically smooth of dimension the dimension of $U \to \mathrm{Bun}_G$.*

Moreover one can prove that the dualizing complex of Bun$_G$ is (non-canonically) isomorphic to $\Lambda[0]$.

One way to construct such charts is to use the following result that is an analog of a result by Drinfeld and Simpson ([**42**]).

***Theorem 8.4.2* ([55]).** — *Let F be an algebraically closed $\overline{\mathbb{F}}_q$-perfectoid field, $\infty \in |\mathfrak{X}_F|$ a closed point and $\mathscr{E}$ a G-bundle on $\mathfrak{X}_F$. Then*

$$\mathscr{E}_{|\mathfrak{X}_F \setminus \{\infty\}}$$

is trivial.

Using this and Beaville-Laszlo gluing one constructs a v-surjective morphism ([**58**, Section III.3])

$$\mathrm{Gr}_G^{B_{dR}} \longrightarrow \mathrm{Bun}_G$$

that allows us to prove that Bun_G is an Artin v-stack. We refer for this to [**58**, Section IV.1.2.1]).

Remark 8.4.3. — *One has to be careful that, contrary to the usual stack of vector bundle on a "usual" curve,* **the stack $\mathbf{Bun}_G$ is not quasi-separated**. *For example, the sheaf of automorphisms of the trivial G-bundle is $\underline{G(E)}$ that is no quasicompact contrary to the algebraic group G. With notations to follow, if $[b] \in B(G)$, the locally closed inclusion*

$$i^b : [*/\widetilde{G}_b] \hookrightarrow \mathrm{Bun}_G$$

of the associated HN strata is not quasicompact. In particular $R(i^b)_*$ does not commute with arbitrary direct sums and thus, à priori, $(i^b)^*$ does not send compact objects to compact objects...although this is the case, see theorem 8.10.1.

Remark 8.4.4. — *Concerning the smooth Artin v-stacks that show up in* [**58**]*, things happen differently for our classifying stacks than in the "usual" case. For example, the v-stack*

$$\left[\ */\underline{G(E)}\ \right]$$

is a smooth Artin v-stack of dimension 0*. A presentation is given by*

$$\begin{tikzcd} G^\diamond/\underline{K} \arrow[d, "\ell\text{-coho. smooth}" description] \\ \left[\ */\underline{G(E)}\ \right] \end{tikzcd}$$

where $K \subset G(E)$ is compact open pro-p and $G^\diamond \to \mathrm{Spa}(E)^\diamond$ being ℓ-cohomologically smooth, $G^\diamond \to *$ is too by composition with $\mathrm{Spa}(E)^\diamond \to *$ ℓ-cohomologically smooth, and thus $G^\diamond/\underline{K} \to *$ is ℓ-cohomologically smooth (see [**128**, Section 24]*).*

In the "classical seting", the Artin algebraic stack

$$BG = [Spec(E)/G]$$

is smooth over $Spec(E)$ with a smooth presentation given by $Spec(E) \to [Spec(E)/G]$. In our situation, the morphism $* \to [*/\underline{G(E)}]$ is not smooth in any sense.

Remark 8.4.5. — *We could use the v-chart $\mathrm{Gr}_G^{B_{dR}} \to \mathrm{Bun}_G$ to analyse étale complexes on Bun_G i.e. the category $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$. This is not what we do in* [**58**]*. As a matter of fact, if $A \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ then its pullback as an element of $D_{\text{ét}}(\mathrm{Gr}_G^{B_{dR}}, \Lambda)$ is very different from the "simple" étale complexes showing up in the geometric Satake correspondence: in general they do not have quasi-compact support and they are not locally constant along the stratification given by the affine Schubert cells.*

8.5. Points of Bun_G

For any small v-stack X we can define

$$|X|$$

as a topological space (see [**128**, Section 12]). The open subsets of $|X|$ are in bijection with the open sub-stacks. As a consequence of the classification of G-bundles on X_F when F is an algebraically closed perfectoid field one obtains the following.

Theorem 8.5.1. — *We have an identification*

$$B(G) \xrightarrow{\sim} |\mathrm{Bun}_G|$$

as sets.

8.6. The topology on $|\mathrm{Bun}_G|$ ([**58**, Section III.2])

8.6.1. Connected components. — The following theorem says, for example, that for $G = \mathrm{GL}_n$ the degree of a vector bundle is a locally constant function and the corresponding open/closed sub-stack Bun_n^d of degree d rank n vector bundles is connected. We refer to [**58**, Section IV.1.2.2] for the following result.

Theorem 8.6.1. — *The function*

$$c_1 : |\mathrm{Bun}_G| \longrightarrow \pi_1(G)_\Gamma$$

is locally constant with connected fibers.

When G_{der} is simply connected the proof of the local constancy of c_1 is reduced to the case of tori by using the factorization $c_1 : |\mathrm{Bun}_G| \to |\mathrm{Bun}_{G/G_{der}}| \to X_*(G/G_{der})_\Gamma \to \pi_1(G)_\Gamma$. This case is easy. For any G the proof is more subtle.

We thus have a decomposition in connected open/closed substacks

$$\mathrm{Bun}_G = \coprod_{c \in \pi_1(G)_\Gamma} \mathrm{Bun}_G^c.$$

8.6.2. HN stratification. — Suppose G is quasi-split to simplify. Let $A \subset T \subset B$ be as usual. The HN polygon defines a map

$$\mathrm{HN} : |\mathrm{Bun}_G| \longrightarrow X_*(A)_{\mathbb{Q}}^+.$$

We refer to [**58**, Section III.2.5] for the following result.

Theorem 8.6.2. — *1. The map*

$$\mathrm{HN} : |\mathrm{Bun}_G| \longrightarrow X_*(A)^+_{\mathbb{Q}}$$

is semi-continuous in the sense that if $X_*(A)^+_{\mathbb{Q}}$ is equipped with the order $\nu_1 \leq \nu_2 \Leftrightarrow \nu_2 - \nu_1 \in \mathbb{Q}_+.\check{\Phi}$ then $\{[b] \mid [\nu_b] \geq \nu\}$ is open.

2. In fact the embedding $B(G) \hookrightarrow \pi_1(G)_\Gamma \times X_*(A)^+_{\mathbb{Q}}$ defines the topology of $|\mathrm{Bun}_G|$ in the sense that for $[b_1], [b_2] \in B(G)$, $[b_1] \leq [b_2]$ in $|\mathrm{Bun}_G|$ if and only if

$$\begin{cases} \kappa(b_1) = \kappa(b_2), \\ \nu_{b_1} \leq \nu_{b_2}. \end{cases}$$

The proof of the semi-continuity of the HN polygon is a nice argument base on the theory of Banach-Colmez spaces and spatial diamonds that allows us to give new more conceptual proofs of the results of [**84**]. More precisely, if $\mathscr{E}$ is a vector bundle on X_S with S qc qs, the morphism

$$\begin{tikzcd} BC(\mathscr{E}) \smallsetminus \{0\}/\pi^{\mathbb{Z}} \arrow[d] \\ S \end{tikzcd}$$

is a proper morphism of spatial diamonds. Its image is thus closed. From this we deduce that

$$\left\{ s \in S \mid H^0(X_{K(s),K(s)^+}, \mathscr{E}_{|X_{K(s),K(s)^+}}) \neq 0 \right\}$$

is closed in $|S|$. Applying this to $\wedge^i \mathscr{E}(d)$ for all $i \geq 1$ and $d \in \mathbb{Z}$ we obtain the semi-continuity result.

The description of the topology on $B(G)$ coming from the one on $|\mathrm{Bun}_G|$ is done in [**16**] for GL_n. For any group G this uses a result of Viehmann ([**139**]).

Recall ([**88**]) that

$$\kappa_{|B(G)_{bsc}} : \underbrace{B(G)_{bsc}}_{\substack{\text{basic} \\ \text{elements}}} \xrightarrow{\sim} \pi_1(G)_\Gamma.$$

This is translated geometrically in the following statement: *any connected component of* Bun_G *has a unique semi-stable point that is thus open.*

Example 8.6.3. — *Consider $G = \mathrm{GL}_2$.*

1. *The unique semi-stable point of Bun_2^0 is $\mathcal{O}^2$. There is then a chain of specializations in Bun_2^0*

$$\mathcal{O}^2 \geqslant \mathcal{O}(1) \oplus \mathcal{O}(-1) \geqslant \mathcal{O}(2) \oplus \mathcal{O}(-2) \geqslant \dots$$

2. *The unique semi-stable point of* Bun_2^1 *is* $\mathcal{O}(\frac{1}{2})$. *There is then a chain of specializations of* Bun_2^1

$$\mathcal{O}(\tfrac{1}{2}) \geqslant \mathcal{O}(2) \oplus \mathcal{O}(-1) \geqslant \mathcal{O}(3) \oplus \mathcal{O}(-2) \geqslant \dots .$$

Example 8.6.4. — *Consider* $G = \mathrm{GL}_3$ *and let us look at* Bun_3^0 *the connected component of degree* 0 *vector bundles. Here is a map of specialization relations:*

$$\begin{tikzcd}
 & & & \mathcal{O}(2) \oplus \mathcal{O}(-1)^2 \arrow[ddr, rightsquigarrow] & \\
 & \mathcal{O}(1) \oplus \mathcal{O}(-\frac{1}{2}) \arrow[urr, rightsquigarrow] \arrow[dr, rightsquigarrow] & & & \\
\mathcal{O}^3 \arrow[ur, rightsquigarrow] \arrow[rr, rightsquigarrow] \arrow[dr, rightsquigarrow] & & \mathcal{O}(1) \oplus \mathcal{O} \oplus \mathcal{O}(-1) \arrow[uur, rightsquigarrow] \arrow[rr, rightsquigarrow] \arrow[ddr, rightsquigarrow] & & \mathcal{O}(2) \oplus \mathcal{O} \oplus \mathcal{O}(-2) \\
 & \mathcal{O}(\frac{1}{2}) \oplus \mathcal{O}(-1) \arrow[ur, rightsquigarrow] \arrow[drr, rightsquigarrow] & & & \\
 & & & \mathcal{O}(1)^2 \oplus \mathcal{O}(-2) \arrow[uur, rightsquigarrow] &
\end{tikzcd}$$

Remark 8.6.5. — *One has* $B(G) = G(\breve{E})/\sim$ *where* $\sim$ *is the* σ*-conjugacy relation. The induced quotient topology on* $B(G)$ *deduced from the usual one on* $G(\breve{E})$ *is the discrete topology and not the one coming from* $|\mathrm{Bun}_G|$. *In fact, for* $b \in G(\breve{E})$, *there exist a neighborhood* U *of* b *such that for all* $b' \in U$, b' *is* σ*-conjugated to* b.

***Remark 8.6.6* (Bun$_G$ and Shimura varieties).** — *Suppose G is quasi-split. For any cocharater μ, Kottwitz set $B(G,\mu)$ ([**86**]) is an open subset of $|\mathrm{Bun}_G|$:*

$$B(G,\mu) = \{\textit{generalizations of the } \mu\textit{-ordinary element in } B(G,\mu)\} \underset{open}{\subset} |\mathrm{Bun}_G|.$$

Here if T is a maximally split torus in G, one can suppose $\mu \in X_*(T)$ whose image in $B(G)$ via

$$X_*(T) \longrightarrow X_*(T)_{\Gamma_E} = B(T) \longrightarrow B(G)$$

is the so-called μ-ordinary element.
This means at the end that the Newton strata of mod p Shimura varieties are parametrized by a finite open subset of $|\mathrm{Bun}_G|$.
*One can go further. If S_{K^p} is a smooth integral model of a PEL type Shimura variety with hyperspecial level at p ([**88**]) and level K^p outside p, with mod p reduction $\overline{S}_{K^p}$, there is a morphism*

$$\pi_{cris} : \underbrace{\overline{S}_{K^p}^{1/p^\infty}}_{perfect\ scheme} \longrightarrow \mathrm{Isoc}_{G_{\mathbb{Q}_p}}$$

with the notations of section 5.10, where G is the global reductive group associated with the Shimura variety. The morphism π_{cris} is the cristalline period morphism given by the crystal of the universal p-divisible group. This induces a morphism of v-stacks

$$\pi_{cris} : \Bigg(\underbrace{\overline{S}_{K^p}^{1/p^\infty}}_{perfect\ scheme} \Bigg)^{\diamond} \longrightarrow \mathrm{Isoc}^{\diamond}_{G_{\mathbb{Q}_p}} \longrightarrow \mathrm{Bun}_G$$

*and the non-vacuity of Newton strata ([**98**]) is equivalent to*

$$\mathrm{Im}\,\pi_{cris} = B(G,\mu) \subset |\mathrm{Bun}_G|.$$

Example 8.6.7. — *The simplest example if of course the case of the Picard stack* $\mathrm{Bun}_{\mathbb{G}_m}$*. In this case,*

$$\mathrm{Bun}_{\mathbb{G}_m} = \big[\ */\underline{E}^\times\ \big] \times \underline{\mathbb{Z}}.$$

In general, for a torus T one has an exact sequence of Picard v-stacks

$$0 \longrightarrow \big[\ */\underline{T(E)}\ \big] \longrightarrow \mathrm{Bun}_T \longrightarrow X_*(T)_{\Gamma_E} \longrightarrow 0.$$

8.7. Some “nice charts” on Bun$_G$ ([**58**, Section V.3])

Let us consider the case of GL_2 and more precisely the connected component of degree 0 rank 2 vector bundles, Bun_2^0. For $d \geq 0$ let

$$\mathcal{M}_d$$

be the moduli stack that sends S to extensions

$$0 \longrightarrow \mathscr{L} \longrightarrow \mathscr{E} \longrightarrow \mathscr{L}' \longrightarrow 0$$

where

1. $\mathscr{E}$ is a rank 2 vector bundle on X_S,
2. $\mathscr{L}$ is a degree $-d$ line bundle on X_S,
3. $\mathscr{L}$ is a degree d line bundle on X_S.

We thus consider "anti-Harder-Narasimhan filtrations" of a given rank 2 vector bundle $\mathscr{E}$. The evident morphism

$$\mathcal{M}_d \longrightarrow \mathrm{Bun}_2^0$$

is ℓ-cohomologically smooth. Moreover, its image (that is open as it is ℓ-coho. smooth) is the set of generalizations of $\mathcal{O}(d) \oplus \mathcal{O}(-d)$ i.e. the $\mathcal{O}(d') \oplus \mathcal{O}(-d')$ with $0 \leq d' \leq d$.

The Picard stack, Bun_1^d is isomorphic to $[*/\underline{E}^\times]$ by sending $\mathscr{L}$ to the pro-étale torsor of isomorphisms between $\mathcal{O}(d)$ and $\mathscr{L}$. Let

$$\widetilde{\mathcal{M}}_d = \mathrm{BC}(\mathcal{O}(-2d)[1])$$

be the absolute Banach-Colmez space that is the moduli of extensions of $\mathcal{O}(d)$ by $\mathcal{O}(-d)$. One has

$$\mathcal{M}_d = \left[\mathrm{BC}(\mathcal{O}(-2d)[1])/\underline{E}^\times \times \underline{E}^\times\right].$$

We have the more general following theorem for any G that uses the so-called Jacobian criterion of smoothness ([**58**, Section IV.4]).

Theorem 8.7.1. — *For any* $[b] \in B(G)$ *one can define a diagram*

$$\begin{tikzcd}
\mathcal{M}_b \arrow[r, "\substack{\ell\text{-coho.}\\ smooth}", "\pi_b"'] \arrow[d] & \mathrm{Bun}_G \\
{[\,*\,/\underline{G_b(E)}\,]} \arrow[u, bend left]
\end{tikzcd}$$

where G_b *is the* σ*-centralizer of* b *and such that if* $\widetilde{\mathcal{M}}_b$ *is defined via the cartesian diagram*

$$\begin{tikzcd}
\widetilde{\mathcal{M}}_b \arrow[r] \arrow[d] & \mathcal{M}_b \arrow[d] \\
* \arrow[r] \arrow[u, bend left] & {[\,*\,/\underline{G_b(E)}\,]} \arrow[u, bend right]
\end{tikzcd}$$

then

$$\widetilde{\mathcal{M}}_b \setminus \{*\}$$

is a spatial diamond. Moreover the image of $\mathcal{M}_b \to \mathrm{Bun}_G$ *is the set of generalizations of* $[b]$.

One of the main point of the preceding result is the spatialness of $\widetilde{\mathcal{M}}_b \setminus \{*\}$. *This is the main reason why we consider* Bun_G *“absolutely” over* $*$ and not its pullback to $\mathrm{Spa}(C)$ for some algebraically closed $\overline{\mathbb{F}}_q$-perfectoid field C since the pullback to $\mathrm{Spa}(C)$ of $\widetilde{\mathcal{M}}_b \setminus \{*\}$ is only locally spatial non quasi-compact.

The ℓ-cohomological smoothness of π_b is deduced from the so-called Jacobian criterion of smoothness ([**58**, Chapter IV.4]).

For $K \subset G_b(E)$ compact open pro-p we can consider

$$f_K^b : [\widetilde{\mathcal{M}}_b/\underline{K}] \longrightarrow \mathrm{Bun}_G$$

that is thus ℓ-cohomologically smooth and set

$$A_K^b := Rf_{K!}^b Rf_K^{b!}\Lambda \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda).$$

The collection of objects $(A_K^b)_{[b],K}$ is a generalization of the “classical set of compact generators”

$$\left(\text{c-Ind}_K^{G(E)}\,\Lambda\right)_K$$

of the category of smooth representations of $G(E)$ with coefficients in Λ.

Theorem 8.7.2. — *The category $D_{\text{ét}}(\mathrm{Bun}_G,\Lambda)$ is compactly generated with $(A^b_K)_{[b],K}$ a set of compact generators.*

Those compact generators are a key tool in [**58**]. As a matter of fact, the construction of the spectral action goes first through its construction on the compact objects of $D_{\text{ét}}(\mathrm{Bun}_G,\Lambda)$.

8.8. Semi-orthogonal decomposition

For S an $\mathbb{F}_q$-perfectoid space there is an equivalence of categories

$$\left\{\begin{array}{c}\text{pro-étale sheaves of }\underline{\mathbb{Q}_p}\text{-vector spaces}\\ \text{locally isomorphic to }\underline{\mathbb{Q}_p^n}\end{array}\right\} \simeq \left\{\begin{array}{c}\text{vector bundles }\mathscr{E}\text{ on }X_S\\ \text{fiberwise on }S\text{ semi-stable of slope }0\end{array}\right\}.$$

This equivalence extends for all G to an isomorphism for $[b]\in B(G)$ basic to an isomorphism of v-stacks ([**58**, Section III.2.3])

$$\underbrace{\left[\ */\underline{G_b(E)}\ \right]}_{\substack{\text{stack of pro-étale}\\ \underline{G_b(E)}\text{-torsors}}} \simeq \mathrm{Bun}_G^b = \underbrace{\mathrm{Bun}_G^{-\kappa(b),ss}}_{\substack{\text{semi-stable locus}\\ \text{in connected component}\\ \text{defined by } c_1=-\kappa(b)}} \xhookrightarrow[\text{open}]{} \mathrm{Bun}_G$$

The semi-stable locus of Bun_G, an open substack, is thus isomorphic to a disjoint union of pro-étale classifying stacks of locally profinite groups:

$$\mathrm{Bun}_G^{ss} = \coprod_{[b]\text{ basic}} \left[\ */\underline{G_b(E)}\ \right].$$

It is easily verified ([**58**, Section V.1]) that there is an equivalence of categories

$$\underbrace{D(G_b(E),\Lambda)}_{\substack{\text{derived category}\\ \text{of smooth representations}}} \xrightarrow{\ \sim\ } D_{\text{ét}}\Big(\left[\ */\underline{G_b(E)}\ \right],\Lambda\Big).$$

One can go further. In fact, if $\mathscr{E}$ is a vector bundle on X_S with constant Harder-Narasimhan polygon fiberwise on S then ([**58**, Section II.2.5]) $\mathscr{E}$ is equipped with a filtration by vector bundles $(\mathscr{E}^{\geqslant\lambda})_{\lambda\in\mathbb{Q}}$ whose graded pieces are vector bundles, and inducing the Harder-Narasimhan filtration of $\mathscr{E}$ fiberwise on S. After replacing S by a pro-étale cover one can even split this filtration. For any $|b]\in B(G)$, not necessarily basic, we deduce that the corresponding Harder-Narasimhan strata associated to $[b]$ is a classifying stack

$$\boxed{\mathrm{Bun}_G^b \simeq \left[*/\widetilde{G}_b\right]}$$

where Bun_G^b is the locally closed sub-stack defined by

$$\mathrm{Bun}_G^b(S) = \Big\{\mathscr{E}\in\mathrm{Bun}_G(S)\mid \forall\mathrm{Spa}(C,C^+)\to S,\ \mathscr{E}_{|X_{C,C^+}}\simeq\mathscr{E}_b\Big\},$$

and $\widetilde{G}_b$ is the v-sheaf of automorphisms of $\mathscr{E}_b$ that can be written as

$$\widetilde{G}_b \simeq \underline{G_b(E)} \rtimes \underbrace{\widetilde{G}_b^\circ}_{\substack{\text{unipotent diamond} \\ \text{succesive extension of} \\ >0 \text{ BC spaces}}} .$$

Example 8.8.1. — *If $G = \mathrm{GL}_n$ and $\mathscr{E}_b = \bigoplus_{i=1}^d \mathcal{O}(\lambda_i)^{m_i}$ then*

$$G_b = \prod_{i=1}^{d} \mathrm{GL}_{m_i}(D_{\lambda_i}),$$

with D_λ the division algebra with invariant λ. Moreover, if $\lambda_1 < \cdots < \lambda_d$, there is a decreasing filtration $\mathrm{Fil}^\bullet \widetilde{G}_b$ with $\mathrm{Fil}^0 = \widetilde{G}_b$, $\mathrm{Fil}^1 = \widetilde{G}_b^\circ$, $\mathrm{Fil}^d = \{1\}$, and for $1 \leq k \leq d-1$,

$$\mathrm{Fil}^k/\mathrm{Fil}^{k+1} = \mathrm{BC}\left(\bigoplus_{i=1}^{d-k} \underbrace{\mathcal{O}(-\lambda_i) \otimes \mathcal{O}(\lambda_{i+k})}_{\substack{\textit{a finite direct sum of} \\ \mathcal{O}(\lambda_{i+k}-\lambda_i)}} \right).$$

One can then prove ([**58**, Section V.2])

$$D(G_b(E), \Lambda) = D_{\text{ét}}\left(\left[\ */\underline{G_b(E)}\ \right], \Lambda\right) \xrightarrow{\sim} D_{\text{ét}}\left(\left[\ */\widetilde{G}_b\ \right], \Lambda\right).$$

This is where the fact that Λ *is prime to p-torsion* shows up (this fact would be false for $\mathbb{F}_p$-coefficients). In fact, the morphism

$$\left[\ */\underline{G_b(E)}\ \right] \longrightarrow \left[\ */\widetilde{G}_b\ \right]$$

is a $\widetilde{G}_b^\circ$-torsor and it thus suffices to prove that for $f : T \to S$ a torsor under a > 0 Banach-Colmez space then

$$f^* : D_{\text{ét}}(S, \Lambda) \longrightarrow D_{\text{ét}}(T, \Lambda)$$

is fully faithfull. By a devissage this is reduced to prove the same thing for T an open perfectoid ball over S, in which case

"the ℓ-contractibility of the open ball" for $\ell \neq p$

implies the result. More precisely, one has to verify that for an integer $d \geq 1$, if

$$f : \mathbb{B}_S^{d,1/p^\infty} \longrightarrow S$$

is the projection of a perectoid ball to its base then

$$f^* : D_{\text{ét}}(S, \Lambda) \longrightarrow D_{\text{ét}}(\mathbb{B}_S^{d,1/p^\infty}, \Lambda)$$

is fully faithful. Since f is ℓ-cohomologically smooth this is reduced to the same statement for $Rf^!$. One then has to proving that

$$Rf_!Rf^! \xrightarrow{\sim} \mathrm{Id}\,.$$

Using proper base change this is reduced to the case when $S = \mathrm{Spa}(C, C^+)$ with C algebraically closed. Since both functors commute with filtered colimits we can suppose that we work with a constructible sheaf of Λ-modules on $\mathrm{Spa}(C, C^+)$. Then an easy dévissage argument reduces the statement to the fact that

$$Rf_!Rf^!\Lambda \xrightarrow{\sim} \Lambda.$$

We deduce the following result.

Theorem 8.8.2. — *The triangulated category* $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ *has a semi-orthogonal decomposition by the collection of triangulated categories*

$$D(G_b(E), \Lambda)$$

when $[b]$ *goes through Kottwitz set* $B(G)$.

This reflects the fact that Bun_G is in some sense obtained by gluing the classifying stacks $[*/\widetilde{G}_b]$ when $[b]$ varies in $B(G)$. The semi-orthogonal decomposition is given by the locally-closed immersion

$$i^b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$$

and the associated couple of adjoint functors

$$\begin{tikzcd} D(G_b(E), \Lambda) \arrow[r, shift left, "(i^b)_!"] & D(\mathrm{Bun}_G, \Lambda). \arrow[l, shift left, "(i^b)^*"] \end{tikzcd}$$

8.9. $D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ ([**58**, Chapter VII])

In those notes, to simplify, we only deal with torsion (prime to p) coefficients Λ. Nevertheless in [**58**, Chapter VII] we define a triangulated category

$$D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$$

for any $\mathbb{Z}_\ell$-algebra Λ. This category is the preceding $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ when Λ is killed by a power of ℓ.

Let Λ be any $\mathbb{Z}_\ell$-algebra. We see it as a condensed ring via the formula $\Lambda := \Lambda_{disc} \otimes_{\mathbb{Z}_\ell^{dics}} \mathbb{Z}_\ell$. This defines a pro-étale sheaf on $\mathrm{Perf}_{\mathbb{F}_p}$, for any $\mathbb{F}_p$-perfectoid space S

$$\Lambda(S) = \varinjlim_{M \subset \Lambda} \underbrace{\mathscr{C}(|S|, M)}_{\text{continuous functions}}$$

where M goes through the set of finite type sub-$\mathbb{Z}_\ell$-modules of Λ. The point now is the following. There is a good notion of solid pro-étale sheaves on any small v-stack X,

$$D_{\text{proét},\blacksquare}(X, \Lambda) \subset D_v(X, \Lambda).$$

This is equipped wqith a formalism of 5 operations

$$(f^*, Rf_*, f_\natural, - \overset{\blacksquare^{\mathbb{L}}}{\otimes}_\Lambda -, R\text{Hom}_\Lambda(-,-))$$

for f a 0-truncated morphism of small v-stacks. Here $f_\natural$ is the relative homology functor, a left adjoint to f^*. One has to be careful that there does not exist functors $Rf_!$ or $Rf^!$ in this context. Here is a little bit more details.

Definition 8.9.1. — *1. A pro-étale sheave of Λ-modules $\mathscr{F}$ on the perfectoid space X is solid if for any $U = \varprojlim_i U_i \to X$ a cofiltered limit of étale affinoid X-perfectoid spaces, if*

$$\Lambda[U]^\blacksquare = \varprojlim_i \Lambda[U_i]$$

as a pro-étale sheaf, then the morphism $\Lambda[U] \to \Lambda[U]^\blacksquare$ induces an isomorphism

$$\mathit{Hom}_\Lambda(\Lambda[U]^\blacksquare, \mathscr{F}) \xrightarrow{\sim} \mathscr{F}(U).$$

2. If X is a small v-stack and $A \in D_v(X, \Lambda)$, A is a solid quasi-pro-étale sheaf if for any $S \to X$ with S a strictly totally disconnected perfectoid space and any $i \in \mathbb{Z}$, $A_{|S} \in D_v(S, \Lambda)$ is of the form $\lambda^* B$ with $B \in D(S_{pro\acute{e}t}, \Lambda)$ such that for all $i \in \mathbb{Z}$, $\mathcal{H}^i(B)$ is solid.

In the preceding definition $\lambda : S_v \to S_{\text{proét}}$ is the morphism from the big v-site to the small pro-étale one; see section 6.5. Using this one obtains for any small v-sheaf X a triangulated category

$$D_{\text{proét},\blacksquare}(X, \Lambda) \subset D_v(X, \Lambda).$$

On of the main points now is that if $f : X \to Y$ is a 0-truncated morphism of small v-stacks then $f^* : D_{\text{qproét},\blacksquare}(Y, \Lambda) \to D_{\text{qproét},\blacksquare}(X, \Lambda)$ commutes with all limits since we can compute it as a pullback functor between big v-sites where this is evident. It thus admits a left adjoint

$$\underbrace{Rf_\natural}_{\text{relative homology}} : D_{\text{proét},\blacksquare}(X, \Lambda) \longrightarrow D_{\text{proét},\blacksquare}(Y, \Lambda),$$

typically, with the notations of definition 8.9.1, if $f : U \to X$,

$$Rf_\natural \Lambda = \Lambda[U]^\blacksquare.$$

This extends more generally to any cofiltered limit of étale sheaves of Λ-modules,

$$Rf_\natural \varprojlim_i \underbrace{\mathscr{F}_i}_{\substack{\text{étale} \\ \text{sheaf}}} = \varprojlim_i Rf_\natural \mathscr{F}_i.$$

Using some projection formula ([**58**, Section VI.3]) one obtains for example that

$$Rf_\natural\Lambda = \underbrace{Rf_\natural\mathbb{Z}_\ell}_{\varprojlim_{k\geq 1} Rf_\natural\mathbb{Z}/\ell^k\mathbb{Z}} \overset{\blacksquare^{\mathbb{L}}}{\otimes}_{\mathbb{Z}_\ell}\Lambda.$$

If we work with $D_{\text{proét},\blacksquare}(\mathrm{Bun}_G,\Lambda)$ we may fall on a semi-orthogonal decomposition by categories

$$\underbrace{D(G_b(E),\Lambda_\blacksquare)}_{\substack{\text{derived category of representations}\\ \text{of } G_b(E) \text{ as a condensed group}\\ \text{in solid } \Lambda\text{-modules}}}.$$

When $\Lambda = \mathbb{Q}_\ell$ this typically contains continuous representations of the topological group $G_b(E)$ in $\mathbb{Q}_\ell$-Banach spaces. We're only interested in smooth representations. To deal with this we need to cut out a sub-triangulated category.

Definition 8.9.2. — *For X a small v-stack, $D_{\mathrm{lis}}(X,\Lambda)$ is the smallest sub-triangulated category of $D_{pro\acute{e}t,\blacksquare}(X,\Lambda)$ stable under all direct sums and that contains $f_\natural\Lambda$ for any $f : Y \to X$ that is separated, representable in locally spatial diamonds and ℓ-cohomologically smooth.*

This definition works well for us because of the following.

Proposition 8.9.3. —

1. *For any $[b] \in B(G)$,*

$$D_{\mathrm{lis}}([*/\underline{G_b(E)}],\Lambda) = D(G_b(E),\Lambda)$$

the derived category of smooth representations with coefficients in Λ.

2. *The triangulated category $D_{\mathrm{lis}}(\mathrm{Bun}_G,\Lambda)$ has a semi-orthogonal decomposition by the collection of categories*

$$\left(D(G_b(E),\Lambda)\right)_{[b]\in B(G)}.$$

This is in fact deduced from one of the main results of [**58**, Chapter VII] that is the following, see [**58**, Section VII.6].

Proposition 8.9.4. — *If f is a separated ℓ-cohomologically smooth morphism of spatial diamonds then for any $k \geq 1$,*

$$Rf_\natural\mathbb{Z}/\ell^k\mathbb{Z} = \underbrace{Rf_! Rf^!\mathbb{Z}/\ell^k\mathbb{Z}}_{\substack{\textit{perfect constructible}\\ \textit{étale complex}}}.$$

This result allows us to proves that

$$D_{lis}(*,\Lambda) = D_{lis}(\mathrm{Spa}(C),\Lambda) = \underbrace{D(\Lambda)}_{\substack{\text{classical derived}\\ \text{cat. of discrete}\\ \Lambda\text{-modules}}}$$

and using this one can prove proposition 8.9.3.

Everything we do in [**58**] works for any coefficients Λ that are $\mathbb{Z}_\ell$-algebras but we advise the reader to restrict itself to torsion coefficients Λ first since it is technically easier to deal with $D_{\text{ét}}$ than D_{lis}.

8.10. Final thoughts

Kottwitz philosophy (following Vogan's one for pure inner forms ([**140**])) that one should not only consider the local Langlands correspondence for a quasi-split group G but for all its extended pure inner forms together, the $(G_b)_{[b],\text{ basic}}$, has been extremely important. One can find traces of this philosophy in [**116**] (the conjecture in [**116**] about the cohomology of Rapoport-Zink spaces was already an important motivation for [**50**] as some kind of analog at p of Schmid realization of discrete series in the L^2-cohomology of symmetric spaces). The main reference now is [**81**]. More precisely, one has the following refinement of the local Langlands correspondence as presented in Chapter 1.

For G quasi-split and $\varphi : W_E \to {}^L G(\overline{\mathbb{Q}}_\ell)$ a cuspidal parameter, via the identification between $X^*(Z(\widehat{G})^{\Gamma_E})$ and the basic elements in $B(G)$, if $[b]$ corresponds to χ then there is a bijection between elements in the supercuspidal packet of $G_b(E)$ associated to φ and

$$\left\{\rho \in \mathrm{Irr}(S_\varphi) \mid \rho_{Z(\widehat{G})^{\Gamma_E}} = \chi\right\}.$$

Finally, it has become more and more clear that the objects of $D_{\text{ét}}(\mathrm{Bun}_G,\Lambda)$ are the natural objects of the local Langlands program and not the smooth representations of $G(E)$ with Λ-coefficients. Let us cite for example the following result deduced from the results of section 8.7 ([**58**, Chapter V]).

Theorem 8.10.1. — *The following is satisfied for* $A \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$*:*

1. *A is compact if and only if it is supported on a quasi-compact open subset and for any* $[b] \in B(G)$ *the object*
$$(i^b)^* A \in D(G_b(E), \Lambda)$$
(the restriction of A to the associated HN strata) is compact i.e. in the thick triangulated sub-category generated by the collection $(c\text{-}\mathrm{Ind}_K^{G_b(E)}\,\Lambda)_K$ *where K is open pro-p in* $G_b(E)$*.*
2. *A is ULA if and only if for any* $[b]$ *and any* $K \subset G_b(E)$ *open pro-p,*
$$[(i^b)^* A]^K \in D(\Lambda)$$
is a perfect complex.

This means that the usual notions of *finite type* and *admissible* smooth representations of p-adic groups, that are at the heart of the classical work of Bernstein on smooth representations of p-adic groups ([**12**]), have natural geometric interpretations in our context.

LECTURE 9

HECKE CORRESPONDENCES

The stack Bun_G does not come alone but equipped with cohomological correspondences. In [**132**] their "ghost" appears under the form of local Shtuka moduli spaces together with their de-Rham and Hodge-Tate period maps.

9.1. The moduli of degree 1 effective divisors on the curve

The moduli of degree 1 effective divisors on the curve first appeared in [**56**]. Let

$$\mathrm{Spa}(\breve{E})^\diamond \longrightarrow *.$$

If S is an $\overline{\mathbb{F}}_q$-perfectoid space then any untilt of S over $\breve{E}$, $S^\sharp$, defines a Cartier divisor

$$S^\sharp \hookrightarrow Y_S.$$

In fact, if $S = \mathrm{Spa}(R, R^+)$, an untilt over $\breve{E}$ is given by an ideal $I \subset W_{\mathcal{O}_E}(R^+)$ generated by a degree 1 distinguished element ξ (that is automatically a regular element) i.e.

$$\xi = \sum_{n\geq 0} [a_n]\pi^n$$

with $a_0 \in R^{\circ\circ} \cap R^\times$ and $a_1 \in (R^+)^\times$. This defines our Cartier divisor

$$V(\xi) \subset Y_S$$

via the embedding of $W_{\mathcal{O}_E}(R^+)$ inside $\mathcal{O}(Y_{R,R^+})$ (one has $W_{\mathcal{O}_E}(R^+) = \mathcal{O}(Y_{R,R^+})^+$). One verifies that composing with the projection defines a degree 1 Cartier divisor

$$S^\sharp \hookrightarrow X_S.$$

This defines a morphism

$$\mathrm{Spa}(\breve{E})^\diamond \longrightarrow \mathrm{Div}^1$$

where we take the following definition of a relative Cartier divisor.

Definition 9.1.1. — *We note* $\mathrm{Div}^1(S)$ *the set of equivalence classes of couples* $(\mathscr{L}, u)$ *where* $\mathscr{L}$ *is a degree* 1 *line bundle on* X_S *and* $u \in H^0(X_S, \mathscr{L})$ *satisfies*

$$\forall s \in S,\ u_{|X_{K(s),K(s)^+}} \neq 0$$

as an element of $H^0\big(X_{K(s),K(s)^+}, \mathscr{L}_{|X_{K(s),K(s)^+}}\big)$.

This morphism is $\varphi^{\mathbb{Z}}$-invariant and induces an isomorphism.

Proposition 9.1.2. — *The preceding morphism induces an isomorphism*

$$\mathrm{Spa}(\breve{E})/\varphi^{\mathbb{Z}} \xrightarrow{\sim} \mathrm{Div}^1.$$

Thus, contrary to the "classical case", Div^1 is note the curve itself. Nevertheless we have the following remark.

Remark 9.1.3. — *We thus have for any* $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$,

$$X_S^\diamond = (S \times_{\mathrm{Spa}(\overline{\mathbb{F}}_q)} \mathrm{Spa}(\breve{E})^\diamond)/\varphi^{\mathbb{Z}} \times \mathrm{Id}$$

and

$$\mathrm{Div}^1_S = (S \times_{\mathrm{Spa}(\overline{\mathbb{F}}_q)} \mathrm{Spa}(\breve{E})^\diamond)/\,\mathrm{Id} \times \varphi^{\mathbb{Z}}$$

and thus

$$|X_S| = |\mathrm{Div}^1_S|$$

and even equivalences of étale sites $(X_S^\diamond)_{\text{ét}} \simeq (\mathrm{Div}^1_S)_{\text{ét}}$. *For example, although* X_S *sits over* $\mathrm{Spa}(E)$ *and but not over* S*, there is still a continuous generalizing map of locally spectral spaces*

$$|X_S| = |\mathrm{Div}^1_S| \longrightarrow |S|$$

"as if X_S *were sitting over* S*".*

Remark 9.1.4. — *One has to be careful that although* Div^1 *is a qc diamond it is not spatial since not qs. Nevertheless* $\mathrm{Div}^1 \to *$ *is representable in locally spatial diamonds proper* ℓ*-cohomologically smooth.*

Remark 9.1.5. — *One can have a look at* [**56**] *for* Div^d *when* $d \geq 1$ *where it is proven that* $\mathrm{Div}^d = (\mathrm{Div}^1)^d/\mathfrak{S}_d$ *as a pro-étale quotient and the symmetrization morphism* $(\mathrm{Div}^1)^d \to \mathrm{Div}^d$ *is quasi-pro-étale surjective.*

Another way to understand Div^1 is to use *the Abel-Jacobi morphism*

$$\begin{array}{rcl} \mathrm{AJ}^1 : \mathrm{Div}^1 & \longrightarrow & \mathrm{Bun}^1_{\mathbb{G}_m} = [*/\underline{E}^\times] \\ D & \longmapsto & \mathcal{O}(D) \end{array}$$

where $\mathcal{O}(D)$ is the line bundle $\mathscr{L}$ of definition 9.1.1. The pullback along AJ^1 of $* \to [*/\underline{E}^\times]$ is the $\underline{E}^\times$-torsor of isomorphisms between $\mathcal{O}(1)$ and $\mathcal{O}(D)$. This is identified with

$$\mathbb{B}^{\varphi=\pi} \smallsetminus \{0\} \simeq \mathrm{Spa}\left(\overline{\mathbb{F}}_q(\!(T^{1/p^\infty})\!)\right),$$

a punctured absolute Banach-Colmez space. Here, if $\mathcal{G}$ is a Lubin-Tate group associated to E over $\overline{\mathbb{F}}_q$, after fixing a coordinate T on $\mathcal{G}$, i.e. a formal group law,

$$\mathrm{Spa}\left(\overline{\mathbb{F}}_q[\![T^{1/p^\infty}]\!]\right) = \underbrace{\varprojlim_{\times\pi} \mathcal{G}}_{\substack{\text{universal cover}\\ \text{of } \mathcal{G}}}$$

and the $E^\times$ action on $\overline{\mathbb{F}}_q(\!(T^{1/p^\infty})\!)$ is deduced from the one on $\mathcal{G}$. For example, if $E = \mathbb{Q}_p$, using $\mathcal{G} = \widehat{\mathbb{G}}_m$, the action of $\mathbb{Z}_p^\times$ is the usual action given by $T^a = (1+T)^a - 1 = \sum_{k\geq 1} \binom{a}{k} T^k$ for $a \in \mathbb{Z}_p^\times$. The action of $p \in \mathbb{Q}_p^\times$ is given by the Frobenius $T \mapsto T^p$.

At the end we obtain

$$\begin{aligned} \mathrm{Div}^1 &= \mathbb{B}^{\varphi=\pi} \smallsetminus \{0\}/\underline{E}^\times \\ &= \mathrm{Spa}\left(\overline{\mathbb{F}}_q(\!(T^{1/p^\infty})\!)\right)/\underline{E}^\times \\ &= \mathrm{Spa}(\breve{E}_\infty^\flat)/\underline{E}^\times \end{aligned}$$

where $\breve{E}_\infty$ is the completion of the extension of $\breve{E}$ obtained by adding the torsion points of a Lubin-Tate group over $\mathcal{O}_E$. Here the variable $T \in \breve{E}_\infty^\flat$ can be taken to be the mod π reduction of a generator of $T_\pi(\mathcal{G})$. In fact, $[\pi]_\mathcal{G}$ is congruent to the q-Frobenius modulo π and thus any element in $T_\pi(\mathcal{G})$ defines an element of $\varprojlim_{\mathrm{Frob}_q} \mathcal{O}_{\breve{E}_\infty}/\pi = \mathcal{O}_{\breve{E}_\infty^\flat}$.

9.2. Drinfeld Lemma

The following is our verion of Drinfeld lemma whose proof is simpler than the classical one. Let us note there is a natural morphism

$$\mathrm{Div}^1 \longrightarrow [*/\underline{W_E}]$$

defined by the $\underline{W_E}$-torsor

$$\begin{tikzcd} \mathrm{Spa}(\widehat{\overline{E}}^\flat) \arrow[d] \arrow[d, bend left=60, dash, "\underline{I_E}"] \arrow[dd, bend right=60, dash, "\underline{W_E}"'] \\ \mathrm{Spa}(\breve{E})^\diamond \arrow[d] \\ \mathrm{Div}^1 = \mathrm{Spa}(\breve{E})^\diamond/\varphi^{\mathbb{Z}}. \end{tikzcd}$$

***Proposition 9.2.1* ("Drinfeld lemma").** — *For any finite set I there is fully faithful functor*

$$\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G \times [*/\underline{W_E}]^I, \Lambda) \hookrightarrow \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G \times (\mathrm{Div}^1)^I, \Lambda)$$

that is an equivalence if $I = \{*\}$ has one element.

There is moreover an equivalence

$$\underbrace{\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda)}_{\substack{\text{stable } \infty\text{-cat.}\\ \text{upgraded to a}\\ \text{condensed}\\ \text{stable } \infty\text{-cat.}}}^{\overbrace{BW_E^I}^{\substack{\text{condensed}\\ \infty\text{-groupoid}}}} \xrightarrow{\sim} \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G \times [*/\underline{W_E}]^I, \Lambda).$$

Here the condensation is to take into account the topology of W_E that is seen as a condensed group and the classifying stack BW_E^I as a condensed ∞-groupoid that is to say an $(\infty, 0)$-category in the topos of condensed sets. More precisely, the functor

$$\begin{array}{rcl} \{\text{profinite sets}\} & \longrightarrow & \text{stable } \infty\text{-categories} \\ P & \longrightarrow & \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G \times \underline{P}, \Lambda) \end{array}$$

is an hypersheaf of stable ∞-categories on profinite sets. This is what we call the "condensed upgrade" of the usual stable ∞-category $\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$. It is a condensed infinite sub-category of the evident condensed infinite category

$$\mathcal{D}_{\text{pro-ét},\blacksquare}(\mathrm{Bun}_G, \Lambda).$$

The condensed ∞-groupoid BW_E^I is the hypersheaf of ∞-groupoids

$$\begin{array}{rcl} \{\text{profinite sets}\} & \longrightarrow & \text{stable } \infty\text{-categories} \\ P & \longrightarrow & B(\underbrace{W_E^I(P)}_{\mathscr{C}(P, W_E^I)}). \end{array}$$

Remark 9.2.2. — *Here we use the notation $\mathcal{D}^{\mathcal{C}}$ for the infinite category of ∞-functors from $\mathcal{C}$ to $\mathcal{D}$, and more generally for the hypersheaf of ∞-categories of ∞-functors between hypersheaves of ∞-categories whose value on an object of our topos U is the limit of the diagram of ∞-categories*

$$\prod_{V \to U} \mathcal{D}(V)^{\mathcal{C}(V)} \rightrightarrows \prod_{W \to V \to U} \mathcal{D}(W)^{\mathcal{C}(V)}$$

where the two ∞-functors are given by $\mathcal{C}(V) \to \mathcal{C}(W) \to \mathcal{D}(W)$ and $\mathcal{C}(V) \to \mathcal{D}(V) \to \mathcal{D}(W)$. When X is a topos, Λ a ring in X and G a group in X, one has an identification

$$\underbrace{\mathcal{D}(X,\Lambda)}_{\substack{\textit{hypersheaf of}\\ \infty\textit{-categories}\\ X\ni U\mapsto\mathcal{D}(U,\Lambda)}}^{\overbrace{BG}^{\substack{\textit{hypersheaf of}\\ \infty\textit{-cat.}\\ X\ni U\mapsto BG(U)}}} = \underbrace{\mathcal{D}(\overbrace{BG}^{\substack{\textit{classifying}\\ \textit{stack}\\ \textit{in } X}},\Lambda)}_{\substack{\textit{hypersheaf}\\ \textit{of } \infty\textit{-cat.}\\ X\ni U\mapsto\mathcal{D}(BG\times U,\Lambda)}}.$$

as hypersheaves of stable ∞-categories on X.

Example 9.2.3. — *If $\mathcal{C}$ is an ∞-category and G a group then $\mathcal{C}^{BG}$ has as objects the objects of $\mathcal{C}$ equipped with an action of G.*

Example 9.2.4. — *If $\mathcal{C}$ is a condensed ∞-category, for all x, y two objects of $\mathcal{C}$, the mapping space $Hom(x, y)$ is a condensed anima. In particular $\pi_0 Hom(x, y) = Hom_{\mathrm{Ho}\,\mathcal{C}}(x, y)$ is a condensed set. If G a topological group then the objects of $\mathcal{C}^{BG}$ are the objects x of $\mathcal{C}$ together with a morphism of condensed groups*

$$G \to \mathrm{Aut}_{\mathrm{Ho}\,\mathcal{C}}(x).$$

The following example is a warm-up for proposition 9.7.2.

***Example 9.2.5* (Compactness and discretness of G-actions: a simple example)**

The ∞-category

$$\mathcal{D}(\Lambda) = \mathcal{D}_{\text{ét}}(*)$$

is naturally upgraded to a condensed ∞-category that associated to the profinite set P,

$$\mathcal{D}(\mathcal{C}(P,\Lambda)) = \mathcal{D}_{\text{ét}}(\underline{P},\Lambda),$$

where $\mathcal{C}(P,\Lambda)$ means the locally constant functions on P with values in Λ. For $A, B \in \mathcal{D}(\Lambda)$, the condensed anima

$$Hom(A, B)$$

is identified with

$$\underline{Hom}(A_{disc}, B_{disc})$$

in $\mathcal{D}(\Lambda_{disc})$ the derived ∞-category of condensed Λ_{disc}-modules. If A is a compact object of $\mathcal{D}(\Lambda)$ then for any B, the condensed anima $Hom(A, B)$ is discrete. But for example if $A = \Lambda^{(\mathbb{N})}$ and $B = \Lambda$ this is $(\Lambda_{disc})^{\mathbb{N}}$ that is not discrete.

If G is a profinite group then

$$\left(\mathcal{D}(\Lambda)^{\omega}\right)^{BG} = \mathrm{colim}_K \left(\mathcal{D}(\Lambda)^{\omega}\right)^{B(G/K)}$$

where K is open distinguished in G: the action on compact objects is "discrete".

Remark 9.2.6. — *For $p = \infty$ the analog of the preceding is the following. Let us consider the Twister projective line*

$$\widetilde{\mathbb{P}}^1_{\mathbb{R}} = \mathbb{P}^1_{\mathbb{C}}/z \sim -\tfrac{1}{\bar{z}}.$$

This can be described as

$$\widetilde{\mathbb{P}}^1_{\mathbb{R}} = \mathbb{A}^2_{\mathbb{C}} \setminus \{(0,0)\}/W_{\mathbb{R}}$$

where $W_{\mathbb{R}}$ is the Weil group. The corresponding torsor $\mathbb{A}^2_{\mathbb{C}} \setminus \{(0,0)\} \to \widetilde{\mathbb{P}}^1_{\mathbb{R}}$ defines a morphism of analytic stacks

$$\widetilde{\mathbb{P}}^1_{\mathbb{R}} \longrightarrow [*/\underline{W_{\mathbb{R}}}].$$

9.3. What we want to do

For each finite set I we equip the ∞-category of ∞-functors

$$\mathcal{D}_{\text{ét}}(\text{Bun}_G, \Lambda) \longrightarrow \mathcal{D}_{\text{ét}}(\text{Bun}_G, \Lambda)^{BW_E^I}$$

with a monoidal structure by setting

$$u \otimes v := u(-) \otimes^{\mathbb{L}}_{\Lambda} v(-).$$

The purpose now is to define a monoidal functor between monoidal stable ∞-categories

$$\boxed{F_I : \big(\text{Rep}_{\Lambda}({}^L G)^I, \otimes\big) \longrightarrow \Big(\underbrace{\mathscr{H}om\big(\mathcal{D}_{\text{ét}}(\text{Bun}_G, \Lambda), \mathcal{D}_{\text{ét}}(\text{Bun}_G, \Lambda)^{BW_E^I}\big)}_{\infty\text{-cat. of } \infty\text{-functors}}, \otimes\Big)}$$

where

$$\text{Rep}_{\Lambda}({}^L G)^I$$

is the category of representations of $({}^L G)^I$ on finite type projective Λ-modules that are algebraic when restricted to $\widehat{G}^I$ and discrete when restricted to W_E^I.

We ask moreover that

- (**Factorization property**) This is functorial in the finite set I in the sense that if $I \to I'$ is a map of finite sets then

$$\begin{tikzcd}
\mathrm{Rep}_\Lambda({}^LG)^I \arrow[r, "F_I"] \arrow[d] & \mathscr{H}om\big(\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G,\Lambda), \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G,\Lambda)^{BW_E^I}\big) \arrow[d] \\
\mathrm{Rep}_\Lambda({}^LG)^{I'} \arrow[r, "F_{I'}"] & \mathscr{H}om\big(\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G,\Lambda), \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G,\Lambda)^{BW_E^{I'}}\big)
\end{tikzcd}$$

  commutes where the left vertical map is induced by the morphism $({}^LG)^{I'} \to ({}^LG)^I$ and the right vertical one by $W_E^{I'} \to W_E^I$.
- (**Linearity**) This is linear over $\mathrm{Rep}_\Lambda W_E^I$ in the sense that if $W \in \mathrm{Rep}_\Lambda W_E^I$ then

$$F_I(W) = - \otimes_\Lambda^{\mathbb{L}} W.$$

Example 9.3.1. — *If $I = \{1,2\}$ and $I' = \{1\}$ the preceding factorization property is the following "**fusion property**". Let $W \in \mathrm{Rep}_\Lambda({}^LG)^2$. We note $\Delta^* W$ its restriction to the diagonal, for example*

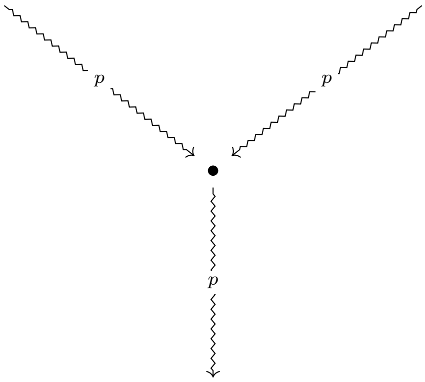
$$\Delta^*(W_1 \boxtimes W_2) = W_1 \otimes W_2.$$

Then, $\mathrm{Res}_{W_E}^{W_E^2} F_{1,2}(W) = F_1(\Delta^* W)$ via the restriction of the W_E^2-action to W_E embedded diagonally inside W_E^2.

Remark 9.3.2. — *The factorization property implies that after forgetting the action of W_E^I the functor*

$$\mathrm{Rep}_\Lambda({}^LG)^I \longrightarrow \mathscr{H}om\big(\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G,\Lambda), \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G,\Lambda)\big)$$

factorizes through the restriction to the diagonal $\mathrm{Rep}_\Lambda({}^LG)^I \longrightarrow \mathrm{Rep}_\Lambda {}^LG$.



The fusion of two copies of the prime number p

9.4. From local to global

To construct our functor F_I we consider *the global Hecke stack*

$$\begin{tikzcd}
 & \mathrm{Hecke}_I \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathrm{Bun}_G & & \mathrm{Bun}_G \times (\mathrm{Div}^1)^I
\end{tikzcd}$$

where for $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$, $\mathrm{Hecke}_I(S)$ is the groupoid of quadruples $(\mathscr{E}_1, \mathscr{E}_2, (D_i)_{i\in I}, u)$ where

- $\mathscr{E}_1$ and $\mathscr{E}_2$ are G-bundles on X_S,
- $(D_i)_{i\in I}$ is a collection of degree 1 effective Cartier divisors on X_S,
- and
$$u : \mathscr{E}_{1|X_S \smallsetminus \cup_{i\in I} D_i} \xrightarrow{\sim} \mathscr{E}_{2|X_S \smallsetminus \cup_{i\in I} D_i}$$
that is *meromorphic along the Cartier divisor* $\sum_{i\in I} D_i$.

Here the meromorphy condition means that after pushing forward by any representation of G, the associated modification of vector bundles $\mathscr{F}_1 \dashrightarrow \mathscr{F}_2$ comes from a morphism $\mathscr{F}_1 \to \mathscr{F}_2(k \sum_i D_i)$ for $k \gg 0$.

Remark 9.4.1. — *(Schematical descrition of the global Hecke stack) Suppose* $S = \mathrm{Spa}(R, R^+)$ *is affinoid perfectoid. Let* $\mathfrak{X}_{R,R^+}$ *be the schematical curve*

$$\mathfrak{X}_{R,R^+} = \mathrm{Proj}\left(\bigoplus_{d\geq 0} \mathcal{O}(Y_{R,R^+})^{\varphi=\pi^d}\right).$$

Then, using GAGA, $\mathrm{Hecke}_I(S)$ *is the groupoid of quadruples where* $\mathscr{E}_1$ *and* $\mathscr{E}_2$ *are étale* G*-torsors on* $\mathfrak{X}_{R,R^+}$*,* $(D_i)_{i\in I}$ *is a collection of effective Cartier divisors on* $\mathfrak{X}_{R,R^+}$ *that give rise to degree* 1 *effective Cartier divisors when pulled-back to* $\mathfrak{X}_{K(s),K(s)^+}$ *for any* $s \in S$*, and*

$$u : \mathscr{E}_{1|\mathfrak{X}_{R,R^+} \smallsetminus \cup_{i\in I} D_i} \xrightarrow{\sim} \mathscr{E}_{2|\mathfrak{X}_{R,R^+} \smallsetminus \cup_{i\in I} D_i}.$$

We want to *upgrade this correspondence to a cohomological one.* This is done in the following way. Let

$$\mathcal{H}ecke_I \longrightarrow (\mathrm{Div}^1)^I$$

be the so-called *local Hecke stack.* This is obtained in the same way as the global Hecke stack but *by replacing* X_S *by its formal completion along the divisor* $\sum_{i\in I} D_i$. Here is a formal definition.

Definition 9.4.2. — *The local Hecke stack is the functor on affinoid perfectoid $\overline{\mathbb{F}}_q$-algebras that sends (R, R^+) to quadruples*

$$(\mathscr{E}_1, \mathscr{E}_2, (D_i)_{i\in I}, u)$$

where

1. *$(D_i)_{i\in I}$ is as before a collection of degree 1 effective "relative" Cartier divisors on $\mathfrak{X}_{R,R^+}$,*
2. *$\mathscr{E}_1$ and $\mathscr{E}_2$ are étale G-torsors on the formal completion of $\mathfrak{X}_{R,R^+}$ along $\sum_{i\in I} D_i$,*
3. *u is a meromorphic isomorphism between $\mathscr{E}_1$ and $\mathscr{E}_2$ outside the special fiber of the formal completion.*

There is thus a morphism *from global to local*

$$\begin{tikzcd}
& \mathrm{Hecke}_I \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[rr, "loc"] & & \mathcal{H}ecke_I \arrow[dr] & \\
\mathrm{Bun}_G & & \mathrm{Bun}_G \times (\mathrm{Div}^1)^I \arrow[rr, "\mathrm{proj}"] & & (\mathrm{Div}^1)^I
\end{tikzcd}$$

The advantage of the local Hecke stack is that *it has an interpretation in terms of loop groups.*

Definition 9.4.3. —

1. *We note*

$$\mathbb{B}^+_{dR,I}, \ \textit{resp. } \mathbb{B}_{dR,I},$$

for the v-sheaf of E-algebras over $(Div^1)^I$ that sends (R, R^+) to the algebra of formal functions on the formal completion of the curve along $\sum_{i\in I} D_i$, resp. the algebra of formal meromorphic functions.

2. *We note*

$$L^+_I G, \ \textit{resp. } L_I G$$

for the v-sheaves of groups over $(\mathrm{Div}^1)^I$ equal to

$$L^+_I(G) = G(\mathbb{B}^+_{dR,I}), \ \textit{resp. } L_I G = G(\mathbb{B}_{dR,I}).$$

3. *We note*

$$\mathrm{Gr}_{G,I} = L_I G / L^+_I G.$$

This generalized B_{dR}-affine Grassmanian are the so-called *factorization B_{dR}-affine Grassmanians.*

One thus has for $f : S \to (\mathrm{Div}^1)^I$ given by $(D_i)_{i\in I}$,

$$\mathbb{B}^+_{dR,I}(S) \times_{(\mathrm{Div}^1)^I(S)} \{f\} = \Gamma\Big(X_S, \varprojlim_{k\geq 0} \mathcal{O}_{X_S}/\prod_{i\in I}\mathscr{I}^k_{D_i}\Big) \tag{4}$$

and

$$\mathbb{B}_{dR,I}(S) \times_{(\mathrm{Div}^1)^I(S)} \{f\} = \Gamma\Big(X_S, \varinjlim_{l\geq 0}\varprojlim_{k\geq 0} \prod_{i\in I}\mathscr{I}^{-l}_{D_i}/\prod_{i\in I}\mathscr{I}^k_{D_i}\Big). \tag{5}$$

More concretely, locally on S affinoid perfectoid the morphism S is given by a collection of untilts $S^{\sharp,i}, i \in I$, of S over E. Write $S = \mathrm{Spa}(R, R^+)$. For $i \in I$, the untilt $S^{\sharp,i}$ is given by some degree one distinguished element

$$\xi_i \in W_{\mathcal{O}_E}(R^+).$$

Then,

- (4) is the $\prod_{i\in I}\xi_i$-adic completion of $W_{\mathcal{O}_E}(R^+)[\frac{1}{\pi}]$,
$$\widehat{W_{\mathcal{O}_E}(R^+)[\tfrac{1}{\pi}]}^{\prod_{i\in I}\xi_i}$$
- (5) is the localization
$$\widehat{W_{\mathcal{O}_E}(R^+)[\tfrac{1}{\pi}]}^{\prod_{i\in I}\xi_i}\left[\frac{1}{\prod_{i\in I}\xi_i}\right].$$

The following result is easy.

Lemma 9.4.4. — *One has an equality of small v-stacks*

$$\mathcal{H}ecke_I = \big[\ L^+_I G\backslash L_I G/L^+_I G\ \big].$$

Remark 9.4.5. — *Take $I = \{1\}$ the set with one element. Compared to* [**132**] *we don't work over* $\mathrm{Spa}(\breve{E})^\diamond$ *(or even* $\mathrm{Spa}(C^\flat)$*, see* [**131**, Definition 19.1.1]*) but over the more natural geometric object* $\mathrm{Div}^1 = \mathrm{Spa}(\breve{E})^\diamond/\varphi^{\mathbb{Z}}$*. This means that the B_{dR}-affine Grassmanian of* [**131**] *that sits over* $\mathrm{Spa}(E)^\diamond$ *is the pullback via* $\mathrm{Spa}(E)^\diamond \to \mathrm{Div}^1$ *of* $\mathrm{Gr}_G := LG/L^+G$.

Finally let us remark the following that is a consequence of Beauville-Laszlo gluing ([**6**]). Let

$$\begin{tikzcd}\mathbb{T} \arrow[d, "L^+_I G"] \\ \mathrm{Bun}_G \times (\mathrm{Div}^1)^I\end{tikzcd}$$

be the étale $L^+ I_G$-torsor of trivializations of $\mathscr{E} \in \mathrm{Bun}_G$ along the formal completion of the curve along $\sum_{i\in I} D_i$ where $(D_i)_{i\in I} \in (\mathrm{Div}^1)^I$. One then has

(Beauville-Laszlo gluing)
$$\mathrm{Hecke}_I = \mathbb{T} \overset{L_I^+ G}{\times}_{(\mathrm{Div}^1)^I} \mathrm{Gr}_{G,I}\,.$$

The following remark is a follow-up to remark 8.6.6.

Remark 9.4.6 (Hecke **and Shimura varieties).** — *We keep the notations of remark 8.6.6. Let* $\mathrm{Sh}_{\infty K^p}$ *be the perfectoid Shimura variety with infinite level at* p *and*

$$\pi_{HT} : \mathrm{Sh}_{\infty K^p} \longrightarrow \underbrace{\mathscr{F}(G_{\mathbb{Q}_p}, \mu^{-1})}_{\substack{\textit{adic flag}\\ \textit{manifold}}}$$

be the associated Hodge-Tate period morphism, see [**130**]*). Here* μ *is the so-called Hodge cocharacter. Let* $G(\mathbb{Z}_p)$ *be our fixed compact hyperspecial subgroup of* $G(\mathbb{Q}_p)$*. For any* p*-adic formal scheme* $\mathcal{S}$ *over* $Spf(\mathbb{Z}_p)$ *with generic fiber* $\mathcal{S}_\eta$ *and special fiber* $\overline{\mathcal{S}}$*, there is specialization map*

$$\mathrm{sp} : \mathcal{S}_\eta^\diamond \longrightarrow \overline{\mathcal{S}}^{1/p^\infty, \diamond} \times \mathrm{Spa}(\mathbb{Q}_p)^\diamond.$$

Let Hecke_μ *be the closed substack of the Hecke stack defined by modifications "of type* μ*". We note* K *for the* p*-adic completion of the reflex field of our Shimura datum associated to our choice of an embedding of* $\overline{\mathbb{Q}}$ *inside* $\overline{\mathbb{Q}}_p$ *(and thus our* p*-adic flag manifold leaves over* K*). We note*

$$\begin{tikzcd}
& \mathrm{Hecke}_\mu \arrow[ld, "p_{1,\mu}"'] \arrow[rd, "p_{2,\mu}"] & \\
\mathrm{Bun}_G & & \mathrm{Bun}_G \times \mathrm{Spa}(K)^\diamond.
\end{tikzcd}$$

We have an identification

$$p_{1,\mu}^{-1}\left(\mathrm{Bun}_G^1\right) = \left[\ \underline{G(\mathbb{Q}_p)} \backslash \mathscr{F}(G, \mu^{-1})^\diamond\ \right].$$

The following diagram is then commutative

$$\begin{tikzcd}
\mathrm{Sh}^\flat_{\infty K^p} \arrow[r, "\pi_{HT}"] \arrow[d] \arrow[d, bend right=40, dash, "\underline{G(\mathbb{Z}_p)}"'] & \mathscr{F}(G_{\mathbb{Q}_p}, \mu^{-1})^\diamond \arrow[d] \arrow[d, bend left=40, dash, "\underline{G(\mathbb{Z}_p)}"] \\
(S_{K^p})_\eta^\diamond \arrow[r] \arrow[dd, "\mathrm{sp}"'] & \left[\ \underline{G(\mathbb{Z}_p)} \backslash \mathscr{F}(G_{\mathbb{Q}_p}, \mu^{-1})^\diamond\ \right] \arrow[d] \\
& \left[\ \underline{G(\mathbb{Q}_p)} \backslash \mathscr{F}(G_{\mathbb{Q}_p}, \mu^{-1})^\diamond\ \right] = \mathrm{Hecke}_\mu \arrow[d, "p_{2,\mu}"] \\
\left(\overline{S}_{K^p}^{1/p^\infty}\right)^\diamond \times \mathrm{Spa}(K)^\diamond \arrow[r, "\pi_{cris} \times \mathrm{Id}"] & \mathrm{Bun}_G \times \mathrm{Spa}(K)^\diamond.
\end{tikzcd}$$

The stratification of $|\mathscr{F}(G_{\mathbb{Q}_p}, \mu^{-1})|$ *by* $B(G, \mu)$ *defined in* [**24**] *is the pull-back of the HN stratification of* $|\mathrm{Bun}_G|$ *via the morphism*

$$\mathscr{F}(G_{\mathbb{Q}_p}, \mu^{-1})^\diamond \longrightarrow \mathrm{Bun}_G \times \mathrm{Spa}(K)^\diamond \xrightarrow{\mathrm{proj}} \mathrm{Bun}_G.$$

9.5. The Satake correspondence

We now want to use the local Hecke stack to define our functor

$$\boxed{F_I(W) : \mathcal{D}_{\text{ét}}(\text{Bun}_G, \Lambda) \longrightarrow \mathcal{D}_{\text{ét}}(\text{Bun}_G \times (\text{Div}^1)^I, \Lambda)}$$

via the formula

$$F_I(W) = Rp_{2*}\big(p_1^*(-) \otimes_\Lambda^{\mathbb{L}} loc^* S_W\big)$$

for $W \in \text{Rep}_\Lambda({}^L G)^I$ and where

$$S_W \in D_{\text{ét}}(\mathcal{H}ecke_I, \Lambda)^b$$

is the so-called *Satake sheaf associated to* W (where the upperscript "b" means bounded i.e. with quasi-compact support on the B_{dR}-affine Grassmanian, that is to say supported on a finite union of closed Schubert cells).

More precisely, we want to define a monoidal functor

$$(\text{Rep}_\Lambda({}^L G)^I, \otimes) \xrightarrow{\otimes} (D_{\text{ét}}(\mathcal{H}ecke_I, \Lambda)^b, *)$$

where the monoidal structure on the right is the one given by the composition of cohomological correspondences that is to say *the convolutions product*

$$\boxed{A * B = Rb_*(a^* A \boxtimes_\Lambda^{\mathbb{L}} B)}$$

where

$$\begin{tikzcd}
 & \left[\, L_I^+ G \backslash L_I G \overset{L_I^+ G}{\times} L_I G / L_I^+ G \,\right] \arrow[dl, "a"'] \arrow[dr, "b"] & \\
\left[\, L_I^+ G \backslash L_I G / L_I^+ G \,\right] \times \left[\, L_I^+ G \backslash L_I G / L_I^+ G \,\right] & & \left[\, L_I^+ G \backslash L_I G / L_I^+ G \,\right]
\end{tikzcd}$$

is the convolution diagram. Here the upper object in this diagram is the moduli of $(D_i)_{i \in I} \in (\text{Div}^1)^I$ together with three G-bundles $(\mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3)$ on the formal completion of the curve along $\sum_{i \in I} D_i$, and meromorphic isomorphism outside $\sum_{i \in I} D_i$ (i.e. formal modifications)

$$\mathscr{E}_1 \dashrightarrow \mathscr{E}_2 \dashrightarrow \mathscr{E}_3.$$

The left hand map a sends this datum to $(\mathscr{E}_1 \dashrightarrow \mathscr{E}_2, \mathscr{E}_2 \dashrightarrow \mathscr{E}_3)$, and the right hand one b to the composite modification $\mathscr{E}_1 \dashrightarrow \mathscr{E}_3$:

$$\begin{tikzcd} & \mathscr{E}_1 \dashrightarrow \mathscr{E}_2 \dashrightarrow \mathscr{E}_3 \arrow[dl] \arrow[dr] & \\ (\mathscr{E}_1 \dashrightarrow \mathscr{E}_2, \mathscr{E}_2 \dashrightarrow \mathscr{E}_3) & & \mathscr{E}_1 \dashrightarrow \mathscr{E}_3. \end{tikzcd}$$

This is given by the following theorem. Here we suppose that Λ is a $\mathbb{Z}_\ell[q^{1/2}]$-algebra. We can now state the geometric Satake equivalence in our context. We will explain later the meaning of all the terms.

Theorem 9.5.1 **(Geometric Satake equivalence)**
Let $\mathrm{Sat}_I(G, \Lambda)$ *be the category of bounded perverse flat ULA sheaves on* $\mathcal{H}ecke_I$.

1. *This is stable under the convolution product* $*$ *and functorial in* I.
2. *There is an equivalence of monoidal categories*

$$(\mathrm{Sat}_I(G, \Lambda), *) \xrightarrow{\sim} \left(\mathrm{Rep}_\Lambda({}^LG)^I, \otimes\right).$$

3. *This equivalence is functorial in* I, *and linear over* $\mathrm{Rep}_\Lambda W_E^I$ *via the identification between* $\mathrm{Rep}_\Lambda W_E^I$ *and the category of étale local systems of* Λ*-modules on* $(\mathrm{Div}^1)^I$.
4. *IF* $I = \{*\}$, *for any* $\mu \in X_*(T) = X^*(\widehat{T})$, *if* $\bar{\mu}$ *is the* Γ_E*-orbit of* μ *and* $W_{\bar{\mu}}$ *is the associated highest weight irreducible representation of* ${}^LG$, *then* $W_{\bar{\mu}}$ *corresponds to*

$$\underbrace{j_{\bar{\mu}!*}\Lambda\Big[\langle \mu, 2\rho \rangle\Big]}_{\substack{\textit{intersection cohomology} \\ \textit{complex of the Schubert cell}}}$$

where $j_{\bar{\mu}}$ *is the inclusion of the open Schubert cell defined by* $\bar{\mu}$ *inside the closed one.*

9.6. About the action of the Hecke correspondences and Drinfeld lemma

À priori we only obtain a functor

$$F_I(W) : \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda) \longrightarrow \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G \times (\mathrm{Div}^1)^I, \Lambda).$$

We now use proposition 9.2.1.

Proposition 9.6.1. — *For $W \in \text{Rep}_\Lambda({}^LG)^I$, the functor $F_I(W)$ takes values in*

$$D_{\text{ét}}(\text{Bun}_G \times [*/\underline{W_E}]^I,\Lambda)$$

via the embedding of proposition 9.2.1.

The proof is done by reduction to the case $I = \{*\}$ that is done in proposition 9.2.1, see [**58**, Section IX.2]. In fact, the case when $W = \boxtimes_{i\in I} W_i$ is reduced to the case when I has one element and one then proceeds by taking a resolution of any W by such representations that are external tensor products.

9.7. Where we use the compact generation of $D_{\text{ét}}(\text{Bun}_G,\Lambda)$

We need the following that follows formally from an adjunction argument, see [**58**, Section IX.2].

Proposition 9.7.1. — *The action of the Hecke correspondences preserves compact objects.*

We need in fact more than that. The following is proven in [**58**, Section IX.5]. This point is necessary to define the spectral action: we first define it on compact objects and then extend it. Let us note that this uses the local charts $\mathcal{M}_b \to \text{Bun}_G$ and the associated compact generators, see section 8.7.

Proposition 9.7.2 **(Discretness of the Weil action on compact objects)**
For any $A \in D_{\text{ét}}(\text{Bun}_G,\Lambda)$ that is a compact object there is an open subgroup K distinguished in W_E such that for any finite set I and any $W \in \text{Rep}_\Lambda({}^LG)^I$,

$$F_I(W)(A) \in \mathcal{D}_{\text{ét}}(\text{Bun}_G,\Lambda)^{B(W_E/K)^I} \subset \mathcal{D}_{\text{ét}}(\text{Bun}_G,\Lambda)^{BW_E^I}.$$

If $\mathcal{C}$ is an ∞-category it gives rise to a category enriched in anima

$$\infty\text{-category} \rightsquigarrow \text{category enriched in anima}$$

where here the ∞-category of anima is the homotopy coherent nerve of the simplicial category of Kan complexes. This is sometimes called the ∞-category of spaces or the ∞-category of ∞-groupoids. Its homotopy category is the one of topological spaces up to weak equivalences that is equivalent to the category of CW complexes up to homotopy. For x, y two objects of an ∞-category we note

$$\underset{\sim\sim\sim}{\text{Hom}}(x,y)$$

for the anima of morphisms between x and y. For example, if we take $\mathcal{D}(\mathcal{A})$ the derived ∞-category of an abelian category $\mathcal{A}$ admitting enough injectives objects, for

$A, B \in \mathcal{D}(\mathcal{A})$

$$\underline{\mathrm{Hom}}(A, B) \longmapsto \tau_{\leq 0} R\mathrm{Hom}(A, B)$$

via the Dold-Kan correspondence

$$\underbrace{\mathcal{A}ni\,(\mathcal{A}b)}_{\substack{\text{animated}\\ \text{ab. groups}}} \xrightarrow{\sim} \underbrace{\mathcal{D}^{\leqslant 0}(\mathbb{Z})}_{\substack{\text{derived}\\ \infty\text{-cat.}\\ \text{of ab. gp.}}}.$$

For example, for $i \in \mathbb{N}$,

$$\pi_i \underline{\mathrm{Hom}}(A, B) \simeq \mathrm{Ext}^{-i}(A, B).$$

The same goes on with condensed ∞-categories:

$$\text{condensed } \infty\text{-category} \rightsquigarrow \text{category enriched in condensed anima}$$

which means that for x, y two objects of a condensed ∞-category, their hom space

$$\underline{\mathrm{Hom}}(x, y)$$

is a condensed anima. In particular, $\pi_0(\underline{\mathrm{Hom}}(x, y))$ is a condensed set and for $i > 0$, $\pi_i(\underline{\mathrm{Hom}}(x, y))$ is a condensed group.

Recall that we see $\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ as a condensed ∞-category. The proof of proposition 9.7.2 uses the following result whose proof uses the local charts $\pi_b : \mathcal{M}_b \to \mathrm{Bun}_G$.

Proposition 9.7.3. — *If $A, B \in \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ seen as a condensed ∞-category with A compact in the usual triangulated category $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ then the condensed anima $\underline{Hom}(A, B)$ is in fact a discrete anima.*

The following remark is an analog of this result for the HN strata of Bun_G whose proof is elementary.

Remark 9.7.4. — *As for* Bun_G*, the* ∞*-category*

$$\mathcal{D}(G(E), \Lambda)$$

is naturally a condensed ∞*-category. It associated to the profinite set* P *the* ∞*-category*

$$\mathcal{D}(G(E), \mathcal{C}(P, \Lambda)) = \mathcal{D}_{\text{ét}}([*/\underline{G(E)}] \times \underline{P}, \Lambda)$$

where $\mathcal{C}(P, \Lambda)$ *is the set of locally constant functions on* P *with values in* Λ*.*
One can already verify by elementary means that for $A \in \mathcal{D}(G(E), \Lambda)^\omega$ *and* $B \in \mathcal{D}(G(E), \Lambda)$*, the condensed anima*

$$\underset{\sim\sim\sim}{Hom}(A, B)$$

is discrete.
In fact, $\pi_i \underset{\sim\sim\sim}{Hom}(A, B)$ *is the animated group*

$$P \mapsto \mathrm{Ext}^i_{\mathcal{C}(P,\Lambda)}(A \otimes^{\mathbb{L}}_{\Lambda} \mathcal{C}(P, \Lambda), B \otimes^{\mathbb{L}}_{\Lambda} \mathcal{C}(P, \Lambda)) = \mathrm{Ext}^i_{\Lambda}(A, B \otimes^{\mathbb{L}}_{\Lambda} \mathcal{C}(P, \Lambda)).$$

The triangulated category $\mathcal{D}(G(E), \Lambda)^\omega$ *is the thick triangulated category generated by the* $\text{c-Ind}_K^{G(E)} \Lambda$ *where* K *is open pro-p. Taking* $A = \text{c-Ind}_K^{G(E)} \Lambda$ *in the preceding formula one finds* 0 *if* $i > 0$ *and*

$$B^K \otimes^{\mathbb{L}}_{\Lambda} \mathcal{C}(P, \Lambda)$$

if $i = 0$*. This last expression is nothing else than*

$$(B^K)_{disc}(P).$$

From this point of view, *the construction of the spectral action relies on the compact generation of* $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$.

Remark 9.7.5. — *There is another construction of the semi-simple Langlands parameters via the cohomology of local Shtuka moduli spaces in* [**58**, Section IX.3]*. It seems simpler and not using* Bun_G*. This is misleading since in fact it relies on a finiteness property of the cohomology of such spaces generalizing the same fact for Rapoport-Zink spaces in* [**50**] *that uses integral models of Rapoport-Zink spaces. The proof of this finiteness result relies on* Bun_G *and the properties of its compact objects. More precisely, we need that*

$$(i^1)^* : D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)^\omega \longrightarrow D(G(E), \Lambda)^\omega.$$

Finally, let us note that the preceding results extend in the context of any $\mathbb{Z}_\ell$-algebra Λ that may not be torsion.

Remark 9.7.6. — *Let* Λ *be any* $\mathbb{Z}_\ell$*-algebra. In the context of* $D_{lis}(\mathrm{Bun}_G, \Lambda)$ *(see section 8.9) the analog of propositions 9.7.2 and 9.7.3 holds:*

1. *For* A *an object of* $D_{lis}(\mathrm{Bun}_G, \Lambda)$ *compact there is open subgroup* $K \subset P_E$*, the wild inertia, distinguished in* W_E *such that for all* I *and any* $W \in \mathrm{Rep}_\Lambda \left({}^L G\right)^I$*,*

$$F_I(W)(A) \in \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda)^{B(W_E/K)^I} \subset \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda)^{BW_E^I}.$$

2. For A, B two objects of $\mathcal{D}_{lis}(\mathrm{Bun}_G, \Lambda)$ with A compact the condensed anima $Hom(A, B)$ is relatively discrete over $\mathbb{Z}_\ell$.

In the preceding remark a condensed set X over $\mathbb{Z}_\ell$ is relatively discrete if it comes by pullback from a sheaf on the topological space $|\mathbb{Z}_\ell| = \mathbb{Z}_\ell(*)$. Suppose that we have a morphism of condensed groups

$$\underline{P} \longrightarrow G$$

where G is relatively discrete over $\mathbb{Z}_\ell$. One verifies that such a morphism has to factorize through an open subgroup of P.

9.8. Final thoughts

One of the main motivations for [**58**] was the so-called Hecke eigensheaf property. This is a bridge between the local Shtuka moduli spaces from [**132**] and the Hecke correspondences. More precisely, let $\varphi : W_E \to {}^L G(\overline{\mathbb{Q}}_\ell)$ be a cuspdial Langlands parameter with G split (to simplify) and suppose we have an S_φ-equivariant object

$$\mathscr{F}_\varphi \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$$

that satisfies:

1. the restriction of the S_φ-action to $Z(\widehat{G})^{\Gamma_E}$ is given $-\alpha \in X^*(Z(\widehat{G})^{\Gamma_E})$ for $\mathscr{F}_{\varphi|\mathrm{Bun}_G^\alpha}$.
2. for any basic $[b] \in B(G)$,
$$(i^b)^* \mathscr{F}_\varphi = \bigoplus_{\substack{\rho \in \mathrm{Irr}(S_\varphi) \\ \rho_{|Z(\widehat{G})^{\Gamma_E}}}} \rho \otimes \pi_{\varphi,\rho}$$
where $\varphi_{\varphi,\rho}$ is irreducible cuspidal and $\varphi \mapsto (\varphi_{\varphi,\rho})$ is a local Langlands correspondence for all extended pure inner forms of G
3. $\mathscr{F}_\varphi$ is an Hecke eigensheaf in the following sense. For μ a minuscule cocharacter of G let us note $\mathrm{Hecke}_\mu \subset \mathrm{Hecke}$ the moduli of modifications of type μ and
$$T_\mu : D_{\text{ét}}(\mathrm{Bun}_G, \Lambda) \longrightarrow D_{\text{ét}}(\mathrm{Bun}_G \times [*/\underline{W_E}], \Lambda)$$
the associated Hecke transform. The Hecke eigensheaf property for $\mathscr{F}_\varphi$ is an S_φ-equivariant isomorphism for all minuscule μ
$$T_\mu(\mathscr{F}_\varphi) \simeq \mathscr{F}_\mu \boxtimes r_\mu \circ \varphi.$$

This eigensheaf property is inspired by the usual Hecke eigensheaf property in the "classical" geometric Langlands program (see [**61**] for example).

Then, Kottwitz conjecture on the cohomology of minuscule local Shimura varieties (this contains the case of Rapoport-Zink spaces, see [**116**]) is satisfied. Thus, this type of Hecke eigensheaf property is an enhancement of statements about the cohomology of Rapoport-Zink spaces.

Finally, let us note that the ability to give a meaning to two copies of the prime number p, by forming

$$\mathrm{Spa}(\mathbb{Q}_p)^\diamond \times \mathrm{Spa}(\mathbb{Q}_p)^\diamond$$

has been a great succes of the theory of diamonds. If E is equal characteristic, $E = \mathbb{F}_q(\!(\pi)\!)$, one has

$$\mathrm{Spa}(E) \times_{\mathrm{Spa}(\mathbb{F}_q)} \mathrm{Spa}(E) = \mathrm{Spa}\left(\mathbb{F}_q[\![X,Y]\!], \mathbb{F}_q[\![X,Y]\!]\right) \smallsetminus V(XY)$$

i.e. we dno't need the theory of diamonds and can double the variable π as X and Y. But in the unequal characteristic case we can not do this and the fact that we can write $(\mathrm{Div}^1)^I$ for any finite set I has been a great success of the theory.

LECTURE 10

THE GEOMETRIC SATAKE EQUIVALENCE

Here are the tools used for the geometric Satake equivalence:

1. The notion of ULA complexes,
2. Hyperbolic localization,
3. Fusion,
4. Degeneration of the B_{dR}-affine Grassmanian to a "classical" Witt vectors affine Grassmanian.

Here, as before, the coefficients Λ are torsion to simplify.

10.1. ULA complexes ([**58**, Section IV.2])

10.1.1. The classical case. — Classically, if $f : X \to S$ is a finite presentation morphism of schemes, we have a good notion of f-ULA complexes in $D_{\text{ét}}(X, \Lambda)$ where here Λ is a Noetherian ring killed by a power of ℓ invertible on S. More precisely, those are the étale complexes "universally without vanishing cycles" i.e. the

$$A \in \underbrace{D^b_{\text{ét,c}}(X, \Lambda)}_{\substack{\text{bounded with constructible} \\ \text{cohomology sheaves}}}$$

such that

$$\forall \ \mathrm{Spec}(V) \longrightarrow S$$

where V is a rank 1 valuation ring, one has

$$R\Phi_{\bar{\eta}}\Big(A_{|X \times_S \mathrm{Spec}(V)}\Big) = 0$$

where $\bar{\eta}$ is a geometric point over the generic point of $\mathrm{Spec}(V)$.

Remark 10.1.1. — *Said roughly this means that for any morphism*

$$\underbrace{\mathcal{C}}_{a \text{ ``curve''}} \longrightarrow S$$

the étale complex $A_{|X\times_S\mathcal{C}}$ *is without vanishing cycles relatively to* $X\times_S\mathcal{C}\to\mathcal{C}$*. Thus, the condition is tested universally for all "curves" mapping to the target* S*.*

One can prove, following Gaitsgory, that this is equivalent to A behaving well with respect to Verdier duality: A is f-ULA if and only if *universally over* S,

$$\boxed{\forall B\in D_{\text{ét}}(S,\Lambda),\ \mathbb{D}_{X/S}(A)\otimes^{\mathbb{L}}_{\Lambda} f^*B\xrightarrow{\sim} R\mathscr{H}om_{\Lambda}(A,Rf^!B).}$$

One can moreover prove that if A is f-ULA then it is bidual with respect to Verdier duality:

$$A\xrightarrow{\sim}\mathbb{D}_{X/S}(\mathbb{D}_{X/S}(A)).$$

10.1.2. The diamond case. — Let $f:X\longrightarrow Y$ be a morphism of locally spatial diamonds (compactifiable of finite dim. trg.). Let $A\in D_{\text{ét}}(X,\Lambda)$. We define a good notion of A to be f-ULA.

Definition 10.1.2. — *$A\in D_{\text{ét}}(X,\Lambda)$ is f-ULA if f-ULA if for any $j:U\to X$ (separated) étale with composite $U\to X\to Y$ quasicompact then $R(f\circ j)_!A$ is a perfect constructible complex when restricted to each quasicompact open subset of Y.*

Here, when Y is a spatial diamond the constructibility condition has to be though of differently from the usual case of algebraic varieties. Perfect constructible complexes are étale complexes of Λ-modules that differ from local systems only via non-overconvergence i.e. a perfect constructible étale complex of Λ-modules is étale locally constant if and only if it is overconvergent.

One can prove that all properties of "classical" algebraic étale local systems adapt in this situation, typically the nice behavior with respect to Verdier duality.

10.1.3. Perverse ULA sheaves on $\mathrm{Gr}_{G,I}$. — Suppose G is split to simplify and let $T\subset B$ be a maximal torus inside a Borel subgroup. We note

$$\begin{tikzcd}\mathrm{Gr}_{G,I}=L_IG/L_I^+G \arrow[d] \\ (\mathrm{Div}^1)^I.\end{tikzcd}$$

One thus has with those notations

$$\mathcal{H}ecke_I=\left[\ L_I^+G\setminus\mathrm{Gr}_{G,I}\ \right].$$

There is a stratification of the local Hecke stack indexed by $(X_*(T)^+)^I$.

Definition 10.1.3. — *For* $(\mu_i)_{i\in I} \in (X_*(T)^+)^I$ *we note*

$$\mathrm{Gr}_{G,I,(\mu_i)_{i\in I}}$$

the associated open Schubert cell and

$$\mathscr{H}ecke_{G,I,(\mu_i)_{i\in I}} = [\ L_I^+G \setminus \mathrm{Gr}_{G,I,(\mu_i)_{i\in I}}\ ].$$

the associated stratum.

Concretely, given a point

$$x : \mathrm{Spa}(C, C^+) \longrightarrow \mathrm{Gr}_{G,I}$$

there is associated a collection of classical points on the curve X_{C,C^+}. This defines a map

$$u : I \longrightarrow |X_{C,C^+}|^{cl}.$$

For each element in the image of u there is associated a relative position, an element of $X_*(T)^+$. We say that our point x is in the stratum defined by $(\mu_i)_{i\in I}$ if for any $z \in \mathrm{Im}(u)$, this relative position is given by $\sum\limits_{\substack{i\in I\\ u(i)=z}} \mu_i$.

One has (for whatever definition of the dimension: Krull or cohomological):

$$\dim \mathrm{Gr}_{G,I,(\mu_i)_{i\in I}} / (\mathrm{Div}^1)^I = \sum_{i\in I} \langle \mu_i, 2\rho \rangle$$

(relative dimension).

Example 10.1.4. — *For* $I = \{1, 2\}$*, if* $\Delta : \mathrm{Div}^1 \hookrightarrow (\mathrm{Div}^1)^2$ *is the diagonal,*

$$\begin{tikzcd}
\big(\mathrm{Gr}_{G,\mu_1} \times \mathrm{Gr}_{G,\mu_2}\big) \times_{(\mathrm{Div}^1)^2} (\mathrm{Div}^1)^2 \setminus \Delta \arrow[r, hook] \arrow[d] & \mathrm{Gr}_{G,I,(\mu_1,\mu_2)} \arrow[d] & \mathrm{Gr}_{G,\mu_1+\mu_2} \arrow[l, hook'] \arrow[d] \\
(\mathrm{Div}^1)^2 \setminus \Delta \arrow[r, hook] & (\mathrm{Div}^1)^2 & \mathrm{Div}^1. \arrow[l, hook', "\Delta"']
\end{tikzcd}$$

with cartesian squares.

Definition 10.1.5. — *Let*

$$D := D_{\text{ét}}^{\text{ULA}}(\mathscr{H}ecke_I, \Lambda)^b$$

be the category of $A \in D_{\text{ét}}(\mathscr{H}ecke_I, \Lambda)$ *with qc support that are ULA relative to the morphism* $\mathscr{H}ecke_I \to (\text{Div}^1)^I$.

We define

$${}^pD^{\leq 0} = \{A \in D \mid \forall x : \text{Spa}(C, C^+) \to \mathscr{H}ecke_I,\ x^*A \in D^{\leq -\sum_{i \in I} \langle \mu_i(x), 2\rho \rangle}(\Lambda)\}$$

where $\mu_i(x) \in X_*(T)^+$, $i \in I$, *gives the relative position at* x, *and*

$${}^pD^{\geq 0} = \{A \in D \mid \mathbb{D}(A) \in {}^pD^{\leq 0}\}.$$

One verifies that this defines a t-structure with heart the abelian category

$$\text{Perv}^{\text{ULA}}(\mathscr{H}ecke_I, \Lambda),$$

see [**58**, section VI.7].

Definition 10.1.6. — *The Satake category*

$$\text{Sat}_I(G, \Lambda)$$

is the category of $A \in \text{Perv}^{\text{ULA}}(\mathscr{H}ecke_I, \Lambda)$ *that are flat perverse in the sense that for all finite presentation* Λ*-module* M, $A \otimes_\Lambda^{\mathbb{L}} M$ *is perverse.*

There is an "easy piece" of this Satake category. In fact, there is an equivalence

$$\text{Rep}_\Lambda W_E^I \xrightarrow{\sim} D_{\text{ét,lcp}}((\text{Div}^1)^I, \Lambda)$$

where the left hand side is the category of complexes of discrete representations of W_E^I with values in Λ-modules that are perfect complex of Λ-modules after forgetting the W_E^I-action. The right hand side is the category of étale complexes that are locally constant with perfect fibers. This is a version of Drinfeld lemma, see [**58**, Section IV.7]. By pullback to the Hecke stack we deduce an inclusion

$$\underbrace{\text{Rep}_\Lambda(W_E^I)}_{\substack{\text{discrete rep. with values in} \\ \text{finite type projective modules}}} \hookrightarrow \text{Sat}_I(G, \Lambda).$$

10.2. Mirkovic Vilonen cycles, the constant term functor, and hyperbolic localization

Suppose G is split to simplify. Let B be a Borel subgroup of G with maximal torus T. We can look at

$$\text{Gr}_{B,I} = L_{B,I}/L_{B,I}^+.$$

One has a decomposition

$$\mathrm{Gr}_{B,I} = \coprod_{\lambda \in X_*(T)} \mathrm{Gr}_{I,B}^{\lambda}$$

given by the projection

$$\mathrm{Gr}_{B,I} \longrightarrow \mathrm{Gr}_{T,I}$$

and the locally constant function

$$|\,\mathrm{Gr}_{T,I}\,| \longrightarrow X_*(T)$$

whose value on the open Schubert cell $|\,\mathrm{Gr}_{T,I}^{(\mu_i)_i}\,|$, for $(\mu_i)_{i\in I} \in X_*(T)^I$, is

$$\sum_{i\in I} \mu_i.$$

The morphism

$$\mathrm{Gr}_{B,I} \longrightarrow \mathrm{Gr}_{G,I}$$

is a bijection at the level of points and induces a locally closed immersion

$$\mathrm{Gr}_{B,I}^{\lambda} \hookrightarrow \mathrm{Gr}_{G,I}$$

for each $\lambda \in X_*(T)$. We note

$$S_\lambda \subset \mathrm{Gr}_{G,I}$$

the image of this locally closed immersion, a so-called Mirkovic-Vilonen cycle (see [**109**]). We have now the following *semi-continuity property*: for any $\mu \in X_*(T)$,

$$\bigcup_{\lambda \leq \mu} S_\lambda$$

is closed.

Remark 10.2.1. — *One has to be careful that the Schubert cells are parametrized by* $(X_*(T)^+)^I$ *but the Mirkovic-Vilonen cycles by* $X_*(T)$*.*

We can now define the constant term functor via the diagram

$$\begin{tikzcd}
\mathrm{Gr}_{B,I} \arrow[r, "\mathfrak{q}"] \arrow[d, "\mathfrak{p}"'] & \mathrm{Gr}_{G,I} \\
\mathrm{Gr}_{T,I} &
\end{tikzcd}$$

Definition 10.2.2. — *The constant term functor is*

$$\mathrm{CT}_B = R\mathfrak{p}_!\mathfrak{q}^* : D_{\text{ét}}(\mathrm{Gr}_{G,I}, \Lambda) \longrightarrow D_{\text{ét}}(\mathrm{Gr}_{T,I}, \Lambda).$$

If $I = \{1\}$ one has $\mathrm{Gr}_{T,I} = X_*(T) \times \mathrm{Div}^1$. If for $\lambda \in X_*(T)$ if one notes $\mathfrak{p}_\lambda : S_\lambda \to \mathrm{Div}^1$ one has

$$\mathrm{CT}_B(A)_{|\{\lambda\}\times \mathrm{Div}^1} = R\mathfrak{p}_{\lambda !}(A_{|S_\lambda})$$

and this is thus given by the relative compactly supported cohomology of Mirkovic-Vilonen cycles.

Remark 10.2.3. — *The constant term map already appears in the "classical" geometric Satake isomorphism (see* [**70**]*). Here, in the geometric Satake context, this corresponds at the end (up to a shift) to restricting a representation of* $({}^LG)^I$ *to* $({}^LT)^I$*.*

One of the main features is the following hyperbolic localization result ([**58**, Section IV.6]). This is an adaptation of results that can be found in [**124**] for schemes.

Theorem 10.2.4 **(Hyperbolic localization).** — *Let* $S \to (\mathrm{Div}^1)^I$ *be a morphism with* S *a small v-stack. If we denote* $\mathfrak{p}_S^{\pm}$, $\mathfrak{q}_S^{\pm}$ *the preceding maps associated to* $B^+ = B$ *and* B^- *its opposite Borel subgroup, pulled back to* S*, one has an isomorphism*

$$\mathrm{CT}_{B,S} := R(\mathfrak{p}_S^+)_!(\mathfrak{q}_S^+)^* A \simeq R(\mathfrak{p}_S^-)_* R(\mathfrak{q}_S^-)^! A$$

for A *a monodromic complex in* $D_{\text{ét}}(\mathrm{Gr}_{G,I} \times_{(\mathrm{Div}^1)^I} S, \Lambda)^b$*. Moreover,*

1. $\mathrm{CT}_{B,S}$ *commutes with base change with respect to any* $T \to S$,
2. *it sends ULA complexes relatively to* S *to ULA complexes i.e. perfect constructible complexes in* $D_{\text{ét}}(S, \Lambda)$,
3. *it commutes with Verdier duality.*

Here the concept of monodromic complex refers to a $\mathbb{G}_m$-action ([**138**]). The only thing to know is that any complex that comes from a complex in $D_{\text{ét}}(\mathscr{H}ecke_I, \Lambda)^b$ is monodromic.

One of the main results is the following ([**58**, Section VI.7.1]) that gives a characterization of the Satake category in terms of the constant term functor. This uses heavily the preceding hyperbolic localization results.

Theorem 10.2.5. — *Let* $S \to (\mathrm{Div}^1)^I$ *with* S *a small v-stack. A complex* $A \in D_{\text{ét}}(\mathscr{H}ecke_I \times_{(\mathrm{Div}^1)^I} S, \Lambda)$ *with bounded support is*

1. *ULA over* S *if and only if*

$$R\pi_{T,S*}\,\mathrm{CT}_B(A)[\deg] \in D_{\text{ét}}(S, \Lambda)$$

is locally constant with perfect fibers.

2. *flat perverse if and only if*

$$R\pi_{T,S*}\,\mathrm{CT}_B(A)[\deg] \in D_{\text{ét}}(S, \Lambda)$$

is étale locally on S *isomorphic to a finite projective* Λ*-module in degree* 0 *where*

$$\pi_{T,S} : \mathrm{Gr}_{T,I} \times_{(\mathrm{Div}^1)^I} S \longrightarrow S.$$

Here

$$\deg : |\operatorname{Gr}_{T,I}| \longrightarrow \mathbb{Z}$$

is given by the preceding function $|\operatorname{Gr}_{T,I}| \to X_*(T)$ composed with $\langle -, 2\rho\rangle$.

Remark 10.2.6. — *When $|I| > 1$ the ULA condition is difficult to express independently of the constant term functor. Nevertheless when $I = \{*\}$ and G is split, for $S \to \operatorname{Div}^1$, one has for $A \in D_{\text{ét}}(\operatorname{Gr}_G \times_{\operatorname{Div}^1} S, \Lambda)$*

$$A \text{ ULA over } S \Longleftrightarrow \forall \mu \in X_*(T)^+,\ [\mu]^* A \in D_{\text{ét}}(S, \Lambda)$$

is locally constant with perfect fiber; where $[\mu]$ is given by the inclusion of the associated Schubert cell. When $S = \operatorname{Div}^1$ this means it is given by a complex of discrete representations of Λ-modules that is a perfect complex of Λ-modules after forgetting the W_E-action.

As a corollary of this theorem one deduces that the Satake category is stable under the involution given by Verdier duality and, if $I = \{*\}$ has one element and G is split, then it contains

$${}^p\mathcal{H}^0(j_{\mu!}\Lambda) \text{ and } {}^p\mathcal{H}^0(Rj_{\mu*}\Lambda)$$

for j_μ the inclusion of the Schubert cell given by $\mu \in X_*(T)^+$.

We moreover prove the following.

Proposition 10.2.7. — *The functor*

$$R\pi_{G*} : \operatorname{Sat}_I(G, \Lambda) \longrightarrow D_{\text{ét}}((\operatorname{Div}^1)^I, \Lambda)$$

given by the pullback to $D_{\text{ét}}(\operatorname{Gr}_{G,I}, \Lambda)$ composed with the pushforward along $\operatorname{Gr}_{G,I} \to (\operatorname{Div}^1)^I$, takes values in complexes A such that for all $i \in \mathbb{Z}$, $\mathcal{H}^i(A)$ is a local system of projective Λ-modules.

The corresponding functor

$$\bigoplus_{i\in\mathbb{Z}} \mathcal{H}^i(R\pi_{G*}) : \operatorname{Sat}_I(G, \Lambda) \longrightarrow \underbrace{\operatorname{Rep}_\Lambda W_E^I}_{\substack{\text{representations with values} \\ \text{in projective finite type} \\ \Lambda\text{-modules}}}$$

is exact, faithful and conservative.

Another part of this result is that this functor satisfies the hypothesis to apply Barr-Beck. We refer to [**58**, Section VI.7.1]. This is the starting point of the reconstruction theorem.

10.3. Fusion ([**58**, Section VI.9])

The stability of the Satake category under the convolution product is verified in [**58**, Section VI.8]. The problem of the commutativity constraint of the convolution product, the canonical isomorphism

$$A * B \simeq B * A,$$

is solved using the interpretation of the convolution product as a fusion product as in [**122**], see [**5**] too.

This is the following. Suppose $I = I_1 \coprod \cdots \coprod I_r$ and let

$$(\mathrm{Div}^1)^{I;I_1,\ldots,I_r} \subset (\mathrm{Div}^1)^I$$

be the open subset where for two divisors D_i, D_j, $i, j \in I$, on the curve are disjoint if $i \in I_k$ and $j \in I_l$ with $k \neq l$. Let

$$\mathrm{Sat}_{I;I_1,\ldots,I_r}(G,\Lambda)$$

be defined as $\mathrm{Sat}_I(G,\Lambda)$ by replacing $\mathscr{H}ecke_I$ by

$$\mathscr{H}ecke_I \times_{(\mathrm{Div}^1)^I} (\mathrm{Div}^1)^{I;I_1,\ldots,I_r}.$$

The main point is now the following.

Proposition 10.3.1. — *The restriction functor*

$$\mathrm{Sat}_I(G,\Lambda) \longrightarrow \mathrm{Sat}_{I;I_1,\ldots,I_r}(G,\Lambda)$$

is fully faithful.

In fact, there is an identification

$$\mathscr{H}ecke_I \times_{(\mathrm{Div}^1)^I} (\mathrm{Div}^1)^{I;I_1,\ldots,I_r} = \prod_{j=1}^{r} \mathscr{H}ecke_{I_j} \times_{(\mathrm{Div}^1)^{I_j}} (\mathrm{Div}^1)^{I;I_1,\ldots,I_r}.$$

External tensor product thus defines a morphism

$$\mathrm{Sat}_{I_1}(G,\Lambda) \times \cdots \times \mathrm{Sat}_{I_r}(G,\Lambda) \longrightarrow \mathrm{Sat}_{I;I_1,\ldots,I_r}(G,\Lambda).$$

The key result now is the following.

Proposition 10.3.2. — *The image of the external tensor product map*

$$\mathrm{Sat}_{I_1}(G,\Lambda) \times \cdots \times \mathrm{Sat}_{I_r}(G,\Lambda) \longrightarrow \mathrm{Sat}_{I;I_1,\ldots,I_r}(G,\Lambda)$$

lies in

$$\mathrm{Sat}_I(G,\Lambda) \subset \mathrm{Sat}_{I;I_1,\ldots,I_r}(G,\Lambda).$$

The proof relies on the construction of an ad-hoc convolution local Hecke stack

$$\mathcal{H}ecke_{I;I_1,\ldots,I_r} \longrightarrow (\mathrm{Div}^1)^I$$

that, given $(D_i)_{i\in I} \in (\mathrm{Div}^1)^I$, parametrizes G-bundles $\mathscr{E}_0, \ldots, \mathscr{E}_r$ on the formal completion of the curve along $\sum_{i\in I} D_i$ together with meromorphic modifications

$$\mathscr{E}_0 \dashrightarrow \mathscr{E}_1 \dashrightarrow \cdots \dashrightarrow \mathscr{E}_r$$

where the modification $\mathscr{E}_{i-1} \dashrightarrow \mathscr{E}_i$ is supported on $\sum_{k\in I_i} D_k$. There are evident morphisms for $1 \leq j \leq r$,

$$p_j : \mathcal{H}ecke_{I;I_1,\dots,I_r} \longrightarrow \mathcal{H}ecke_{I_j}$$

that sends the preceding datum to the restriction of $\mathscr{E}_{j-1} \dashrightarrow \mathscr{E}_j$ to the formal completion along $\sum_{k\in I_j} D_k$. There is moreover a morphism

$$m : \mathcal{H}ecke_{I;I_1,\dots,I_r} \longrightarrow \mathcal{H}ecke_I$$

that sends the preceding datum to the composite $\mathscr{E}_0 \dashrightarrow \mathscr{E}_r$. One can then define

$$Rm_* \left(p_1^* A_1 \otimes^{\mathbb{L}}_{\Lambda} \cdots \otimes^{\mathbb{L}}_{\Lambda} p_r^* A_r\right).$$

One verifies this is in $\mathrm{Sat}_I(G,\Lambda)$ as soon as $A_1,\dots,A_r$ are in the Satake category, and that this restricts to the preceding external tensor product.

We thus have constructed a fusion product

$$\boxed{\text{Fusion} : \mathrm{Sat}_{I_1}(G,\Lambda) \times \cdots \times \mathrm{Sat}_{I_r}(G,\Lambda) \longrightarrow \mathrm{Sat}_I(G,\Lambda).}$$

This defines a map

$$\mathrm{Sat}_I(G,\Lambda) \times \mathrm{Sat}_I(G,\Lambda) \longrightarrow \mathrm{Sat}_{I\coprod I}(G,\Lambda)$$

that we compose with the pullback via the diagonal map $\Delta : I \to I \coprod I$ giving rise to

$$\mathrm{Sat}_{I\coprod I}(G,\Lambda) \longrightarrow \mathrm{Sat}_I(G,\Lambda).$$

At the end this defines the *fusion product*

$$* : \mathrm{Sat}_I(G,\Lambda) \times \mathrm{Sat}_I(G,\Lambda) \longrightarrow \mathrm{Sat}_I(G,\Lambda).$$

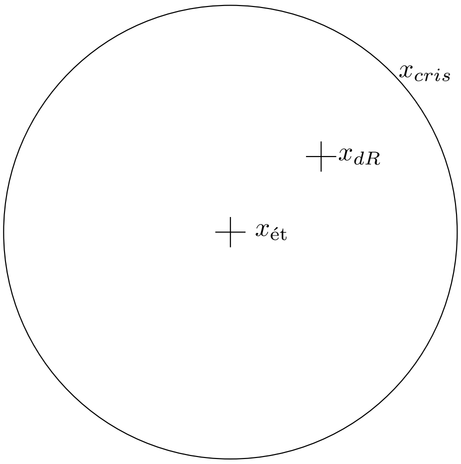
This evidently satisfies the commutativity constraint. By some formal arguments this coincides with the convolution product that thus satisfies the commutativity constraint.

10.4. Degeneration of the B_{dR}-affine Grassmanian to a "classical" Witt vectors affine Grassmanian

We use the diagram

$$\mathrm{Spa}(\overline{\mathbb{F}}_q)^\diamond \hookrightarrow \mathrm{Spa}(\mathcal{O}_{\breve{E}})^\diamond \hookleftarrow \mathrm{Spa}(\breve{E})^\diamond \tag{6}$$

Let us look at the following picture.



In this picture, if $S = \mathrm{Spa}(R, R^+)$ is an $\overline{\mathbb{F}}_q$-perfectoid space with untilt $S^\sharp$ given by the degree 1 distinguished element $\xi \in \mathcal{O}(Y_S)^+$:

- the open disk represents
$$\mathcal{Y}_S = \mathrm{Spa}\left(W_{\mathcal{O}_E}(R^+), W_{\mathcal{O}_E}(R^+)\right) \smallsetminus V([\varpi]),$$
an $\mathcal{O}_E$-analytic adic space with diamond $S \times_{\mathrm{Spd}(\overline{\mathbb{F}}_q)} \mathrm{Spa}(\mathcal{O}_{\breve{E}})^\diamond$ (if $E = \mathbb{F}_q(\!(\pi)\!)$ then $\mathcal{Y}_S = \mathbb{D}_S^*$ a "true" open disk with associated variable π),
- the boundary of the open disk is the crystalline divisor $x_{cris} = V([\varpi])$,
- the de Rham divisor $x_{dR} = V(\xi)$ is given by the untilt,
- the étale divisor $x_{\text{ét}} = V(\pi)$ is the origin of the disk,
- $Y_S = \mathcal{Y}_S \smallsetminus V(\pi)$ is the open punctured disk.

The preceding degeneration diagram (6) then consists in *making x_{dR} degenerate to the origin of the disk $x_{\text{ét}}$*,

$$x_{dR} \rightsquigarrow x_{\text{ét}}.$$

The left hand side and middle terms are not diamonds but only v-sheaves. This is one of the reasons why we deal with "any small v-stack S“ in [**58**, Chapter VI] as a base for the local Hecke stack and not a locally spatial diamond. The preceding induces a diagram of small v-sheaves

$$\mathrm{Spa}(\overline{\mathbb{F}}_q)^\diamond/\varphi^{\mathbb{Z}} \hookrightarrow \mathrm{Spa}(\mathcal{O}_{\breve{E}})^\diamond/\varphi^{\mathbb{Z}} \hookleftarrow \mathrm{Div}^1$$

Suppose G is split as before. We chose a reductive integral model over $\mathcal{O}_E$. For any $S \to (\mathrm{Spa}(\mathcal{O}_{\breve{E}})^\diamond/\varphi^{\mathbb{Z}})^I$ with S a small v-stack, one can define a local Hecke stack $\mathcal{H}ecke_{I,S}$ that sits over S. All the preceding results are still valid in this context with such an S as a base. One has

$$\boxed{\mathrm{Gr}_{G,\mathrm{Spa}(\mathcal{O}_{\breve{E}})^{\diamond}} \times_{\mathrm{Spa}(\mathcal{O}_{\breve{E}})} \mathrm{Spa}(\overline{\mathbb{F}}_q)^{\diamond} = \mathrm{Gr}^{Witt,\diamond}_{G,\overline{\mathbb{F}}_q}}$$

where Gr^{Witt}_{G} is the Witt vector affine Grassmanian of Zhu ([**145**] and [**15**]), an $\mathbb{F}_q$-perfect scheme that is a, increasing union of perfection of projective varieties over $\mathbb{F}_q$. Now, we can use the results of [**128**, Section 27] to relate étale cohomology of schemes over $\overline{\mathbb{F}}_q$ and the one of their diamonds.

One can then exploit the results of [**144**] and use classical technics from classical algebraic geometry/geometric representation theory. Here are two typical examples. The first one is used in the proof of proposition 10.2.7 to obtain some cohomological bounds for $\mathrm{CT}_B(A)$.

Proposition 10.4.1. — *For any* $\lambda \in X_*(T)$ *and* $\mu \in X_*(T)^+$*, the scheme* $S_\lambda \cap \mathrm{Gr}^{Witt}_{G,\leq\mu}$ *is affine equidimensional of dimension* $\langle \rho, \mu + \lambda \rangle$*.*

The second one is used to prove the following result ([**58**, Section VI.7]) that is used in the reconstruction theorem.

Proposition 10.4.2. — *For any* $\mu \in X_*(T)^+$*, if* j_μ *is the inclusion of the associated open Schubert cell inside the closed one, there exists an integer* $a(\mu)$ *such that for any* Λ *a torsion* $\mathbb{Z}_\ell$*-algebra, the kernel and cokernel of*

$$^p\mathcal{H}^0(j_{\mu!}\Lambda) \longrightarrow {}^p\mathcal{H}^0(Rj_{\mu*}\Lambda)$$

are killed by $\ell^{a(\mu)}$*.*

In fact, this is reduced to the same type of statement for $\mathrm{Gr}^{Witt}_{G,\leqslant\mu}$ where this is deduced from the isomorphism

$$^p\mathcal{H}^0(j_{\mu!}\mathbb{Q}_\ell) \xrightarrow{\sim} {}^p\mathcal{H}^0(Rj_{\mu*}\mathbb{Q}_\ell).$$

The proof of this statement is "classical" (see for example [**144**, Lemma 2.1]) but uses as a key point *the decomposition theorem* applied to a Demazure resolution. There is thus no known proof that could be given purely in the analytic world without reduction to the case of algebraic varieties.

10.5. Final thoughts

The "classical" geometric Satake isomorphism is due to Mirkovic and Vilonen ([**109**]). Many works have been undertaken to generalize it/find more natural proofs in different directions. A good introduction to the subject is [**147**]. The work of Richartz ([**123**]) has been determinant, introducing the concept of ULA complexes and fusion in the domain, making the proof of the commutativity constraint of the convolution product more natural. Zhu is the one that first introduced the geometric Satake isomorphism in an arithmetic context ([**144**]) via his work on the Witt vectors affine Grassmanian ([**145**] completed by Bhatt and Scholze in [**15**] who proves that the algebraic spaces showing up in Zhu's work are in fact schemes).

LECTURE 11

THE SPECTRAL ACTION

The spectral action is one of the main results of [**58**]. Here we explain how to build it out of the collection of monoidal ∞-functors

$$F_I : \mathrm{Rep}_\Lambda({}^LG)^I \longrightarrow \mathrm{Hom}\left(\mathcal{D}, \mathcal{D}^{BW_E^I}\right)$$

where $\mathcal{D} = \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ as a condensed stable ∞-category. One of the key point is to use the animated point of view and see the ∞-groupoid $B\Gamma$ for Γ a group as a sifted homotopy colimit of finite sets.

11.1. Background on infinite categories

We fix a "sufficiently large" regular cardinal κ. All our categories and sets are κ-small. Here:

- an ∞-category means an $(\infty, 1)$-category i.e. a quasi-category, which is nothing else than a particular type of simplicial set: the *weak Kan simplicial sets.*
- an ∞-groupoid or $(\infty, 0)$-category means a *Kan simplicial set.* The basic example of an ∞-groupoid is
$$BG$$
where G is a group. This is the nerve of the category with one object with automorphisms G, $(BG)_n = G^n$.

Example 11.1.1. — *If $\mathcal{C}$ is a "usual" 1-category then its nerve $N\mathcal{C}$ with $(N\mathcal{C})_n = Hom([n], \mathcal{C})$ is an ∞-category.*

If C is an ∞-category we note

$$\underbrace{\mathrm{Ho}(C)}_{\text{homotopy category}}$$

for the category whose objects are C_0 and if $C_0 \overset{d_0}{\underset{d_1}{\rightrightarrows}} C_1$ then

$$\mathrm{Hom}_{\mathrm{Ho}(C)}(x, y) = \{f \in C_1 \mid d_0(f) = x,\ d_1(f) = y\}/\sim$$

where $\sim$ is the equivalence relation

$$f \sim g \Leftrightarrow \exists z \in C_2 \begin{cases} d_2(z) = s_0(x) \\ d_1(z) = f \\ d_0(z) = g. \end{cases}$$

The ∞-category C is an ∞-groupoid if and only if $\mathrm{Ho}(C)$ is a groupoid.

If C is an ∞-category:

- we call the elements of C_0 *the objects of* C
- if $x, y \in C_0$ *the maps from* x *to* y are by definition the $f \in C_1$ satisfying $d_0 f = x$, $d_1 f = y$.

By definition:

- an ∞*-functor* between to infinity categories C and D is a morphism of simplicial sets from C to D
- a *natural transformation between* ∞*-functors* $F, G : C \to D$ is a diagram

$$\begin{tikzcd} C \times \Delta_0 \arrow[r, "\mathrm{Id} \times \delta_0"] \arrow[dr, "F"'] & C \times \Delta_1 \arrow[d] & C \times \Delta_0 \arrow[l, "\mathrm{Id} \times \delta_1"'] \arrow[dl, "G"] \\ & D & \end{tikzcd}$$

- If C and D are infinity categories then the simplicial set

$$\underbrace{\underline{\mathrm{Hom}}(C, D)}_{\substack{\text{internal maps} \\ \text{in the category of simplicial sets}}}$$

  is a weak Kan complex that we call the ∞-category of functors from F to D. Its objects are ∞-functors as defined earlier and the morphisms are natural transformations as defined before. We call it the ∞-category of ∞-functors from C to D and note it sometimes D^C.
- if x, y are two objects of the ∞-category C then

$$\mathrm{Hom}(x, y)$$

  is the sub-simplicial complex

$$\mathrm{Hom}(x, y)_n = \{c \in C_{n+1} \mid v_0 c = \cdots = v_n c = x,\ v_{n+1} c = y\}$$

  where $v_i : C_n \to C_0$ is the i-th vertex corresponding to the inclusion $[0] \hookrightarrow [n{+}1]$ sending 0 to i. This is in fact a Kan complex, *the* ∞*-groupoid of morphisms from* x *to* y sometimes called *the mapping space from* x *to* y when we see it as a Kan complex up to (weak) homotopy. Let us note that there exists other versions of this Kan complex but they are all homotopy equivalent. Moreover, the composition

$$\mathrm{Hom}(x, y) \times \mathrm{Hom}(y, z) \longrightarrow \mathrm{Hom}(x, z)$$

as a morphism of Kan simplicial sets is only well defined up to homotopy. This point of view leads to the one of ∞-categories as categories enriched in the category of spaces (i.e. the category of Kan simplical sets up to homotopy) but this is not the one we use. Let us finally note that

$$\mathrm{Hom}_{\mathrm{Ho}(C)}(x, y) = \pi_0 \mathrm{Hom}(x, y).$$

By definition, an ∞-functor between ∞-categories $F : C \to D$

1. is an equivalence if there exists an ∞-functor $G : D \to C$ such that $F \circ G$ is isomorphic to Id_D and $G \circ F$ is isomorphic to Id_C,
2. is fully faithful if for x, y two objects of C, the map of Kan simplicial sets $\mathrm{Hom}(x, y) \to \mathrm{Hom}(F(x), F(y))$ is an homotopy equivalence.

The following is satisfied (see [**101**, Remark 1.2.11.1]):

- F is an equivalence if and only if the induced functor $\mathrm{Ho}\, C \to \mathrm{Ho}\, D$ is an equivalence,
- F is fully faithful if and only if the induced functor $\mathrm{Ho}\, C \to \mathrm{Ho}\, D$ is fully faithful.

11.1.1. Homotopy coherent nerve. — There is a construction ([**101**, Definition 1.1.5.5], [**126**]) called *homotopy coherent nerve*

$$\mathrm{N}^{\mathrm{hc}} : \underbrace{\mathrm{sSet}\text{-}\mathrm{Cat}}_{\substack{\text{categories enriched}\\ \text{in simplicial sets}}} \longrightarrow \underbrace{\mathrm{sSet}}_{\substack{\text{simplicial}\\ \text{sets}}}$$

where a category enriched in simplicial sets is such that for any objects x, y, $\mathrm{Hom}(x, y)$ has a structure of simplicial set and the composition

$$\mathrm{Hom}(x, y) \times \mathrm{Hom}(y, z) \to \mathrm{Hom}(x, z)$$

is a morphism of simplicial sets. Here we recall that if S and T are simplicial sets, $S \times T$ is such that $(S \times T)_n = S_n \times T_n$ i.e. $S \times T$ is defined using the diagonal functor $\Delta \to \Delta \times \Delta$, Δ being the simplex category. For basic facts about simplicial objects we advise to look at [**67**].

This construction is such that if for all $x, y \in \mathrm{Ob}\, \mathcal{C}$, the simplicial set $\mathrm{Hom}(x, y)$ is a Kan simplicial set then $\mathrm{N}^{\mathrm{hc}}\, \mathcal{C}$ is a weak Kan complex. We thus have a construction

$$\boxed{\mathrm{N}^{hc} : \underbrace{\text{Kan-sSet-Cat}}_{\substack{\text{categories enriched in}\\ \text{Kan simplicial sets}}} \longrightarrow (\infty, 1)\text{-categories.}}$$

If $\mathcal{C}$ is a category enriched in Kan simplicial sets then for $x, y \in \mathrm{Ob}(\mathcal{C})$, the Kan simplicial sets

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \quad \text{and} \quad \mathrm{Hom}_{\mathrm{N}^{hc}(\mathcal{C})}(x, y)$$

are homotopy equivalent.

The construction of the homotopy coherent nerve is done via a functor

$$\mathcal{P} : \underbrace{\Delta}_{\substack{\text{simplex}\\ \text{category}}} \longrightarrow \text{sSet-Cat}$$

called *the path category.* Then,

$$\mathrm{N}^{\mathrm{hc}}(C)_n = \mathrm{Hom}_{\text{sSet-Cat}}(\mathcal{P}([n]), C).$$

It satisfies:

1. $\mathrm{N}^{\mathrm{hc}}(C)_0 = \mathrm{Ob}\, C$,
2. For $x, y \in C_0$,
$$\mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}(C)}(x, y)_0 = \mathrm{Hom}_C(x, y)_0$$
and there is a natural homotopy equivalence between
$$\mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}(C)}(x, y) \quad \text{and} \quad \mathrm{Hom}_C(x, y)$$
i.e. a canonical isomorphism betwwen those two Kan simplicial sets in the ∞-category of Kan simplicial sets up to homotopy.
3. The homotopy category of $\mathrm{N}^{\mathrm{hc}}(C)$ is identified with the category whose objects are the objects of C and morphisms between x and y are given by $\pi_0\mathrm{Hom}(x, y)$.

Example 11.1.2. — *1. The category of simplicial sets is enriched in simplicial sets: if S and T are simplicial sets, $\underline{Hom}(S, T)$ is the simplical set $[n] \mapsto Hom_{sSet}(X \times \Delta_n, Y)$.*

2. The category of topological spaces is enriched in simplicial sets by setting, for X and Y two topological spaces,

$$Hom(X, Y)_n = Hom(X \times |\Delta_n|, Y).$$

3. If $\mathcal{C}$ is a dg-category then this gives rise the the following category enriched in simplicial sets. Recall the Dold-Kan correspondence given by the simplicialization functor

$$\Gamma : \mathrm{CoCh}_{\leq 0}(\mathrm{Ab}) \xrightarrow{\sim} s\,\mathrm{Ab}$$

We can then set for $X, Y \in \mathrm{Ob}(\mathcal{C})$,

$$Hom(X, Y) = \Gamma\, \tau_{\leq 0} Hom_{\mathcal{C}}(X, Y).$$

This defines a morphism of 2-categories

$$\text{dg-Cat} \longrightarrow \textit{sSet-Cat.}$$

Composed with the preceding homotopy coherent nerve construction we obtain a construction

$$\text{dg-Cat} \longrightarrow \infty\text{-Cat}\,.$$

4. *If $\mathcal{A}$ is an abelian category then the category of cochain complexes of elements of $\mathcal{A}$ is naturally a dg-category. The associated ∞-category is the one of cochain complexes i.e. its objects are cochain complexes and its morphisms are morphisms of cochain complexes with*

$$\pi_0 Hom(A^\bullet, B^\bullet) = Hom(A^\bullet, B^\bullet)/\,\text{homotopy}\,.$$

5. *If S and T are weak Kan simplicial sets then $\underline{Hom}(S,T)$ is a weak Kan simplicial set. Thus, à priori*

N(*the simplicially enriched cat. of weak Kan complexes*) = "*an* $(\infty, 2)$-*category*".

Nevertheless, there is a construction

$$\text{Core} : \infty\text{-categories} \longrightarrow \infty\text{-groupoids}$$

that sends a weak Kan complex to the biggest sub-Kan complex (i.e. we only keep the 1*-morphisms that are isomorphisms). Then, if we take the simplicial category whose objects are the weak Kan complexes with morphisms between S and T*

$$\text{Core}\,\underline{Hom}(S,T),$$

its homotopy coherent nerve is what we call the ∞-category of ∞-categories.

11.1.2. Dwyer-Kan/Lurie localization. — There is another construction. If C is an ∞-category and $S \subset C_1$ a set of maps one can define its localization in the sense of Lurie

$$S^{-1}C = \text{an } \infty\text{-category},$$

see [**100**, Section 1.3.4]. One has

$$\text{Ho}(\underbrace{S^{-1}C}_{\substack{\text{Lurie}\\ \text{localization}}}) = \underbrace{S^{-1}\,\text{Ho}(C)}_{\substack{\text{Gabriel-Zisman}\\ \text{localization}}}\,.$$

There is another construction due to Dwyer-Kan ([**43**]) that produces form a category C and a set of maps S in C a simplicial category $L(C,S)$ whose objects are the same as the one of C and such that the associated category whose objects are the one of $L(C,S)$ with morphisms between x and y, $\pi_0\text{Hom}(x,y)$ is $S^{-1}C$. This means that the usual Gabriel-Zisman localization has an upgrade that constructs a simplicial category. When applied to model categories in the sense of Quillen together with their weak equivalences, this can be used to produce some localizations of ∞-categories by applying the homotopy coherent nerve construction but we prefer to use Lurie's point of view.

Here is a use of this process. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{K}(\mathcal{A})$ be the ∞-category of cochain complexes of objects of $\mathcal{A}$, see 11.1.2 point 4. Let S be the set of quasi-isomorphism. One can then define

$$\mathcal{D}(\mathcal{A}) = S^{-1}\mathcal{K}(\mathcal{A})$$

the derived ∞-category of $\mathcal{A}$. Its homotopy category is the usual derived category of $\mathcal{A}$.

11.1.3. The ∞-category of spaces. — If C and D are Kan simplicial sets then the simplicial set $\underline{\mathrm{Hom}}(C,D)$ is again a Kan simplicial set. We note

$$\mathcal{A}ni$$

for the associated ∞-category by applying the homotopy coherent nerve construction to the simplistically enriched category of Kan simplicial sets. This is the ∞-category of animated sets. This ∞-category has some other names, typically this is the ∞-category of ∞-groupoids,

$$\underbrace{\mathcal{A}ni}_{\substack{\infty\text{-categories } C\\ \text{s.t. Ho } C \text{ is a}\\ \text{groupoid}}} \subset \infty\text{-category of } \infty\text{-categories.}$$

Via the geometric realization functor this is equivalent to the ∞-category of spaces (an inverse to the geometric realization functor being given by the singular set functor from spaces to simplicial sets).

11.1.4. Homotopy limits and colimits. — Let $p : K \to C$ be a map of simplicial sets. One can define ([**101**, Section 1.2.9]) two simplicial sets

$$C_{/p} \text{ and } C_{p/}$$

that generalizes the notion overcategory/undercategory. When C is an ∞-category, $C_{/p}$ and $C_{p/}$ are again ∞-categories ([**101**, Proposition 1.2.9.3]).

If C is a usual category, an object x of C is said to be final if for any y in C, $\mathrm{Hom}(y,x)$ is a set with one element. The same goes on for the notion of initial object; x is initial i for any y, $\mathrm{Hom}(x,y)$ is a set with one element.

If C is an infinity category, by definition, an object x of C is final, resp. initial, if for any object y of C, the Kan complex $\mathrm{Hom}(y,x)$, resp. $\mathrm{Hom}(x,y)$, is contractible. One of the basic results of the domain is that ([**101**, Proposition 1.2.12.9]), if non-empty, *the sub-category of C spanned by the final, resp. initial, objects is a contractible Kan complex.* As a corollary, if x and y are final, resp. initial, objects of C then they are canonically isomorphic in $\mathrm{Ho}\, C$.

By definition, a homotopy limit, resp. colimit, of $p : K \to C$ is a final, resp. initial, object of $C_{/p}$, resp. $C_{p/}$.

Let us give som examples:

- If f is a map in the ∞-category C between x and y then it corresponds to a morphism $p : \Delta_0 \to C$. A limit of p is a homotopical fiber of $[f]$ in $\mathrm{Ho}\, C$. A colimit of p is a homotopical cofiber of $[f]$ in $\mathrm{Ho}\, C$. Thus, limits and colimits in ∞-categories allows us to give a canonical (up to a contractible set of isomorphisms) sense to homotopical fibers and cofibers.
- For example, if $\mathcal{A}$ is an abelian category, f a morphism in the ∞-category $\mathcal{D}(\mathcal{A})$, then a colimit of f is a cone of f. Thus, ∞-categories make cone of

morphisms canonical (up to a contractible set of isomorphisms) contrary to usual triangulated categories.

- If X is a Kan simplicial set then
$$\operatorname*{colim}_{[n]\in\Delta} X_n \xrightarrow{\sim} X$$
in $\mathcal{A}ni$. *This means that in the category of spaces up to homotopy, any space is a homotopical colimit of discrete sets, and thus of finite sets.*
- If I is any small category and we are given a functor $F : I \to \operatorname{Ho} C$, a lift of F to a morphism of simplicial set $\widetilde{F} : N(I) \to C$, $N(I)$ being the nerve of I, let us give a meaning to a homotopical limit, resp. colimit, of F.
- Let I be a small category and $(X_i)_{i\in I}$ be a fibered topos over I. Let Λ be a ring. Then, working in the ∞-category of ∞-categories, $\lim_{i\in I} \mathcal{D}(X_i, \Lambda)$ is equivalent to $\mathcal{D}(\mathcal{A})$ where $\mathcal{A}$ is the abelian category of cartesian sheaves of Λ-modules on $(X_i)_{i\in I}$.

11.2. The animation slogan

11.2.1. Sifted colimits. — Recall that if I is a small category then I is said to be *filtered* if colimits (with values in usual 1-categories) indexed by I *commute with finite limits*. This is well known to be equivalent to:

1. for any $i, j \in I$ there exists $k \in I$ with $\operatorname{Hom}(i, k) \neq \emptyset$ and $\operatorname{Hom}(j, k) \neq \emptyset$,
2. for two morphisms $\begin{tikzcd} i \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & j \end{tikzcd}$ in I there exist a morphism $f : j \to k$ in I such that $f \circ u = f \circ v$.

By definition, a small category is **1-sifted** if colimits (with values in usual 1-categories) indexed by I *commute with finite products*. This is of course the case if I is filtered. Another example is the case of *reflexive co-equalizers* which correspond to the diagram $\tau_{\leq 1}\Delta$
$$\begin{tikzcd} i \arrow[r, shift left=2, "u"] \arrow[r, shift right=2, "v"'] & j \arrow[l, dashed, "s" description] \end{tikzcd}$$
where s is a joint section of u and v i.e. $u\circ s = \mathrm{Id} = v \circ s$. In fact, for a finite collection of morphisms $\left(X_\alpha \rightrightarrows Y_\alpha \right)_\alpha$, the morphism
$$\operatorname{coeq}\Big(\textstyle\prod_\alpha X_\alpha \rightrightarrows \prod_\alpha Y_\alpha \Big) \longrightarrow \displaystyle\prod_\alpha \operatorname{coeq}\left(X_\alpha \rightrightarrows Y_\alpha \right)$$
has an explicit inverse induced by $\prod_\alpha s_\alpha$ if $s_\alpha : Y_\alpha \to X_\alpha$ is a joint section of $X_\alpha \rightrightarrows Y_\alpha$.

11.2.2. Animation ([137, Section 5.1.4]). — Let $\mathcal{C}$ be a cocomplete category. By definition an object x of $\mathcal{C}$ is

- compact if $\mathrm{Hom}(x,-)$ commutes with filtered colimits,
- projective compact if $\mathrm{Hom}(x,-)$ commutes with sifted colimits.

We note $\mathcal{C}^{\mathrm{fp}}$, resp. $\mathcal{C}^{\mathrm{sfp}}$, for the category of compact, resp. compact projective, objects of $\mathcal{C}$.

Animation slogan: If $\mathcal{C}$ is a category

1. admitting small colimits,
2. generated under small colimits by its *compact projective* objects,

its animation

$$\mathcal{A}ni(\mathcal{C})$$

is the ∞-category freely generated under sifted colimits by its compact projective objects.

Let us begin with one remark first. If $\mathcal{C}$ is generated under (small) colimits by its compact projective objects then it is generated under sifted colimits by its compact projective objects. There is then an equivalence

$$\mathrm{sInd}(\mathcal{C}^{\mathrm{sfp}}) \xrightarrow{\sim} \mathcal{C}$$

where the left hand term is defined analogously as the Ind-category but by replacing filtered colimits by sifted one.

Here is how is defined the ∞-category freely generated under sifted colimits. Look at the ∞-category of functors

$$\mathrm{Hom}(N(\mathcal{C}^{\mathrm{sfp}})^{op}, \mathcal{A}ni)$$

where $N(\mathcal{C}^{\mathrm{sfp}})^{op}$ is the nerve of $\mathcal{C}^{\mathrm{sfp}}$. There is a Yoneda fully faithful embedding ([101, Lemma 5.1.3])

$$N(\mathcal{C}^{\mathrm{sfp}}) \hookrightarrow \mathrm{Hom}\left(N(\mathcal{C}^{\mathrm{sfp}})^{op}, \mathcal{A}ni\right).$$

The ∞-category $\mathrm{Hom}(N(\mathcal{C}^{\mathrm{sfp}})^{op}, \mathcal{A}ni)$ is cocomplete and one can add all sifted colimits to this embedding to obtain $\mathrm{sInd}(\mathcal{C}^{\mathrm{sfp}})$.

Fundamentally, *the idea of animation is to reconstruct some usual ∞-categories from some basic pieces that is a category of compact projective objects.* Let us give some examples:

- If $\mathcal{C} = \mathcal{A}b$, the category of abelian groups, $\mathcal{A}ni(\mathcal{A}b)$ is identified via the Dold-Kan correspondence with $\mathcal{D}^{\leq 0}(\mathbb{Z})$.
- If $\mathcal{C}$ is the category of rings one finds back the usual ∞-category of rings associated to the simplicial model category of simplicial rings defined and studied by Quillen ([115]).

- If X is a topological space then, in the ∞-category of spaces (that is equivalent to $\mathcal{A}ni$),

  $$X \simeq \operatorname*{colim}_{\substack{[n]\in\Delta \\ E\subset \mathrm{Sing}_n(X) \\ \text{finite}}} E.$$

  In the usual category of sets this colimit would be $\pi_0(X)$. *The animation of this colimit, seen as a homotopy colimit, allows us to recover the higher homotopy invariants of X, the $\pi_i(X)$ for $i > 0$.*

11.3. The moduli of Langlands parameters

11.3.1. Definition. — Let Λ be any $\mathbb{Z}_\ell$-algebra. We see it as a condensed ring via the formula $\Lambda := \Lambda \otimes_{\mathbb{Z}_\ell^{disc}} \mathbb{Z}_\ell$ that maps a profinite set P to functions from P to Λ whose image is contained in a finite type sub-$\mathbb{Z}_\ell$-module and are continuous.

Definition 11.3.1. — *We note $Z^1(W_E, \widehat{G})$ for the functor on $\mathbb{Z}_\ell$-algebras that sends Λ to condensed 1-cocycles $W_E \to \widehat{G}(\Lambda)$ where the topological group W_E is seen as a condensed group and $\widehat{G}(\Lambda)$ is a condensed group.*

This means that if we fix an embedding $\widehat{G} \hookrightarrow \mathrm{GL}_n$ then we have a 1-cocycle $W_E \to \widehat{G}(\Lambda)$ such that the associated map $W_E \to \mathrm{GL}_n(\Lambda)$ is given coordinate-wise by maps $W_E \to \Lambda$ whose image are contained in a finite type sub-$\mathbb{Z}_\ell$-module and are continuous.

11.3.2. Representability theorem ([58, Chapter VIII], [33]). — Such a condensed cocycle has to factorize through some quotient W_E' of W_E by some open subgroup of P_E distinguished in W_E that acts trivially on $\widehat{G}$. One can then prove that there is a dense subgroup $W \subset W_E'$ such that

$$W/P_E = \mathbb{Z}\left[\tfrac{1}{p}\right] \rtimes \sigma^{\mathbb{Z}} \subset \widehat{\mathbb{Z}}^{(p)} \rtimes \mathbb{Z} = W_E/P_E$$

where σ acts as multiplication by p on $\mathbb{Z}\left[\frac{1}{p}\right]$, and satisfying

$$Z^1(W_E', \widehat{G}_{\mathbb{Z}_p}) = Z^1(W, \widehat{G}_{\mathbb{Z}_\ell})$$

where in the definition of $Z^1(W, \widehat{G}_{\mathbb{Z}_\ell})$ there is no topological condition on the cocycle. It is easy to verify that $Z^1(W, \widehat{G}_{\mathbb{Z}[\frac{1}{p}]})$ is represented by an affine scheme of finite type over $\mathrm{Spec}(\mathbb{Z}\left[\frac{1}{p}\right])$. From this it is easy to verify that *$Z^1(W_E, \widehat{G})$ is representable by a disjoint union of finite type affine schemes.* We can go further.

Theorem 11.3.2. — *The stack of Langlands parameters*

$$\mathrm{LocSys}_{\widehat{G}} = \left[\ Z^1(W_E, \widehat{G})/\widehat{G}\ \right]$$

is locally complete intersection of relative dimension 0 *over Spec*$(\mathbb{Z}\left[\frac{1}{p}\right])$.

We refer to [**33**] for a detailed study of the geometry of this stack.

Example 11.3.3 **(Principal block).** — *Suppose* G *is split. Consider the moderate quotient* $W_E/P_E \simeq \widehat{\mathbb{Z}}^{(p)}(1) \rtimes \sigma^{\mathbb{Z}}$. *There is associated*

$$\left[\ \mathit{Hom}\ \left(\mathbb{Z}\left[\tfrac{1}{p}\right] \rtimes \mathbb{Z}, \widehat{G}_{\mathbb{Z}[\frac{1}{p}]}\right)/\widehat{G}_{\mathbb{Z}[\frac{1}{p}]}\ \right] \underbrace{\subset}_{\mathit{open/closed}} \mathrm{LocSys}_{\widehat{G}}\,.$$

Here one has

$$\mathit{Hom}\ \left(\mathbb{Z}\left[\tfrac{1}{p}\right] \rtimes \mathbb{Z}, \widehat{G}_{\mathbb{Z}[\frac{1}{p}]}\right) = \left\{(g_1, g_2) \in \widehat{G}_{\mathbb{Z}[\frac{1}{p}]} \times \widehat{G}_{\mathbb{Z}[\frac{1}{p}]}\ \middle|\ g_1 g_2 g_1^{-1} = g_2^p\right\}$$

and the theorem says that this is locally complete intersection of relative dimension $\dim G$ *over Spec*$(\mathbb{Z}\left[\frac{1}{p}\right])$.

Over $\mathbb{Q}$ the structure of the stack becomes much more simple: it is isomorphic to a moduli space of Weil-Deligne representations and is in particular fibered over the nilpotent cone of $\widehat{G}$.

11.3.3. Coarse moduli space and semi-simple Langlands parameters ([58, Section VIII.3]). —

As a consequence of Mumford's numerical criterion coupled with some results of Richardson ([**120**]) one obtains the following result.

Proposition 11.3.4. — *The* $\overline{\mathbb{F}}_\ell$, *resp.* $\overline{\mathbb{Q}}_\ell$, *points of the coarse moduli space Spec*$\left(\mathcal{O}\left(Z^1(W_E, \widehat{G})\right)^{\widehat{G}}\right)$ *are in bijection with the conjugacy classes of semi-simple Langlands parameters* $W_E \to {}^L(G(\overline{\mathbb{F}}_\ell))$, *resp.* $W_E \to {}^L G(\overline{\mathbb{Q}}_\ell)$.

For $\overline{\mathbb{Q}}_\ell$, semi-simple Langlands parameters correspond to representations (ρ, N) of the Weil Deligne group satisfying $N = 0$. Over $\overline{\mathbb{F}}_\ell$ their definition is a slightly more complicated.

Remark 11.3.5. — *Of course, as usual, Spec*$\left(\mathcal{O}\left(Z^1(W_E, \widehat{G})\right)^{\widehat{G}}\right)$ *can be interpreted as a moduli of pseudo-parameters i.e. pseudo-representations when* $G = \mathrm{GL}_n$.

11.3.4. The moduli as an infinite derived stack. —

Definition 11.3.6. — *An infinite derived stack $\mathfrak{X}$ is an ∞-functor*

$$\mathfrak{X} : \mathcal{A}ni(\text{Ring}) \longrightarrow \underbrace{\infty - \text{Gpd}}_{\mathcal{A}ni}$$

that safisfies descent in the sense that if $R \to R'$ is a faithfully flat morphism of animated rings ([**137**]*) with associated cosimplicial animated ring $(R_n)_{[n]\in\Delta}$, $R_n = \underbrace{R' \otimes^{\mathbb{L}}_R \cdots \otimes^{\mathbb{L}}_R R'}_{n\text{-}times}$ then*

$$\mathfrak{X}(R) \longrightarrow \lim_{[n]\in\Delta} \mathfrak{X}(R_n)$$

is an equvialence.

Definition 11.3.7. — *We note $\text{LocSys}^{der}_{\widehat{G}/\mathbb{Z}_\ell}$ for the infinite derived stack that is the stack associated to the prestack that sends a $\mathbb{Z}_\ell$-animated ring R to*

$$Hom_{BW_E}(\underbrace{BW_E}_{\substack{\text{Condensed}\\ \text{anima}}}, \overbrace{B\ \underbrace{{}^LG(R \otimes_{\mathbb{Z}_\ell^{disc}} \mathbb{Z}_\ell)}_{\substack{\text{Condensed animated}\\ \text{group}}}}^{\text{Condensed anima}}).$$

One then has the following.

Proposition 11.3.8. — *The infinite derived stack of Langlands parameters is underived: for any animated $\mathbb{Z}_\ell$-algebra R,*

$$\text{LocSys}^{der}_{\widehat{G}/\mathbb{Z}_\ell}(R) \xrightarrow{\sim} N\,\text{LocSys}_{\widehat{G}/\mathbb{Z}_\ell}(\pi_0(R)).$$

This result is a consequence of theorem 11.3.2. Take for example the open/closed substack of parameters $\mathbb{Z}[\frac{1}{p}] \rtimes \mathbb{Z} \longrightarrow \widehat{G}_{\mathbb{Z}[\frac{1}{p}]}$ (the so-called tame parameters). Suppose G is split to simplify. This is given by the fiber over $e \in G(\mathbb{Z}[\frac{1}{p}])$ of the morphism

$$\begin{array}{ccc} \widehat{G}_{\mathbb{Z}[\frac{1}{p}]} \times \widehat{G}_{\mathbb{Z}[\frac{1}{p}]} & \longrightarrow & \widehat{G}_{\mathbb{Z}[\frac{1}{p}]} \\ (g_1, g_2) & \longmapsto & g_2 g_1 g_2^{-1} g_1^{-p}. \end{array}$$

According to theorem 11.3.2 the fibers of this morphism at any point of the section $\text{Spec}(\mathbb{Z}[\frac{1}{p}]) \to \widehat{G}$ have dimension $\dim G$. Since the source and the target of this morphism are regular schemes we deduce that this morphism is flat in a neighborhood of this section. The pullback of e thus coincides with the derived pullback.

Remark 11.3.9. — *In a non-arithmetic context, see for example* [**62**]*, the moduli of Langlands parameters is not a locally complete intersection of the good dimension and proposition 11.3.8 fails. In those non-arithmetic situations the good geometric object is the derived stack which makes the geometry more complicated. Proposition 11.3.8 is one instance where things simplify in* [**58**] *compared to the geometric Langlands program over a compact Riemann surface for example (*[**1**]*).*

Another instance where things simplify is that the singular support condition of Arinkin-Gaitsgory ([**1**]*, the fact that the singular support of our coherent complexes have to be contained in the nilpotent cone) disappears over* $\overline{\mathbb{Q}}_\ell$*, see* [**58**, Chapter VIII.2.2].

11.4. The spectral action ([58, Chapter X])

11.4.1. A sifted category related to excursion operators. — Let us define the following category.

Definition 11.4.1. — *For* W *a group we note* $\mathcal{C}_W$ *for the category of couples* $(I, F(I) \to W)$ *where*

1. I *is a finite set,*
2. $F(I)$ *is the free group on* I *and* $F(I) \to W$ *is a morphism of groups,*
3. *morphisms between* $(I, F(I) \to W)$ *and* $(I', F(I') \to W)$ *are given by morphisms of groups* $F(I) \to F(I')$ *such that the diagram*

$$\begin{tikzcd} F(I) \arrow[dd] \arrow[dr] & \\ & W \\ F(I') \arrow[ur] & \end{tikzcd}$$

commutes.

Lemma 11.4.2. — *The category* $\mathcal{C}_W$ *is sifted.*

In fact, for two objects $F(I) \xrightarrow{u} W$ and $v : F(I') \xrightarrow{v} W$ of $\mathcal{C}_W$ there is a diagram

$$\begin{tikzcd} F(I) \arrow[dr] \arrow[drr, bend left, "u"] & & \\ & F(I \coprod I') \arrow[r, "u*v"] & W \\ F(I') \arrow[ur] \arrow[urr, bend right, "v"'] & & \end{tikzcd}$$

Thus, for $x, y \in \mathrm{Ob}\,\mathcal{C}_W$ one can find $z \in \mathrm{Ob}\,\mathcal{C}_W$ such that $\mathrm{Hom}(x, z) \neq \emptyset$ and $\mathrm{Hom}(y, z) \neq \emptyset$.

If we have two morphisms

$$\begin{tikzcd}
F(I) \arrow[dd, shift left] \arrow[dd, shift right] \arrow[dr] & \\
& W \\
F(I') \arrow[ur] &
\end{tikzcd}$$

the image of $F(I)$ in $F(I') \times_W F(I')$ is a finite type subgroup of the free group $F(I) \times F(I)$. It is thus isomorphic to $F(J)$ for a finite set J. From this we deduce that for two morphisms $x \rightrightarrows y$ in $\mathcal{C}_W$, there is a factorization of those two morphisms

$$\begin{tikzcd}
x \arrow[r] \arrow[rr, bend left] \arrow[rr, bend right] & z \arrow[r, shift left] \arrow[r, shift right] & y
\end{tikzcd}$$

where $z \rightrightarrows y$ is a reflexive coequalizer.
Those two properties prove that $\mathcal{C}_W$ is sifted.

For a finite set I we note $\Sigma I = \vee_{i \in I} S^1$ (a "bouquet de cercles") that is identified with $BF(I)$ as an ∞-groupoid.

Corollary 11.4.3. — *We have*

$$\operatorname*{colim}_{(I, F(I) \to W) \in \mathcal{C}_W} \Sigma I \xrightarrow{\sim} BW$$

in the ∞-category of ∞-groupoids i.e. the ∞-groupoid BW is a sifted colimit of ΣI for finite sets I.

Remark 11.4.4. — *We used the animation slogan here already: the free groups on a finite set are the compact projective objects of the category of groups and any group is a sifted colimit of such groups.*

11.4.2. The moduli space of Langlands parameters as a sifted limit of derived stacks. — We take the notations at the beginning of section 11.3.2. Let us fix Q a finite quotient of W that acts on $\widehat{G}$. One can write $[Z^1(W, \widehat{G})/\widehat{G}]$ as an Hom stack in the ∞-derived sense:

$$\left[Z^1(W, \widehat{G})/\widehat{G}\right] = \mathrm{Hom}_{BQ}\left(BW, B(\widehat{G} \rtimes Q)\right).$$

As an application of corollary 11.4.3 one obtains the following.

Proposition 11.4.5. — *There is an isomorphism of derived stacks over Spec*$(\mathbb{Z}[\frac{1}{p}])$,

$$\Big[\ Z^1(W,\widehat{G})/\widehat{G}\ \Big] \xrightarrow{\sim} \varprojlim_{(I,F(I)\to W)\in\mathcal{C}_W} [\ \widehat{G}^I\ /\ \widehat{G}\ ]$$

where if $\tau : I \to W$, *the action of* $\widehat{G}$ *on* $\widehat{G}^I$ *is given by* $g.(h_i)_{i\in I} = (gh_ig^{-\tau_i})_{i\in I}$.

Corollary 11.4.6. — *One has an isomorphism of* $\mathbb{Z}\big[\frac{1}{p}\big]$*-algebras*

$$\mathcal{O}\left(Z^1(W,\widehat{G})\right) = \varinjlim_{(I,F(I)\to W)\in\mathcal{C}_W} \mathcal{O}\left(\widehat{G}^I\right)$$

where the right and colimit is sifted.

11.4.3. Excursion operators. — We can now make the link with V. Lafforgue's point of view ([**92**]).

Definition 11.4.7. — *The algebra of excursion operators is*

$$\mathrm{Exc} := \varinjlim_{(I,F(I)\to W)\in\mathcal{C}_W} \mathcal{O}\left(\widehat{G}^I\right)^{\widehat{G}}.$$

Geometric invariant theory then gives the following. Sifted colimits do not commute with applying $H^0(\widehat{G},-)$ unless we work rationally since then the category of algebraic representations of $\widehat{G}$ is semi-simple.

Proposition 11.4.8. — *There is a morphism of* $\mathbb{Z}\big[\frac{1}{p}\big]$*-algebras*

$$\mathrm{Exc} \longrightarrow \mathcal{O}\left(Z^1(W,\widehat{G}) /\!\!/ \widehat{G}\right)$$

that is a homeomorphism and an isomorphism after tensoring with $\mathbb{Q}$.

Thus, after inverting some integer $N \gg 1$ with N, this becomes an isomorphism. In fact, in [**58**, Section VIII.5], we give an explicit N depending on the root datum for G. Typically, we can take $N = 1$ for GL_n, i.e. this is an isomorphism integrally for the linear group, and $N = 2$ for classical groups.

11.4.4. The spectral action. — We no come to one of the main results of [**58**]: the spectral action. This is the following. We use the same notations as before.

Theorem 11.4.9. — *Let Λ be a $\mathbb{Q}$-algebra. Let $\mathcal{C}$ be a stable Λ-linear ∞-category that is idempotent complete. The following datum are equivalent:*

1. *A functorial in the finite set I monoidal ∞-functor*
$$F_I : \mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I \longrightarrow \mathrm{End}(\mathcal{C})^{BW^I}$$
that is linear over $\mathrm{Rep}_\Lambda Q^I$.
2. *A monoidal action of the monoidal stable ∞-category $\mathrm{Perf}\left(\left[\ Z^1(W,\widehat{G})/\widehat{G}\ \right]\right)$ on $\mathcal{C}$.*

The proof makes use of the animation slogan: *we replace BW by a sifted homotopy colimit of finite sets*. More precisely, if X is an animated set sitting over BQ both objects in the points (1) and (2) make sense for X:

1. A functorial in the finite set I monoidal ∞-functor
$$F_I : \mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I \longrightarrow \mathrm{End}(\mathcal{C})^{X^I}$$
that is linear over $\mathrm{Rep}_\Lambda Q^I$.
2. A monoidal action of the monoidal stable ∞-category
$$\mathrm{Perf}\left(\left[\ \mathrm{Hom}_{BQ}\left(X, B(\widehat{G} \rtimes Q)\right)\right]\right)$$
on $\mathcal{C}$.

There is an evident construction that attaches to an object of point (2) one of point (1), functorially for X over BQ. One verifies moreover that both constructions commute with sifted colimits of animated sets (here we have to use that Λ is a $\mathbb{Q}$-algebra). The result is then reduced to the case when X is a finite set. In this case the map $X \to BQ$ is trivial:

- The first construction is equivalent to a functorial in I monoidal functor $\mathrm{Rep}_\Lambda(\widehat{G})^I \to \mathrm{End}(\mathcal{C})^{X^I}$.
- The second one is equivalent to a monoidal action of $\mathrm{Perf}(B\widehat{G}^X)$.

The equivalence of the two constructions in this case is then a consequence of Yoneda lemma.

As in section 11.4.3 one can extend the result integrally over $\mathbb{Z}[\frac{1}{pN}]$ for some explicit integer N depending on G, see [**58**, Section X.3].

11.5. Recollection

At the end we obtain a monoidal action of $\mathrm{Perf}(\mathrm{SysLoc}_{\widehat{G}/\overline{\mathbb{Q}}_\ell})$ on $D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ or $\mathrm{Perf}(\mathrm{SysLoc}_{\widehat{G}/\overline{\mathbb{Z}}_\ell})$ on $D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$ for $l \nmid N$ with N an explicit integer. This produces a morphism of rings

$$\boxed{\mathcal{O}\left(Z^1(W,\widehat{G}) /\!\!/ \widehat{G}\right) \longrightarrow \underbrace{\mathfrak{Z}}_{\substack{\text{Bernstein}\\ \text{center}}} \left(D_{\text{lis}}(\text{Bun}_G, \overline{\mathbb{Q}}_\ell)\right).}$$

Let π be a smooth $\overline{\mathbb{Q}}_\ell$-irreducible representation of $G(E)$. Let $i^1 : [*/\underline{G(E)}] \hookrightarrow \text{Bun}_G$ be the inclusion of the semi-stable locus in Bun_G^0. Let $\mathscr{F}_\pi$ be the local system on $[*/\underline{G(E)}]$ associated to π. The preceding morphism of rings applied to $(i^1)_!\mathscr{F}_\pi$ produces a $\overline{\mathbb{Q}}_\ell$-valued character of the ring of functions on the coarse moduli space of Langlands parameters. Using proposition 11.3.4 we find the semi-simple parameter φ_π^{ss}.

11.6. En résumé

$$\begin{tikzcd}
& \mathrm{Bun}_G \arrow[d, squiggly, "\text{Hecke corr. upgrade}"] & & \\
& \mathrm{Hecke} \arrow[dl] \arrow[dr] & & \\
\mathrm{Bun}_G & {} \arrow[d, squiggly, "\text{Local Hecke stack shows up}"] & \mathrm{Bun}_G \times (\mathrm{Div}^1)^I & \\
& \mathrm{Hecke} \arrow[dl] \arrow[dr] \arrow[rr, "loc"] & & \mathcal{H}ecke \arrow[d] \\
\mathrm{Bun}_G & {} \arrow[d, squiggly, "\text{geometric Satake input: upgrade to a coho. corr.}"] & \mathrm{Bun}_G \times (\mathrm{Div}^1)^I \arrow[r, "\mathrm{proj}"] & (\mathrm{Div}^1)^I \\
& (\mathrm{Hecke}, loc^* S_W)_{W \in \mathrm{Rep}({}^L G)^I} \arrow[dl] \arrow[dr] & \begin{array}{c} \text{geo. Satake} \\ \mathrm{Rep}_\Lambda({}^L G)^I \xrightarrow{\sim} \mathrm{Sat}_I(G, \Lambda) \end{array} \arrow[l, squiggly] & \\
\mathrm{Bun}_G & {} \arrow[d, squiggly, "\text{action of coho. corr.}"] & \mathrm{Bun}_G \times (\mathrm{Div}^1)^I & \\
& F_I : \mathrm{Rep}_\Lambda({}^L G)^I \to \mathcal{E}nd\left(\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda) \times (\mathrm{Div}^1)^I\right) \arrow[d, squiggly, "\text{Drinfeld lemma}"] & & \\
& F_I : \mathrm{Rep}_\Lambda({}^L G)^I \to \mathcal{E}nd\left(\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda)\right)^{BW_E^I} \arrow[d, squiggly, dash] & & \\
& {} & &
\end{tikzcd}$$

$$\begin{tikzcd}[row sep=small]
 & {} \arrow[d, rightsquigarrow] & \\
 & \substack{\text{continuity action on compact objects}\\ +\mathcal{D}_{\text{ét}}(\mathrm{Bun}_G,\Lambda)^{\omega} \text{ stable under Hecke}} \arrow[d, rightsquigarrow] & \\
 & F_I : \mathrm{Rep}(\widehat{G} \rtimes Q)^I \longrightarrow \mathcal{E}nd(\mathcal{C})^{BW^I} \arrow[d, rightsquigarrow] & \\
 & \text{BW=animated colim of finite sets} \arrow[d, rightsquigarrow] & \\
\substack{\text{Jacobian criterion}\\ \text{local charts}\\ \pi_b : \mathcal{M}_b \to \mathrm{Bun}_G} \arrow[uuur, rightsquigarrow, bend left] \arrow[dddr, rightsquigarrow, bend right] & \text{spectral action on } \mathcal{C} \text{ of } \mathrm{Perf}\left(\underbrace{\mathrm{Hom}_{BQ}(BW, \widehat{G} \rtimes Q)}_{\infty\text{-derived-stack}}\right) \arrow[d, rightsquigarrow] & \\
 & \text{derived stack is underived} \arrow[d, rightsquigarrow] & \\
 & {} & \\
 & \text{spectral action on } \mathcal{C} \text{ of } \mathrm{Perf}([Z^1(W, \widehat{G})/\widehat{G}]) \arrow[d, rightsquigarrow] & \\
 & \mathcal{D}_{\text{ét}}(Bun_G, \Lambda) \text{ compactly generated} \arrow[d, rightsquigarrow] & \\
 & \text{spectral action on } \mathcal{D}_{\text{ét}}(\mathrm{Bun}_G, \Lambda) \text{ of } \mathrm{Perf}([Z^1(W_E, \widehat{G})/\widehat{G}]) \arrow[d, rightsquigarrow] & \\
 & \text{morphism } \mathcal{O}\left(Z^1(W_E, \widehat{G}) /\!\!/ \widehat{G}\right) \to \mathfrak{Z}\left(D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)\right) \arrow[dd, rightsquigarrow] & \substack{\text{Semi-orthogonal}\\ \text{decomposition}\\ \text{of } D_{\text{ét}}(\mathrm{Bun}_G, \Lambda) \text{ by}\\ \left(D(G_b(E), \Lambda)\right)_{[b] \in B(G)}} \arrow[dl, rightsquigarrow] \\
 & {} & \\
 & \text{morphism } \mathcal{O}\left(Z^1(W_E, \widehat{G}) /\!\!/ \widehat{G}\right) \to \mathfrak{Z}(D(G_b(E), \Lambda)) \arrow[d, rightsquigarrow] & \\
 & \substack{\text{geometric invariant theory}\\ \text{Mumford numerical criterion}\\ +\text{Richardson}} \arrow[d, rightsquigarrow] & \\
 & \text{semi-simple Langlands parameters} &
\end{tikzcd}$$

11.7. Final thoughts

The first appearance in the domain of factorization objects, i.e. the finite set I, is due to Beilinson and Drinfeld in the context of $\mathcal{D}$-modules ([**7**]). V. Lafforgue was the one to realize that their use in an étale context could be used to define semi-simple global Langlands parameters for function fields ([**91**]). The idea that one could build a spectral action is due to Gaitsgory. The ∞-categorical construction of this spectral action is in [**58**].

11.8. Some more final thoughts

The stack of Langlands parameters is locally complete intersection over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{p}])$. In the p-adic Langalands program another stack of parameters shows up: the Emerton-Gee stack ([**45**]) as a formal scheme over $\mathrm{Spf}(\mathbb{Z}_p)$. Some pieces of the Emerton-Gee stack are linked to $\mathrm{LocSys}_{\widehat{G}}$, typically $\mathfrak{X}_d^{ss,\underline{\lambda}}$ that is p-adic for any sequence of Hodge-Tate weights $\underline{\lambda}$ via the classification of semi-stable p-adic Galois representations in terms of filtered (φ, N)-modules.

On the other side, recent work ([**125**]) has shown that, using motivic objects, one can make the geometric Satake equivalence integral independent of ℓ.

One can hope to obtain a motivic version

$$D_{\mathrm{mot}}(\mathrm{Bun}_G, \mathbb{Z})$$

that allows us to glue the different $\ell \neq p$ and even goes into the p-adic Langlands direction.

Finally, one can hope that the ideas of [**58**] adapted in a non-perfect context, that is to say working with non-perfect prisms, may lead to some new interesting results.

PART II

THE JACOBIAN CRITERION OF SMOOTHNESS: A COURSE AT THE HAUSDORFF INSTITUTE

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial \mathbf{f}}{\partial x_1} & \cdots & \dfrac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \nabla^{\mathrm{T}} f_1 \\ \vdots \\ \nabla^{\mathrm{T}} f_m \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Preface

This are the notes of the course given by the author at the Hausdorff center at the occasion of the special program “the arithmetic of the Langlands program” in 2023. The author would like to thank all of the participants. The idea of this course is to take some part of [**58**] and explain it in details, together with some parts of [**128**] as an example of the techniques used by the authors.

LECTURE 1

THE JACOBIAN CRITERION OF SMOOTHNESS: PRESENTATION OF THE RESULT

The Jacobian criterion of smoothness is a key tool in our joint work with Scholze on the geometrization of the local Langlands correspondence.

This allows us to construct ”nice charts“ on Bun_G the stack of G-bundles on the curve. Here the charts are for the “smooth topology“ (we will precise later what this means).

1.1. Algebraic classical analog

Let X be a smooth projective curve over the field k. The datum is the following:

$$\begin{tikzcd} Z \arrow[d, "\text{quasi-projective smooth morphism of schemes}" description] \\ X \end{tikzcd}$$

From this we define a moduli space.

Definition 1.1.1. — *$\mathcal{M}_Z \longrightarrow$ Spec(k) represents the functor*

$$S \longmapsto \left\{ \textit{sections } s \quad \begin{tikzcd} & Z \arrow[d] \\ X \times_k S \arrow[ur, bend left, "s"] \arrow[r] & X \end{tikzcd} \right\}$$

It is easily verified that $\mathcal{M}_Z$ is representable by a quasi-projective k-scheme.

Example 1.1.2. — • *Let $\mathscr{E}$ be a vector bundle on X and $Z = \mathbb{V}(\mathscr{E})$ be its geometric realization. Then, $\mathcal{M}_Z$ is representable by the affine space $\mathbb{V}(H^0(X, \mathscr{E}))$.*

- *$\mathscr{E}$ = v.b. on X and $d \geq 1$ an integer. Let $Z = \mathrm{Gr}_d(\mathscr{E})$ be the Grassmianian of quotients of $\mathscr{E}$ that are locally free of rank d. Then*

$$\mathcal{M}_Z(S) = \Big\{ \textit{quotients of } \underbrace{\mathscr{E}_{|X\times_k S}}_{\textit{pullback of } \mathscr{E} \textit{ via } X\times_k S\to X} \textit{ that are locally free of rank } d \Big\}.$$

One has an open immersion

$$\mathcal{M}_Z \underset{open}{\subset} Quot_{\mathscr{E}/X/k}$$

where

$$Quot_{\mathscr{E}/X/k}(S) = \Big\{ \textit{coherent quotients of } \mathscr{E}_{|X\times_k S} \textit{ that are flat over } S \Big\}.$$

We now define what we call the smooth part of $\mathcal{M}_Z$.

Definition 1.1.3. — *We note $\mathcal{M}_Z^{sm} \subset \mathcal{M}_Z$ the open sub-functor defined by*

$$\mathcal{M}_Z^{sm}(S) = \Big\{ \textit{sections } s \textit{ satisfying: } \forall t \in S,\ H^1\big(X \otimes_k k(t), (s^* T_{Z/X})_{|X\otimes_k k(t)}\big) = 0 \Big\}$$

where

- *— $T_{Z/X}$ is the relative tangent bundle (that is well defined as a vector bundle on Z since $Z \to X$ is smooth)*
- *— $s^*T_{Z/X}$ is a vector bundle on $X \times_k S$*
- *— $(s^*T_{Z/X})_{|X\otimes_k k(t)}$ is the pullback of this vector bundle via $X \otimes_k k(t) \to X \times S$ via $Spec(k(t)) \to S$.*

The Jacobian criterion of smoothness in this context is the following proposition.

Proposition 1.1.4 **(Jacobian criterion of smoothness, classical case)**
The morphism

$$\mathcal{M}_Z^{sm} \longrightarrow Spec(k)$$

is smooth.

Proof. — Let

$$\begin{tikzcd} & Z \arrow[d] \\ X \times_k S \arrow[r] \arrow[ur, bend left, "s"] & X \end{tikzcd}$$

lie in $\mathcal{M}_Z^{sm}(S)$. Note

$$\pi : X \times_k S \longrightarrow S$$

the projection. Using

- $R\pi_*(s^*T_{Z/X}) \in \mathrm{Perf}^{[0,1]}(\mathcal{O}_S)$ (perfect complex with amplitude in $[0,1]$) since π is proper and flat,
- coupled with the vanishing condition fiberwise,

- coupled with the proper base change in coherent cohomology (aka Zariski formal function theorem: if S is noetherian then for $t \in S$, $R\pi_*(s^*T_{Z/X})_t \otimes^{\mathbb{L}}_{\mathcal{O}_{S,t}} \widehat{\mathcal{O}}_{S,t} = R\widehat{\pi}_{t*}\widehat{s^*T_{Z/X}}$ where $\widehat{\pi}_t : X\widehat{\otimes}_k\widehat{\mathcal{O}}_{S,t} \to \mathrm{Spf}(\widehat{\mathcal{O}}_{S,t}))$

one deduces that $R^1\pi_*(s^*T_{Z/X}) = 0$ and thus if S is affine then

$$H^1(X \times_k S, s^*T_{Z/X}) = 0.$$

Now, if $S \hookrightarrow S'$ is a nilpotent immersion of affine schemes defined by $\mathcal{I}$ with $\mathcal{I}^2 = (0)$ and we are looking for s' as in the following diagram

$$\begin{tikzcd}
X \times_k S' \arrow[r, dotted, "s'"] & Z \arrow[d] \\
X \times_k S \arrow[u, hook] \arrow[ur, "s"] \arrow[r] & X
\end{tikzcd}$$

the sheaf on $(X \times_k S)_{Zar}$ of liftings of s to s' is a $s^*T_{Z/X} \otimes_{\mathcal{O}_{X\times_k S}} \pi^*\mathcal{I}$-torsor. Using the projection formula, $R\pi_*(s^*T_{Z/X} \otimes_{\mathcal{O}_{X\times_k S}} \pi^*\mathcal{I}) = R\pi_*(s^*T_{Z/X}) \otimes^{\mathbb{L}} \mathcal{I}$, one has

$$H^1(X \times_k S, s^*T_{Z/X} \otimes_{\mathcal{O}_{X\times_k S}} \pi^*\mathcal{I}) = 0$$

and we conclude. ☐

Remark 1.1.5. — *We used the infinitesimal criterion for formal smoothness of Grothendieck. This is not available in the perfectoid world since there are no infinitesimals. This is why the proof of our Jacobian criterion of smoothness is much more difficult in the perfectoid world.*

1.2. Example of application of the classical Jacobian criterion

Let X/k be a smooth projective curve as before, $n \geq 1$, and

$$\mathrm{Bun}_n$$

the stack of rank n vector bundles on X,

$$\mathrm{Bun}_n(S) = \big\{\text{rank } n \text{ vector bundles on } X \times_k S\big\}$$

where the notation $\{\dots\}$ here means ”the groupoid of“ and not the set. Let $\mathcal{O}(1)$ be an ample line bundle on X and $r \geq 1, N \geq 0$ integers. Let

$$U_{r,N} \longrightarrow \mathrm{Spec}(k)$$

be the moduli whose values on S is a morphism

$$u : \mathcal{O}_{X\times_k S}(-N)^r \twoheadrightarrow \mathscr{E},$$

with $\mathscr{E}$ locally free of rank n s.t. fiberwise on S, $\underbrace{\mathscr{H}om(\ker u, \mathscr{E})}_{\text{vector bundle}}$ has no H^1. Then,

$$\big(U_{r,N} \longrightarrow \mathrm{Bun}_n\big)_{r,N}$$

is a set of smooth charts of Bun_n.

1.3. The Jacobian criterion of smoothness

1.3.1. Background on the curve ([57], [58, Chapter II]). — Let E be a local field and $\mathbb{F}_q = \mathcal{O}_E/\pi$ its residue field. We have either:

1. $[E:\mathbb{Q}_p] < +\infty$,
2. or $E = \mathbb{F}_q(\!(\pi)\!)$.

Let (R, R^+) be an $\mathbb{F}_q$-affinoid perfectoid algebra and

$$\underbrace{X_{R,R^+}}_{E\text{-analytic adic space}} = \text{adic curve over } \mathrm{Spa}(E) \text{ attached to } (R,R^+).$$

Recall that the adic curve is

$$\boxed{X_{R,R^+} = \underbrace{Y_{R,R^+}}_{\substack{\text{Stein } E\text{-analytic}\\ \text{adic space}}} /\varphi^{\mathbb{Z}}}$$

where

- the quotient $Y_{R,R^+} \longrightarrow Y_{R,R^+}/\varphi^{\mathbb{Z}}$ is for the analytic topology i.e. this is a local isomorphism,
- the group of deck transformations is $\varphi^{\mathbb{Z}}$.

One has

$$\boxed{Y_{R,R^+} = \mathrm{Spa}(W_{\mathcal{O}_E}(R^+), W_{\mathcal{O}_E}(R^+)) \smallsetminus V(\pi[\underbrace{\varpi}_{\substack{\text{pseudo}\\ \text{-uniformizer}}}])}$$

→ remove two divisors stable under φ to Y_{R,R^+}, (π) and $([\varpi])$, after removing those two fixed divisors the action of $\varphi^{\mathbb{Z}}$ is without fixed points totally discontinuous
— (π) = étale divisor,
— $([\varpi])$ = crystalline divisor.

Here

$$W_{\mathcal{O}_E}(R^+) = \text{ramified Witt vectors}$$

that is to say $R^+\hat{\otimes}_{\mathbb{F}_q}\mathcal{O}_E = R^+[\![\pi]\!]$ if $E = \mathbb{F}_q(\!(\pi)\!)$ (in equal characteristic the Teichmüller is additive, $[-] : R^+ \hookrightarrow W_{\mathcal{O}_E}(R^+)$), and the usual ramified Witt vectors if $E|\mathbb{Q}_p$. Moreover,

$$\varphi\left(\sum_{n\geq 0}[a_n]\pi^n\right) = \sum_{n\geq 0}[a_n^q]\pi^n.$$

There is an "ample" line bundle $\mathcal{O}_{X_{R,R^+}}(1)$ on X_{R,R^+} that is trivial when pulled back to Y_{R,R^+}. It corresponds to the φ-equivariant line bundle

$$\Big(\mathcal{O}_{Y_{R,R^+}}, \underbrace{\pi^{-1}\varphi}_{\substack{\text{Frobenius}\\ \text{structure}}}\Big)$$

on Y_{R,R^+}. Set

$$\underbrace{\mathbb{B}(R,R^+)}_{\substack{\text{Fontaine's}\\ \text{period ring}}} = \mathcal{O}\big(\underbrace{Y_{R,R^+}}_{\text{Stein}}\big).$$

Ampleness of $\mathcal{O}(1)$: Kedlaya and Liu have proven that $\forall \mathscr{E}$ a vector bundle on X_{R,R^+}, for $n \gg 0$, $\mathscr{E}(n)$ is generated by its global sections and $H^1(X_{R,R^+}, \mathscr{E}(n)) = 0$. See [**58**, Section II.2.6] for example where we retake Kedlaya-Liu's arguments.

If

$$\underbrace{P}_{\substack{\text{graded alg.}\\ \text{of periods}}} = \bigoplus_{d\geq 0} \underbrace{H^0\big(X_{R,R^+}, \mathcal{O}(d)\big)}_{\mathbb{B}(R,R^+)^{\varphi=\pi^d}=\{f \mid \varphi(f)=\pi^d f\}}$$

set

$$\mathfrak{X}_{R,R^+} = \mathrm{Proj}(P) = \text{“algebraic curve”}.$$

<u>Fact</u>: there is a canonical morphism of ringed spaces

$$X_{R,R^+} \longrightarrow \mathfrak{X}_{R,R^+}$$

that induces a GAGA equivalence by pullback

$$\big\{\text{vector bundles on } \mathfrak{X}_{R,R^+}\big\} \xrightarrow{\sim} \big\{\text{vector bundles on } X_{R,R^+}\big\}$$

This morphism of ringed spaces is constructed in the following way: for $t \in \mathbb{B}(R,R^+)^{\varphi=\pi}$

$$Y_{R,R^+} \setminus V(t) \longrightarrow \mathrm{Spec}\big(\mathbb{B}(R,R^+)[\tfrac{1}{t}]^{\varphi=\mathrm{Id}}\big) \underbrace{\hookrightarrow}_{\substack{\text{affine}\\ \text{open}}} \mathfrak{X}_{R,R^+}$$

is induced by

$$\mathbb{B}(R,R^+)[\tfrac{1}{t}]^{\varphi=\mathrm{Id}} \hookrightarrow \mathbb{B}(R,R^+)[\tfrac{1}{t}] \longrightarrow \mathcal{O}\big(Y_{R,R^+} \setminus V(t)\big).$$

When t varies

$$Y_{R,R^+} = \bigcup_t Y_{R,R^+} \setminus V(t)$$

and this glues to a morphism of ringed spaces

$$Y_{R,R^+} \longrightarrow \mathfrak{X}_{R,R^+}$$

that is φ-invariant. This defines our morphism of ringed spaces.

Remark 1.3.1. — *When we say that $\mathbb{B}(R,R^+)$ is a Fontaine's period ring we can give a more precise content to this sentence. Suppose $E=\mathbb{Q}_p$. Let $(R^\sharp,R^{\sharp,+})$ be an untilt of (R,R^+). There is then a natural morphism*

$$\mathbb{B}(R,R^+)\longrightarrow B^+_{cris}(R^{\sharp,+}):=H^0(Spec(R^{\sharp,+}/p)/Spec(\mathbb{Z}_p),\mathcal{O}_{cris})\left[\tfrac{1}{p}\right].$$

inducing an isomorphism $\mathbb{B}(R,R^+)^{\varphi=\pi^d}\xrightarrow{\sim}B^+_{cris}(R^{\sharp,+})^{\varphi=\pi^d}$ for all $d\geq 0$.

1.4. The Jacobian criterion

Let us now come to the main result we're interested in. Consider a diagram

$$\begin{tikzcd}
Z \arrow[r, hook, "\exists"] \arrow[d, "\substack{\text{smooth morphism}\\ \text{of sous-perfectoid spaces}}"'] & \mathbb{P}^n_{X_{R,R^+}} \arrow[dl] & \substack{\text{analytically open}\\ \text{inside Zariski closed}} \arrow[d] \arrow[ll, bend right] \\
X_{R,R^+} & & \substack{\text{quasi-projectivity}\\ \textit{assumption}}
\end{tikzcd}$$

In fact there is a good notion of smooth morphisms of sous-perfectoid spaces ([**58**, Sectiob IV.4.1]). Recall in fact that X_{R,R^+} is not perfectoid but

$$X_{R,R^+}\hat{\otimes}_E\widehat{\overline{E}}$$

is perfectoid. More generally, by definition, (A,A^+) is sous-perfectoid if there exists a morphism $(A,A^+)\to(B,B^+)$ with (B,B^+) affinoid perfectoid and an A-linear continuous section $\begin{tikzcd} A \arrow[r] & B \arrow[l, bend right] \end{tikzcd}$. Sous-perfectoid implies sheafy and they are heavily used in the theory.

Remark 1.4.1. — *The sous-perfectoid space X_{R,R^+} has a nice formula for its diamond. Namely,*

$$X_S^\diamond=(X_S\hat{\otimes}_E\widehat{\overline{E}})^\flat/\underline{Gal(\overline{E}|E)}$$

is canonically identified with

$$\underbrace{S\times_{\mathrm{Spa}(\overline{\mathbb{F}}_q)}\mathrm{Spd}(E)/\varphi_S^{\mathbb{Z}}}_{Y_S^\diamond}\times\mathrm{Id}$$

Here is now the main object of our study.

Definition 1.4.2. — *Define*

$$\begin{aligned} \mathcal{M}_Z : \mathrm{Perf}_{R,R^+} &\longrightarrow \mathrm{Sets} \\ S &\longmapsto \left\{ \textit{sections } s \quad \begin{tikzcd} & Z \arrow[d] \\ X_S \arrow[ur, bend left, "s"] \arrow[r] & X_{R,R^+} \end{tikzcd} \right\} \end{aligned}$$

Remark 1.4.3. — *If Z is Zariski closed inside some $\mathbb{P}^n_{X_{R,R^+}}$ it is of the form $\mathfrak{Z}^{ad}$ for some Zariski closed $\mathfrak{Z} \to \mathfrak{X}_{R,R^+}$ (GAGA). Then, GAGA applies to give for $S = \mathrm{Spa}(A, A^+)$ affinoid perfectoid with a morphism $S \to \mathrm{Spa}(R, R^+)$*

$$\left\{ \textit{sections } s \quad \begin{tikzcd} & \mathfrak{Z} \arrow[d] \\ \mathfrak{X}_{A,A^+} \arrow[ur, bend left, "s"] \arrow[r] & \mathfrak{X}_{R,R^+} \end{tikzcd} \right\} \xrightarrow{\sim} \left\{ \textit{sections } s \quad \begin{tikzcd} & Z \arrow[d] \\ X_S \arrow[ur, bend left, "s"] \arrow[r] & X_{R,R^+} \end{tikzcd} \right\}$$

This means that in this case we can compute $\mathcal{M}_Z$ using the adic or the algebraic curve.

The first basic result is the following.

Proposition 1.4.4. — *$\mathcal{M}_Z$ is representable by a locally spatial diamond with*

$$\mathcal{M}_Z \longrightarrow \mathrm{Spa}(R, R^+)$$

compactifiable of finite dim trg.

The "compactifiable of finite dim trg." property will be explained later. We now define, by analogy with the "classical case", the smooth locus.

Definition 1.4.5. — *$\mathcal{M}_Z^{sm} \subset \mathcal{M}_Z$ is the open sub-diamond such that*

$$\mathcal{M}_Z^{sm}(S) = \left\{ \begin{matrix} \textit{sections } s \textit{ s.t. } \forall t \in S, \ \big(\underbrace{s^* \overbrace{T_{Z/X_{R,R^+}}}^{v.b.\ on\ Z}}_{v.b.\ on\ X_S} \big)_{|X_{K(t),K(t)^+}} \\ \textit{has} > 0 \textit{ H.N. slopes} \end{matrix} \right\}.$$

Here we use the fact that for a smooth morphism of sous-perfectoid spaces one can define its relative tangent bundle as a vector bundle.

The fact that $\mathcal{M}_Z^{sm}$ is open inside $\mathcal{M}_Z$ is a consequence of the semi-continuity of the H.N. polygon of a vector bundle on the curve, see [**58**, Section II.2].

Remark 1.4.6 **(link with the classical Jacobian criterion of smoothness)**
For (K, K^+) an $\mathbb{F}_q$-affinoid perfectoid field and $\mathscr{E}$ a vector bundle on X_{K,K^+}, one has

$$\mathscr{E} \text{ has } > 0 \text{ H.N. slopes } \Leftrightarrow \forall (L, L^+)|(K, K^+),\ H^1(X_{L,L^+}, \mathscr{E}_{|X_{L,L^+}}) = 0$$

where (L, L^+) is an affinoid perfectoid field.

The purpose of those lectures is then to prove the following.

Theorem 1.4.7 (**Jacobian criterion of smoothness**)
The morphism

$$\mathcal{M}_Z^{sm} \longrightarrow \operatorname{Spa}(R, R^+)$$

is ℓ-cohomologically smooth of dimension $\deg(s^*T_{Z/X_{R,R^+}})$ at a section s.

We will explain later what ℓ-cohomologically smooth means. This roughly means that some form of relative Poincaré duality is satisfied for ℓ-torsion étale cohomology.

Example 1.4.8. — 1. *Let $\mathscr{E}$ be a vector bundle on X_{R,R^+} and $Z = \mathbb{V}(\mathscr{E})$. Then*

$$\mathcal{M}_Z = BC(\mathscr{E})$$

and

$$\mathcal{M}_Z^{sm} = BC(\mathscr{E}) \times_{\operatorname{Spa}(R,R^+)} U$$

where $U \subset \operatorname{Spa}(R, R^+)$ is the open subset where $\mathscr{E}$ has > 0 H.N. slopes.

2. *Let $\mathscr{E} =$ be a vector bundle on X_{R,R^+} and $Z = \mathbb{P}(\mathscr{E})$. One has a decomposition*

$$\mathcal{M}_Z = \coprod_{d \in \mathbb{Z}} \mathcal{M}_Z^d$$

where $\mathcal{M}_Z^d$ is the open/closed subset where $s^*\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1)$ has degree d ($s =$section). One has moreover

$$\begin{array}{ccc} \overbrace{\mathcal{P}ic^d}^{\substack{\text{Picard stack} \\ \text{of deg. } d \text{ line bundles}}} & \xrightarrow{\sim} & \overbrace{[*/\underline{E}^\times]}^{\substack{\text{classifying stack} \\ \text{of pro-étale } \underline{E}^\times\text{-torsors}}} \\ \mathscr{L} & \longmapsto & \operatorname{Isom}(\mathcal{O}(d), \mathscr{L}). \end{array}$$

From this we deduce that

$$\boxed{\mathcal{M}_Z^d \simeq U/\underline{E}^\times \subset \underbrace{\overbrace{BC\big(\mathscr{E}^\vee(d)\big) \smallsetminus \{0\}}^{\substack{\text{punctured} \\ \text{BC- space}}}/\underline{E}^\times}_{\substack{\text{"projective space"} \\ \text{associated to a BC space}}}}$$

where $U \subset BC\big(\mathscr{E}^\vee(d)\big) \smallsetminus \{0\}$ *is the open sub-diamond defined by*

$$\begin{tikzcd}
U(S) \arrow[d, equal] \\
\Big\{u : \mathscr{E}_{|X_S} \to \mathcal{O}_{X_S}(d) \ \Big| \ \forall t \in S,\ u_{|X_{K(t),K(t)^+}} : \mathscr{E}_{|X_{K(t),K(t)^+}} \to \mathcal{O}_{X_{K(t),K(t)^+}}(d) \textit{ is surjective}\Big\} \arrow[d, hook] \\
\Big\{u : \mathscr{E}_{|X_S} \to \mathcal{O}_{X_S}(d) \ \Big| \ \forall t \in S,\ u_{|X_{K(t),K(t)^+}} : \mathscr{E}_{|X_{K(t),K(t)^+}} \to \mathcal{O}_{X_{K(t),K(t)^+}}(d) \textit{ is non-zero}\Big\} \arrow[d, equal] \\
\big(BC\big(\mathscr{E}^\vee(d)\big) \smallsetminus \{0\}\big)(S)
\end{tikzcd}$$

3. *Let*

$$Z = X_{R,R^+} \times_{\mathrm{Spa}(E)} W$$

where $W \subset \mathbb{P}^n_E$ *is smooth equal to* $V(f_i)_{i\in I}$, f_i = *homogeneous polynomial in* $n+1$*-variables with coefficients in* E. *Then,*

$$\mathcal{M}_Z = \coprod_{d\in\mathbb{N}} \mathcal{M}_Z^d$$

where

$$\mathcal{M}_Z^d = \Big\{(x_0,\dots,x_n) \in U \subset \big(\mathbb{B}^{\varphi=\pi^d}\big)^{n+1} \smallsetminus \{(0,\dots,0)\} \mid \underbrace{\forall i \in I,\ f_i(x_0,\dots,x_n) = 0}_{\textit{alg. equation in a BC space}}\Big\}/\underline{E}^\times$$

where U *is the open subset of* $\big(\mathbb{B}^{\varphi=\pi^d}\big)^{n+1} \smallsetminus \{(0,\dots,0)\}$ *defined as before:* $U(S)$ *is the set of*

$$(x_0,\dots,x_n) \in \big(\mathbb{B}(S)^{\varphi=\pi^d}\big)^{n+1}$$

satisfying

$$\forall \textit{aff. perf. field } (K,K^+)|E,\ \forall \mathrm{Spa}(K^\flat, K^{\flat,+}) \to S,$$

$$(\theta_K(x_0),\dots,\theta_K(x_n)) \in K^{n+1} \smallsetminus \{(0,\dots,0)\}$$

where $\theta_K : \mathbb{B}(K^\flat, K^{\flat,+}) \to K$ *is Fontaine's* θ *map.*

Thus:

- The Banach-Colmez spaces are the linear objects of the category of diamonds, the analogs of affine spaces. For $Z = \mathbb{V}(\mathscr{E})$ the Jacobian criterion of smoothness is ”easy“, established first much before the Jacobian criterion ([**58**, Section II.3]).
- Here we look more generally at solutions of algebraic equations inside Banach-Colmez spaces.
- This is already needed if we take $Z = \mathrm{Gr}_d(\mathscr{E})$ with $d > 1$ since the Plücker embedding of $\mathrm{Gr}_d(\mathscr{E})$ inside $\mathbb{P}(\wedge^d \mathscr{E})$ is defined by quadratic equations.

1.5. The application to Bun_G

Here is how we use the Jacobian criterion in our work ([**58**, Section V.3]). Let G be a reductive group over E and P a parabolic subgroup of G.

Let $\mathscr{E}$ be a G-bundle on X_{R,R^+} (one can give a meaning to this as an étale G^{ad}-torsor over X_{R,R^+}, at the end this is the same as an étale G-torsor over the scheme $\mathfrak{X}_{R,R^+}$, the "algebraic curve"). Take

$$Z = P\backslash\mathscr{E} \longrightarrow X_{R,R^+}.$$

Then,

$$\mathcal{M}_Z = \text{moduli of reductions of } \mathscr{E} \text{ to } P$$

i.e.

$$\mathcal{M}_Z(S) = \Big\{\mathscr{E}_P \text{ a } P\text{-torsor} + \text{iso. } \mathscr{E}_P \overset{P}{\times} G \xrightarrow{\sim} \mathscr{E}\Big\}.$$

One then has

$$\mathcal{M}_Z^{sm}(S) = \Big\{\mathscr{E}_P \mid \underbrace{\mathscr{E}_P \overset{P}{\times} \mathfrak{g}/\mathfrak{p}}_{\text{v.b.}} \text{ has } > 0 \text{ H.N. slopes fiberwise}/S\Big\}$$

where $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{p} = \mathrm{Lie(P)}$ and $P \to \mathrm{GL}(\mathfrak{g}/\mathfrak{p})$ is given by the adjoint representation.

The Jacobian criterion implies the following: the morphism

$$\begin{array}{ccc} \mathrm{Bun}_P^{sm} & \longrightarrow & \mathrm{Bun}_G \\ \mathscr{E}_P & \longmapsto & \mathscr{E}_P \overset{P}{\times} G \end{array}$$

is ℓ-coho. smooth where Bun_P^{sm} is the open substack of Bun_P formed by P-bundles $\mathscr{E}_P$ such that $\mathscr{E}_P \overset{P}{\times} \mathfrak{g}/\mathfrak{p}$ has > 0 H.N. slopes fiberwise.

This is the result we use to construct the local charts $\pi_b : \mathcal{M}_b \to \mathrm{Bun}_G$ in [**58**, Section V.3]. Let us finish with a remark.

Remark 1.5.1. — *One can imagine that "$X_S = X \times S$" although this formula has no meaning: the curve exists only after pullback to any $\mathbb{F}_q$-perfectoid space S but not absolutely over* $\mathrm{Spa}(\mathbb{F}_q)$. *Still, this is a good way to think about the situation by analogy with the classical case of usual algebraic curves.*

LECTURE 2

ÉTALE/PRO-ÉTALE/v-SHEAVES

We now discuss the Grothendieck topologies showing up in the domain and the associated cohomological formalism.

2.1. We lied to you

We explain here that, in general, the usual unbounded derived categories of sheaves are not the good objects to consider bu rather their left completion.

Let us explain what this ”left completeness means”. For this left X be a (Grothendieck) topos and Λ a ring. We can consider the following triangulated categories

$$\underbrace{D^+(X,\Lambda)}_{\text{good object}} \subset \underbrace{\qquad\overbrace{D(X,\Lambda)}^{\substack{\text{usual derived category}\\ \text{of complexes of } \Lambda\text{-modules on } X}}\qquad}_{\substack{\text{not good object in general, does}\\ \text{not satisfy hyperdescent in general, not left complete}}}$$

Let

$$X^{\mathbb{N}}$$

be the topos of projective systems of objects of X i.e. functors $(\mathbb{N},\leq) \to X$. For each integer $n \geq 0$ there is a ”stage n“ morphism of topoi $i_n : X \to X^{\mathbb{N}}$ with $i_n^{-1}(U_k)_{k\geq 0} = U_n$ where $\cdots \to U_k \to \cdots U_1 \to U_0$ is a projective system. We now take the following definition.

Definition 2.1.1. — *We note*

$$\widehat{D}(X,\Lambda) \subset D(X^{\mathbb{N}},\Lambda)$$

for the sub-category of $A \in D(X^{\mathbb{N}},\Lambda)$ *satisfying*

1. $\forall n \geq 0,\ \underbrace{A_n}_{i_n^{-1}A} \in D^{\geq -n}(X,\Lambda)$,
2. $\forall n \geq 0,\ \tau_{\geq -n}A_{n+1} \underbrace{\xrightarrow{\sim}}_{\substack{in\ D^+(X,\Lambda)\\ i.e.\ quasi\text{-}iso.}} A_n$

There are two adjoint functors

$$\begin{tikzcd}\widehat{D}(X,\Lambda) \arrow[r, shift left, "R\varprojlim"] & D(X,\Lambda) \arrow[l, shift left, "\tau"]\end{tikzcd}$$

where

- $R\varprojlim$ is the derived functor of projective limits from Λ-modules in $X^{\mathbb{N}}$ to Λ-modules in X,
- τ sends A to the projective system of truncations $(\tau_{\geq -n}A)_{n\geq 0}$.

We now have the following definition.

Definition 2.1.2. — *1. The category* $D(X,\Lambda)$ *is left complete if those two adjoint functors are equivalences i.e.*

$$\forall A \in D(X,\Lambda),\quad A \xrightarrow{\sim} R\varprojlim_{n\geq 0} \tau_{\geq -n}A$$

2. The category $\widehat{D}(X,\Lambda)$ *is called the left completion of* $D(X,\Lambda)$.

Remark 2.1.3. — *Although canonically defined as a composite of two functors applied to* A, $R\varprojlim_{n\geq 0}\tau_{\geq -n}A$ *can be though of as a homotopy limit*

$$h\varprojlim_{n\geq 0} \tau_{\geq -n}A$$

where by definition a homotopy limit of a projective system $(B_n)_{n\geq 0}$ *in a triangulated category admitting countable products is* $C(f)[-1]$ *where* $C(f)$ *is a cone of* $f : \prod_{n\geq 0} B_n \to \prod_{n\geq 0} B_n$ *sending* $(x_n)_{n\geq 0}$ *to* $(u_{n+1}(x_{n+1}) - x_n)_{n\geq 0}$, $u_{n+1} : B_{n+1} \to B_n$. *One can thus give a meaning to the notion of a left complete triangulated category equipped with a* t*-structure and admitting countable products.*

For the next proposition recall that the topos X is replete if for any projective system $(\mathscr{F}_n)_{n\geq 0} \in X^{\mathbb{N}}$ such that for all n, $\mathscr{F}_{n+1} \to \mathscr{F}_n$ is surjective,

$$\varprojlim_{n\geq 0} \mathscr{F}_n \longrightarrow \mathscr{F}_0$$

is surjective (where we use the word surjective in the topos X as an abuse of terminology for epimorphism). For example, if k is a field with $[k^s : k] = +\infty$ then $\widetilde{\mathrm{Spec}(k)}_{\text{ét}}$ is not replete. In fact, if $(k_i)_{i\geq 0}$ is a sequence of finite degree separable extensions of k with $k_{i+1}|k_i$ then

$$\emptyset = \varprojlim_{i\geq 0} \mathrm{Spec}(k_i) \longrightarrow \mathrm{Spec}(k)$$

is not surjective in $\widetilde{\mathrm{Spec}(k)}_{\text{ét}}$. The topos of sets is replete. The following proposition is well known, this is particular case of some standard convergence results of Postnikov towers (see [**101**, Proposition 7.2.1.10] for example).

Proposition 2.1.4. — *1. (finite coho. dim. $\Rightarrow$ left complete) If $X = \widetilde{S}$ with S a site and $\forall \mathscr{F} \in X$ a Λ-module, $\exists d \geq 0$ s.t. $\forall U \in S$ $\exists (V_i \to U)_{i\in I}$ a cover in S s.t.*

$$\forall i \in I,\ H^q(V_i, \mathscr{F}) = 0 \text{ for } q > d$$

then $D(X, \Lambda)$ is complete.

2. (replete implies left complete) If X is replete then $D(X, \Lambda)$ is left complete.

Example 2.1.5. — *If X is a finite type k-scheme, $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$, ℓ any prime number, and $\mathrm{cd}_\ell(k) < +\infty$ then $D(X_{\text{ét}}, \Lambda)$ is left complete. This is for example the case when k is algebraically closed or when k is a finite field.*

Finally, left completeness has the following advantage. Let $f : X \to Y$ be a morphism of topoi. Then the usual derived functor $Rf_* : D^+(X, \Lambda) \to D^+(Y, \Lambda)$ extends to a functor

$$\widehat{D}(X, \Lambda) \xrightarrow{Rf_*} \widehat{D}(Y, \Lambda) \xrightarrow{R\varprojlim} D(Y, \Lambda)$$

and thus if $D(X, \Lambda)$ is left complete we obtain a functor

$$Rf_* : D(X, \Lambda) \to D(Y, \Lambda)$$

that is right adjoint to f^*.

Finally, left completness is evidently interpreted in terms of ∞-categories. In fact, if $\mathcal{D}(X, \Lambda)$ is the stable ∞-category with homotopy category $D(X, \Lambda)$ one has

$$\widehat{D}(X, \Lambda) = \underbrace{\mathrm{Ho}}_{\text{homotopy cat.}} \underbrace{\lim_{n\geq 0}}_{\substack{\text{limit in} \\ \text{the } \infty\text{-cat.} \\ \text{of stable } \infty\text{-cat.}}} \mathcal{D}^{\geq -n}(X, \Lambda).$$

One of the advantages of left completed derived categories that explains why they are the good objects is the following.

Proposition 2.1.6. — *The correspondence* $\operatorname{Ob} X \ni U \mapsto \widehat{\mathcal{D}}(U,\Lambda)$ *is an hypersheaf of stable* ∞*-categories.*

This means that if $U_\bullet \to V$ is an hypercover in the topos X then

$$\widehat{\mathcal{D}}(V,\Lambda) \xrightarrow{\sim} \lim_{[n]\in\Delta} \widehat{\mathcal{D}}(U_n,\Lambda)$$

is an equivalence.

2.2. Étale/pro-étale/v-topology for perfectoid spaces

2.2.1. The étale topology on perfectoid spaces. —

2.2.1.1. *Finite étale morphisms of perfectoid spaces.* — *Recall*: there is a good notion of vector bundles on perfectoid spaces. Let X be a perfectoid space. By definition a vector bundle on X is a locally free of finite rank $\mathcal{O}_X$-module. Then, if $X = \operatorname{Spa}(A, A^+)$ is affinoid perfectoid

$$\Gamma(X,-) : \{\text{v.b. on } X\} \xrightarrow{\sim} \{A\text{-modules that are projective of finite type}\}$$

with inverse given by $P \mapsto P\otimes_A \mathcal{O}_X$. This property is what we call "a good notion".

From this and the purity theorem we deduce that there is a good notion of finite étale morphisms of perfectoid spaces such that for X perfectoid,

$$\big\{\text{finite étale perf. spaces}/X\big\} \xrightarrow{\sim} \left\{\begin{array}{c}\text{finite locally free } \mathcal{O}_X\text{-algebras } \mathcal{A} \text{ s.t.}\\ \text{the quad. form } \operatorname{tr}_{\mathcal{A}/\mathcal{O}_X} : \mathcal{A}\times\mathcal{A}\to\mathcal{O}_X\\ \text{is perfect}\end{array}\right\}$$

where here by perfect we mean $\mathcal{A} \xrightarrow{\sim} \mathcal{A}^\vee$. Moreover, with this definition GAGA applies: for (A, A^+) affinoid perfectoid

$$\big\{\text{finite étale}/\operatorname{Spec}(A)\big\} \xrightarrow{\sim} \big\{\text{finite étale}/\operatorname{Spa}(A,A^+)\big\}.$$

2.2.1.2. *A result by Huber.* — The following result by Huber about étale morphisms of noetherian analytic adic spaces is a key point for the definition of an étale morphism of perfectoid spaces. For morphisms of analytic noetherian adic spaces there is a "good" notion of étale and smooth morphisms analogous to the one for morphisms of schemes using the infinitesimal criterion for formal smoothness couples with some locally (topologically) of finite type hypothesis ([**78**, Chapter 1]).

***Proposition 2.2.1* (Huber, see** [**78**, Lemma 2.2.8]**)**

Let $f : X \to Y$ be a morphism between adic spectra of strongly noetherian Tate rings. Then, for any $y \in Y$ there exists a nbd. V of y and a factorization in the category of Noetherian analytic adic spaces

$$\begin{tikzcd}
f^{-1}(V) \arrow[r, hook, "\textit{open immersion}"] \arrow[d, "f"'] & W \arrow[dl, "\textit{finite étale}"] \\
V &
\end{tikzcd}$$

sketch of proof. — Write $X = \mathrm{Spa}(B, B^+)$ and $Y = \mathrm{Spa}(A, A^+)$. General results about étale morphisms of analytic noetherian adic spaces show that one can write

$$B = A\langle X_1, \dots, X_n\rangle/(f_1, \dots, f_n)$$

with

$$\det\left(\frac{\partial f_i}{\partial X_j}\right)_{1\le i,j\le n} \mod (f_1, \dots, f_n) \in B^\times$$

(and B^+ is the integral closure of the image of $A^+\langle X_1, \dots, X_n\rangle$). An approximation result “à la Elkik” ([**44**]) shows that if we “modify slightly” the equations $f_1, \dots, f_n$ then the obtained topological A-algebra B' is isomorphic to B. We can thus suppose that

$$f_1, \dots, f_n \in A[X_1, \dots, X_n]$$

(this is a typical algebraization result). Up to replacing $A[X_1, \dots, X_n]/(f_1, \dots, f_n)$ by its localization with respect to the image of an element of $1 + A^{\circ\circ}[X_1, \dots, X_n]$ we can suppose that the Jacobian becomes invertible and thus that we have a finite type étale A-algebra C with a finite set of elements $g_1, \dots, g_n \in C$ such that

$$X = \big\{x \in \mathrm{Spec}(C)^{ad} \mid |g_1(x)| \le 1, \dots, |g_n(x)| \le 1\big\}.$$

where $(-)^{ad}$ is the analytification functor

$$(-)^{ad} : \left\{\begin{matrix}\text{finite type}\\ \mathrm{Spec}(A)\text{-schemes}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{loc. of finite type}\\ \mathrm{Spa}(A, A^+)\text{-adic spaces}\end{matrix}\right\}$$

that sends $\mathbb{A}^n_{\mathrm{Spec}(A)}$ to $\mathbb{A}^{n,ad}_{\mathrm{Spa}(A,A^+)}$ (affine schematical space to affine adic space). Since $\mathcal{O}_{Y,y}$ is Henselian, the étale $\mathrm{Spec}(\mathcal{O}_{Y,y})$-scheme

$$\mathrm{Spec}(C) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\mathcal{O}_{Y,y})$$

splits as a disjoint union of a finite étale scheme together with an étale scheme over $\mathrm{Spec}(\mathcal{O}_{Y,y})$ with image in $\mathrm{Spec}(\mathcal{O}_{Y,y})$ not containing the closed point. Since $\mathcal{O}_{Y,y} = \varinjlim_{V \ni y} \mathcal{O}(V)$ with V a rational neighborhood of y, a finite presentation argument shows that up to replacing (A, A^+) by a rational localization (A', A'^+) such that $y \in \mathrm{Spa}(A', A'^+)$, and C by $C \otimes_A A'$, we can suppose that

$$\mathrm{Spec}(C) = \mathrm{Spec}(C_1) \coprod \mathrm{Spec}(C_2)$$

with

1. $\mathrm{Spec}(C_2) \to \mathrm{Spec}(A)$ finite étale,
2. and $\mathrm{supp}(y) \notin \mathrm{Im}\big(\mathrm{Spec}(C_2) \to \mathrm{Spec}(A)\big)$.

Now, the image of $\mathrm{Spec}(C_2)^{ad} \cap \big\{|g_1| \leq 1, \dots, |g_n| \leq 1\big\}$ in $\mathrm{Spa}(A, A^+)$ is a quasi-compact open subset and thus, since pro-constructible, its closure is its set of specializations in the analytic adic space $\mathrm{Spa}(A, A^+)$. Since in an analytic affinoid adic space $z_1 \geq z_2 \Rightarrow \mathrm{supp}(z_1) = \mathrm{supp}(z_2)$, y does not lie in this closure. Up to another rational localization we can thus suppose that

$$\mathrm{Spec}(C_2)^{ad} \cap \{|g_1| \leq 1, \dots, |g_n| \leq 1\} = \emptyset.$$

The morphism f then factorizes as

$$\begin{tikzcd}
\mathrm{Spec}(C_1)^{ad} \cap \{|g_1| \leq 1, \dots, |g_n| \leq 1\} \arrow[r, hook, "\text{open immersion}"] & \mathrm{Spec}(C_1)^{ad} \arrow[d, "\text{finite étale}"] \\
& \mathrm{Spa}(A, A^+). \quad \square
\end{tikzcd}$$

Thus, the fact that such a result is true for noetherian analytic adic spaces but not for schemes is, at the end, a consequence of the fact that the local rings are Henselian. For schemes, Zariski's main theorem says that we can only find a compactification of a spearated étale morphism that is finite but no étale in general.

2.2.1.3. *Étale morphisms of perfectoid spaces.* — Motivated by Huber's definition we take the following.

Definition 2.2.2. — *A morphism of perfectoid spaces $X \to Y$ is étale if locally on X and Y it can be written as a composite*

$$\begin{tikzcd}
X \arrow[r, hook, "\textit{open immersion}"] \arrow[d] & \overline{X} \arrow[dl, "\textit{finite étale morphism of perf. spaces}"] \\
X &
\end{tikzcd}$$

This is a good definition. Typically:

- étale morphisms are open
- a morphism between étale X-perfectoid spaces, X perfectoid, is étale.

This type of result is reduced via the tilting equivalence to characteristic p. For X affinoid perfectoid space of char.p, if $X = \varprojlim_i X_i$ with X_i affinoid top. of finite type over $\mathrm{Spa}(\mathbb{F}_p(\!(\varpi)\!))$, and the limit is cofiltered, then

$$2 - \varinjlim_i \{\text{qc qs, étale}/X_i\} \xrightarrow{\sim} \{\text{qc qs, étale}/X\}.$$

2.2.1.4. *The étale site of a perfectoid space.* — The following definition is now evident.

Definition 2.2.3. — *X perfectoid space*

$$X_{\text{ét}} := \{\textit{small étale site of étale perf. spaces over } X\}.$$

Coverings: families $(U_i \to V)_i$ such that $V = \cup_i \operatorname{Im}(U_i \to V)$.

We now set the following. Here $\Lambda =$ is a torsion ring.

Definition 2.2.4. —

$$\begin{array}{rcl} D^+_{\text{ét}}(X,\Lambda) & := & D^+(X_{\text{ét}},\Lambda) \\ D_{\text{ét}}(X,\Lambda) & := & \widehat{D}(X_{\text{ét}},\Lambda) \ \ (\textit{left completion of } D(X_{\text{ét}},\Lambda)) \end{array}$$

We will give later a "geometric incarnation of $D_{\text{ét}}(X,\Lambda)$", i.e. not using the abstract left completion process, using the pro-étale topology on X.

2.2.2. The pro-étale topology on perfectoid spaces ([128, Section 8]). —

2.2.2.1. *Pro-étale morphisms.* —

Recall: The category $\underbrace{\text{Aff. Perf.}}_{\text{affinoid perfetoid spaces}}$ admits cofiltered limits:

$$\varprojlim_i \operatorname{Spa}(A_i, A_i^+) = \operatorname{Spa}(A_\infty, A_\infty^+)$$

where if $\varpi =$ image of a p.u. of some A_i,

$$A_\infty^+ = \widehat{\varinjlim_i A_i^+}, \quad A_\infty = A_\infty^+[\tfrac{1}{\varpi}]$$

(ϖ-adic completion).

Of course this not true for "classical" Noetherian analytic adic spaces that don't admit such limits; perfectoid spaces are much more flexible. This is one of the advantages of perfectoid spaces.

Definition 2.2.5. — *1. A morphism of affinoid perfectoid spaces is affinoid pro-étale if it can be written as*

$$\varprojlim_{i\geq i_0} X_i \longrightarrow X_{i_0}$$

where

- *(a) the limit is cofiltered with smallest index i_0,*
- *(b) each X_i is affinoid perfectoid,*
- *(c) the transition morphisms in the projective system are étale.*

2. A morphism $X \to Y$ is pro-étale if locally on X and Y it is affinoid pro-étale.

Example 2.2.6. — *For X perfectoid and $x \in X$, the loalization of X at x,*

$$\mathrm{Spa}(K(x), K(x)^+) = \varprojlim_{U\ni x} U \hookrightarrow X$$

is pro-étale. In particular, in general, pro-étale morphisms, even the surjective one, are not open. For example,

$$\coprod_{x\in X} \mathrm{Spa}(K(x), K(x)^+) \longrightarrow X$$

is pro-étale surjective but not open in general.

Example 2.2.7. — *For $X = \mathrm{Spa}(A, A^+)$ aff. perf. and $I \subset A$ an ideal, $V(I) \subset |X|$ is represented by an aff. perf. space pro-étale inside X,*

$$V(I) = \varprojlim_{\substack{n\geq 1\\ f_1,\ldots,f_n\in I}} \{|f_1| \leq 1, \ldots, |f_n| \leq 1\}$$

Those are the so-called Zaiski closed subsets.

2.2.2.2. *The pro-étale site of a perfectoid space.* — Here X is a perfectoid space.

Definition 2.2.8. — *We note $X_{\text{pro-ét}}$ for the small site whose objects are perfectoid spaces that are pro-étale over X.*
The coverings families are the families $(U_i \to V)_{i\in I}$ of X-morphisms satisfying the strong surjectivity property

$$\forall W \underset{qc\ open}{\subset} V,\ \exists \underbrace{J}_{finite} \subset I \text{ and } \forall j \in J,\ Z_j \underset{qc\ open}{\subset} U_j \text{ s.t. } W = \bigcup_{j\in J} \mathrm{Im}(Z_j \to V).$$

Remark 2.2.9. — *The definition of a covering is subtle, like for the fpqc topology for schemes, since pro-étale morphisms are not open in general. In fact, if* $(U_i \to V)_{i\in I}$ *is a family of morphisms of perfectoid spaces such that* $\forall i,\ U_i \to V$ *is open then*

$$\forall W \underset{qc\ open}{\subset} V,\ \exists \underbrace{J}_{finite} \subset I \text{ and } \forall j \in J,\ Z_j \underset{qc\ open}{\subset} U_j \text{ s.t. } W = \bigcup\nolimits_{j\in J} \mathrm{Im}(Z_j \to V)$$

$$\Updownarrow$$

$$V = \bigcup\nolimits_{i\in I} \mathrm{Im}(U_i \longrightarrow V) \text{ i.e. } \coprod\nolimits_{i\in I} U_i \longrightarrow V \text{ is surjective}$$

i.e. for families of open morphisms strongly surjective $\Leftrightarrow$ *surjective in the usual "naive" sense.*

For example, if $\mathbb{B}^{1/1/p^\infty}_{K,K^+}$ is the one dimensional perfectoid closed ball over the affinoid perfectoid field (K, K^+) then

$$\mathrm{Spa}(K, K^+) \coprod \mathbb{B}^{1,1/p^\infty}_{K,K^+} \smallsetminus \{0\} \longrightarrow \mathbb{B}^{1,1/p^\infty}_{K,K^+},$$

where $\mathrm{Spa}(K, K^+) \to \mathbb{B}^{1,1/p^\infty}_{K,K^+}$ is given by the inclusion of the origin of the ball, is pro-étale surjective but not a pro-étale cover.

Definition 2.2.10. — *We note*

$$\begin{array}{rcl} D^+_{pro\text{-}\acute{e}t}(X,\Lambda) & := & D^+(X_{pro\text{-}\acute{e}t},\Lambda) \\ D_{pro\text{-}\acute{e}t}(X,\Lambda) & := & D(X_{pro\text{-}\acute{e}t},\Lambda) \end{array}$$

Since $\widetilde{X}_{\text{pro-ét}}$ is replete, $D(X_{\text{pro-ét}}, \Lambda)$ is left complete.

Here is the verification that $\widetilde{X}_{pro\text{-}\acute{e}t}$ *is replete (see* [**14**]*).* Let $(\mathscr{F}_n)_{n\geq 0}$ be a projective system of pro-étale sheaves on X with surjective transition morphisms. Let U be an object of $X_{\text{pro-ét}}$ and $s \in \mathscr{F}_0(U)$. Up to replacing U by an affinoid perfectoid cover and s by its restriction to this cover we can suppose that U is affinoid perfectoid. Set $U_0 = 0$ and $s_0 = s$. Suppose by induction that we have constructed $U_n \to U$ an affinoid pro-étale cover and $s_n \in \mathscr{F}_n(U_n)$ such that $s_n \mapsto s_{|U_n}$. Any pro-étale cover of U_n is dominated by an affinoid pro-étale cover formed by one element. We can thus find $U_{n+1} \to U_n$ an affinoid pro-étale cover and $s_{n+1} \in \mathscr{F}_{n+1}(U_{n+1})$ such that $s_{n+1} \mapsto s_{n|U_{n+1}}$. Let now $U_\infty = \varprojlim_{n\geq 0} U_n$ that is an affinoid pro-étale cover of U. The collection $(s_{n|U_\infty})_{n\geq 0}$ lies in $\varprojlim_{n\geq 0} \mathscr{F}_n(U_\infty)$ and is mapped to $s_{|U_\infty}$. This proves the result.

2.2.3. The v-topology on perfectoid spaces. — The v-topology is an analog of the fpqc topology for schemes.

Definition 2.2.11. — *Let X be a perfectoid space. We note X_v the big site whose objects are Perf_X and coverings are families of morphisms $(U_i \to V)_{i\in I}$ of X-perfectoid spaces that are strongly surjective as in Definition 2.2.8.*

As for the pro-étale site, $\widetilde{X}_v$ is replete and we set.

Definition 2.2.12. — *Define*

$$\begin{aligned} D_v^+(X,\Lambda) &= D^+(X_v,\Lambda) \\ D_v(X,\Lambda) &= D(X_v,\Lambda) \end{aligned}$$

Remark 2.2.13. — *$X_{\text{ét}}$ and $X_{\textit{pro-ét}}$ are equivalent to small sites. This is not the case for X_v and to do cohomology one needs to fix some set-theoretical bounds by fixing a "sufficiently large regular cardinal" κ and consider only κ-small perfectoid spaces in the sense that the cardinal of $|X|$ and A for any $\mathrm{Spa}(A,A^+) \subset X$ affinoid perfectoid is less than κ. Doing this the v-site of κ-small perfectoid spaces is equivalent to a small site, and if κ is taken sufficiently large then all results and constructions do not depend on κ. We refer to* [**128**, Section 4].

2.2.4. Comparison of étale/pro-étale/v for perfectoid spaces ([128, Section 14]).

— Let X be a perfectoid space. There are evident continuous morphisms of sites

$$X_v \xrightarrow{\lambda} X_{\text{pro-ét}} \xrightarrow{\nu} X_{\text{ét}}$$

Proposition 2.2.14. — *The functors*

$$D_{\text{ét}}(X,\Lambda) \xrightarrow{\nu^*} D_{\textit{pro-ét}}(X,\Lambda) \xrightarrow{\lambda^*} D_v(X,\Lambda)$$

satisfy:

1. *ν^* is fully faithful and* $\mathrm{Id} \xrightarrow{\sim} R\nu_*\nu^*$
2. *λ^* is fully faithful and* $\mathrm{Id} \xrightarrow{\sim} R\lambda_*\lambda^*$.

Using left completness this is reduced to proving that for $\mathscr{F}$ an étale sheaf of Λ-modules on X and $\mathscr{G}$ a pro-étale sheaf of Λ-modules on X, $\forall q \in \mathbb{N}$,

$$\begin{aligned} H^q(X_{\text{ét}},\mathscr{F}) &\xrightarrow{\sim} H^q(X_{\text{pro-ét}},\nu^*\mathscr{F}) \\ H^q(X_{\text{pro-ét}},\mathscr{G}) &\xrightarrow{\sim} H^q(X_v,\lambda^*\mathscr{G}) \end{aligned}$$

This is analogous to the following "classical" result for schemes: let X be a scheme and consider

$$X_{fpqc} \xrightarrow{\lambda} \underbrace{X_{\text{pro-ét}}}_{\text{Bhatt-Scholze pro-étale site}} \xrightarrow{\nu} X_{\text{ét}}.$$

Then one can compute the étale cohomology of an étale torsion sheaf using the pro-étale or even the fpqc site: for $\mathscr{F}$ an étale sheaf of Λ-modules on X one has

$$H^q(X_{\text{ét}},\mathscr{F}) \xrightarrow{\sim} H^q(X_{\text{pro-ét}},\nu^*\mathscr{F}) \xrightarrow{\sim} H^q(X_{fpqc},\lambda^*\nu^*\mathscr{F}).$$

This type of results itself is an abelian generalization of well known results about the non-abelian H^1. Typically, if G is a smooth X-group scheme then any fpqc G-torsor on X is representable by a smooth X-scheme (smooth morphisms satisfy fpqc descent). It has thus a section over an étale cover of X and is thus an étale G-torsor. From this one can deduce that

$$H^1_{\text{ét}}(X,G) \xrightarrow{\sim} H^1_{fpqc}(X,G).$$

Here is the key point in the proof of $H^\bullet_{\text{ét}}(X,\mathscr{F}) \xrightarrow{\sim} H^\bullet_{\text{pro-ét}}(X,\nu^*\mathscr{F})$. This is the following result ([**128**, Section 8]).

Proposition 2.2.15. — *Let X be affinoid perfectoid.*

1. *The functor $\varprojlim$ induces an equivalence*

$$\text{Pro category (affinoid perfectoid, étale}/X) \xrightarrow{\sim} \{\text{affinoid pro-étale}/X\}$$

2. *If $U = \varprojlim_i U_i$ with U_i affinoid perfectoid étale over X then*

$$\nu^*\mathscr{F}(U) = \varinjlim_i \mathscr{F}(U_i).$$

Point (2) is an easy consequence of point (1) that reduces the problem to an "algebraic statement" like in Bhatt-Scholze ([**14**]). Point (1) is a " decompletion argument" that is obtained by dévissage from the following using a "graph of a morphism" argument.

Proposition 2.2.16. — *If $(A_i,A_i^+)_{i\in I}$ is a filtered inductive system of affinoid complete Tate rings and $A_\infty^+ = \widehat{\varinjlim_{i\in I} A_i^+}$, $A_\infty = A_\infty^+[\frac{1}{\varpi}]$ then*

$$2-\varinjlim_{i\in I}\{\text{finite étale } A_i\text{-alg.}\} \xrightarrow{\sim} \{\text{finite étale } A_\infty\text{-alg.}\}$$

This result can either be deduced from [**44**] applied to the henselian pair $(\varinjlim_i A_i^+,(\varpi))$ or using the purity theorem about almost finite étale algebras.

2.2.5. Description of the essential image of $\nu^*: D_{\text{ét}} \hookrightarrow D_{\text{pro-ét}}$. —

2.2.5.1. *Strictly totally disconnected perfectoid spaces.* — Let us recall the following ([**128**]).

Definition 2.2.17. — *A perfectoid space is strictly totally disconnected if it is quasi-compact quasi-separated and any étale cover has a section.*

It is immediately checked that any s.t.d. perfectoid space is affinoid perfectoid (a section of $\coprod_i U_i \to X$ with $(U_i)_i$ an aff. perf. covering gives a decomposition $X = \coprod_i V_i$ with V_i open/closed in U_i and thus aff. perf.).

Proposition 2.2.18. — *Let X be a qc qs perfectoid space. The following are equivalent:*

1. *X is strictly totally disconnected*
2. *$\forall \mathscr{F}$ étale sheaf of abelian groups on X and $q > 0$, $H^q_{\text{ét}}(X, \mathscr{F}) = 0$*
3. *for all $x \in X$, $K(x)$ is algebraically closed and the connected components of $|X|$ have a unique closed point i.e. are of the form $\mathrm{Spa}(K(x), K(x)^+)$ for some $x \in X$.*

If X is a qc qs perfectoid space there is a continuous surjective map

$$\underbrace{|X|}_{\text{spectral space}} \longrightarrow \underbrace{\pi_0(|X|)}_{\text{profinite space}}$$

and thus a surjective morphism of v-sheaves

$$X \longrightarrow \underline{\pi_0(X)}$$

where for P a profinite set $\underline{P} = v$-sheaf on Perf s.t.

$$\underline{P}(S) = \mathscr{C}(|S|, P).$$

If $P = \varprojlim_i \underbrace{P_i}_{\text{finite set}}$ then $\underline{P} = \varprojlim_i \underbrace{\underline{P_i}}_{\text{constant sheaf wt. value } P_i}$.

Thus, any qc qs perfectoid space X is fibered naturally over a profinite set with connected fibers.

Then,

$$\begin{gathered} X \text{ strictly totally disconnected} \\ \Updownarrow \\ \text{the fibers of } X \to \underline{\pi_0(X)} \text{ are of the form } \mathrm{Spa}(C, C^+) \text{ wt. } C \text{ alg.closed} \end{gathered}$$

Remark 2.2.19. — *There is a stronger notion than s.t.d. perf. spaces: strictly w-local perf. spaces. For this one adds the condition that $X \to \underline{\pi_0(X)}$ has a section*

$$\begin{tikzcd} X \arrow[r] & \underline{\pi_0(X)} \arrow[l, bend right] \end{tikzcd}.$$

In this case one can really think about strictly w-local perf. spaces as "amalgamations" of $\operatorname{Spa}(C(x), C(x)^+)$, $x \in P$, along a profinite set P with $C(x)$ alg. closed.

Proposition 2.2.20. — *Any perfectoid space X admits a pro-étale cover $(U_i \to X)_i$ such that for all i, $U_i \to X$ is open and U_i is strictly totally disconnected.*

The proof consist in giving a meaning to

$$\text{“} \varprojlim_{\substack{U \to X \\ \text{étale cover} \\ U \text{ aff. perf.}}} U\text{”} \to \text{ has no sense from the set theoretical point of view}$$

This is done via an induction process that stops at some point; we don't even need any transfinite induction!

Remark 2.2.21. — *There is no explicit formula for such a strictly totally disconnected pro-étale cover in general, typically for the ball $\mathbb{B}^{1,1/p^\infty}_{K,K^+}$ over the affinoid perfectoid field (K, K^+). The only "most general case" where one can give such an explicit forumla is for $\operatorname{Spa}(K, K^+) \times \underline{P} = \operatorname{Spa}(\mathscr{C}(P, K), \mathscr{C}(P, K^+))$ where P is a profinite set. For such a space one can take $\operatorname{Spa}(\widehat{\overline{K}}, \widehat{\overline{K}}^+) \times \underline{P} \to \operatorname{Spa}(K, K^+) \times \underline{P}$ as a s.t.d. pro-étale cover.*

2.2.5.2. *Description of the image.* —

The functor $\nu^* : D_{\text{ét}}(X, \Lambda) \hookrightarrow D_{\text{pro-ét}}(X, \Lambda)$ is far from being essentially surjective in general.

For example, if $X = \operatorname{Spa}(C, \mathcal{O}_C)$ with C alg. closed this is the embedding

$$\boxed{\nu^* : D(\Lambda) \hookrightarrow D(\text{condensed } \Lambda_{disc}\text{-modules}).}$$

More generally, if $X = \mathrm{Spa}(K, K^+)$ is the spectrum of an affinoid perfectoid field this is the embedding

$$\begin{tikzcd}
D(\Lambda\text{-modules} + \text{discrete } \mathrm{Gal}(\overline{K}|K) \text{ linear action}) \arrow[d, hook, "\nu^*"] \\
D(\text{condensed } \Lambda_{disc}\text{-modules} + \text{linear action of } \underbrace{\mathrm{Gal}(\overline{K}|K)}_{\text{condensed group}}\ )
\end{tikzcd}$$

The starting point is the following remark.

Proposition 2.2.22. — *If X is a s.t.d. perf. space then*

1. *any qc open subset is strictly totally disconnected*
2. *the projection $X_{\text{ét}} \to |X|$ induces an equivalence of topoi $\widetilde{X_{\text{ét}}} \xrightarrow{\sim} |X|^{\sim}$*
3. *$\nu^* : D_{\text{ét}}(X, \Lambda) \xrightarrow{\sim} D(|X|, \Lambda)$*
4. *for any étale sheaf of ab. gp. $\mathscr{F}$ on X and $U \to X$ étale with U perfectoid qc qs, $H^q_{\text{ét}}(U, \mathscr{F}) = 0$ for $q > 0$.*

The proof is easy using Prop. 2.2.18. The vanishing of cohomlogy (point (4)) implies that $D(X_{\text{ét}}, \Lambda)$ is left complete, see point (1) of Prop. 2.1.4.

Remark 2.2.23. — *In the scheme context, see* [**14**], *any scheme X admits a pro-étale cover $(U_i \to X)_{i \in I}$ with U_i strictly totally disconnected schemes (which means in this context that U_i is affine with connected components spectra of strictly Henselian local rings). In this context Prop. 2.2.22 is false: a qc open subset of a s.t.d. scheme may not be s.t.d.. For example, if K is a complete non-archimedean field for a rank 1 valuation with residue field k_K separably closed then Spec($\mathcal{O}_K$) is strictly totally disconnected but the open subset Spec(K) $\hookrightarrow$ Spec($\mathcal{O}_K$) may not be s.t.d. i.e. K may not be separably closed. One of the points is that spectral spaces associated to analytic adic spaces satisfy: for any $x \in$ this space, the set of generalizations of x is a chain. This is not true for schemes in general. What is true for X a s.t.d. scheme is that if $\mathscr{F}$ an étale sheaf of abelian groups on X then $H^q_{\text{ét}}(X, \mathscr{F}) = 0$ when $q > 0$.*

We now have the following result that describes the image of ν^* and even $\lambda^* \nu^*$.

Proposition 2.2.24. — *For X a perfectoid space*

1. *There are equivalences*

$$\begin{tikzcd}
D_{\text{ét}}(X,\Lambda) \arrow[d, "\nu^*"', "\wr"] \\
\left\{A \in D_{\textit{pro-ét}}(X,\Lambda) \mid \forall S \to X,\ S \text{ s.t.d. } , A_{|S} \in D_{\text{ét}}(S,\Lambda) = D(|S|,\Lambda)\right\} \arrow[d, "\lambda^*"', "\wr"] \\
\left\{A \in D_{v}(X,\Lambda) \mid \forall S \to X,\ S \text{ s.t.d. } , A_{|S} \in D_{\text{ét}}(S,\Lambda) = D(|S|,\Lambda)\right\}
\end{tikzcd}$$

2. *If $(U_i \to X)_{i\in I}$ is a pro-étale cover with $\forall i$, U_i is s.t.d. then this reduces to*

$$\begin{tikzcd}
D_{\text{ét}}(X,\Lambda) \arrow[d, "\nu^*"', "\wr"] \\
\left\{A \in D_{\textit{pro-ét}}(X,\Lambda) \mid \forall i \in I,\ A_{|U_i} \in D_{\text{ét}}(U_i,\Lambda) = D(|U_i|,\Lambda)\right\} \arrow[d, "\lambda^*"', "\wr"] \\
\left\{A \in D_{v}(X,\Lambda) \mid \forall i \in I, A_{|U_i} \in D_{\text{ét}}(U_i,\Lambda) = D(|U_i|,\Lambda)\right\}
\end{tikzcd}$$

The fact that ν^* is fully faithful on $D^+_{\text{ét}}$ coupled with the fact that $\widetilde{X}_{\text{pro-ét}}$ is replete and thus $D(X_{\text{pro-ét}},\Lambda)$ is left complete, coupled with the fact that for all S perfectoid strictly totally disconnected $D(S_{\text{ét}},\Lambda))$ is complete implies that we get a ”geometric concrete incarnation“ of the abstractly defined left completion $\widehat{D}(X_{\text{ét}},\Lambda)$.

2.3. Étale/quasi-pro-étale/v-topology for locally spatial diamonds

After having investigated the case of perfectoid spaces, we investigate the case of locally spatial diamonds.

2.3.1. A Key descent result. — The following result is one of the most difficult in [**128**].

Proposition 2.3.1. — *Separated étale morphisms of perfectoid spaces satisfy descent for the v-topology.*

This means the following: if $(U_i \to X)_i$ is a v-cover of perf. spaces then

$$\begin{tikzcd}
\big\{\text{separated étale perf. spaces } /X\big\} \arrow[d, "\wr"] \\
\{\text{separated étale perf. spaces } /\coprod_i U_i \ + \ \text{descent datum w.r.t.} \coprod_i U_i \to X\}\,.
\end{tikzcd}$$

By a descent datum we mean cartesian separated étale perfectoid spaces over the diagram

$$\begin{tikzcd}
\coprod_{i,j,k} U_i \times_X U_j \times_X U_k \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \coprod_{i,j} U_i \times_X U_j \arrow[r, shift left=1] \arrow[r, shift right=1] & \coprod_i U_i
\end{tikzcd}$$

For finite étale morphism of perfectoid spaces this result is "easy" since vector bundles on perfectoid spaces satisfy v-descent:

Proposition 2.3.2 **([132, Lemma 17.1.8]).** — *If $(U_i \to X)_{i\in I}$ is a v-cover of perfectoid spaces,*

$$\begin{tikzcd}
\textit{vector bundles on } X \arrow[d, "\wr"] \\
\textit{vector bundles on } \coprod_i U_i \ + \textit{descent datum}
\end{tikzcd}$$

One of the key tools of the descent results for the v-topology on perf. spaces is that after a pro-étale covering one can suppose that our perfectoid spaces are s.t.d. and the use of the following key remark:

Let $X = \mathrm{Spa}(A, A^+)$ be s.t.d. and let $Y \to X$ be a morphism with $Y = \mathrm{Spa}(B, B^+)$ affinoid perfectoid then:

1. $\mathrm{Spec}(B^+/\varpi) \longrightarrow \mathrm{Spec}(A^+/\varpi)$ is flat
2. if $Y \to X$ is surjective i.e. a v-cover then

$$\mathrm{Spec}(B^+/\varpi) \longrightarrow \mathrm{Spec}(A^+/\varpi) \text{ is faithfully flat}$$

Example 2.3.3. — *Here is an application of Proposition 2.3.1 that we use all the time. Let X be a perf. space.*

1. *Let G be a finite group and $T \to X$ be a $\underline{G}$-torsor for the v-topology. Then T is represented by an étale separated perf. space over X and is thus an étale $\underline{G}$-torsor;*

$$\boxed{H^1_{\text{ét}}(X,G) \xrightarrow{\sim} H^1_v(X,G).}$$

2. *Suppose now that G is locally profinite. If $T \to X$ is a $\underline{G}$-torsor for the v-topology then $T \xrightarrow{\sim} \varprojlim\limits_{\substack{K\subset G \\ \text{compact open subgp.}}} \underline{K}\backslash T$ and $\underline{K}\backslash T$ is represented by a separated étale X-perfectoid space. Thus, $T \to X$ is represented by a pro-étale perfectoid space and is thus a pro-étale $\underline{G}$-torsor,*

$$\boxed{H^1_{\textit{pro-ét}}(X,\underline{G}) \xrightarrow{\sim} H^1_v(X,\underline{G}).}$$

2.3.2. Locally spatial diamonds ([132]). —

Recall the following: A diamond is a pro-étale sheaf X on $\mathrm{Perf}_{\mathbb{F}_p}$ (i.e. on the big pro-étale site) satisfying:

$\exists \widetilde{X}$ perf. space and $R \subset \widetilde{X} \times \widetilde{X}$ an eq. relation representable by a perf. space s.t.

1. $\widetilde{X} \rightrightarrows X$ are pro-étale
2. $X \simeq \widetilde{X}/R$ as pro-étale sheaves on the big pro-étale site $(\mathrm{Perf}_{\mathbb{F}_p})_{\text{pro-ét}}$.

Thus, a diamond is an "algebraic space for the pro-étale topology" on $\mathrm{Perf}_{\mathbb{F}_p}$. One can verify that a diamond is in fact a v-sheaf (analogous result to : any algebraic space in the sense of Artin is in fact an fpqc sheaf (Gabber)).

The category of diamonds is "too large" to work with. For Artin "classical" algebraic spaces it is for example usual to assume that our algebraic spaces are quasi-separated to remove pathological objects like $\mathbb{G}_{a,k}/\mathbb{Z}$, k a field of char. 0.

Definition 2.3.4. — *A spatial diamond is a diamond X satisfying*

1. *X is qc qs as a v-sheaf*
2. *each point of $|X|$ has a basis of qc open nbd.*

For X a spatial diamond the first basic result is that in fact

$$|X| \text{ is spectral.}$$

The typical example of a locally spatial diamond is $X^\diamond$ for X a locally Noetherian analytic adic space where

$$|X| = |X^\diamond|.$$

But those are not the only one locally spatial diamonds we deal with: we deal with "more exotic ones" like punctured absolute BC's.

Spatial diamonds share a lot of properties with analytic adic spaces, typically:

- in the spectral space $|X|$ the set of generalizations of a point form a chain,
- for $f : X \to Y$ a morphism of spatial diamonds, $|f| : |X| \to |Y|$ is generalizing.
- Any morphism between spatial diamonds is qc qs.

One has to be careful still: any morphism between perfectoid spaces is locally separated (since any morphism of affinoid perfectoid spaces is separated) but this is not the case for morphisms of locally spatial diamonds: *they are only locally quasi-separated in general.*

Finally let us cite the following.

Proposition 2.3.5. — *Let X be a spatial diamond, $Z \subset |X|$ be a pro-constructible generalizing subset. The v-sheaf*

$$\mathrm{Perf}_{\mathbb{F}_p} \ni S \longmapsto \{S \to X \mid \mathrm{Im}(|S| \to |X|) \subset Z\}$$

is represented by a spatial diamond Y with $Y \hookrightarrow X$ quasi-compact quasi-pro-étale and $|Y| = Z$.

Thus, if if X is a qs finite type adic space over $\mathrm{Spa}(K)$, K a non-archimedean field, any $Z \subset |X| = |X^\diamond|$ defines a sub spatial diamond of $X^\diamond$. We can thus speak about the étale cohomology of such a subset even if this is not a rigid analytic space: this has a nice geometric structure in the world of diamonds.

2.3.3. The étale site of a locally spatial diamond.

— Let X be a locally spatial diamond. Since separated étale morphisms of perfectoid spaces descend for the v-topology and thus the pro-étale one, there is a good notion of a locally separated étale morphism of locally spatial diamond. As for perfectoid spaces, those are open morphisms, and we thus have a "good" notion of étale coverings.

Definition 2.3.6. — *For X a loc. spatial diamond we note*

$$X_{\text{ét}}$$

for the small site of locally separated étale loc. spatial diamonds over X. A family of morphisms $(U_i \to V)_i$ in $X_{\text{ét}}$ is a cover if $\coprod_i |U_i| \to |V|$ is surjective.

If X is a perfectoid space one recovers the preceding étale site of a perfectoid space; the definition is thus coherent.

Definition 2.3.7. — *For X a loc. spatial diamond we note*

$$\begin{aligned} D^+_{\text{ét}}(X,\Lambda) &= D^+(X_{\text{ét}},\Lambda) \\ D_{\text{ét}}(X,\Lambda) &= \widehat{D}(X_{\text{ét}},\Lambda) \ \ (\textit{left completion}) \end{aligned}$$

Finally let us cite.

Proposition 2.3.8. — *If X is an adic space locally of finite type over* $\mathrm{Spa}(K,K^+)$*, K a complete non-archimedean field, the continuous morphism of sites*

$$(X^\diamond)_{\text{ét}} \longrightarrow \underbrace{X_{\text{ét}}}_{\substack{\textit{étale site} \\ \textit{as defined by Huber}}}$$

induces an equivalence of topoi

$$(X^\diamond)^{\sim}_{\text{ét}} \xrightarrow{\sim} \widetilde{X}_{\text{ét}}.$$

In particular one can compute étale cohomology of rigid analytic spaces using diamonds.

2.3.4. Quasi-pro-étale morphisms. — Contrary to (separated) étale morphisms that satisfy descent wrt v-covers, pro-étale morphisms do not even descend along pro-étale morphisms and we need to take care of this. This is for example the case for the Kummer morphism $z \mapsto z^2$ from the perfectoid closed ball to itself that is not pro-étale but becomes pro-étale after a pro-étale localization of the target.

Definition 2.3.9. — *A morphism $X \to Y$ of perfectoid spaces is quasi-pro-étale if there exists a pro-étale cover $\widetilde{Y} \to Y$ s.t. $X \times_Y \widetilde{Y} \to \widetilde{Y}$ is pro-étale.*

This definition is a little bit abstract and hopefully we have this geometric caracterization.

Proposition 2.3.10. — *For $f : X \to Y$ a morphism of perfectoid spaces the following are equivalent:*

1. *f is quasi-pro-étale.*
2. *If $\widetilde{Y} \to Y$ is a pro-étale cover with $\widetilde{Y}$ a disjoint union of s.t.d. perf. spaces then $X \times_Y \widetilde{Y} \to \widetilde{Y}$ is pro-étale.*
3. *If $(U_i)_i$ is a cover of X by qc qs open subsets s.t. $f_{|U_i} : U_i \to Y$ is separated,*

   $$f_{|U_i} : U_i \to Y \text{ has profinite geometric fibers}$$

   which means: $\forall \mathrm{Spa}(C, C^+) \to Y$, $\forall i$, $U_i \times_Y \mathrm{Spa}(C, C^+)$ is isomorphic to $\mathrm{Spa}(C, C^+) \times \underline{P}$ with P a profinite set.

Thus, quasi-pro-étale morphisms are the morphisms with locally on the source profinite geometric fibers This characterization of quasi-pro-étale morphisms is very well adapted to the study of moduli spaces.

2.3.5. The quasi-pro-étale site.

— Let us take the following definition.

Definition 2.3.11. —
1. *A morphism $X \to Y$ of locally spatial diamonds is quasi-pro-étale if $\forall S$ s.t.d. perfectoid space and $S \to Y$, $X \times_Y S \to S$ is a pro-étale morphism of perfectoid spaces.*
2. *For X a loc. spatial diamond*

   $$X_{\textit{q-pro-ét}} = \text{small site of quasi-pro-étale loc. spatial diamonds } /X$$

   where the covers are defined as for the pro-étale site of a perf. space using the "strong surjectivity condition".
3. *For X a loc. spatial diamond*

   $$\begin{aligned} D^+_{\textit{pro-ét}}(X, \Lambda) &= D^+(X_{\textit{q-pro-ét}}, \Lambda) \\ D_{\textit{pro-ét}}(X, \Lambda) &= D(X_{\textit{q-pro-ét}}, \Lambda). \end{aligned}$$

If X is a perf. space then the continuous morphism of sites $X_{\text{q-pro-ét}} \to X_{\text{pro-ét}}$ induces an equivalence of topoi

$$\widetilde{X}_{\text{q-pro-ét}} \xrightarrow{\sim} \widetilde{X}_{\text{pro-ét}}$$

and thus our definition of $D_{\text{pro-ét}}$ is coherent.

Proposition 2.3.12. — *Propositions 2.2.14 and 2.2.24 remain valid by replacing the perfectoid space X by a locally spatial diamond:*

$$\begin{tikzcd}
D_{\text{ét}}(X,\Lambda) \arrow[r, hook, "\nu^*"] \arrow[dr, hook] & D_{\textit{pro-ét}}(X,\Lambda) \arrow[d, "\lambda^*"] \\
& D_v(X,\Lambda)
\end{tikzcd}$$

where here $D_v(X,\Lambda) = D(X_v,\Lambda)$ where X_v is the big site Perf/X equipped with the localized v-topology on Perf. And the essential image of $\lambda^* \circ \nu^*$ is given by

$$\big\{ A \in D_v(X,\Lambda) \mid \forall S \textit{ s.t.d. perf. space }, \ \forall S \to X, \ A_{|S} \in D(|S|,\Lambda) \big\}.$$

Remark 2.3.13. — *The functor $\lambda^* : D_{\textit{pro-ét}}(X,\Lambda) \to D_v(X,\Lambda)$ is in general not fully-faithful and does not satisfy $\mathrm{Id} \xrightarrow{\sim} R\lambda_*\lambda^*$. That being said, this is the case when restricted to*

$$D_{\textit{pro-ét},\blacksquare}(X,\Lambda) \subset D_{\textit{pro-ét}}(X,\Lambda)$$

and we have a diagram

$$\begin{tikzcd}
D_{\text{ét}}(X,\Lambda) \arrow[r, hook, "\nu^*"] \arrow[dr, hook] & D_{\textit{pro-ét},\blacksquare}(X,\Lambda) \arrow[d, hook, "\lambda^*"] \\
& D_v(X,\Lambda)
\end{tikzcd}$$

LECTURE 3

SMALL v-STACKS

3.1. $D_{\text{ét}}(X, \Lambda)$ for X a small v-sheaf

3.1.1. Small v-sheaves. — Let X be a v-sheaf of sets on the big site $(\text{Perf}_{\mathbb{F}_p})_v$.

Definition 3.1.1. — *The v-sheaf X is small if there exists a perfectoid space U and an epimorphism of v-sheaves $U \to X$.*

One has to be careful that there exists non-small v-sheaves. For example, $\underline{\{s, \eta\}}$ is non-small where $\underline{\{s, \eta\}}(S) = \{\text{open subsets of } S\}$.

For example, diamonds are small v-sheaves since if X is a diamond, $X \simeq \widetilde{X}/R$ with $\widetilde{X}$ perfectoid and $R \subset \widetilde{X} \times \widetilde{X}$ a pro-étale equivalence relation and thus $\widetilde{X} \to X$ is a v-cover.

Finally, in [**58**], at some points we need to work with more general objects than locally spatial diamonds.

Here is a slogan that is used in [**128**] and [**58**].

The good category of geometric objects we work with is the category of small v-sheaves equipped with morphisms that are relatively representable in loc. spatial diamonds

Example 3.1.2. — *1. If k is a characteristic p discrete field then* Spa(k) *is a small v-sheaf, not representable by a loc. spatial diamond*

2. *(Locally profinite sets) If P is a locally profinite set then $\underline{P}$ is a small v-sheaf not representable by a loc. spatial diamond although*

$$\begin{tikzcd}
\underline{P} \arrow[d, "\textit{rel. rep. in perf. spaces}"'] & \textit{not a loc. spatial diamond} \arrow[l, bend right, red] \\
\mathrm{Spa}(\mathbb{F}_p) &
\end{tikzcd}$$

3. *(Formal schemes) If $\mathfrak{X}$ is an $\mathbb{F}_p$-formal scheme we can associated to it a small v-sheaf $\mathfrak{X}^\diamond$ by taking the analytic sheaf associated to the presheaf*

$$(R, R^+) \mapsto \mathfrak{X}(\textit{Spf}(R^+))$$

This small v-sheaf is not representable by a locally spatial diamond. For example, if $\mathfrak{X} = \textit{Spf}(\mathbb{F}_p[\![x_1, \dots, x_d]\!])$, $\mathfrak{X}^\diamond(S) = (\Gamma(S, \mathcal{O}_S)^{\circ\circ})^d$,

$$\mathfrak{X}^\diamond \times_{\mathrm{Spa}(\mathbb{F}_p)} S \simeq \underbrace{\mathring{\mathbb{B}}_S^{d,1/p^\infty}}_{\textit{open perf. ball}/S}$$

but $\mathfrak{X}^\diamond$ is not a locally spatial diamond. The small v-sheaf

$$\mathfrak{X}^\diamond \smallsetminus (\mathfrak{X}_{red})^\diamond \qquad \underbrace{\subset}_{\substack{\textit{represented by}\\ \textit{an open immersion}}} \qquad \mathfrak{X}^\diamond$$

is always representable by a perfectoid space for any $\mathfrak{X}$, for example

$$\begin{aligned} & \mathrm{Spf}(\mathbb{F}_p[\![x_1, \dots, x_d]\!])^\diamond \smallsetminus \textit{Spec}(\mathbb{F}_p)^\diamond \\ = \; & \underbrace{\mathrm{Spa}\big(\mathbb{F}_p[\![x_1^{1/p^\infty}]\!], \dots, x_d^{1/p^\infty}, \mathbb{F}_p[\![x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}]\!]\big) \smallsetminus V(x_1, \dots, x_d)}_{\textit{qc qs perfectoid space}} \end{aligned}$$

$$\begin{tikzcd}
\textit{qc qs perf. space} \arrow[r, bend right, red] & \mathfrak{X}^\diamond \smallsetminus (\mathfrak{X}_{red})^\diamond \arrow[r, hook] & \mathfrak{X}^\diamond \arrow[d, "\textit{rel. rep. in perf. spaces}"] & \textit{not a loc. spatial diamond} \arrow[l, bend right, red] \\
& & \mathrm{Spa}(\mathbb{F}_p) &
\end{tikzcd}$$

4. *(Absolute positive BC spaces). For (D, φ) an $\overline{\mathbb{F}}_q$-isocrystal relative to E, i.e. $D = \underbrace{\breve{E}}_{\widehat{E^{un}}}$-vector space and φ is a $\underbrace{\sigma}_{\textit{Frob of } E^{un}|E}$-linear automorphism, with ≤ 0 slopes the functor*

$$\begin{aligned} BC(D, \varphi) : \mathrm{Perf}_{\overline{\mathbb{F}}_q} &\longrightarrow \mathrm{Sets} \\ S &\longmapsto H^0(X_S, \mathscr{E}(D, \varphi)) \end{aligned}$$

is

(a) representable by a formal scheme ($\mathcal{G}^\diamond$, $\mathcal{G}$ a formal p-divisible group $/\overline{\mathbb{F}}_q$) when the slopes $\in \big[-[E:\mathbb{Q}_p],0[$.

(b) representable by a formal scheme $\times$ a locally profinite set when the slopes $\in [-[E:\mathbb{Q}_p],0]$,

(c) only a small v-sheaf for any slopes.

If $* = \mathrm{Spa}(\overline{\mathbb{F}}_q)$ there is a "zero section" $* \hookrightarrow BC(D,\varphi)$. In general, for any slope, one has the picture for any (D,φ)

$$\begin{tikzcd}
\textit{spatial diamond} \arrow[r, bend right] & BC(D,\varphi)\smallsetminus\{0\} \arrow[r, hook] & BC(D,\varphi) \arrow[d, "\textit{rel. rep. in loc. spatial diamonds}"] & \textit{not a loc. spatial diamond} \arrow[l, bend right] \\
 & & * &
\end{tikzcd}$$

5. *(Absolute negative BC spaces) (D,φ) has >0 slopes,*

$$\begin{array}{rcl} BC((D,\varphi)[1]) : \mathrm{Perf}_{\overline{\mathbb{F}}_q} & \longrightarrow & \mathrm{Sets} \\ S & \longmapsto & H^1(X_S, \mathscr{E}(D,\varphi)) \end{array}$$

is a small v-sheaf and we have the same picture

$$\begin{tikzcd}
\textit{spatial diamond} \arrow[r, bend right] & BC((D,\varphi)[1])\smallsetminus\{0\} \arrow[r, hook] & BC((D,\varphi)[1]) \arrow[d, "\textit{rel. rep. in loc. spatial diamonds}"] & \textit{not a loc. spatial diamond} \arrow[l, bend right] \\
 & & * &
\end{tikzcd}$$

For example,

$$\underbrace{BC(\mathcal{O}(-1)[1])}_{\substack{\textit{small } v\textit{-sheaf} \\ \textit{not a diamond}}} \times_{\mathrm{Spa}(\mathbb{F}_q)} \mathrm{Spa}(E)^\diamond \simeq \underbrace{(\mathbb{G}_{a/E})^\diamond/\underline{E}}_{\textit{loc. spatial diamond}}$$

and the spatial diamond $BC(\mathcal{O}(-1)[1])\smallsetminus\{0\}$ is an absolute version of

$$\Omega^\diamond \Big/ \begin{pmatrix} 1 & \underline{E} \\ 0 & 1 \end{pmatrix}.$$

Proposition 3.1.3. — *If X is a small v-sheaf there exists a v-hypercover*

$$U_\bullet \longrightarrow X$$

such that for all $n \geq 0$, U_n is a disjoint union of strictly totally disconnected perfectoid spaces.

Thus, any U as in the definition of a small v-sheaf can be replaced, up to a pro-étale cover, by a disjoint union of strictly totally disconnected perfectoid spaces. Now,

$$U \times_X U \quad \underbrace{\subset}_{\text{sub } v\text{-sheaf}} \quad \underbrace{U \times U}_{\text{perf. space}}$$

We use the following:

Any sub-v-sheaf of a diamond is a diamond and in particular a small v-sheaf.

That is deduced from the following:

If X is a strictly totally disconnected perfectoid space then any pro-constructible generalizing subset of $|X|$ is representable by a perfectoid space pro-étale inside X.

Here is how to use the preceding. Let $\mathscr{F} \subset X$ be a sub-v-sheaf of X a strictly totally disconnected perfectoid space. For each S affinoid perfectoid and each element of $\mathscr{F}(S)$ there is an associated morphism $S \to X$ to which is associated $\mathrm{Im}(|S| \to |X|)$ that is pro-constructible generalizing. Thus, applying the preceding result, for each element of $\mathscr{F}(S)$, S affinoid perfectoid, is associated an affinoid perfectoid $Z \subset X$ that is pro-étale inside X. When S and the element of $\mathscr{F}(S)$ vary this forms a subset of the set Σ of such $Z \subset X$. Then, using the v-sheaf property, there is a quasi-pro-étale surjection $\coprod_{Z \in \Sigma} Z \to \mathscr{F}$.

3.1.2. $D_{\text{ét}}(X, \Lambda)$.

— Let X be a small v-sheaf. Let $X_v = (\mathrm{Perf})_v/X$ be the v-site of X whose underlying category is the one of perfectoid spaces over X.

Definition 3.1.4. — *Set*

$$D_{\text{ét}}(X, \Lambda) = \left\{ \begin{matrix} A \in D(X_v, \Lambda) \mid \forall S \to X \ S \ \textit{perf. s.t.d.} \\ A_{|S} \in D_{\text{ét}}(S, \Lambda) = D(|S|, \Lambda) \end{matrix} \right\}.$$

One recovers the preceding category $D_{\text{ét}}(X, \Lambda)$ for X a locally spatial diamond.

Since $D(X_v, \Lambda)$ and $D_{\text{ét}}(S, \Lambda)$ for S perfectoid strictly totally disconnected are left complete, $D_{\text{ét}}(X, \Lambda)$ is left complete.

The main remark is now the following. Let

$$S_\bullet \longrightarrow X$$

be a v-hypercover such that for all $n \geq 0$, S_n is a disjoint union of strictly totally disconnected perfectoid spaces. Then, pull back from the topos $\widetilde{X_v}$ to the topos of cartesian sheaves on $S_{\bullet,v}$ induces

$$D_{\text{ét}}(X,\Lambda) \xrightarrow{\sim} \underbrace{D_{cart}(|S_\bullet|,\Lambda)}_{\substack{\text{derived cat. of cartesian}\\ \text{sheaves of }\Lambda\text{-mod. on}\\ \text{the simplicial top. space }|S_\bullet|}}$$

Moreover if $A \in D_{\text{ét}}(X,\Lambda)$ corresponds to $\mathscr{F}_\bullet$ then

$$R\Gamma(X,A) \xrightarrow{\sim} \Gamma(|S_\bullet|,\mathscr{F}_\bullet).$$

→ étale cohomology of perfectoid spaces / locally spatial diamonds / small v-sheaves is simpler than étale cohomology of schemes: everything is reduced to simplicial cartesian sheaves on top. spaces !

We have in fact the ore general formula for $S_\bullet \to X$ a v-hypercover by perfectoid space

$$D_{\text{ét}}(X,\Lambda) \xrightarrow{\sim} \underbrace{\widehat{D}_{cart}(S_{\bullet,\text{ét}},\Lambda)}_{\substack{\text{left completion of}\\ \text{derived cat. of cartesian étale sheaves}\\ \text{on }S_\bullet}}$$

We will need the following Lemma.

Lemma 3.1.5. — *The inclusion $D_{\text{ét}}(X,\Lambda) \subset D_v(X,\Lambda)$ admits a right adjoint $R_{X_{\text{ét}}} : D_v(X,\Lambda) \to D_{\text{ét}}(X,\Lambda)$.*

Proof. — One can apply Freyd's adjunction theorem (or its upgrade to presentable ∞-categories by Lurie ([**101**, Corollary 5.5.2.9])) and the result is then a consequence of the fact that $D_{\text{ét}}(X,\Lambda)$ is stable under colimits. A slightly more constructive proof consists in replacing X by a v-hypercover $S_\bullet$ with S_n a disjoint union of strictly totally disconnected perfectoid spaces for all $n \geq 0$. Then,

$$R_{X_{\text{ét}}} = R\,\mathrm{Cart}\,R(\nu_{S_\bullet} \circ \lambda_{S_\bullet})_*$$

where

$$\big((\nu_{S_\bullet} \circ \lambda_{S_\bullet})^*, (\nu_{S_\bullet} \circ \lambda_{S_\bullet})_*\big) : \widetilde{S}_{\bullet,v} \longrightarrow |S_\bullet|^{\sim}$$

is a morphism of simplicial topoi and Cart is the cartesianification functor that is the right adjoint of the inclusion of cartesian sheaves on $|S_\bullet|$ inside all sheaves on $|S_\bullet|$ (that exists again thanks to Frey's adjunction). ☐

There is no explicit formula in general for the Cartesianification functor and the preceding coontruction is, in general, an abstract construction.

Example 3.1.6. — *If* $* = \mathrm{Spa}(\overline{\mathbb{F}}_q)$ *as a small v-sheaf then one has* $\underbrace{D(\Lambda)}_{\substack{\textit{usual derived cat.}\\ \textit{of }\Lambda\textit{-modules}}} \xrightarrow{\sim} D_{\text{ét}}(*,\Lambda)$. *This is a consequence of the fact that if* $C = \widehat{\overline{\overline{\mathbb{F}}_p((T))}}$ *then* $\mathrm{Spa}(C) \times_{\mathrm{Spa}(\overline{\mathbb{F}}_p)} \mathrm{Spa}(C)$ *is a connected perfectoid space isomorphic to a projective limit with finite* *étale transition morphisms of open punctured disks over C (write $C = \widehat{\cup_{r\geq 0} K_r}$ with $K_r \subset K_{r+1}$ and $K_r|\overline{\mathbb{F}}_p((T))$ separable of finite degree, $K_r \simeq \overline{\mathbb{F}}_p((T))$).*

3.2. $D_{\text{ét}}(\mathfrak{X}, \Lambda)$ for $\mathfrak{X}$ a small v-stack

3.2.1. Small v-stacks. — We lied: the category of small v-sheaves is not enough for [**58**].

The good category of geometric objects we work with is the category of small v-stacks equipped with morphisms that are 0-truncated representable in loc. spatial diamonds (compactifiable loc. of finite dim trg.)

We now have the following definition.

Definition 3.2.1. — *A stack $\mathfrak{X}$ on* $\mathrm{Perf}_{\mathbb{F}_p}$ *equipped with the v-top. is small if $\exists S \to \mathfrak{X}$ and $T \to S \times_{\mathfrak{X}} S$ that are v-surjective with S and T perfectoid spaces.*

Thus, $\mathfrak{X}$ is a rule that sends S an $\mathbb{F}_p$-perfectoid space to a groupoid $\mathfrak{X}(S)$ such that $S \mapsto \mathfrak{X}(S)$ is a fibered category over $\mathrm{Perf}_{\mathbb{F}_p}$ satisfying: if $T \to S$ is a v-cover of aff. perf. spaces then

$$\mathfrak{X}(S) \longrightarrow \underbrace{2-\varprojlim}_{\substack{\text{objects of } \mathfrak{X}(T)\\ +\text{ descent datum}}} \Big[\, \mathfrak{X}(T) \rightrightarrows \mathfrak{X}(T \times_S T) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \mathfrak{X}(T \times_S T \times_S T)\Big]$$

is an equivalence of categories.

To say that $S \to \mathfrak{X}$, S a perfectoid space, is v-surjective means that for all T a perfectoid space, $\forall T \to \mathfrak{X}$, $\exists \widetilde{T} \to T$ a v-cover and a morphism $\widetilde{T} \to S$ such that $\widetilde{T} \to T \to \mathfrak{X}$ and $\widetilde{T} \to S \to \mathfrak{X}$ are isomorphic as elements of the groupoid $\mathfrak{X}(\widetilde{T})$

$$\begin{tikzcd}
& \widetilde{T} \arrow[d, two heads, "\exists"', "v\text{-cover}"] \arrow[dl, bend right, "\exists"'] & & \\
& T \arrow[d] \arrow[r, squiggly] & \widetilde{T} \arrow[r, bend left, ""{name=U}] \arrow[r, bend right, ""'{name=D}] & \mathfrak{X} \\
S \arrow[r] & \mathfrak{X}. & &
\arrow[from=U, to=D, Rightarrow, shorten=2pt, "\exists"']
\end{tikzcd}$$

Example 3.2.2. — *1. If S is a small v-sheaf and $H \to S$ is v-sheaf in groups that is small we can consider the classifying stack*

$$\mathfrak{X} = [S/H] \longrightarrow S.$$

This is the small v-stack over S such that for a perfectoid space T over S, $\mathfrak{X}(T)$ is the groupoid of $H\times_S T$-v-torsors over T.

2. *If $\mathfrak{X}$ is a small v-stack, S an $\mathbb{F}_p$-perfectoid space and $x\in\mathfrak{X}(S)$ we can consider the small v-sheaf in groups $\underline{Aut}(x)\to S$. The morphism $x:S\to\mathfrak{X}$ then factorizes canonically as a morphism of small v-stacks*

$$\big[S/\underline{Aut}(x)\big]\longrightarrow\mathfrak{X}.$$

3. *The stack Bun_G of G-bundles on the curve is small.*

As for small v-sheaves we have the following.

Proposition 3.2.3. — *If $\mathfrak{X}$ is a small v-stack then $\exists$ a v-hypercover*

$$S_\bullet\longrightarrow\mathfrak{X}$$

such that for all $n\geq 0$, S_n is a disjoint union of strictly totally disconnected perfectoid spaces.

3.2.2. $D_{\text{ét}}(\mathfrak{X},\Lambda)$. — Let $\mathfrak{X}$ be a small v-stack. We note

$$\widetilde{\mathfrak{X}}_v=2-\varprojlim_{\substack{S\to\mathfrak{X}\\ \text{perf. space}}}\widetilde{S}_v$$

for the topos of cartesian v-sheaves on $\mathfrak{X}$. This is the topos whose objects are small v-sheaves $\mathscr{F}$ together with a morphism $\mathscr{F}\to\mathfrak{X}$. A morphism between $\mathscr{F}\to\mathfrak{X}$ and $\mathscr{F}'\to\mathfrak{X}$ is given by a morphism $\mathscr{F}\to\mathscr{F}'$ of v-sheaves together with an isomorphism between the associated objects of the groupoid $\mathfrak{X}(\mathscr{F})$.

Definition 3.2.4. — *We note for $\mathfrak{X}$ a small v-stack*

$$D_{\text{ét}}(\mathfrak{X},\Lambda)=\left\{A\in D(\widetilde{\mathfrak{X}}_v,\Lambda)\ \middle|\ \begin{array}{c}\forall S\to\mathfrak{X},\ S\textit{ s.t.d. perf. space,}\\ A_{|S}\in D_{\text{ét}}(S,\Lambda)=D(|S|,\Lambda)\end{array}\right\}$$

This is left complete by construction since $\widetilde{S}_v$ is replete and $D_{\text{ét}}(S,\Lambda)$ is left complete for S a strictly totally disconnected perfectoid space.

As before for small v-sheaves, if

$$S_\bullet\longrightarrow\mathfrak{X}$$

is a v-hypercover by strictly totally disconnected perfectoid spaces then

$$D_{\text{ét}}(\mathfrak{X},\Lambda)\xrightarrow{\sim}D_{cart}(|S_\bullet|,\Lambda).$$

More generally for a v-hypercover by locally spatial diamonds $S_\bullet$ one has

$$D_{\text{ét}}(X,\Lambda) \xrightarrow{\sim} \underbrace{\widehat{D}_{cart}(S_{\bullet,\text{ét}},\Lambda)}_{\substack{\text{left completion of}\\ \text{derived cat. of cartesian étale sheaves}\\ \text{on } S_\bullet}}$$

Let now H be a locally pro-p topological group, typically $G(E)$ where G is an affine algebraic group over E. We consider the small v-stack

$$[\,*\,/\underline{H}]$$

where $* = \mathrm{Spa}(\overline{\mathbb{F}}_p)$ is the final object of the v-topos. If M is a Λ-module it defines a v-sheaf by setting for $S \in \mathrm{Perf}_{\mathbb{F}_p}$

$$\underline{M}(S) = \{f : |S| \to M \mid f \text{ is locally constant}\}.$$

Recall that we set

$$\underline{H}(S) = \mathscr{C}(|S|, H).$$

Suppose now that M is equipped with a smooth action of H. Then, $\underline{M}$ is equipped with an action of $\underline{H}$. In fact, if S is qc, $f : |S| \to M$ is locally constant, $g : |S| \to H$ is continuous: there exists $K \subset H$ compact open such that $f : |S[\to M^K$. Then, the composite $|S| \xrightarrow{g} H \to H/K$ is loc. constant and thus

$$\begin{array}{rcl} |S| & \longrightarrow & M \\ s & \longmapsto & g(s).f(s) \end{array}$$

is locally constant. This defines an action of the v-sheaf $\underline{H}$ on the v-sheaf $\underline{M}$ and thus a v-sheaf on $[*/\underline{H}]$. This v-sheaf is étale since isomorphic to $\underline{M}$ after pull-back to $*$ via the v-cover $* \to [*/\underline{H}]$. This defines an exact functor

$$\boxed{\{\text{smooth rep. of } H \text{ on } \Lambda\text{-modules}\} \longrightarrow \{\text{étales sheaves of } \Lambda\text{-modules on } [*/\underline{H}]\}}$$

and thus an exact functor

$$\underbrace{D(H,\Lambda)}_{\substack{\text{derived cat. of}\\ \text{smooth rep. of}\\ H \text{ wt. coeff. in } \Lambda}} \longrightarrow D_{\text{ét}}([*/\underline{H}],\Lambda).$$

We prove the following theorem.

Theorem 3.2.5. — *If Λ is killed by a power of prime number different from p then the preceding functor*

$$D(H,\Lambda) \longrightarrow D_{\text{ét}}([*/\underline{H}],\Lambda)$$

is an equivalence.

Proof. — One uses the v-hypercover

$$S_\bullet \longrightarrow [*/\underline{H}]$$

where if $C = \widehat{\overline{\overline{\mathbb{F}}_p((T))}}$ for $n \geq 0$

$$S_n = \underbrace{\mathrm{Spa}(C) \times_{\mathrm{Spd}(\overline{\mathbb{F}}_p)} \cdots \times_{\mathrm{Spd}(\overline{\mathbb{F}}_p)} \mathrm{Spa}(C)}_{(n+1)\text{-times, connected}} \times \underline{H}^n.$$

One obtains an identification

$$D_{\text{ét}}([*/\underline{H}],\Lambda) = \widehat{D}(H,\Lambda).$$

Now, the category of Λ-modules with a linear smooth H action is the category of Λ-modules in the topos of discrete H-sets (i.e. sets equipped with an action of H such that the stabilizer of a point is open). Any object in this topos has a cover formed of discrete H-sets of the form H/K for K compact open. Now, the cohoology of H/K with values in the smooth module M is $H^\bullet(K,M) := \varinjlim_{K' \subset K} H^\bullet(K/K', M^{K'})$ that is zero in > 0 degrees. ☐

Remark 3.2.6. — *The proof gives that we always have an equivalence* $\widehat{D}(H,\Lambda) \xrightarrow{\sim} D_{\text{ét}}([*/\underline{H}],\Lambda)$ *and that if* H *has a basis of compact open subgroups* K *such that* $cd_\Lambda(K) < +\infty$ *(cohomological dimension of the category* Λ*-modules equipped with a smooth action of* K*) then* $D(H,\Lambda)$ *is left complete.*

3.2.3. ∞-categorical point of view. — At the end we can apply the ∞-categorical point of view in the preceding although this is not strictly necessary.

Proposition 3.2.7. — *There exists a unique v-hypersheaf of presentable stable ∞-categories on* $\mathrm{Perf}_{\mathbb{F}_p}$,

$$S \mapsto \mathcal{D}_{\text{ét}}(S,\Lambda)$$

such that if S is a strictly totally disconnected perfectoid space then $\mathcal{D}_{\text{ét}}(S,\Lambda) = \mathcal{D}(|S|,\Lambda)$. *One has for $\mathfrak{X}$ a small v-stack*

$$\mathcal{D}_{\text{ét}}(\mathfrak{X},\Lambda) = \varprojlim_{S \to \mathfrak{X}} \mathcal{D}_{\text{ét}}(S,\Lambda)$$

with $\mathrm{Ho}\,\mathcal{D}_{\text{ét}}(\mathfrak{X},\Lambda) = D_{\text{ét}}(\mathfrak{X},\Lambda)$.

LECTURE 4

COHOMOLOGICAL OPERATIONS

4.1. The four operations $(f^*, Rf_*, R\mathscr{H}om, \otimes^{\mathbb{L}}_{\Lambda})$

4.1.1. (Rf_*, f^*) **in general.** —

4.1.1.1. *Morphisms of small v-sheaves.* — Let

$$f : X \longrightarrow Y$$

be a morphism of small v-stacks. There is an evident continuous morphism of topoi

$$(f_v^*, f_{v*}) : \widetilde{X}_v \longrightarrow \widetilde{Y}_v$$

that is a particular case of the following: if $\mathcal{T}$ is a topos and $g : U \to V$ is a morphism in $\mathcal{T}$ there is a morphism of localized topos

$$(g^*, g_*) : \mathcal{T}/U \longrightarrow \mathcal{T}/V.$$

This induces a couples of adjoint functors (use the left complete property to see that Rf_{v*} extends to the whole derived category and not only the D^+)

$$\begin{tikzcd} D(X_v, \Lambda) \arrow[r, shift right, "Rf_{v*}"'] & D(Y_v, \Lambda) \arrow[l, shift right, "f_v^*"'] \end{tikzcd}$$

Now the point is the following.

Proposition 4.1.1. — *1. f_v^* sends $D_{\text{ét}}(X, \Lambda)$ to $D_{\text{ét}}(Y, \Lambda)$ and induces a functor*

$$f^* : D_{\text{ét}}(Y, \Lambda) \longrightarrow D_{\text{ét}}(X, \Lambda)$$

2. f^* admits a right adjoint Rf_*

$$\begin{tikzcd} D_{\text{ét}}(X, \Lambda) \arrow[r, shift left, "Rf_*"] & D_{\text{ét}}(Y, \Lambda) \arrow[l, shift left, "f^*"] \end{tikzcd}$$

Proof. — Point (1) is evident since we work "in a big topos" and f_v^* is just a restriction functor. More precisely, if S is a strictly totally disconnected perfectoid space with a morphism $S \to X$, and $B \in D(Y_v, \Lambda)$ then $(f_v^* B)_{|S} = B_{|S}$ via the composite $S \to X \to Y$. For point (2) one can take

$$\boxed{Rf_* = R_{Y_{\text{ét}}} \circ (Rf_{v*})_{|D_{\text{ét}}(X,\Lambda)}.}$$

where $R_{Y_{\text{ét}}}$ is defined in lemma 3.1.5. □

Thus, there is no explicit formula in general for $Rf_* : D_{\text{ét}}(X,\Lambda) \to D_{\text{ét}}(Y,\Lambda)$.

There is an evident case when there is an explicit formula for Rf_*.

If X and Y are represented by locally spatial diamonds, via the identifications $D_{\text{ét}}(X,\Lambda) = \widehat{D}(X_{\text{ét}},\Lambda)$, $D_{\text{ét}}(Y,\Lambda) = \widehat{D}(Y_{\text{ét}},\Lambda)$ (left completion), one has

$$Rf_* = Rf_{\text{ét}*} \text{ and } f^* = f^*_{\text{ét}}$$

where $f_{\text{ét}} : X_{\text{ét}} \to Y_{\text{ét}}$ is the continuous morphism of étale sites.

In fact, it suffices to verify that $f^* = f^*_{\text{ét}}$ that is evident, the equality $Rf_* = Rf_{\text{ét}*}$ follows by adjunction.

Example 4.1.2. — *If $f : X \to Y$ is a morphism of locally of finite type K-adic spaces, K a non-archimedean field, then via the identifications $D_{\text{ét}}(X^\diamond,\Lambda) = D(X_{\text{ét}},\Lambda)$, $(f^{\diamond *}, Rf^\diamond_*)$ is the usual couple of adjoint functors $(f^*_{\text{ét}}, Rf_{\text{ét}*})$ defined by Huber ([**78**]).*

4.1.1.2. 0*-truncated morphisms of small v-stacks.* — Let

$$f : \mathfrak{X} \longrightarrow \mathfrak{Y}$$

be a morphism of small v-stacks. Suppose it is 0-truncated; this means that if $\mathscr{F}$ is a v-sheaf together with a morphism $\mathscr{F} \to \mathfrak{Y}$ then the v-stack

$$\mathfrak{X} \times_{\mathfrak{Y}} \mathscr{F}$$

is a v-sheaf in the sense that it is a fibered category in "discrete groupoids" i.e. groupoids where objects have no automorphisms, that is to say a set. Another way to say it is that it is relatively representable in v-sheaves. This is for example the case if $\mathfrak{X}$ is itself is a small v-sheaf.

There is associated a morphism of topoi of cartesian sheaves

$$\boxed{(f_v^*, f_{v*}) : \widetilde{\mathfrak{X}}_v \longrightarrow \widetilde{\mathfrak{Y}}_v}$$

This morphism of topoi exists even when f is not 0-truncated but there is a simpler expression for f_{v*} when f is 0-truncated (and this is the only case we use in [**58**]). More precisely, the category of perfectoid spaces over $\mathfrak{X}$, $\mathrm{Perf}/\mathfrak{X}$, whose objects are perfectoid spaces $S \to \mathfrak{X}$ together with morphisms given by

$$\mathrm{Hom}\big(S \xrightarrow{x} \mathfrak{X}, S' \xrightarrow{x'} \mathfrak{X}\big) = \{(f,u) \mid f : S \to S',\ u : f^*x' \xrightarrow{\sim} x\}$$

$$\begin{tikzcd}
S \arrow[rd, "x"{name=X}] \arrow[rd, bend right=40, "f^*x'"'{name=F}] \arrow[dd, "f"'] & \\
& \mathfrak{X} \\
S' \arrow[ru, "x'"'] &
\arrow[from=F, to=X, Rightarrow, "u"']
\end{tikzcd}$$

is equiped with an evident functor by composing with f

$$\mathrm{Perf}/\mathfrak{X} \longrightarrow \mathrm{Perf}/\mathfrak{Y}$$

that induces

$$f_v^* : \underbrace{2 - \varprojlim_{T \to \mathfrak{Y}} \widetilde{T}_v}_{\widetilde{\mathfrak{X}}_v} \longrightarrow \underbrace{2 - \varprojlim_{S \to \mathfrak{X}} \widetilde{S}_v}_{\widetilde{\mathfrak{Y}}_v}$$

i.e. the value (as an element of $\widetilde{S}_v$) of the cartesian sheaf $f_v^*\mathscr{F}$ on $S \to \mathfrak{X}$ is given by the value of $\mathscr{F}$ on $S \to \mathfrak{X} \xrightarrow{f} \mathfrak{Y}$ i.e.

$$(f_v^*\mathscr{F})_{|S} = \mathscr{F}_{|S}$$

via $S \to \mathfrak{X} \to \mathfrak{Y}$.

The functor f_{v*} sends the cartesian sheaf $\mathscr{F}$ to the cartesian sheaf whose value on $T \to \mathfrak{Y}$ is the pushforward via $\widetilde{\mathrm{Perf}_v}/\mathfrak{X} \times_{\mathfrak{Y}} S \to S$ of the value (as a v-sheaf sitting over the v-sheaf $\mathfrak{X} \times_{\mathfrak{Y}} S$) of $\mathscr{F}$ restricted to $\mathfrak{X} \times_{\mathfrak{Y}} S$ (this defines a cartesian sheaf since we are working with big topoi and pullback is nothing else than restriction) i.e.

$$(f_{v*}\mathscr{F})_{|S} = f_{S,v*}(\mathscr{F}_{|\mathfrak{X}\times_{\mathfrak{Y}} S})$$

where $f_S : \mathfrak{X} \times_{\mathfrak{Y}} S \longrightarrow S$.

We thus obtain a couple of adjoint functors

$$D_v(\mathfrak{X}, \Lambda) \mathrel{\substack{\xleftarrow{\ f_v^*\ } \\ \xrightarrow[\ Rf_{v*}\ ]{}}} D_v(\mathfrak{Y}, \Lambda)$$

It is immediately checked that f_v^* sends $D_{\text{ét}}(\mathfrak{Y}, \Lambda)$ to $D_{\text{ét}}(\mathfrak{X}, \Lambda)$. This is not the case for Rf_{v*} in general. We set

$$Rf_* = R_{\mathfrak{Y}_{\text{ét}}} \circ Rf_{v*|D_{\text{ét}}(\mathfrak{X},\Lambda)}$$

This defines a couple of adjoint functors

$$D_{\text{ét}}(\mathfrak{X},\Lambda) \underset{Rf_*}{\overset{f^*}{\rightleftarrows}} D_{\text{ét}}(\mathfrak{Y},\Lambda)$$

As before, there is in general no explicit formula for Rf_*.

Example 4.1.3. — *If $H' \subset H$ is a closed subgroup of H that is locally pro-p, then $f : [*/\underline{H'}] \to [*/\underline{H}]$. Then, $f^* = \mathrm{Res}^H_{H'}$ (exact functor, extends immediately to the derived category) and $Rf_* = \mathrm{Ind}^H_{H'}$ (smooth induction, exact functor).*

4.1.2. The case of a qc qs morphism representable in locally spatial diamonds. —

There is a particular case when on can compute Rf_*. This is the following.

***Proposition 4.1.4* (Quasi-compact base change)**
Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a qc qs morphism of small v-stacks representable in locally spatial diamonds i.e. $\forall S \to Y$ with S a locally spatial diamonds $X \times_Y S$ is a locally spatial diamond. Suppose that Λ is killed by a power of a prime to p integer. Let $A \in D^+_{\text{ét}}(\mathfrak{X},\Lambda)$:

1. *$Rf_{v*}A \in D_{\text{ét}}(\mathfrak{Y},\Lambda)$ and is equal to Rf_*A.*
2. *If S is a loc. spatial diamond, $S \to \mathfrak{Y}$ and*
$$f_S : \mathfrak{X} \times_{\mathfrak{Y}} S \longrightarrow S,$$
inducing
$$(f_S)_{\text{ét}} : (\mathfrak{X} \times_{\mathfrak{Y}} S)_{\text{ét}} \longrightarrow S_{\text{ét}}$$
one has for $A \in D^+_{\text{ét}}(\mathfrak{X},\Lambda)$
$$R(f_S)_{\text{ét}*}\big(A_{|\mathfrak{X}\times_{\mathfrak{Y}} S}\big) \xrightarrow{\sim} (Rf_*A)_{|S}$$
via the identifications $D((\mathfrak{X} \times_{\mathfrak{Y}} S)_{\text{ét}},\Lambda) = D_{\text{ét}}(\mathfrak{X} \times_{\mathfrak{Y}} S,\Lambda)$ and $D(S_{\text{ét}},\Lambda) = D_{\text{ét}}(S,\Lambda)$.

The proof uses Huber's quasi-compact base change ([**78**, Theorem 4.3.1]). The hypothesis that Λ is killed by a power of ℓ with $\ell \neq p$ is essential. In fact, already the étale cohomology of the one dimensional ball over $C|\mathbb{Q}_p$ algebraically closed, $H^\bullet_{\text{ét}}(\mathbb{B}^1_C,\mathbb{F}_p)$, depends on C and thus qc base change does not hold in this situation.

Here is a strinking application.

Corollary 4.1.5. — *Let $j : \mathfrak{U} \hookrightarrow \mathfrak{X}$ be an open immersion of small v-stacks. Suppose that j is qc qs. Then for A an* $\underbrace{\text{étale } v\text{-sheaf of } \Lambda\text{-modules}}_{\substack{\text{an object of } D_{\text{ét}}(\mathfrak{U},\Lambda) \\ \text{concentrated in deg. } 0}}$ *on $\mathfrak{U}$ with Λ killed by a power of $\ell \neq p$,*

$$\boxed{R^i j_* A = 0 \text{ for } i > 0.}$$

Proof. — The proof consist in computing the pullback of $R^i j_* A$ via $\mathrm{Spa}(C, C^+) \to \mathfrak{X}$ using the qc base change theorem. If U is a qc open subset of $\mathrm{Spa}(C, C^+)$, $j' : U \hookrightarrow \mathrm{Spa}(C, C^+)$, since any qc open subset of $\mathrm{Spa}(C, C^+)$ is strictly totally disconnected one has $R^i j'_* = 0$ for $i > 0$. ☐

The quasi-compactness assertion is essential. For example, if $j : \mathbb{B}^{1,\diamond}_{K,K^+} \smallsetminus \{0\} \hookrightarrow \mathbb{B}^{1,\diamond}_{K,K^+}$ is the inclusion of the punctured closed ball over the affioid field (K, K^+) inside the ball then $R^1 j_* \mathbb{F}_\ell \neq 0$ if ℓ is invertible in K.

Remark 4.1.6. — *The stack* Bun_G *is not quasi-separated and thus Corollary 4.1.5 does not apply to open subs-tacks of* Bun_G*. For example, $j : [*/\underline{G(E)}] \hookrightarrow \mathrm{Bun}_G$ and to any π a smooth rep. of $G(E)$ one can associate $\mathscr{F}_\pi$ an étale v-sheaf on $[*/\underline{G(E)}]$. Then in general $R^i j_* \mathscr{F}_\pi \neq 0$ for $i > 0$.*

4.1.3. $\otimes^{\mathbb{L}}_{\Lambda}$ and $R\mathscr{H}om(-,-)$.

— Let $\mathfrak{X}$ be a small v-stack. It is easy to verify that $- \otimes^{\mathbb{L}}_{\Lambda} -$ on $D_v(\mathfrak{X}, \Lambda) \times D_v(\mathfrak{X}, \Lambda)$ sends $D_{\text{ét}}(\mathfrak{X}, \Lambda) \times D_{\text{ét}}(\mathfrak{X}, \Lambda)$ to $D_{\text{ét}}(\mathfrak{X}, \Lambda)$.

Now, for $A \in D_{\text{ét}}(\mathfrak{X}, \Lambda)$ we can look at the functor

$$\begin{array}{ccc} D_{\text{ét}}(\mathfrak{X}, \Lambda) & \longrightarrow & D_{\text{ét}}(\mathfrak{X}, \Lambda) \\ B & \longmapsto & A \otimes^{\mathbb{L}}_{\Lambda} B \end{array}$$

This commutes with colimits and thus has a right adjoint (Freyd's adjunction theorem)

$$C \longmapsto R\mathscr{H}om_{\Lambda}(A, C).$$

Once again, like Rf_*, there is no explicit formula in general for this functor.

If $S_\bullet \to \mathfrak{X}$ is a v-hypercover by a disjoint union of strictly totally disconnected perfectoid spaces then via

$$D_{\text{ét}}(\mathfrak{X}, \Lambda) \xrightarrow{\sim} D_{cart}(|S_\bullet|, \Lambda),$$

one has

$$R\mathscr{H}om_\Lambda(A,B) = R \underbrace{\mathrm{Cart}}_{\substack{\text{Cartesianification}\\ \text{functor}}} \underbrace{R\mathscr{H}om_\Lambda(A,B)}_{\substack{\text{usual } R\mathscr{H}om\\ \text{in } D(|S_\bullet|,\Lambda)}}$$

As for Rf_*, if $\mathfrak{X} = X$ is a locally spatial diamond then $R\mathscr{H}om_\Lambda(A,B)$ is the usual derived functor computed in $\widehat{D}(X_{\text{ét}},\Lambda)$.

4.2. The two operations $(Rf_!, Rf^!)$

4.2.1. Huber's canonical compactification. —

4.2.1.1. *Classical context. — Recall:* Let

$$f : X \to Y$$

be a morphism of adic spaces locally of finite type over $\mathrm{Spa}(K,K^+)$, an affinoid analytic field. We say that f is proper if it is qc separated and universally closed. This is equivalent to saying that f is qc qs and $\forall (R,R^+)$ topologically of finite type over (K,K^+),

$$\begin{tikzcd}
\mathrm{Spa}(R,R^\circ) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\mathrm{Spa}(R,R^+) \arrow[r] \arrow[ur, dashed, "\exists !"] & Y
\end{tikzcd}$$

This last property is called partially proper. Thus,

$$\text{proper} \iff \text{quasi-compact quasi-separated and partially proper.}$$

Separated partially proper morphisms are exactly the good one for which the derived functor of $f_{\text{ét}!}$ is the good notion for relatic cohomology with proper support. More precisely, if $\mathscr{F}$ is an étale sheaf on X and $U \to Y$ is étale then

$$\Gamma(U, f_{\text{ét}!}\mathscr{F}) = \{s \in \Gamma(X \times_Y U, \mathscr{F}) \mid \mathrm{supp}(s) \xrightarrow{f_{|\,\mathrm{supp}(s)}} U \text{ is proper}\}.$$

Then,

$$Rf_{\text{ét}!} : D(X_{\text{ét}},\Lambda) \longrightarrow D(Y_{\text{ét}},\Lambda)$$

is "the good relative cohomology with proper support" functor.

Example 4.2.1. *— Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal schemes locally formally of finite type over Spf($\mathcal{O}_K$). Let $\mathfrak{X}_\eta$, resp. $\mathfrak{Y}_\eta$, be their generic fiber as adic spaces locally of finite type over* $\mathrm{Spa}(K,\mathcal{O}_K)$*. Then if $f_\eta : \mathfrak{X}_\eta \longrightarrow \mathfrak{Y}_\eta$,*

$$\forall Z \text{ irred. comp. of } \mathfrak{X}_{red},\ f_{red|Z} \text{ proper} \iff f_\eta \text{ partially proper.}$$

For $f : X \to Y$ as before Huber says that f is *taut* if $\forall V \subset Y$ open qc qs and $U \subset f^{-1}(V)$ qc then $\overline{U}$ is qc. Let f be separated and taut. Then Huber defines a canonical compactification

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[d, "f"'] & \overline{X}^{/Y} \arrow[dl, "\bar{f}"] \\ Y & \end{tikzcd}$$

where j is an open immersion and $\bar{f}$ is separated partially proper. Then he defines

$$Rf_{\text{ét}!} = R\bar{f}_{\text{ét}!} \circ j_!$$

and proves Poincaré duality in this context when f is moreover smooth; this last point proves that this is the good definition for relative cohomology with proper support.

When $X = \text{Spa}(B, B^+)$ and $Y = \text{Spa}(A, A^+)$ define

$$(B^+)' = \text{integrale closure of } f^*(A^+) + B^{\circ\circ}.$$

Then,

$$X = \text{Spa}(B, B^+) \xhookrightarrow[\text{open immersion}]{} \overline{X}^{/Y} = \text{Spa}(B, (B^+)')$$

is an open immersion.

In fact, since f is of finite type, by definition, there exists an open surjective morphism $A\langle T_1, \dots, T_n\rangle \to B$ such that B^+ is the integral closure of the image of $A^+\langle T_1, \dots, T_n\rangle$. Thus $A^+/A^{\circ\circ} \to B^+/B^{\circ\circ}$ is integral over a finite type $A^+/A^{\circ\circ}$-algebra. Now, if $g_1, \dots, g_n \in B^+$ is a lift of a set of elements $\bar{g}_1, \dots, \bar{g}_n \in B^+/B^{\circ\circ}$ such that $B^+/B^{\circ\circ}$ is integral over $A^+/A^{\circ\circ}[\bar{g}_1, \dots, \bar{g}_n]$ then

$$X = \{|g_1| \leq 1, \dots, |g_n| \leq 1\} \subset \overline{X}^{/Y}$$

Remark 4.2.2. — *Of course,* $|\overline{X}^{/Y}| \setminus |X|$ *is made of rank* > 1 *valuations only.*

Example 4.2.3. — *Take* $X = \mathbb{B}^1_K = \mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle) \to \mathrm{Spa}(K, \mathcal{O}_K) = Y$. *Then,*

$$\overline{X}^{/Y} = \mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K + K^{\circ\circ}\langle T\rangle).$$

One has $|\overline{X}^{/Y}| = |X| \cup \{x\}$ *where*

$$v\Big(\sum_{n\geq 0} a_n T^n(x)\Big) = \inf\Big\{(v(a_n), -n) \in \underbrace{\Gamma_K \times \mathbb{Z}}_{\substack{+lexicographic\\ order}} \quad | \ n \geq 0\Big\}$$

with x *a specialization of the Gauss norm.*

$X =$ *adic space associated to a classical rigid space i.e.* $\mathrm{Sp}(R) \mapsto \mathrm{Spa}(R, R^\circ)$
$\overline{X}^{/Y}$ $=$ *adic space not associated to a classical rigid space i.e.* $\mathrm{Spa}(R, R^+)$ *with* $R^+ \neq R^\circ$.

For Berkovich spaces one considers only rank 1 valuations and $|\mathbb{B}^1_K|$ is compact. Thus, for Berkovich spaces, the cohomology with compact support equals the cohomology, and this is thus not the good definition of cohomology with compact support: $\partial\mathbb{B}^{1,an}_K \neq \emptyset \Leftrightarrow \mathbb{B}^{1,ad}_K \to \mathrm{Spa}(K, \mathcal{O}_K)$ not partially proper. We need to consider non-overconvergent étale sheaves like $i_{x*}\Lambda$ to define the cohomology with compact support. This is why we can not define cohomology with compact support in general for K-Berkovich spaces X such that $\partial(X/K) \neq \emptyset$, typically for affinoid Berkovixh spaces X where $|X|$ is compact and thus $\Gamma_c(X, -) = \Gamma(X, -)$. In fact if $X^{Berk} = \mathcal{M}(A)$ and $X^{ad} = \mathrm{Spa}(A, A^\circ)$ then there is an equivalence of topoi

$$\widetilde{(X^{Berk})_{\text{ét}}} \xrightarrow{\sim} \{\text{overconvergent étale sheaves on } X^{ad}\}$$

where overconvergent means that for $x \in X$, if $x : \mathrm{Spa}(C, C^+) \to X$, C alg. closed,

$$x^*\mathscr{F}$$

is a constant sheaf on $|\mathrm{Spa}(C, C^+)|$ with value its stalk at the generic point $\mathrm{Spa}(C, \mathcal{O}_C)$; equivalently, if U is a qc open subset of X^{ad} then

$$\varinjlim_{U \subset\subset V} \Gamma(V, \mathscr{F}) \xrightarrow{\sim} \Gamma(U, \mathscr{F})$$

where $U \subset\subset V$ means $\overline{U} \subset V$. Morale of the story: even to define compactly supported cohomology for overconvergent sheaves like $\underline{\Lambda}$ we need to go through non-overconvergent sheaves when f is not partially proper.

4.2.2. Compactifiable morphisms of small v-stacks. — Let

$$f : \mathfrak{X} \longrightarrow \mathfrak{Y}$$

be a 0-trunncated morphism of small v-stacks. Define

$$\underbrace{\overline{\mathfrak{X}}}_{\substack{\text{absolute compactification}\\ \text{of } \mathfrak{X}\to *}} = v\text{-stack}$$

such that $\overline{\mathfrak{X}}(R, R^+) = \mathfrak{X}(R, R^\circ)$. This is the "absolute compactification of $\mathfrak{X}$ over $*$". Then, one verifies that

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, hook] \arrow[d, "f"] & \overbrace{\overline{\mathfrak{X}}}^{\text{small}v\text{-stack}} \arrow[d, "\bar{f}=0\text{-truncated}"] \\
\mathfrak{Y} \arrow[r, hook] & \underbrace{\overline{\mathfrak{Y}}}_{\text{small}v\text{-stack}}
\end{tikzcd}$$

And define

$$\underbrace{\overline{\mathfrak{X}}^{/\mathfrak{Y}}}_{\substack{\text{relative}\\ \text{compactification}}} = \overline{\mathfrak{X}} \times_{\overline{\mathfrak{Y}}} \mathfrak{Y}$$

There is a diagram

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, hook, "j"] \arrow[d, "f"'] & \overline{\mathfrak{X}}^{/\mathfrak{Y}} \arrow[dl, "\overline{f}^{/\mathfrak{Y}}"] \\
\mathfrak{Y}. &
\end{tikzcd}$$

We now take the result that says that for separated taut morphisms of adic spaces locally of finite type over $\mathrm{Spa}(K, K^+)$, j is an open immersion as definition.

Definition 4.2.4. — *The morphism f is compactifiable if it is separated and $j : \mathfrak{X} \hookrightarrow \overline{\mathfrak{X}}^{/\mathfrak{Y}}$ is representable by an open immersion.*

One has f qc compactifiable $\Rightarrow \overline{f}^{/\mathfrak{Y}}$ is proper $\Rightarrow$ we really have a canonical compactification. In general, if f is not qc then $\overline{f}^{/\mathfrak{Y}}$ is only partially proper.

Remark 4.2.5. — *One has to be careful that f representable in loc. spatial diamonds compactifiable does only imply that $\overline{f}^{/\mathfrak{Y}}$ is representable in diamonds but à priori non locally spatial one..although in all cases when we apply this compactification construction this is the case.*

Example 4.2.6. — *If $f : X \to Y$ is a separated taut morphism of adic spaces locally of finite type over* $\mathrm{Spa}(K, K^+)$ *then $f^\diamond : X^\diamond \to Y^\diamond$ is compactifiable.*

4.2.3. Geometric transcendance degree. — We need to bound some cohomological dimension to have a "good" $Rf_!$.

Let $C'|C$ be an extension of complete algebraically closed non-archimedean fields. Define the topological transcendance degree

$$\operatorname{tr.c}(C'/C) \in \mathbb{N} \cup \{+\infty\}$$

as the minimum of the cardinal of I where there exists $(x_i)_{i\in I} \in C'^I$ such that the sub-field $\overline{C(x_i)_{i\in I}}$ (algebraic closure) of C' is dense in C'.

This number is "well behaved" when finite ([**136**]): if $C \subset C' \subset C''$ and $\operatorname{tr.c}(C'/C) < +\infty$ then $\operatorname{tr.c}(C'/C) \leq \operatorname{tr.c}(C''/C)$. But it may happen (the answer to this is not known) that $\operatorname{tr.c}(C'/C) = +\infty$ and $\operatorname{tr.c}(C''/C) < +\infty$. Since we don't know we set

$$\underbrace{\widetilde{\operatorname{tr.c}}(C'/C)}_{\substack{\text{topological}\\ \text{transcendance degree}}} \quad = \inf\nolimits_{C''/C'} \operatorname{tr.c}(C''/C).$$

Then $C'|C \mapsto \widetilde{\operatorname{tr.c}}(C'/C)$ is monotonic sub-additive: for $C''|C'|C$

$$\begin{aligned} \widetilde{\operatorname{tr.c}}(C'/C) &\leq \widetilde{\operatorname{tr.c}}(C''/C) \\ \widetilde{\operatorname{tr.c}}(C''/C) &\leq \widetilde{\operatorname{tr.c}}(C''/C') + \widetilde{\operatorname{tr.c}}(C'/C). \end{aligned}$$

We now take the following definition for the *geometric transcendence degree.*

Definition 4.2.7. — *1. Let $f : X \to Y$ be a morphism of diamonds. Set*

$$\underbrace{\operatorname{dim.trg}}_{\substack{\textit{geometric}\\ \textit{transcendance degree}}} \quad f = \sup_{x\in X} \widetilde{\operatorname{tr.c}}(C(x)|C(f(x)))$$

where $\operatorname{Spa}(C(x), C(x)^+) \to X$ *and* $\operatorname{Spa}(C(f(x), C(f(x))^+) \to Y$ *are quasi-pro-étale with x and $f(x)$ in their image and we have an extension*

$$\begin{tikzcd} \operatorname{Spa}(C(x), C(x)^+) \arrow[r] \arrow[d, dashed] & X \arrow[d, "f"] \\ \operatorname{Spa}(C(f(x), C(f(x)^+) \arrow[r] & Y. \end{tikzcd}$$

2. if $f : \mathfrak{X} \to \mathfrak{Y}$ is representable in diamonds set

$$\operatorname{dim.trg}(f) = \sup_{\substack{S\to Y\\ S \text{ diamond}}} \dim .trg(X \times_{\mathfrak{Y}} S \to S).$$

Example 4.2.8. — *Let (D, φ) be an isocrystal with ≤ 0 slopes. Then*

$$f : BC(D, \varphi) \longrightarrow *$$

is of finite geometric transcendence degree. As a matter of fact, there exists $d \geq 0$, V a finite dimensional E-vector space, such that for any S there exists a pro-étale surjection

$$\mathring{\mathbb{B}}_S^{d,1/p^\infty} \times \underline{V} \longrightarrow BC(D, \varphi) \times_{\mathrm{Spa}(\overline{\mathbb{F}}_q)} S$$

and $\dim .trg(f) \leq d$.

Proposition 4.2.9 **(Key cohomological bound)**

Let $f : X \to \mathrm{Spa}(C, C^+)$ be a spatial diamond:

1. *For all maximal point $x \in X$,*
$$X_x \simeq \mathrm{Spa}(C', \mathcal{O}_{C'})/\underline{G_x}$$
for a profinite group $G_x \subset \mathrm{Aut}(C')$ satisfying
$$cd_\ell(G_x) \leq dim.\ trg.(f).$$
2. *One has*
$$\dim |X| \leq dim.\ trg.(f)$$
3. *For all $\mathscr{F}$ étale sheaf of Λ-modules on X with Λ killed by a power of $\ell \neq p$ one has*
$$H^i_{\text{ét}}(X, \mathscr{F}) = 0 \quad for \quad i > 2 dim.\ trg.(f).$$
4. *If f is compactifiable then*
$$H^i_{\text{ét,c}}(X, \mathscr{F}) = 0 \quad for\ i > 3 \dim .trg.(f).$$
(in fact $2 \dim .trg.(f)$ if $\overline{X}^{/\mathrm{Spa}(C,C^+}$, that is à priori only a diamond, is moreover spatial).

Here $\dim |X|$ is the usual dimension of a spectral space: the maximal length of a chain of specializations.

The proof is identical to the one for schemes: if X is a finite type k-scheme, k alg. closed, and $\mathscr{F}$ an étale torsion sheaf on X then $H^i_{\text{ét}}(X, \mathscr{F}) = 0$ for $i > 2 \dim X$:

1. One has first that (Tsen's theorem: k alg. closed implies $k(T)$ is (C1) and thus $cd(G_{k(T)}) \leq 1$; and thus if $K|k$ is finite type then $cd(G_K) \leq tr.deg(K/k)$) for all $x \in X$, $cd(\mathrm{Spec}(k(x))) \leq \dim(X)$
2. We use the projection $\mu : X_{\text{ét}} \to X_{Zar}$
3. We use (Grothendieck: Noetherian induction on open subsets of $|X|$) that if S is a Noetherian topological space then $cd(S) \leq \dim(S)$.

LECTURE 5

COHOMOLOGICALLY SMOOTH MORPHISMS AND ARTIN v-STACKS

5.1. The two operations $(Rf_!, Rf^!)$

5.1.1. $Rf_!$ for f representable in spatial diamonds. — We are seeking to define $(Rf_!, Rf^!)$ for f representable in locally spatial diamonds compactifiable of finite dim. trg.. Let us begin first with the case when f is qc i.e.f *is representable in spatial diamonds.*

From now on Λ *is killed by a power of $\ell \neq p$.* This is used to bound the cohomological dimensions of $Rf_!$ by 3dim. trg.(f) (and even 2dim. trg.(f) if $\overline{f}^{/\mathfrak{Y}}$ is representable in loc. spatial diamonds), see proposition 4.2.9.

Definition 5.1.1. — *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of small v-sheaves $\underbrace{\text{representable in spatial diamonds}}_{\text{and thus qc}}$ compactifiable of finite* dim. trg. *We define*

$$Rf_! = R(\underbrace{\overline{f}^{/\mathfrak{Y}}}_{\substack{\text{proper morphism}\\ \text{representable in qc qs diamonds}}})_* \circ j_! : D_{\text{ét}}(\mathfrak{X}, \Lambda) \longrightarrow D_{\text{ét}}(\mathfrak{Y}, \Lambda).$$

where $j : \mathfrak{X} \hookrightarrow \overline{\mathfrak{X}}^{/\mathfrak{Y}}$.

The finite dim. trg. implies that $Rf_!$ *commutes with direct sums* and has finite cohomological dimension. As a matter of fact, $Rf_!$ is first defined as a functor $D^+_{\text{ét}}(\mathfrak{X}, \Lambda) \to D^+_{\text{ét}}(\mathfrak{Y}, \Lambda)$ and then extended by left completion. Thanks to the finite cohomological dimension hypothesis, for any $i \in \mathbb{Z}$ there exists $n \in \mathbb{Z}$ such that for any $A \in D^+_{\text{ét}}(\mathfrak{X}, \Lambda)$, $\mathcal{H}^i(Rf_!A)$ depends only on $\tau_{\geq n}A$. This implies that to verify that $R_f! : D_{\text{ét}}(\mathfrak{X}, \Lambda) \longrightarrow D_{\text{ét}}(\mathfrak{Y}, \Lambda)$ commutes with direct sums it suffices to do it for the functor $D^{\geqslant 0} \to D^{\geqslant 0}$ which is clear.

5.1.2. $Rf_!$ for f a morphism of loc. spatial diamonds. — $f : X \to Y$ a morphism of locally spatial diamonds, compactifiable of finite dim. trg..

Definition 5.1.2. — *We define*

$$Rf_! : D^+_{\text{ét}}(X,\Lambda) \longrightarrow D^+_{\text{ét}}(Y,\Lambda)$$

as $R(\overline{f}^{/Y})_{\text{ét}!} \circ j_!$ *where* $(\overline{f}^{/Y})_{\text{ét}!}$ *is the functor from sheaves of* Λ*-modules on* $\underbrace{(\overline{X}^{/Y})_{\text{ét}}}_{\substack{\textit{small site of}\\ \textit{loc. separated}\\ \textit{étale morphisms}}}$ *to sheaves of* Λ*-modules on* $Y_{\text{ét}}$ *that is the subfunctor* $f_{\text{ét}*}$ *of section with relative proper support.*

The extension to the full derived category $D_{\text{ét}}(X,\Lambda)$ is delicate. For Rf_*, the extension from $D^+_{\text{ét}}$ to $D_{\text{ét}}$ is straightforward since Rf_* commutes with cofiltered limits and we can use the formula $Rf_*A = \varprojlim_{n\geq 0} Rf_*\tau_{\geq -n}A$ that makes sense since our target category is left complete. This is not the case of $Rf_!$ that commutes with filtered colimits but not with cofiltered limit.

We have to use a process of left Kan extension to solve this, this can only be done in the ∞-categorical setting $\to$ *since $Rf_!$ has to commute with colimits this has to commute with left Kan extensions*: this property forces the definition of $Rf_!$ as a left Kan extension.

Definition 5.1.3. — *Let* $\mathcal{D}_{\text{ét,prop/Y}}(X,\Lambda)$ *be the presentable stable* ∞*-category of* $A \in \mathcal{D}_{\text{ét}}(X,\Lambda)$ *such that there exists* $U \subset X$ *such that* $U \to Y$ *is qc. and* $j_!j^*A \xrightarrow{\sim} A$*. The funtor* $Rf_!$ *is the left Kan extension of*

$$R(\overline{f}^{/Y})_* \circ j_! : \mathcal{D}_{\text{ét,prop/Y}}(X,\Lambda) \longrightarrow \mathcal{D}_{\text{ét}}(Y,\Lambda)$$

to $\mathcal{D}_{\text{ét}}(X,\Lambda)$*.*

Thus, for $A \in \mathcal{D}_{\text{ét}}(X,\Lambda)$, since $X \to Y$ is taut since compactifiable, one can write $A = \varinjlim_i A_i$ (filtered colimit as a complex of v-sheaves) with $k_{i!}k_i^*A_i \xrightarrow{\sim} A_i$ with $k_i : U_i \hookrightarrow X$ and $U_i \to Y$ qc.. Take simply $A_i = k_{i!}k_i^*A$.

Then,

$$Rf_!A = \underbrace{\varinjlim_i R(\overline{f_{|U_i}}^{/Y})_* j_{i!}A_i}_{\text{homotopy colimit}}$$

where $j_i : U_i \hookrightarrow \overline{U_i}^{/Y}$ and $\overline{f_{|U_i}}^{/Y} : \overline{U_i}^{/Y} \to Y$ is proper. Here the homotopy limit is only defined up to a non-canonical isomorphism in the usual triangulated category

$D_{\text{ét}}(Y,\Lambda)$. This defines $Rf_!A$ only up to a non-canonical isomorphism as a cone. To have a definition of $Rf_!$ as a functor we need to upgrade to ∞-categories where this limit, the so-called process of Kan extension, is defined canonically.

5.1.3. $Rf_!$ for f representable in locally spatial diamonds. — Let $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ be a morphism of small v-stacks representable in locally spatial diamonds compactifiable of finite dim. trg..

We use the preceding with the proper base change theorem to construct $Rf_!$. Let

$$T_\bullet \longrightarrow \mathfrak{Y}$$

be a v-hypercover with T_n a locally spatial diamond for all n. We note

$$S_\bullet = \mathfrak{X} \times_{\mathfrak{Y}} T_\bullet.$$

One has ($S \mapsto \mathcal{D}_{\text{ét}}(S,\Lambda)$ is a v-hypersheaf on locally spatial diamonds)

$$\begin{aligned} \mathcal{D}_{\text{ét}}(\mathfrak{X},\Lambda) &\xrightarrow{\sim} \varprojlim_{[n]\in\Delta} \mathcal{D}_{\text{ét}}(S_n,\Lambda) \\ \mathcal{D}_{\text{ét}}(\mathfrak{Y},\Lambda) &\xrightarrow{\sim} \varprojlim_{[n]\in\Delta} \mathcal{D}_{\text{ét}}(T_n,\Lambda). \end{aligned}$$

For each $n \geq 0$, we have the ∞-functor

$$Rf_{n!} : \mathcal{D}_{\text{ét}}(S_n,\Lambda) \longrightarrow \mathcal{D}_{\text{ét}}(T_n,\Lambda).$$

Proper base change (that is an immediate application of quasi-compact base change) implies this extends to an ∞-functor

$$\boxed{Rf_! : \varprojlim_{[n]\in\Delta} \mathcal{D}_{\text{ét}}(S_n,\Lambda) \longrightarrow \varprojlim_{[n]\in\Delta} \mathcal{D}_{\text{ét}}(T_n,\Lambda).}$$

All of this being done, proper base change applies.

***Theorem 5.1.4* (proper base change).** — *$f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ morphism of small v-stacks representable in loc. spatial diamonds compactifiable of finite dim. trg., Λ killed by a power of $\ell \neq p$. Consider a cartesian diagram of small v-stacks*

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, "f'"] \arrow[d, "g'"'] & \mathfrak{Y}' \arrow[d, "g"] \\ \mathfrak{X} \arrow[r, "f"] & \mathfrak{Y} \end{tikzcd}$$

The, for all $A \in D_{\text{ét}}(\mathfrak{X},\Lambda)$ one has

$$g^*Rf_!A \xrightarrow{\sim} Rf'_!g'^*A.$$

5.1.4. $Rf^!$. —

Since $Rf_!$ commutes with direct sums (see the after definition 5.1.1) there exists a right adjoint $Rf^!$ by an application of Freyd's adjunction theorem. We can thus take the following definition.

Definition 5.1.5. — *For $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ a morphism of small v-stacks representable in locally spatial diamonds, compactifiable of finite dim. trg.*

$$Rf^! : D_{\text{ét}}(\mathfrak{Y}, \Lambda) \longrightarrow D_{\text{ét}}(\mathfrak{X}, \Lambda)$$

is the right adjoint of $Rf_!$.

5.2. Annexe: the catalog of operations

Consider a cartesian diagram

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, "f'"] \arrow[d, "g'"'] & \mathfrak{Y}' \arrow[d, "g"] \\ \mathfrak{X} \arrow[r, "f"] & \mathfrak{Y} \end{tikzcd}$$

where all morphisms are compactifiable representable in locally spatial diamonds of finite dim. trg. and Λ is killed by a power of $\ell \neq p$.

Relative tautological adjunction	$R\mathscr{H}om(A, Rf_*B) \xrightarrow{\sim} Rf_*R\mathscr{H}om(f^*A, B)$
Tautological base change map (iso. if g qc qs: qc base change)	$f^*Rg_* \longrightarrow Rg'_*f'^*$
Relative proper adjunction	$R\mathscr{H}om(Rf_!A, B) \xrightarrow{\sim} Rf_*R\mathscr{H}om(A, Rf^!B)$
Proper base change	$f^*Rg_! \xrightarrow{\sim} Rg'_!f'^*$
Dual proper base change	$Rg'_*Rf'^! \xrightarrow{\sim} Rf^!Rg_*$
Projection formula	$Rf_!(f^*A \otimes^{\mathbb{L}}_{\Lambda} B) \xrightarrow{\sim} A \otimes^{\mathbb{L}}_{\Lambda} Rf_!B$
Expectational pull-back of Hom	$R\mathscr{H}om(f^*A, Rf^!B) \xrightarrow{\sim} Rf^!R\mathscr{H}om(A, B)$

Those formulas are deduced from the proper base change and the projection formula. For example, for the "dual proper base change",

$$\begin{aligned}\mathrm{Hom}(A, Rf^!Rg_*B) \quad &\underbrace{=}_{\text{adjunction}} \quad \mathrm{Hom}(g^*Rf_!A, B) \underbrace{=}_{\text{proper BC}} \mathrm{Hom}(Rf'_!g'^*A, B)\\ &\underbrace{=}_{\text{adjunction}} \quad \mathrm{Hom}(A, Rg'_*Rf'_!B)\end{aligned}$$

and thus Yoneda lemma implies the result. The same goes on for the exceptional pullback of Hom.

5.3. Cohomologically smooth morphisms

5.3.1. Definition. — The definition of cohomologically smooth morphisms is more subtle than what one may think: we have to force the property to be stable under base change. Here $\ell \neq p$.

Definition 5.3.1. — *Let $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ be a separated morphism of small v-stacks representable in locally spatial diamonds. Then, f is ℓ-coho. smooth if*

1. *It is compactifiable of finite dim. trg.,*
2. *For any $S \to \mathfrak{Y}$ with S strictly totally disconnected, if*
$$f_S : \mathfrak{X} \times_{\mathfrak{Y}} S \longrightarrow S$$
then there exists $D \in D_{\text{ét}}(\mathfrak{X} \times_{\mathfrak{Y}} S, \mathbb{F}_\ell)$ invertible and an isomorphism of functors from $D_{\text{ét}}(S, \mathbb{F}_\ell)$ to $D_{\text{ét}}(\mathfrak{X} \times_{\mathfrak{Y}} S, , \mathbb{F}_\ell)$,
$$Rf_S^!(-) \simeq D \otimes_{\mathbb{F}_\ell} f_S^*(-).$$

Now, let us remark that for any f as before there is a natural morphism obtained by playing with the different adjunctions

$$Rf^!(\Lambda) \otimes^{\mathbb{L}}_{\Lambda} f^*(-) \longrightarrow Rf^!(-)$$

Then f is ℓ-cohomologically smooth iff for all $S \to \mathfrak{Y}$ with S a strictly totally disconnected perfectoid space,

1. $Rf_S^!(\mathbb{F}_\ell) \otimes f_S^*(-) \longrightarrow Rf_S^!(-)$ is an iso.
2. $Rf_S^! \mathbb{F}_\ell$ is invertible.

Here invertible means invertible with respect to the monoidal structure $\otimes^{\mathbb{L}}_{\Lambda}$. This is in fact equivalent to be étale locally isomorphic to $\mathbb{F}_\ell[2d]$ for some $d \in \frac{1}{2}\mathbb{Z}$ that we call the dimension of f as a locally constant function on $|S|$.

We can descend the preceding and prove ([**128**, Section 23]):

Theorem 5.3.2. — *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be separated ℓ-cohomologically smooth. Then, if Λ is killed by a power of ℓ,*
$$Rf^!(\Lambda) \otimes^{\mathbf{L}}_{\Lambda} f^*(-) \xrightarrow{\sim} Rf^!(-)$$
as functors from $D_{\text{ét}}(\mathfrak{Y}, \Lambda)$ to $D_{\text{ét}}(\mathfrak{X}, \Lambda)$, and $Rf^!\Lambda$ is invertible in $D_{\text{ét}}(\mathfrak{X}, \Lambda)$. Moreover, the formation of $Rf^!(\Lambda)$ is compatible with base change.

Now, the function "dimension of f" is a locally constant function $|\mathfrak{X}| \to \frac{1}{2}\mathbb{Z}$.

5.3.2. Examples. —

5.3.2.1. *First easy examples.* — Here are some evident examples.

Example 5.3.3. — 1. *Any separated étale morphism of locally spatial diamonds is separated ℓ-cohomologically smooth.*

2. *Any perfectoid ball $\mathbb{B}^d \to *$ is separated ℓ-cohomologically smooth.*
3. *If $f : X \to Y$ is a separated smooth morphism of noetherian adic spaces then $f^\diamond : X^\diamond \to Y^\diamond$ is separated ℓ-cohomologically smooth (*[**78**, Section 7]*).*
4. *Let k be a discrete field,* $\mathrm{Spd}(k(T)) \to \mathrm{Spd}(k)$ *is separated ℓ-cohomologically smooth.*

5.3.2.2. *Open B_{dR}-Schubert cells.* — The following case is used in the geometric Satake correspondence.

Example 5.3.4. — *Let G be a split reductive group over E. For a dominant coweight μ let*

$$\mathrm{Gr}_{G,\mu} \longrightarrow \mathrm{Spa}(E)^\diamond$$

be the open Schubert cell of the associated B_{dR}-affine Grassmanian. This is an ℓ-coho. smooth morphism.

This is proven using a Bialynicki-Birula morphism. More precisely, if μ can be written as a sum of dominant minuscule cocharacters there is associated to such a writing a Bialynicki-Birula morphism

$$\mathrm{Gr}_{G,\mu} \quad \underbrace{\longrightarrow}_{\substack{\text{iterated loc. trivial} \\ \text{fibration in } \mathbb{A}^{1,\diamond}}} \quad (\underbrace{G/P_\mu}_{\substack{\text{flag variety} \\ \text{smooth}/E}})^\diamond \longrightarrow \mathrm{Spa}(E)^\diamond.$$

In general, one can consider the B_{dR}-affine flag manifold

$$\mathcal{F}\ell_G = G(\mathbb{B}_{dR})/I_{dR}$$

where $I_{dR} \subset G(\mathbb{B}_{dR}^+)$ is the reciprocal image of $B^\diamond$ via $\theta : G(\mathbb{B}_{dR}^+) \to G^\diamond$. Here $\mathbb{B}_{dR}^+$ is the sheaf of ring on $\mathrm{Spa}(E)^\diamond$ that associates to (R, R^+) affinoid perfectoid over $\mathbb{F}_q$ an untilt $(R^\sharp, R^{\sharp,+})$ and an element of the completion of $W_{\mathcal{O}_E}(R^+)\left[\frac{1}{p}\right]$ along $\ker\theta$, $\theta : W_{\mathcal{O}_E}(R^+) \to R^{\sharp,+}$. Let $\widetilde{W}$ be the affine Weyl group. If $w \in \widetilde{W}$ maps to μ and is of maximal length among those mapping to μ then there is a diagram

$$\begin{tikzcd}
\mathcal{F}\ell_{G,w} \arrow[r, "\ell\text{-coho.sm.}"] \arrow[d, "\ell\text{-coho.sm.}"', "\mathrm{BL}"] & \mathrm{Gr}_{G,\mu} \\
\mathcal{F}\ell_\mu &
\end{tikzcd}$$

where the Bialynicki-Birula morphism is associated to a writing of w as a product of minimal elements in the affine Weyl group. The property of being ℓ-cohomologically smooth is ℓ-cohomologically smooth local on the source and thus we deduce that $\mathrm{Gr}_{G,\mu} \to \mathrm{Spa}(E)^\diamond$ is ℓ-cohomologically smooth.

5.3.2.3. *Quotient by a pro-p group.* —

Here is a new example where we leave the "usual world" of rigid spaces even further.

Proposition 5.3.5. — *Let $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ representable in loc. spatial diamonds. $K =$ pro-p group such that $\underline{K}$ acts on $\mathfrak{X}$ over $\mathfrak{Y}$ and $\underline{K} \times \mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is qc. 0-truncated.*
Then,

$$\mathfrak{X} \to \mathfrak{Y} \ \ell\text{-coho. smooth} \Longrightarrow \mathfrak{X}/\underline{K} \to \mathfrak{Y} \ \ell\text{-coho. smooth.}$$

One has to be careful that $\mathfrak{X} \to \mathfrak{X}/\underline{K}$ is not ℓ-coho. smooth in general (unless K is finite in which case this is finite étale).

Let P be a profinite set. Then $\underline{P} \longrightarrow * = \mathrm{Spa}(\overline{\mathbb{F}}_p)$ is not ℓ-coho. smooth unless P is finite.

Here is how to verify this last fact. For S a perfectoid space one has

$$D_{\text{ét}}(\underline{P} \times S, \Lambda) = D_{\text{ét}}(S, \underbrace{\mathscr{C}(P, \Lambda)}_{\substack{\text{loc. constant} \\ \text{functions}}})$$

and

$$D_{\text{ét}}(\underline{P}, \Lambda) = D(\mathscr{C}(P, \Lambda)).$$

Let us note $f : \underline{P} \longrightarrow *$.

- The functor $Rf_* : D(\mathscr{C}(P, \Lambda)) \to D(\Lambda)$ is the evident one given by the morphism of rings $\Lambda \to \mathscr{C}(P, \Lambda)$.
- The functor f^* is $- \otimes^{\mathbb{L}}_{\Lambda} \mathscr{C}(P, \Lambda)$.
- f is proper and thus $Rf_* = Rf_!$
- One has $Rf^!(-) = R\mathrm{Hom}_{\Lambda}(\mathscr{C}(P, \Lambda), -)$.

In particular, $Rf^!\Lambda = \mathscr{D}(P, \Lambda) =$ distributions on P with values in Λ as a $\mathscr{C}(P, \Lambda)$-module. This is a projective of finite type module iff P is finite.

Example 5.3.6. — $* = \mathrm{Spa}(\overline{\mathbb{F}}_q)$. *Then* $\mathrm{Spa}(\breve{E})^\diamond \to *$ *is ℓ-coho. smooth and*

$$\underbrace{\mathrm{Div}^1}_{\substack{\textit{moduli of deg. } 1 \\ \textit{eff. divisors} \\ \textit{on the curve}}} = \mathrm{Spa}(\breve{E})^\diamond / \varphi^{\mathbb{Z}} \longrightarrow *$$

is proper ℓ-coho. smooth.

In fact, let $\breve{E}_\infty$ be the completion of the extension generated by torsion points of a Lubin-Tate group, a perfectoid field with $\breve{E}_\infty^\flat = \overline{\mathbb{F}}_q(\!(T^{1/p^\infty})\!)$. Then, $\mathrm{Spa}(\breve{E}_\infty^\flat) \to *$ is ℓ-cohomologically smooth since representable in perfectoid open punctured disks. We thus have a diagram

$$\begin{tikzcd}
\mathrm{Spa}(\breve{E}_\infty^\flat) \arrow[r, "\text{not } \ell\text{-coho. sm.}"] \arrow[rrrd, bend right, "\ell\text{-coho. sm. as a composite}"'] & \mathrm{Spa}(\breve{E})^\diamond \arrow[r, equal] \arrow[rrd, bend right=10, "\ell-\text{coho. sm.}"] & \mathrm{Spa}(E_\infty^\flat)/\mathcal{O}_E^\times \arrow[r, "\text{finite étale}"] & \mathrm{Spa}(\breve{E}_\infty^\flat)/1+\pi\mathcal{O}_E \arrow[d, "\ell\text{-coho. sm. by Prop. 5.3.5}"] \\
 & & & *
\end{tikzcd}$$

that proves that $\mathrm{Spa}(\breve{E})^\diamond \to *$ is ℓ-cohomologically smooth and thus $\mathrm{Div}^1 \to *$ too since $\mathrm{Spa}(\breve{E}) \to \mathrm{Spa}(\breve{E})^\diamond/\varphi^{\mathbb{Z}}$ is representable in local isomorphisms.

As an application one finds back Tate-Nakayama duality for discrete finite $\mathrm{Gal}(\overline{E}|E)$-modules killed by a power of ℓ.

Example 5.3.7 **([80]).** — *Let* $X \simeq \mathring{\mathbb{B}}_{\breve{E}}^{d-1}$ *be the generic fiber of the Lubin-Tate space associated to* GL_d *and* $(X_K)_{K\subset GL_d(\mathcal{O}_E)} \to X$ *be the Lubin-Tate tower. Let* $X_\infty = \varprojlim_K X_K$*, a perfectoid space over* $\mathrm{Spa}(\breve{E}_\infty)$*. Then, for each* K*,* $X_K^\diamond \to \mathrm{Spa}(\breve{E})^\diamond$ *is* ℓ*-cohomologically smooth as the diamond of a smooth morphism of rigid spaces. But going to the limit,* $X_\infty^\flat \to \mathrm{Spa}(\breve{E}_\infty^\flat)$ *is not* ℓ*-cohomologically smooth. In fact, as an application of the Jacobian criterin of smoothness Ivanov and Weinstein prove that the (partially proper) open subset* $U \subset X_\infty$ *where there is no complex multiplication is such that* $U^\flat \to \mathrm{Spa}(\breve{E}_\infty^\flat)$ *is* ℓ*-cohomologically smooth.*

5.3.2.4. *Banach-Colmez spaces.* — We speak here about the linear case of the Jacobian criterion of smoothness.

Theorem 5.3.8 **(Linear case of the Jacobian criterion)**
Let $\mathscr{E}$ *be a vector bundle on* X_S*.*

1. *If* $\forall s \in S$*,* $\mathscr{E}_{|X_{K(s),K(s)^+}}$ *has* > 0 *H.N. slopes then*
$$BC(\mathscr{E}) \longrightarrow S$$
is ℓ*-cohomologically smooth of dimension* $\deg(\mathscr{E})$*.*
2. *If* $\forall s \in S$*,* $\mathscr{E}_{|X_{K(s),K(s)^+}}$ *has* < 0 *H.N. slopes then*
$$BC(\mathscr{E}[1]) \longrightarrow S$$
is ℓ*-cohomologically smooth of dimension* $-\deg(\mathscr{E})$*.*

For point (1), we prove that up to replacing S by an étale cover, one can find an exact sequence $0 \to \mathscr{E}' \to \mathscr{E}'' \to \mathscr{E} \to 0$ where $\mathscr{E}''$ is $\simeq$ to $\oplus_i \mathcal{O}(\lambda_i)$ with $0 < \lambda_i \leq 1$ and $\mathscr{E}'$ is fiberwise s.s. of slope 0. This implies that after replacing S by a strictly totally disconnected perfectoid space $BC(\mathscr{E})$ is a quotient of an open perfectoid ball by the action of $\underline{E^n}$ for some $n \geq 0$.

$$\begin{tikzcd}
\mathring{\mathbb{B}}_S^{1/p^\infty,d} \arrow[r] \arrow[dr, "\ell\text{-coho.sm.}"'] & \mathring{\mathbb{B}}_S^{1/p^\infty,d}/\underline{\mathcal{O}_E^n} \arrow[r, "\text{étale}"] \arrow[d, "\substack{\ell\text{-cohomologicallysm.}\\ \text{by Prop. 5.3.5}}"] & \mathring{\mathbb{B}}_S^{1/p^\infty,d}/\underline{E^n} \arrow[r, equal] & BC(\mathscr{E}) \arrow[dll, bend left, "\ell\text{-coho.sm.}"] \\
 & S & &
\end{tikzcd}$$

5.3.3. Openness of smooth morphisms. —

The following result is quite important.

If $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of small v-sheaves representable in locally spatial diamonds, separated ℓ-cohomologically smooth morphism then $\mathrm{Im}(f) \subset \mathfrak{Y}$ is represented by an open sub-stack.

We will give the proof later using constructible sheaves. It is important for the following reason.

Corollary 5.3.9. — *Let $\mathfrak{X}$ be a small v-stack and consider a family $(U_i \to \mathfrak{X})_{i \in I}$ of morphisms toward $\mathfrak{X}$ where each U_i is a locally spatial diamond, $U_i \to \mathfrak{X}$ is representable in locally spatial diamonds separated ℓ-cohomologically smooth. Then, the family $(U_i \to \mathfrak{X})_{i \in I}$ is a v-cover if and only if and only if it is “in the naïve sense” that is to say $\coprod_{i \in I} |U_i| \to |\mathfrak{X}|$ is surjective.*

5.4. Smooth base change

Start with

$$f : \mathfrak{X} \longrightarrow \mathfrak{Y}$$

a 0-truncated morphism of small v-stacks and let $A \in D_{\text{ét}}(\mathfrak{X}, \Lambda)$. As we said before, it is difficult to compute

$$Rf_* A \in D_{\text{ét}}(\mathfrak{Y}, \Lambda)$$

in general unless f is qc qs (quasi-compact base change). There is another case that is very useful and allows us to compute this in terms of ”smooth charts“.

***Proposition 5.4.1* (Smooth base change).** — *Consider a cartesian diagram*

$$\begin{tikzcd}\mathfrak{X}' \arrow[r, "f'"] \arrow[d, "g'"'] & \mathfrak{Y}' \arrow[d, "g"] \\ \mathfrak{X} \arrow[r, "f"] & \mathfrak{Y}\end{tikzcd}$$

where

- — f *is a* 0*-truncated morphism of small* v*-stacks,*
- — g *is representable in locally spatial diamonds separated* ℓ*-cohomologically smooth*

Then, for any $A \in D_{\text{ét}}(X, \Lambda)$*,*

$$g^* Rf_* A \xrightarrow{\sim} Rf'_* g'^* A.$$

Proof. — Use the "dual proper base change formula"

$$Rf'_* Rg'^! A \xrightarrow{\sim} Rg^! Rf_* A$$

coupled with g and g' separated ℓ-cohomologically smooth and $D_{g'} \xrightarrow{\sim} f'^* D_g$ (the formation of the dualizing complex is compatible with base change). ☐

In the same vein, we have the smooth base change of Hom' using the exceptional pull-back of Hom's.

***Proposition 5.4.2* (Smooth base change of Hom)**
For $f : \mathfrak{X} \to \mathfrak{Y}$ *a separated* ℓ*-cohomologically smooth morphism of small* v*-stacks and* $A, B \in D_{\text{ét}}(\mathfrak{Y}, \Lambda)$ *one has*

$$f^* R\mathscr{H}om_\Lambda(A, B) \xrightarrow{\sim} R\mathscr{H}om_\Lambda(f^* A, f^* B).$$

We can thus compute smooth locally those operations that were non explicit before.

5.5. Artin v-stacks

5.5.1. The example of classifying stacks. — Recall that by definition $* = \mathrm{Spa}(\overline{\mathbb{F}}_q)$.

Proposition 5.5.1. — *Let G be an affine algebraic group over E and $\mathfrak{X} = [*/\underline{G(E)}]$ seen as a small v-stack. Its diagonal is representable in locally spatial diamonds and there exists $f : U \to \mathfrak{X}$ with*

1. *U a locally spatial diamonds*
2. *f v-surjective*
3. *f separated ℓ-cohomologically smooth.*
4. *$U \to *$ separated ℓ-cohomologically smooth.*

In fact, take $U = G^{ad,\diamond}/\underline{K} \to [*/\underline{G(E)}]$ for $K \subset G(E)$ a compact open pro-p subgroup. There is a diagram

$$\begin{tikzcd}
G^{ad,\diamond}/\underline{K} \arrow[r, "{\begin{array}{c}\ell\text{-coho.sm.}\\ \text{since } G^{ad}\to\mathrm{Spa}(E)\\ \text{sm.}\Rightarrow G^{ad,\diamond}\to\mathrm{Spa}(E)^\diamond\\ \ell\text{-coho.sm.}\end{array}}"] \arrow[d, "\ell-\text{coho.sm.}"', "{\begin{array}{c}\text{since } G^{ad,\diamond}\to *\\ \text{is } \ell\text{-coho.sm.}\\ +\text{Prop. 5.3.5}\end{array}}"] & {[\mathrm{Spa}(E)^\diamond/\underline{K}]} \arrow[r, "{\begin{array}{c}\ell\text{-coho.sm. since}\\ \mathrm{Spa}(E)^\diamond\to * \text{ is}\\ \ell\text{-coho.sm.}\end{array}}"] & {[*/\underline{K}]} \arrow[d, "{\begin{array}{c}\text{separated}\\ \text{étale}\end{array}}"] \\
* & & {[*/\underline{G(E)}]}
\end{tikzcd}$$

This proves the assertions of the proposition and gives our first example of an Artin v-stack.

5.5.2. Artin v-stacks. —

5.5.2.1. *Definition.* —

The preceding leads to the following definition.

Definition 5.5.2. — *An artin v-stack is a small v-stack $\mathfrak{X}$ such that*

1. *Its diagonal is representable in locally spatial diamonds.*
2. *There exists a locally spatial diamond U and a separated surjective ℓ-cohomologically smooth morphism $U \to \mathfrak{X}$.*

If one can take $U \to *$ separated ℓ-cohomologically smooth we say that $\mathfrak{X}$ is ℓ-cohomologically smooth. If this is the case then this is true for any $U \to \mathfrak{X}$ that is separated ℓ-cohomologically sm.. One can then define its dualizing complex $D_{\mathfrak{X}}$ as an invertible object in $D_{\text{ét}}(\mathfrak{X}, \Lambda)$ canonically.

Theorem 5.5.3. — *The small v-stack* Bun_G *of G-bundles on the curve is an Artin v-stack ℓ-cohomologically smooth with* $D_{\mathrm{Bun}_G} \simeq \Lambda$.

We give two proofs in [**58**] of this result. The first one uses Beauville-Laszlo uniformization ([**58**, IV.1.2]). The second one uses the charts $\pi_b : \mathcal{M}_b \to \mathrm{Bun}_G$ that we build using the Jacobian criterion of smoothness, see [**58**, Section V.3].

5.5.2.2. *Cohomological operations on Artin v-stacks using smooth charts.* —

For a 0-truncated morphism of Artin v-stacks

$$f : \mathfrak{X} \to \mathfrak{Y}$$

and $A \in D_{\text{ét}}(\mathfrak{X}, \Lambda)$, Rf_*A is computable using the smooth base change theorem and smooth charts. More precisely, if V is a locally spatial diamond and $V \to \mathfrak{Y}$ is separated ℓ-cohomologically smooth then if

$$f_V : U := \mathfrak{X} \times_{\mathfrak{Y}} V \longrightarrow V$$

one has

$$(Rf_*A)_{|V} = R(f_V)_{\text{ét}*}\big(\underbrace{A_{|U}}_{\in \widehat{D}(U_{\text{ét}},\Lambda)} \big)$$

and

$$(f_V)_{\text{ét}} : U_{\text{ét}} \longrightarrow V_{\text{ét}}$$

is the morphism of small étale sites induced by f_V. One can even go further:

In the same vein, if $\mathfrak{X}$ is an Artin v-stack one can compute smooth locally $R\mathscr{H}om(A, B)$ for $A, B \in D_{\text{ét}}(\mathfrak{X}, \Lambda)$. In fact, if $U \to \mathfrak{X}$ is separated ℓ-cohomologically smooth with U a locally spatial diamonds then

$$\underbrace{R\mathscr{H}om_\Lambda}_{\substack{\text{abstract } R\mathscr{H}om_\Lambda \\ \text{defined for any small} \\ v\text{-stack}}} (A, B)_{|U} = \underbrace{R\mathscr{H}om_\Lambda}_{\substack{\text{usual concrete} \\ R\mathscr{H}om_\Lambda \text{ in } D(U_{\text{ét}},\Lambda)}} (A_{|U}, B_{|U}).$$

LECTURE 6

ÉTALE CONSTRUCTIBLE SHEAVES

6.1. Constructible sheaves on spectral spaces

Let X be a spectral space and $\Lambda =$ a Noetherian ring.

Recall the following:

$$\begin{tikzcd}
\text{constructible sets in } X \arrow[d, equal] \\
\text{Boolean algebra generated by quasi-compact open subsets} \arrow[d, equal] \\
\coprod_{\text{finite}} \underbrace{\text{locally closed constructible sets}}_{U \smallsetminus V \text{ with } U \text{ and } V \text{ open qc.}}
\end{tikzcd}$$

Then, if X_{cons} is the topology generated by constructible subsets, i.e. the topology whose closed subsets are the pro-constructible subsets, X_{cons} is compact totally disconnected space i.e. profinite space whose closed/open subsets are exactly the constructible subsets of X.

Definition 6.1.1. — *A sheaf of Λ-module $\mathscr{F}$ on the spectral space X is constructible if there exists a finite partition of X, $X = \bigcup_i Z_i$, in locally closed constructible subsets such that for all i, $\mathscr{F}_{|Z_i}$ is a constant sheaf with value a Λ-module of finite type.*

The category of constructible sheaves is a sub-abelian category of the category of sheaves of Λ-modules on X. There are other characterizations of constructible sheaves:

- $\mathscr{F}$ is constructible iff it is a successive extension of $j_!\underline{M}$ where $j : Z \hookrightarrow X$ with Z locally closed constructible and M of finite type. Thus, the category of constructible sheaves of Λ-modules is the thick sub-category generated by the $j_!\underline{M}$ as before.

- $\mathscr{F}$ is constructible iff its pullback to X_{cons} is locally constant, locally isomorphism to a finite type Λ-module

Remark 6.1.2. — *If X is a spectral space and $Z \subset X$ is constructible then Z is a neighborhood of any maximal point of X lying in Z. In fact, if x is such a point, $\{x\} = X_x$ and thus $\cap_{U \ni x}(U \cap X \setminus Z) = \emptyset$ where U is a qc open neighborhood of x. Since $U \cap X \setminus Z$ is constructible, the compacity of X_{cons} then implies that a finite sub-intersection is empty.*

For example, if $X = |\mathrm{Spa}(A, A^+)|$ where A is topologically of finite type over a non-archimedean field K then any Tate classical point of X that is contained in Z has a neighborhood contained in Z. For example, if $Z \subset |\mathbb{B}^d_K|$ containing the origin 0 then $\mathbb{B}^d(0, \varepsilon) \subset Z$ for some $\varepsilon > 0$.

Proposition 6.1.3. —

1. *The constructible sheaves are exactly the compact objects of the category* $\mathrm{Shv}_\Lambda(X)$
2. *Any sheaf of Λ-modules is a filtered colimit of constructible sheaves and*

$$\varinjlim : \underbrace{\mathrm{Ind}\left(\mathrm{Shv}_\Lambda(X)_{cons}\right)}_{\textit{Ind-category}} \xrightarrow{\sim} \mathrm{Shv}_\Lambda(X).$$

One of the great properties of constructible sheaves is the following.

Proposition 6.1.4. — *If $X = \varprojlim_i X_i$ with X_i spectral and the transition morphisms are qc qs then*

$$2 - \varinjlim_i \mathrm{Shv}_\Lambda(X_i)_{cons} \xrightarrow{\sim} \mathrm{Shv}_\Lambda(X)_{cons}.$$

At the end for any X spectral one can write $X = \varprojlim_i X_i$ with X_i a finite (T0) space. Then,

$$\mathrm{Ind}\Big(2 - \varinjlim_i \underbrace{\mathrm{Shv}_\Lambda(X_i)}_{\text{sheaves on a finite ordered set}}\Big) \xrightarrow{\sim} \mathrm{Shv}_\Lambda(X)$$

This gives a combinatorial description of sheaves of Λ-modules on X.

6.2. Overconvergent étale sheaves

Let X be a spatial diamond.

Definition 6.2.1. — *An étale sheaf $\mathscr{F}$ on X is overconvergent if $\forall \bar{x}$ a geometric point of X and $\forall \bar{y}$ a generalization of $\bar{x}$,*

$$\mathscr{F}_{\bar{x}} \xrightarrow{\sim} \mathscr{F}_{\bar{y}}.$$

This is equivalent to saying that

$$\forall \bar{x} : \mathrm{Spa}(C, C^+) \longrightarrow X,$$

with C algebraically closed as usual, the sheaf

$$x^* \mathscr{F}$$

on $|\mathrm{Spa}(C, C^+)|$ is constant.

Recall that if S is a spectral space such that for all $s \in S$, S_s is a chain then S has a biggest Hausdorff quotient

$$\underbrace{S^B}_{\substack{\text{Berkovich quotient}\\ =\text{compact}\\ \text{Hausdorff}}} = S/\sim$$

where $\sim$ is the equivalence relation generated by the specialization order. Equivalently,

$$s \sim t \Longleftrightarrow s^{max} = t^{max}$$

where s^{max} is the maximal generalization of s (if S is the top. space of an analytic adic space and $S_s = |\mathrm{Spa}(K(s), K(s)^+)|$ then s^{max} is the maximal point of $\mathrm{Spa}(K(s), K(s)^+)$ given by the rank 1 valuation $\mathrm{Spa}(K(s), \mathcal{O}_{K(s)})$). The quotient map

$$\beta : S \longrightarrow S^B$$

induces identifications

$$\begin{tikzcd}
\{\text{sheaves on } S^B\} \arrow[r, "\beta^*", "\sim"'] & \{\text{overconvergent sheaves on } S\} \arrow[d, equal] \\
& \left\{\text{sheaves } \mathscr{F} \text{ on } S \text{ s.t. } \forall U \underbrace{\subset}_{\substack{\text{q.c.}\\ \text{open}}} X,\ \varinjlim_{\substack{V \subset X \text{ open}\\ \overline{U} \subset V}} \mathscr{F}(V) \xrightarrow{\sim} \mathscr{F}(U)\right\}
\end{tikzcd}$$

Here the relation $\overline{U} \subset V$ is sometimes denoted $U \subset\subset V$.

Example 6.2.2. — *If $S = |\mathrm{Spa}(A, A^+)|$ with (A, A^+) an affinoid Tate ring, a sheaf $\mathscr{F}$ on S is overconvergent iff $\forall f_1, \ldots, f_n \in A$ that generate the unit ideal, $\forall g \in A$,*

$$\boxed{\varinjlim_{k\geq 1}\Gamma\Big(\underbrace{S\Big\langle\frac{\varpi f_1^k,\dots,\varpi f_n^k}{g^k}\Big\rangle}_{\substack{\textit{basis of nbd. of}\\ \overline{S\langle\frac{f_1,\dots,f_n}{g}\rangle}^S\\ \textit{when } k \textit{ varies}}},\mathscr{F}\Big)\xrightarrow{\sim}\Gamma\Big(\underbrace{S\Big\langle\frac{f_1,\dots,f_n}{g}\Big\rangle}_{\textit{rational open subset}},\mathscr{F}\Big)}$$

If

$$\underbrace{S}_{\substack{\text{s.t.d.}\\ \text{perf. space}}}\xrightarrow{\substack{\text{quasi-pro-étale}\\ \text{surjective}}}X$$

then the overconvergent étale sheaves on X are identified with the sheaves $\mathscr{F}$ on $X_{\text{ét}}$ such that $\mathscr{F}_{|S}$ comes from a sheaf on $|S|^B$.

There is a partially proper étale site together with a continuous morphism of sites

$$\pi : X_{\text{ét}} \longrightarrow X_{\text{p.p.ét}}$$

that induces an equivalence of topoi

$$\widetilde{X}_{\text{p.p.ét}}\xrightarrow{\sim}\big\{\text{overconvergent sheaves on } X_{\text{ét}}\big\}.$$

Thus, we have the following equivalent description of overconvergent étale sheaves.

Étale sheaves $\mathscr{F}$ on X satisfying $\mathscr{F}_{\bar{x}}\xrightarrow{\sim}\mathscr{F}_{\bar{y}}$ if $\bar{x}\leq\bar{y}$
Étale sheaves $\mathscr{F}$ on X satisfying: $\mathscr{F}_{\vert S}$ comes from a sheaf on the Berkovich spectrum $\vert\mathcal{M}(R)\vert = S^B$ if $\mathrm{Spa}(R,R^+) = S \longrightarrow X$ is a quasi-pro-étale cover with S a strictly totally disconnected perf. spaces
Sheaves on the partially proper étale site of X

6.3. Étale constructible sheaves on spatial diamonds

Let X be a spatial diamond and Λ a Noetherian ring.

Definition 6.3.1. — *A sheaf $\mathscr{F}$ of Λ-modules on $X_{\text{ét}}$ is constructible if $\forall S \to X$ with S a strictly totally disconnected perfectoid space, $\mathscr{F}_{|S}$ is constructible as a sheaf on $|S|$.*

The following result is elementary.

Proposition 6.3.2. — *Let $\mathscr{F}$ be a sheaf of Λ-modules on $X_{\text{ét}}$.*

1. *$\mathscr{F}$ constructible $\Leftrightarrow$ $\exists$ finite partition $|X| = \bigcup_i Z_i$ with Z_i locally closed constructible and $\forall f : S \to X$ with S a strictly totally disconnected perfectoid space, $\forall i$, $\mathscr{F}_{|f^{-1}(Z_i)}$ is isomorphic to $\underline{M}$ with M a finite type Λ-module*
2. *$\mathscr{F}$ constructible $\Leftrightarrow$ $\mathscr{F}$ is a compact object of $\mathrm{Shv}_\Lambda(X_{\text{ét}})$*
3. *$\mathrm{Shv}_\Lambda(X_{\text{ét}})$ is compactly generated and*
$$\varinjlim : \mathrm{Ind}\left(\mathrm{Shv}_\Lambda(X_{\text{ét}})_{cons}\right) \xrightarrow{\sim} \mathrm{Shv}_\Lambda(X_{\text{ét}}).$$
4. *If $X = \varprojlim_i X_i$, a cofiltered limit with qc transition morphisms of spatial diamonds,*
$$2 - \varinjlim_i \mathrm{Shv}_\Lambda(X_{i,\text{ét}})_{cons} \xrightarrow{\sim} \mathrm{Shv}_\Lambda(X_{\text{ét}})_{cons}.$$

Example 6.3.3. — *Consider $X = \mathbb{B}_K^{1,\diamond}$ the 1-dim. closed ball over the non-archi. field K and $j : (\mathbb{B}_K^1 \smallsetminus \{0\})^\diamond \hookrightarrow \mathbb{B}_K^{1,\diamond}$ the inclusion of the punctured ball. For a radius $\rho \in]0,1[\cap|K|$ let $j_\rho : \{\rho \leq |z| \leq 1\}^\diamond \hookrightarrow \mathbb{B}_K^{1,\diamond}$ be the inclusion of the qc annulus with radii $\{\rho, 1\}$. Then*

$$j_!\Lambda = \varinjlim_{\underset{>}{\rho \to 0}} j_{\rho !}\Lambda$$

is a writing of $j_!\Lambda$ as an ind-constructible étale sheaf on $\mathbb{B}_K^{1,\diamond}$.

Here is a key remark/application.

Lemma 6.3.4 **(loc. systems=overconvergent constructible sheaves)** *Let $\mathscr{F} \in \mathrm{Shv}_\Lambda(X_{\text{ét}})$. The following are equivalent:*

1. *$\mathscr{F}$ is constructible and overconvergent*
2. *$\mathscr{F}$ is étale locally isomorphic to $\underline{M}$ with M a finitely generated Λ-module*

In fact, we can suppose that X is a strictly totally disconnected perfectoid space. Now for $x \in X$ one has

$$\mathrm{Spa}(K(x), K(x)^+) = \varprojlim_{\substack{U \ni x \\ \text{open aff. perf.} \\ \text{nbd. of } x}} U$$

and thus

$$2 - \varinjlim_{\substack{U \ni x \\ \text{open aff. perf.} \\ \text{nbd. of } x}} \mathrm{Shv}_\Lambda(U_{\text{ét}})_{cons} \xrightarrow{\sim} \mathrm{Shv}_\Lambda(\mathrm{Spa}(K(x), K(x)^+)_{\text{ét}})_{cons}.$$

Now, $\mathscr{F}$ overconvergent exactly means that for all $x \in X$, if $x : \mathrm{Spa}(K(x), K(x)^+) \to X$, then $x^*\mathscr{F}$ is étale locally constant.

Here is another example.

Example 6.3.5. — *Let $\mathscr{F}$ be constructible on $X_{\text{ét}}$. Then, $\mathscr{F}$ is locally constant in any nbd. of a maximal point of $|X|$. For example, if $X = Y^\diamond$ where Y is a qc qs rigid analytic analytic space then any étale constructible sheaf on Y is locally constant on an open subset U of Y containing all maximal points and in particular all classical Tate points of Y.*

This is exactly where Berkovich theory breaks down from the cohomological point of view: it does not see the difference between constructible sheaves and local systems

6.3.1. Étale perfect-constructible complexes on spatial diamonds.

— Let X be a spatial diamond. Here Λ is any ring.

Definition 6.3.6. — *An object $A \in D_{\text{ét}}(X, \Lambda)$ is perfect constructible if $\forall S \to X$ with S a strictly totally disconnected perfectoid space there exists a finite partition $|S| = \bigcup_i Z_i$ in locally closed constructible subsets such that for all i, $(\underbrace{A_{|S}}_{\in D(|S|,\Lambda)})_{|Z_i}$ is constant with value a perfect complex of Λ-modules via $D(\Lambda) \to D(Z_i, \Lambda)$.*

One can characterize them in sheafy/fiberwise terms and descend the stratification along which they are locally constant. This is the following result.

Proposition 6.3.7. — *The following is satisfied.*

1. *The object $A \in D_{\text{ét}}(X, \Lambda)$ is perfect constructible iff*
   - *(a) it is bounded,*
   - *(b) $\forall i \in \mathbb{Z}$, $\mathcal{H}^i(A)$ is a constructible étale sheaf on X,*
   - *(c) $\forall x : \mathrm{Spa}(C, C^+) \to X$, $x^* A \in D(\Lambda)$ is a perfect complex.*
2. *The object $A \in D_{\text{ét}}(X, \Lambda)$ is perfect constructible iff there exists a finite stratification $|X| = \bigcup_i Z_i$ by locally closed constructible subsets such that $\forall f : S \to X$ with S a strictly totally disconnected perfectoid space such that $\forall i$, $(\underbrace{A_{|S}}_{\in D(|S|,\Lambda)})_{|f^{-1}(Z_i)}$ is constant with value a perfect complex of Λ-modules.*

Remark 6.3.8. — *One has to be careful that for $Z \subset |X|$ locally closed constructible, Z does not define a "sub-spatial diamond" of X (unless it is open) and the expression "$A_{|Z}$"does not make any sense. The only sub-spaces of $|X|$ defining sub-spatial diamonds are the pro-constructible generalizing subsets.*

Let us come to the main point why we are interested in perfect constructible complexes and the way they will be related to our cohomological operations. We first have the following compactness characterization.

Proposition 6.3.9. — *Let X be a spatial diamond satisfying: $\exists N \in \mathbb{N}$ s.t. $\forall U \to X$ separated étale qc, $\forall \mathscr{F} \in \mathrm{Shv}_\Lambda(X_{\text{ét}})$, $H^i_{\text{ét}}(U, \mathscr{F}) = 0$ for $i > N$.*

Then,

$$D_{\text{ét}}(X, \Lambda) = D(X_{\text{ét}}, \Lambda)$$

i.e. $D(X_{\text{ét}}, \Lambda)$ is left complete and $D_{\text{ét}}(X, \Lambda)$ is compactly generated with compact objects the perfect constructible étale complexes.

The finite cohomological dimension hypothesis is satisfied if for example there exists $f : X \to S$ qc. of finite dim. trg. with S a strictly totally disconnected perfectoid space.

Finally the triangulated category of étale perfect constructible complexes has a description as successive extensions of some "simple" perfect constructible objects.

Proposition 6.3.10. — *For X a spatial diamond, the triangulated category*

$$D_{\text{ét}}(X,\Lambda)_{p.c.}$$

is the thick triangulated sub-category generated by the objects

$$j_!\mathscr{L}$$

where j is the inclusion of a locally closed constructible subset Z of $|X|$ and $\mathscr{L}$ an object of $D_{\text{ét}}(U,\Lambda)$, U an open nbd. of Z, étale locally isomorphic to a perfect complex of Λ-modules.

One has to be careful in the preceding statement: such a Z has no structure of a spatial diamond, unless it is open, and $j_!\mathscr{L}$ *is a notation* for $j'_!i_*i^*\mathscr{L}$ where $Z \subset U$, $j' : U \hookrightarrow X$, $i : Z \hookrightarrow X$ and $i_*i^*\mathscr{L}$ is a notation for an object sitting in an exact triangle

$$i_*i^*\mathscr{L} \to \mathscr{L} \to k_!k^*\mathscr{L} \xrightarrow{+1}$$

where $k : U \setminus Z \hookrightarrow U$.

LECTURE 7

TOWARD f-ULA ÉTALE COMPLEXES

7.1. First applications of étale perfect constructible complexes

7.1.1. Toward f-ULA complexes. — A first result toward the definition of f-ULA complexes is the following.

Proposition 7.1.1. — *Let $f : X \to Y$ be a separated ℓ-cohomologically smooth morphism of spatial diamonds. Then, for $A \in D_{\text{ét}}(X, \Lambda)$ perfect constructible, $Rf_! A$ is perfect constructible.*

Proof. — We use proposition 6.3.9. Using the proper base change theorem we can base change and suppose that Y is a strictly totally disconnected perfectoid space. It then suffices to prove that $Rf_! A$ is compact. But this follows from the fact that since f is ℓ-cohomologically smooth, $Rf_!$ has a right adjoint that commutes with direct sums and the fact that A is compact and thus $\mathrm{Hom}(A, -)$ commutes with direct sums. ☐

We used the following elementary but essential lemma.

Lemma 7.1.2. — *Let F be an additive functor between additive categories admitting arbitrary direct sums. If F as a right adjoint that commutes with arbitrary direct sums then F sends compact objects to compact objects.*

From the preceding we deduce the following result.

Proposition 7.1.3. — *Let $f : X \longrightarrow Y$ be a proper ℓ-cohomologically smooth morphism of spatial diamonds and $\mathscr{F} \in D_{\text{ét}}(X, \Lambda)$ be étale locally isomorphic to $\underline{M}$ where M is a perfect complex of Λ-modules. Then, $Rf_* \mathscr{F}$ is étale locally isomorphic to the constant complex associated to a perfect complex of Λ-modules.*

In fact, it remains to see that $Rf_*\mathscr{F}$ is overconvergent. This is a consequence of the following lemma and Poincaré duality.

Lemma 7.1.4. — *Let X be a spatial diamond and $A \in D_{\text{ét}}(X, \Lambda)$ be perfect constructible. Then, $RHom_\Lambda(A, \Lambda)$ is overconvergent.*

Remark 7.1.5. — *Proposition 7.1.3 remains true under the weaker assumption that $\mathscr{F}$ is perfect constructible and "overconvergent along f" in the sense that $\forall y, y' \in Y$ with $y \geq y'$, if $j : X_y \hookrightarrow X_{y'}$ then $\mathscr{F}_{|X_{y'}} \xrightarrow{\sim} j_*(\mathscr{F}_{|X_y})$. Using qc base change this is equivalent to*

$$\forall \bar{x},\ \forall \bar{s} \geq f(\bar{x}), \quad \mathscr{F}_{\bar{x}} \xrightarrow{\sim} \Gamma(X_{\bar{x}} \times_{S_{f(\bar{x})}} S_{\bar{s}}, \mathscr{F}).$$

7.1.2. Openness of cohomologically smooth morphisms.

— The following result was announced before. We can now explain its proof.

Proposition 7.1.6. — *Separated ℓ-cohomologically smooth morphisms are open.*

In fact, let $f : X \to S$ be a separated ℓ-cohomologically smooth morphism of spatial diamonds. Since the image of f is generalizing it suffices to prove it is constructible. since f is qc separated ℓ-cohomologically smooth,

$$A = Rf_! Rf^! \mathbb{F}_\ell$$

is perfect constructible. Let us prove that

$$\text{Im}(|f|) = \{s \in S \mid A_{\bar{s}} \neq 0\}$$

where $\bar{s}$ is a geo. point over s. The formation of A is compatible with base change on S and thus we can suppose that $S = \text{Spa}(C, C^+)$. Then, $A_{\bar{s}}$ is $R\Gamma(S, Rf_! Rf^! \mathbb{F}_\ell)$. Suppose that the closed point s of S is not in the image of f. Then, f factorizes as $j \circ g$ where $j : S \setminus \{s\} \hookrightarrow S$. Thus, $Rf_! = j_! Rg_!$. But since any non-empty closed subset of S contains s, $\Gamma(S, -) \circ j_! = 0$. We thus have $A_{\bar{s}} = 0$.

Suppose now that s is in the image of f. One computes, using the cohomological smoothness of f,

$$Rf_! Rf^! j_! \mathbb{F}_\ell = j_! j^* Rf_! Rf^! \mathbb{F}_\ell.$$

In particular, if one has a distinguished triangle

$$j_! j^* A \to A \to B \xrightarrow{+1}$$

then B is concentrated on $\{s\}$ with stalk $A_{\bar{s}}$ at s. But now,

$$\text{Hom}(B, \mathbb{F}_\ell) = \text{Hom}(Rf^! i_* \mathbb{F}_\ell, Rf^! i_* \mathbb{F}_\ell)$$

where $i_* \mathbb{F}_\ell$ is a notation for a cone of $j_! \mathbb{F}_\ell \to \mathbb{F}_\ell$. This is identified with endomorphisms of the étale local system $Rf^! \mathbb{F}_\ell$ restricted to $f^{-1}(s)$. Since $f^{-1}(s) \neq \emptyset$ the identity is such a non-zero endomorphism and $B \neq 0$.

7.2. f-ULA complexes: the classical scheme case

Motivation: If S is a base scheme and $f : X \to S$ is a finite presentation morphism of schemes, the question is “What is a family of coherent sheaves parametrized by X/S?” The answer given by Grothendieck is: this is a coherent sheaf on X that is flat over S.
Another question is “What is an étale complex of Λ-modules parametrized by X/S ?” The answer is “this an f-ULA étale complex on X”.

Here Λ is a Noehterian ring killed by a power of ℓ invertible on our schemes. All our schemes are qc qs. If X is a scheme we note

$$D_{\text{ét}}(X, \Lambda)$$

for the left completion of $D(X_{\text{ét}}, \Lambda)$, that is equal to $D(X_{\text{ét}}, \Lambda)$ if for example X is of finite type over a field k satisfying $\text{cd}_\ell(k) < +\infty$.

Recall the following definition. For a scheme S and a geometric point $\bar{s}$ of S we note

$$S_{\bar{s}} = \text{Spec}\Big(\underbrace{\mathcal{O}_{S,\bar{s}}}_{\substack{\text{strict henselization} \\ \text{at } \bar{s}}} \Big).$$

If $\bar{s} : \text{Spec}(K) \to S$ with K separably closed,

$$S_{\bar{s}} = \varprojlim U$$

$$\begin{tikzcd} & U \arrow[d, "\text{étale}"] \\ \text{Spec}(K) \arrow[ur] \arrow[r, "\bar{s}"] & S \end{tikzcd}$$

the étale localization of S at $\bar{s}$.

Recall that a specialization $\bar{s'}$ of $\bar{s}$ is the datum of a geometric point $\bar{s'}$ of S together with a factorization of $\bar{s} : \text{Spec}(K) \to S$ via $S_{\bar{s'}} \to S$. This is possible if and only if $s \geq s'$ as points of S. There is then induced a canonical pro-étale morphism

$$S_{\bar{s}} \longrightarrow S_{\bar{s'}}.$$

For any étale sheaf $\mathscr{F}$ on S, its stalk at $\bar{s}$ is

$$\mathscr{F}_{\bar{s}} = \Gamma(S_{\bar{s}}, \mathscr{F}_{|S_{\bar{s}}}).$$

Since $S_{\bar{s}}$ is the spectrum of a strictly Henselian ring, $\Gamma(S_{\bar{s}}, -)$ is exact and for $A \in D_{\text{ét}}(X, \Lambda)$, $\Gamma(S_{\bar{s}}, A_{S_{\bar{s}}}) = R\Gamma(S_{\bar{s}}, A_{|S_{\bar{s}}})$ is the complex of stalks of A at $\bar{s}$, $A_{\bar{s}}$. There is a specialization map

$$\mathscr{F}_{\bar{s'}} \longrightarrow \mathscr{F}_{\bar{s}}.$$

Recall that a complex $A \in D^b_{\text{ét}}(X, \Lambda)$ with constructible cohomology is étale locally constant on X if and only if for all geometric point $\bar{x}$ of X together with a specialization $\bar{x'}$,

$$A_{\bar{x'}} \xrightarrow{\sim} A_{\bar{x}}$$

i.e. all specialization maps are isomorphisms.

We now use the following notation

$$\underbrace{D_{\text{ét}}(X, \Lambda)_{p.c.}}_{\substack{\text{perfect constructible}\\ \text{complexes}}}$$

for the subcategory of $D_{\text{ét}}(X, \Lambda)$ formed by complexes that are bounded with constructible cohomology sheaves and whose stalks at geometric points are perfect complexes of Λ-modules. When $D_{\text{ét}}(X, \Lambda) = D(X_{\text{ét}}, \Lambda)$ those are exactly the compact objects of $D_{\text{ét}}(X, \Lambda)$.

Definition 7.2.1 **(Classical defintion of ULA complexes)**
Let $f : X \to S$ be a finite presentation morphism of schemes.

1. *A complex $A \in D_{\text{ét}}(X, \Lambda)$ is f-locally acyclic if it is perfect constructible and $\forall \bar{x}$ a geometric point of X, $\forall \bar{s}$ a generalization of $f(\bar{x})$ there is an isomorphism*

$$\boxed{A_{\bar{x}} = R\Gamma(X_{\bar{x}}, A) \xrightarrow{\sim} R\Gamma(X_{\bar{x}} \times_{S_{f(\bar{x})}} \bar{s}, A).}$$

2. *A is f-universally locally acyclic if it is locally acyclic after any base change $S' \to S$.*

Remark 7.2.2. — *We wrote $X_{\bar{x}} \times_{S_{f(\bar{x})}} \bar{s}$ for the pullback via $\bar{s} : Spec(L) \to S_{f(\bar{x})}$ of $X_{\bar{x}} \to S_{f(\bar{x})}$. Thus, $R\Gamma(X_{\bar{x}} \times_{S_{f(\bar{x})}} \bar{s}, A)$ is the étale cohomology of the L-scheme $X_{\bar{x}} \times_{S_{f(\bar{x})}} \bar{s}$ with coefficients in (the pullback of) A.*

Remark 7.2.3. — *When S is a curve this is equivalent to $\forall \bar{s}$ a geometric closed point of S, if $X_{\bar{s}} = X \times_S S_{\bar{s}}$, for a choice $\bar{\eta}$ of a geometric point over the generic point of $S_{\bar{s}}$*

$$R\Phi_{\bar{\eta}}(A_{|X_{\bar{s}}}) = 0 \quad \text{in } D_{\text{ét}}(X \times_S Spec(k(\bar{s})))$$

where $R\Phi$ is the vanishing cycle functor

$$R\Phi_{\bar{\eta}} : D_{\text{ét}}(X_{\bar{s}}, \Lambda) \longrightarrow D_{\text{ét}}(X \times_S Spec(k(\bar{s})), \Lambda).$$

Thus, locally acyclic complexes are a generalization of complexes without vanishing cycles when the base is a curve. The terminology locally acyclic means "without any vanishing cycles".

In fact one can verify the following.

Proposition 7.2.4. — *A perfect constructible is f-ULA if and only if for every morphism $g : Spec(V) \to S$ with V a Henselian rank 1 valuation ring and a choice of a separable closure of* $\mathrm{Frac}(V)$*,*

$$R\Phi_{\bar\eta}\big(A_{|X\times_S Spec(V)}\big) = 0.$$

Example 7.2.5. — *A complex $A \in D^b_c(X_{\text{ét}}, \Lambda)$ is f-ULA for $f = \mathrm{Id}$ if and only if A is étale locally constant.*

Thus, ULA complexes are some kind of relative notion for locally constant. Here is the diagram associated to the preceding proposition.

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r, "\bar x", "\substack{\text{closed}\\ \text{géà. point}}"'] & X\times_S \bar s \arrow[r, hook, "\bar i"] \arrow[d] & X\times_S \mathrm{Spec}(\overline{V}) \arrow[d] & X_{\bar\eta} \arrow[l, hook', "\bar j"'] \arrow[d] \\
 & \bar s \arrow[r, hook] & \mathrm{Spec}(\overline{V}) & \bar\eta \arrow[l, hook']
\end{tikzcd}$$

The local acyclicity condition asks that $A_{\bar x}$ is identified with the cohomology of the Milnor fiber (schematical or rigid analytic, see [**9**], [**10**] and [**78**])

$$\begin{aligned}
A_{\bar x} \xrightarrow{\sim} \Big(\bar i^{*} R\bar j_{*}(A_{|X_{\bar\eta}})\Big)_{\bar x} \quad &= \quad \underbrace{R\Gamma\Big(\big(X\times_S \mathrm{Spec}(\overline{V})\big)_{\bar x}[\tfrac{1}{t}], A\Big)}_{\substack{\text{cohomology of the schematical}\\ \text{Milnor fiber over } \bar x}} \\
&= \quad R\Gamma\Big(\underbrace{sp^{-1}(\bar x)}_{\substack{\text{rigid analytic}\\ \text{Milnor fiber}\\ \text{as a tube}}}, \big(A_{|X_{\hat{\bar\eta}}}\big)^{ad}\Big)
\end{aligned}$$

Recall now the following classical theorem. The first part implies the smooth base change theorem.

Theorem 7.2.6. — *Let $f : X \to S$ be a smooth morphism of schemes.*

1. *If A is an étale local system i.e. is étale locally isomorphic to $\underline{M}$ with M a free Λ-module of finite type then A is f-ULA.*
2. *Being ULA is smooth local on the source and on the target.*

Remark 7.2.7. — *The fact that being ULA is smooth local implies ther is a "good notion" of f-ULA complexes in $D_{\text{ét}}(\mathfrak{X}, \Lambda)$ where $f : \mathfrak{X} \to \mathfrak{Y}$ is a representable finite type morphism of Artin stacks. We will use the same fact for Artin v-stacks later.*

Let us note the following now. Proper base change implies that if we have a sequence

$$X \xrightarrow{g} Y \xrightarrow{f} S$$

and $A \in D_{\text{ét}}(X, \Lambda)$ that is $f \circ g$-ULA then g proper $\Rightarrow Rg_* A$ is f-ULA.

Example 7.2.8. — *If A is f-ULA and f is proper then $Rf_* A$ is étale locally constant, étale locally isomorphic to $\underline{M^\bullet}$ where $M^\bullet$ is a perfect complex of Λ-modules.*

We refer to [**99**] for the following.

***Theorem 7.2.9* (Gabber).** — *Locally acyclic implies universally locally acyclic for finite type morphisms of Noetherian schemes.*
In particular, if $A \in D_{\text{ét}}(X, \Lambda)_{pc}$ where X is a finite type k-schemes, k a field, A is ULA with respect to $X \to Spec(k)$ and thus for any k-scheme S, $A_{|X \times_{Spec(k)} S}$ is locally acyclic relatively to $X \times_{Spec(k)} S \to S$.

Gaitsgory realized that ULA complexes behave well with respect to Verdier duality ([?]).

Theorem 7.2.10 **(ULA complexes behave well with respect to Verdier duality)**
Let A be f-ULA where $f : X \to S$ is of finite presentation. Then,

1. *The formation of the Verdier dual $\mathbb{D}_{X/S}(A)$ behaves well with respect to base change: for $S' \to S$,*
$$\mathbb{D}_{X/S}(A)_{|X\times_S S'} = \mathbb{D}_{X\times_S S'/S'}(A_{|X\times_S S'})$$
and $A_{|X\times_S S'}$ is ULA relatively to $X \times_S S' \to S'$
2. *For $B \in D^b_c(X_{\text{ét}}, \Lambda)$ one has*
$$\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_{\Lambda} f^*B \xrightarrow{\sim} R\mathscr{H}om_{\Lambda}(f^*A, Rf^!B).$$

This has lead at the end to a new “Verdier duality point of view” on ULA complexes.

Theorem 7.2.11 **(Verdier duality characterization of ULA complexes, Lu-Zheng)**
For $A \in D_{\text{ét}}(X, \Lambda)$ one has if p_1 and p_2 are the two projections from $X \times_S X$ to X

$$\begin{aligned} A \text{ is } f\text{-ULA} \quad &\Longleftrightarrow \quad \left[\ \mathbb{D}_{X/S}(A) \boxtimes^{\mathbb{L}}_{\Lambda} A \xrightarrow{\sim} R\mathscr{H}om_{\Lambda}(p_1^*A, Rp_2^!A)\ \right] \\ &\Longleftrightarrow \quad X \xrightarrow{A} S \text{ is a left adjoint in the 2-cat.of correspondences} \end{aligned}$$

The second categorical condition will be explained later

This definition is compact since it does not involve something like “for any $S' \to S$...“ or ”$\forall \mathrm{Spec}(V) \to S$”, we just have to test that one morphism is an isomorphism.

Let us cite as an example an immediate corollary of this point of view that is difficult to obtain without it.

Corollary 7.2.12. — *If A is f-ULA then it is dualizable, $A \xrightarrow{\sim} \mathbb{D}_{X/S}(\mathbb{D}_{X/S}(A))$.*

The following 3 definitions are then equivalent:

Classical definition (Grothendieck): Complexes without vanishing cycles	$\forall\ \mathrm{Spec}(\underbrace{V}_{\substack{\text{rank one}\\\text{valuation ring}}}) \to S,$ $R\Phi_{\bar\eta}\big(A_{\mid X\times_S \mathrm{Spec}(V)}\big) = 0$
Modern definition (Gaitsgory): Complexes that behave nicely with respect to Verdier duality	$\mathbb{D}_{X/S}(A)_{\mid X\times_S S'} = \mathbb{D}_{X\times_S S'/S'}(A_{\mid X\times_S S'})$ $\mathbb{D}_{X/S}(A)\otimes^{\mathbb{L}}_{\Lambda} f^*B \xrightarrow{\sim} R\mathscr{H}om_{\Lambda}(f^*A, Rf^!B)$ universally $/S$
Ultra modern 2-categorical super chic definition (Lu-Zheng)	The 1-morphism $X \xrightarrow{A} S$ is a left adjoint in the 2-category of correspondences

Let us cite another example of application of this ULA formalism.

Corollary 7.2.13. — *Let $f : X \to S$ and $g : Y \to S$ be two morphisms of finite presentation and $A \in D_{\text{ét}}(X,\Lambda), B \in D_{\text{ét}}(Y,\Lambda)$ be ULA over S. Then,*

1. *The complex $A \boxtimes^{\mathbb{L}}_{\Lambda} B$ is ULA over S and there is an isomorphism*
$$\mathbb{D}_{X/S}(A)\boxtimes^{\mathbb{L}}_{\Lambda}\mathbb{D}_{Y/S}(B) \xrightarrow{\sim} \mathbb{D}_{X\times_S Y/S}(A\boxtimes^{\mathbb{L}}_{\Lambda} B).$$
2. *One has the Künneth formula: if $h : X\times_S Y \to S$,*
$$Rf_*A\otimes^{\mathbb{L}}_{\Lambda} Rg_*B \xrightarrow{\sim} Rh_*\big(A\boxtimes^{\mathbb{L}}_{\Lambda} B\big).$$

Point (2) is immediately deduced from point (1) by applying Künneth formula with compact support (that is a formal consequence of the projection formula and is always true without any ULA hypothesis) that implies
$$Rf_!\mathbb{D}_{X/S}(A)\otimes^{\mathbb{L}}_{\Lambda} Rg_!\mathbb{D}_{Y/S}(B) \xrightarrow{\sim} Rh_!\big(\mathbb{D}_{X/S}(A)\boxtimes^{\mathbb{L}}_{\Lambda}\mathbb{D}_{Y/S}(B)\big)$$
and an application of $R\mathscr{H}om(-,\Lambda)$ coupled with point (1) and the biduality of ULA complexes.

LECTURE 8

f-ULA ÉTALE COMPLEXES ON SPATIAL DIAMONDS

8.1. ULA complexes on locally spatial diamonds: the "classical" definition

8.1.1. First step of finding the classical definition: mimicking the classical definition. — Let X be a locally spatial diamond.

For $\bar{x} : \mathrm{Spa}(C, C^+) \to X$ a geometric point we define

$$X_{\bar{x}}$$

as for a scheme,

$$X_{\bar{x}} = \varprojlim U$$

$$\begin{tikzcd} & U \arrow[d, "\text{separated étale}"] \\ \mathrm{Spa}(C, C^+) \arrow[ur] \arrow[r, "\bar{x}"] & S \end{tikzcd}$$

where U is a spatial diamond. If $Y \to X$ is quasi-pro-étale where Y is a perfectoid space then we can lift $\bar{x}$ to a geometric point $\bar{y} : \mathrm{Spa}(C, C^+) \to Y$. Then, $Y_{\bar{y}} \xrightarrow{\sim} X_{\bar{x}}$ where $Y_{\bar{y}} = \mathrm{Spa}\Big(\widehat{\overline{K(y)}}, \widehat{\overline{K(y)}}^{+}\Big)$ with $y \in |Y|$ the image of the closed point of $\mathrm{Spa}(C, C^+)$ and $\overline{K(y)}$ the algebraic closure of $K(y)$ inside C.

Suppose

$$f : X \to S$$

is a morphism of locally spatial diamonds and

$$A \in D_{\text{ét}}(X, \Lambda).$$

For such a $\bar{x}$ let $\bar{s}$ be a generalization of $f(\bar{x})$ i.e. $\bar{s}$ is a geometric point of $S_{f(\bar{s})}$. There is a diagram of strictly local perfectoid spaces

$$\begin{tikzcd}
 & & X_{\bar{x}} \arrow[d] \\
\mathrm{Spa}(C, C^+) \arrow[r] \arrow[rr, bend right, "\bar{s}"'] & S_{\bar{s}} \arrow[r, hook, "\substack{\text{qc open}\\ \text{subset}}"] & S_{f(\bar{x})}
\end{tikzcd}$$

Since Λ is killed by a power of $\ell \neq p$, qc base change applies and there are maps

$$A_{\bar{x}} \longrightarrow R\Gamma(\underbrace{X_{\bar{x}} \times_{S_{f(\bar{x})}} S_{\bar{s}}}_{\substack{\text{qc open}\\ \text{subset of } X_{\bar{x}}\\ \text{thus s.t.d.}}}, A) = \Gamma(X_{\bar{x}} \times_{S_{f(\bar{x})}} S_{\bar{s}}, A) \xrightarrow{\sim} R\Gamma(X_{\bar{x}} \times_{S_{f(\bar{x})}} \mathrm{Spa}(C, C^+), A)$$

The classical definition in terms of the cohomology of Milnor fibers translates here in an overconvergence like statement:

$$\boxed{A_{\bar{x}} \xrightarrow{\sim} \Gamma(X_{\bar{x}} \times_{S_{f(\bar{x})}} S_{\bar{s}}, A).}$$

This involves no "higher degree cohomology classes" since, contrary to schemes, any qc open subset of a strictly local perfectoid spaces is strictly local.

In fact, if R is a strictly Henselian local ring then, $\mathscr{F}$ an étale sheaf of abelian groups on $\mathrm{Spec}(R)$, in general, there exists qc open subsets U of $\mathrm{Spec}(R)$ such that $H^i_{\text{ét}}(U, \mathscr{F}) \neq 0$ for some $i > 0$.

The set of open subsets of $X_{\bar{x}}$ is totally ordered $\Rightarrow$ a sheaf on $X_{\bar{x}}$ is the same as a contravariant functor

$$\mathscr{F} : \{\text{qc non-empty open subsets of } X_{\bar{x}}\} \to \text{Sets}$$

Then the preceding condition is that if $U \subset V \subset X_{\bar{x}}$ are pullback of qc non-empty open subsets of $S_{f(\bar{x})}$ then

$$\mathscr{F}(V) \xrightarrow{\sim} \mathscr{F}(U)$$

that is thus identified with $\mathscr{F}(X_{\bar{x}})$.

We now have the following lemma whose proof consists in taking the base change along $X_{\bar{x}} \to S$.

Lemma 8.1.1. — *The complex $A \in D_{\text{ét}}(X, \Lambda)$ is overconvergent along f after any base change/S iff A is overconvergent.*

We thus put the condition to be overconvergent in our definition of ULA complexes.

8.1.2. Second step of finding the classical definition: putting Verdier duality in the machine. — Contrary to schemes the preceding condition is not enough to define ULA complexes: it involves only degree 0 cohomology classes.

Before going further let us remark the following: let

$$f : X \to S$$

be a (compactifiable of finite dim. trg.) morphism of locally spatial diamonds, and

$$A \in D_{\text{ét}}(X, \Lambda)$$

satisfy the "overconvergence along f" condition of the preceding section. Let $j : U \to X$ be a separated étale morphism of locally spatial diamonds such that $f \circ j : U \to X$ is qc qs. Moreover, $A_{|U}$ is "overconvergent along $f \circ j$".

Lemma 8.1.2. — *For such a $j : U \to X$, $R(f \circ j)_*(A_{|U}) \in D_{\text{ét}}(S, \Lambda)$ is overconvergent.*

Proof. — We can use the qc base change Theorem to compute $\big(R(f \circ j)_*(A_{|U})\big)_{\bar{s}}$. ☐

We can now come to the main point.

Lemma 8.1.3. — *Suppose that after any base change over S to a strictly totally disconnected perfectoid space, for any $B \in D_{\text{ét}}(S, \Lambda)$ one has*

$$\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_{\Lambda} f^*B \xrightarrow{\sim} R\mathscr{H}om(A, Rf^!B).$$

Then for any $j : U \to X$ separated étale such that $f \circ j$ is qc,

$$R(f \circ j)_!(A_{|U}) \in D_{\text{ét}}(S, \Lambda)$$

is perfect constructible.

Proof. — Using the proper base change theorem we can suppose that S is a strictly totally disconnected perfectoid space. Replacing f by $f \circ j$ and A by $A_{|U}$ we can suppose that $j = \text{Id}$ and f is qc and thus X is qc. We then have for any B,

$$\text{Hom}(Rf_!A, B) = H^0(X, \mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_{\Lambda} f^*B).$$

Since X is qc we deduce that $Rf_!A$ is a compact object of $D_{\text{ét}}(S, \Lambda)$ and thus perfect constructible. ☐

Let us now remark that this "perfect constructible" property is compatible with the following Lemma that will play an important role later.

***Lemma 8.1.4* (Key ULA Lemma).** — *Let X be a spatial diamond and $A \in D_{\text{ét}}(X, \Lambda)$ perfect constructible. Then $R\mathscr{H}om_\Lambda(A, \Lambda)$ is overconvergent and its formation is compatible with base change: for $f : X' \to X$ a morphism of spatial diamonds,*

$$f^* R\mathscr{H}om_\Lambda(A, \Lambda) \xrightarrow{\sim} R\mathscr{H}om_\Lambda(f^*A, \Lambda).$$

In fact, the complex A has a finite filtration whose graded pieces are of the form

$$j_!\mathscr{L}$$

where $j : Z \hookrightarrow X$ is locally closed constructible and $\mathscr{L}$, defined in a neighborhood of Z, is étale locally constant associated to a perfect complex of Λ-modules. Then,

$$R\mathscr{H}om(j_!\mathscr{L}, \Lambda) = Rj_*\mathscr{L}^\vee$$

where $\mathscr{L}^\vee$ is étale locally constant associated to a perfect complex of Λ-modules. The result is then a consequence of the qc base change Theorem that says this is $j_*\mathscr{L}^\vee$.

8.1.3. The classical definition of ULA. — Let $f : X \longrightarrow S$ is a morphism of locally spatial diamonds, compactifiable locally of finite dim.trg.. Lead by the preceding facts we take the following definition.

***Definition 8.1.5* (Classical definition of ULA)**
A complex $A \in D_{\text{ét}}(X, \Lambda)$ is ULA if

1. *It is overconvergent.*
2. *After any base change over S, for any $j : U \to X$ a separated étale morphism of loc. spatial diamonds such that $f \circ j$ is qc,*

$$R(f \circ j)_!(A_{|U}) \in D_{\text{ét}}(S, \Lambda)_{p.c.}$$

   in the sense that is is perfect constructible when restricted to any spatial open subsets.

Let us remark the following:

- contrary to the scheme case we don't ask that A itself is perfect constructible and as a matter of fact, in the cases we will consider this will almost never be the case. In fact since we ask that A is overconvergent this would imply that A is étale locally constant and we don't want to put such a restriction.
- there is a definition of LA complexes where we ask that A is only overconvergent along f and property (2) is satisfied only over S, not necessarily after any base change, but we don't need it.
- it is enough to check property (2) after base change to any strictly totally disconnected perfectoid space.

8.1.4. Basic properties.

— The following is evident and shows that ULA complexes are a natural generalization of local systems.

Proposition 8.1.6. — *The following is statisfied.*

1. *If f is separated ℓ-cohomologically smooth then any complex étale locally constant with value a perfect complex of Λ-modules is ULA.*
2. *Being ULA is ℓ-cohomologically smooth local on the source and the target.*

The second point will allow us to define a notion of ULA complexes for Artin v-stacks.

Proposition 8.1.7. — *Let*

$$X \xrightarrow{g} Y \xrightarrow{f} S.$$

If $A \in D_{\text{ét}}(X,\Lambda)$ is $f\circ g$-ULA and g is proper then Rg_*A is f-ULA.

Finally let us remark the following.

Proposition 8.1.8. — *Let A be f-ULA, $f : X \to S$, and $B \in D_{\text{ét}}(X,\Lambda)_{pc}$. Then for any $j : U \to X$ separated étale such that $f\circ j$ is quasicompact then*

$$R(f\circ j)_!((A \otimes^{\mathbb{L}}_{\Lambda} B)_{|U})$$

is perfect constructible.

With the notations of proposition 6.3.10 it is sufficient to do it for $j_!\mathscr{L}$, and this is an easy reduction.

Definition 8.1.9. — *Let $\mathfrak{X}$ ba an Artin v-stack and $A \in D_{\text{ét}}(\mathfrak{X},\Lambda)$. We say that A is ULA (relatively to $\mathfrak{X}\to *$) if for one (and thus all) morphism*

$$f : U \longrightarrow \mathfrak{X}$$

separated ℓ-cohomologically smooth surjective with U a locally spatial diamond, for all S a locally spatial diamond,

$$(f^*A)_{|U\times S} \in D_{\text{ét}}(U\times S,\Lambda)$$

if ULA relatively to $U\times S \to S$.

8.1.5. An example.

— Let us prove the following as an exercise.

***Theorem 8.1.10* (ULA $\Leftrightarrow$ admissible for classifying stacks of locally pro-p groups)**
Let G be an affine algebraic group over E. Let $\mathfrak{X} = [*/\underline{G(E)}]$. Then, via the equivalence

$$D(G(E), \Lambda) \xrightarrow{\sim} D_{\text{ét}}(\mathfrak{X}, \Lambda),$$

a complex $\pi^\bullet$ is ULA iff for all compact open pro-p subgroup K of $G(E)$, $(\pi^\bullet)^K$ is a perfect complex of Λ-modules.

Proof. — Let $\mathscr{F}_{\pi^\bullet} \in D_{\text{ét}}(\mathfrak{X}, \Lambda)$ be associated to $\pi^\bullet$. Chose $K \subset G(E)$ compact open pro-p. Since $[*/\underline{K}] \to [*/\underline{G(E)}]$ is étale surjective, $\mathscr{F}_{\pi^\bullet}$ is ULA iff $\mathscr{F}_{\pi^\bullet_{|K}} \in D_{\text{ét}}([*/\underline{K}], \Lambda)$ is ULA.

Let $\mathcal{G}$ be a flat model of G that is of finite type over $\mathcal{O}_E$, $\widehat{\mathcal{G}}$ be its π-adic completion and $\widehat{\mathcal{G}}_\eta$ be its generic fiber as a finite type adic space over $\mathrm{Spa}(E)$. We can suppose that $K \subset \mathcal{G}(\mathcal{O}_E)$. Let $X = \big(\widehat{\mathcal{G}}_\eta\big)^\diamond$, a spatial diamond separated ℓ-cohomologically smooth over $*$. Then,

$$X/\underline{K} \longrightarrow [*/\underline{K}]$$

is a surjective ℓ-cohomologically smooth morphism. Let $\mathscr{G} \in D_{\text{ét}}(X/\underline{K}, \Lambda)$ be the pull-back of $\mathscr{F}_{\pi^\bullet_{|K}}$.

Let S be a strictly totally disconnected perfectoid space and

$$j : U \longrightarrow X/\underline{K} \times S$$

be a separated étale morphism such that the composite

$$U \longrightarrow X/\underline{K} \times S \longrightarrow S$$

is quasi-compact. We can form the cartesian diagram

$$\begin{tikzcd}
T \arrow[r] \arrow[d, "/\underline{K}"'] \arrow[ddr, bend right=60, "\text{qc qs smooth}"'] & X \times S \arrow[d] \\
U \arrow[r] & X/\underline{K} \times S \arrow[d] \\
 & S
\end{tikzcd}$$

Let $f : X/\underline{K} \times S \to S$ be the projection. We have to compute

$$R(f \circ j)_!(\mathscr{G}_{|U}) \in D_{\text{ét}}(S, \Lambda).$$

Let us note $g : T \longrightarrow S$ that is a qc. ℓ-cohomologically smooth morphism invariant under the action of $\underline{K}$ on T. Let us note

$$\widetilde{f} : X/\underline{K} \times S \longrightarrow [S/\underline{K}].$$

We have

$$R(f\circ j)_!(\mathscr{G}_{|U}) = \big[(R\widetilde{f}_!\Lambda)\otimes^{\mathbb{L}}_{\Lambda}\pi^{\bullet}\big]^{K}.$$

After forgetting the action of $\underline{K}$, the image of $R\widetilde{f}_!\Lambda$ under $D_{\text{ét}}([S/\underline{K}],\Lambda)\longrightarrow D_{\text{ét}}(S,\Lambda)$ is $Rg_!\Lambda$ equipped with its action of K. Since g is qc smooth, $Rg_!\Lambda$ is perfect constructible and thus the action of K on $Rg_!\Lambda$ is smooth. Let $K'\subset K$ be compact open such that K' acts trivially on $Rg_!\Lambda$. Then,

$$\big[(R\widetilde{f}_!\Lambda)\otimes^{\mathbb{L}}_{\Lambda}\pi^{\bullet}\big]^{K}\subset Rg_!\Lambda\otimes^{\mathbb{L}}_{\Lambda}(\pi^{\bullet})^{K'}$$

that is a direct factor. Thus, $(\pi^{\bullet})^{K'}$ is a perfect complex of Λ-modules $\Rightarrow R(f\circ j)_!(\mathscr{G}_{|U})$ is perfect constructible.

The reciprocal is obtained in the following way. If $h : X/\underline{K}\to *$ then

$$Rh_!\Lambda = \big[R\Gamma_c(X,\Lambda)\otimes^{\mathbb{L}}_{\Lambda}\pi^{\bullet}\big]^{K}$$

with $R\Gamma_c(X,\Lambda)\in D(\Lambda)$ a perfect complex equipped with a smoth action of K. The result then follows, after taking K smaller so that it acts trivially on $R\Gamma_c(X,\Lambda)$, from the fact that if M is a non-zero perfect complex of Λ-modules with $H^i(M)\simeq\Lambda$ for some $i\in\mathbb{Z}$ and $N\in D(\Lambda)$ then N is perfect iff $M\otimes^{\mathbb{L}}_{\Lambda}N$ is perfect. ☐

Remark 8.1.11. — *We will give another more conceptual proof later using the 2-categorical characterization of ULA complexes.*

8.2. Behavior with respect to Verdier duality

Let

$$f : X\to S$$

be a morphism of locally spatial diamonds, compactifiable locally of finite dim.trg..

Theorem 8.2.1. — *Let A be f-ULA.*

1. *The formation of the dualizing complex $\mathbb{D}_{X/S}(A)$ is compatible with base change: for any $S'\to S$,*
$$\mathbb{D}_{X/S}(A)_{|X\times_S S'} = \mathbb{D}_{X\times_S S'/S'}\big(A_{|X_{X\times_S S'}}\big).$$
2. *For any $B\in D_{\text{ét}}(S,\Lambda)$*
$$\mathbb{D}_{X/S}(A)\otimes^{\mathbb{L}}_{\Lambda}f^{*}B\xrightarrow{\sim}R\mathscr{H}om(A,Rf^{!}B).$$

Let us give a full proof of point (1). Let us start with a Lemma. Here by $\mathbb{B}^d_S$ we mean the spatial diamond that sends (R,R^+) to morphisms $\mathrm{Spa}(R,R^+)\to S$ together with an element of $(R^+)^d$. Let us moreover recall that cofiltered limits of spatial diamonds as v-sheaves are again spatial diamonds.

Lemma 8.2.2. — *Let $X \to S$ be a morphism where X is affinoid perfectoid and S is a spatial diamond. Then one caw write*

$$X = \varprojlim_i U_i,$$

a colfiltered limit where U_i is a quasi-compact open subset inside a ball $\mathbb{B}_S^{d_i}$ for some integer d_i.

Proof. — For I a set we note

$$\mathbb{B}_S^I$$

for the spatial diamond over S such that $\mathbb{B}_S^I(R, R^+)$ is the set of morphisms $\mathrm{Spa}(R, R^+) \to S$ together with an element of $(R^+)^I$. When S is perfectoid this is representable by a perfectoid space, if $S = \mathrm{Spa}(A, A^+)$ then $\mathbb{B}_S^I = \mathrm{Spa}(A\langle X_i\rangle_{i\in I}, A^+\langle X_i\rangle)$ where $A^+\langle X_i\rangle_{i\in I}$ is the ϖ-adic completion of $A^+[X_i]_{i\in I}$ and $A\langle X_i\rangle_{i\in I}$ is $A^+\langle X_i\rangle_{i\in I}[\frac{1}{\varpi}]$.

Take $I = \mathcal{O}(X)$. Then there is an evident monomorphism of v-sheaves over S

$$X \hookrightarrow \mathbb{B}_S^I.$$

We have

$$\mathbb{B}_S^I = \varprojlim_{\substack{J \subset I \\ \text{finite}}} \mathbb{B}_S^J.$$

For any $J \subset I$ finite, the image $Z_J \subset |\mathbb{B}_S^J|$ of $X \hookrightarrow \mathbb{B}_S^I \twoheadrightarrow \mathbb{B}_S^J$ is pro-constructible generalizing and thus an intersection of quasi-compact open subsets of $\mathbb{B}_S^J$. We thus have

$$X = \varprojlim_{\substack{J \subset I \\ \text{finite}}} \varprojlim_{\substack{U \supset Z_J \\ \text{qc open}}} U.$$

This proves the result. ☐

Let us go with another Lemma.

Lemma 8.2.3. — *Let X be a spatial diamond satisfying: $\exists N$ such that for any $U \to X$ separated étale qc and any $\mathscr{F}$ an étale sheaf of Λ-modules on X, $H^i_{\text{ét}}(U, \mathscr{F}) = 0$ for $i > N$. A morphism $A \to B$ in $D_{\text{ét}}(X, \Lambda)$ is an isomorphism if and only if for any $U \to X$ separated étale qc, $R\Gamma(U, A) \xrightarrow{\sim} R\Gamma(U, B)$.*

Proof. — We have $D_{\text{ét}}(X, \Lambda) = D(X_{\text{ét}}, \Lambda)$. Now, for any geometric point $\bar{x} : \mathrm{Spa}(C, C^+) \to X$ and any $D \in D(X_{\text{ét}}, \Lambda)$,

$$\varinjlim \qquad R\Gamma(U, D) = D_{\bar{x}}$$

$$\begin{tikzcd} & U \arrow[d, "\text{sep. étale. qc.}"] \\ \mathrm{Spa}(C, C^+) \arrow[ur] \arrow[r] & X \end{tikzcd}$$

and the result follows. ☐

Proof of point (1) of Theorem 8.2.1. — The assertion is local on X and S and we may assume that X and S are spatial diamonds. There exists a v-cover $S'' \to S'$ that is a strictly totally disconnected perfectoid space. Now, a morphism $D_1 \to D_2$ in $D_{\text{ét}}(X \times_S S', \Lambda)$ is an isomorphism if and only if it is an isomorphism after restriction to $X \times_S S''$. This is a consequence of the fact that $D_{\text{ét}} \subset D_v$. The result for $S'' \to S$ and $S'' \to S'$ thus implies the result for $S' \to S$. We can thus suppose that S' is a strictly totally disconnected perfectoid space.

Since S' is strictly totally disconnected and $X \times_S S' \to S'$ is quasicompact of finite dim.trg., $X \times_S S'$ satisfies the hypothesis of Lemma 8.2.3 and thus, if $g : X \times_S S' \to X$, we have to prove that for any $U \to X \times_S S'$ separated étale qc,

$$R\Gamma(U, g^*\mathbb{D}_{X/S}(A)) \xrightarrow{\sim} R\Gamma(U, \mathbb{D}_{X\times_S S'/S'}(g^*A)).$$

We can now apply Lemma 8.2.2 to write

$$S' = \varprojlim_i S'_i$$

(cofiltered limit) where S'_i is quasi-compact open inside a finite dimension ball over S. Since

$$2 - \varinjlim_i \{\text{separated étale qc}\}/S'_i \xrightarrow{\sim} \{\text{separated étale qc}\}/S',$$

we can find an index i and some separated étale qc morphism $V \to S'_i$ such that $U \to S'$ is a pullback of $V \to S'_i$. We have a diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & V \arrow[d] \arrow[dr] & \\
X \times_S S' \arrow[r] \arrow[d] & X \times_S S'_i \arrow[r] \arrow[d] & X \arrow[d] \\
S' \arrow[r] & S'_i \arrow[r] & S
\end{tikzcd}$$

The result we want to prove is immediate when $S' \to S$ is ℓ-cohomologically smooth. This is in particular the case for $S'_i \to S$. We are thus reduce, up to replacing S by S'_i and X by V, to proving that for any cartesian diagram of spatial diamonds

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
S' \arrow[r, "g"] & S
\end{tikzcd}$$

with $X \to S$ compactifiable of finite dim.trg., and any $A \in D_{\text{ét}}(X, \Lambda)$ that is ULA over S,

$$R\Gamma(X', g'^*\mathbb{D}_{X/S}(A)) \xrightarrow{\sim} R\Gamma(X', \mathbb{D}_{X'/S'}(g'^*A)).$$

For this is suffices to prove the result after applying Rf'_*. We have

$$\begin{aligned} Rf'_* g'^* R\mathscr{H}om_\Lambda(A, Rf^!\Lambda) &\underbrace{=}_{\text{qc BC}} g^* Rf_* R\mathscr{H}om_\Lambda(A, Rf^!\Lambda) \\ &= g^* R\mathscr{H}om_\Lambda(Rf_! A, \Lambda). \end{aligned}$$

We have moreover,

$$\begin{aligned} Rf'_* R\mathscr{H}om_\Lambda(g'^* A, Rf'^!\Lambda) &= R\mathscr{H}om_\Lambda(Rf'_! g'^* A, \Lambda) \\ &= R\mathscr{H}om_\Lambda(g^* Rf_! A, \Lambda). \end{aligned}$$

The result is thus a consequence of the fact that $Rf_!A$ is perfect constructible and Lemma 8.1.4. $\square$

LECTURE 9

2-CATEGORICAL CHARACTERIZATION OF f-ULA ÉTALE COMPLEXES

9.1. The 2-category of correspondences

Let S be a locally spatial diamond.

Definition 9.1.1. — *Define $\mathcal{C}_S$ for the 2-category whose objects are morphisms $X \to S$ that are compacitfiable of finite dim. trg. and*

1. *for $X \to S$ and $Y \to S$, the 1-morphisms between X/S and Y/S are given by an object of $D_{\text{ét}}(X \times_S Y, \Lambda)$,*
2. *the 2-morphisms from A to B elements of $D_{\text{ét}}(X \times_S Y, \Lambda)$*

$$\begin{tikzcd} X \arrow[r, bend left, "A"{name=U}] \arrow[r, bend right, "B"'{name=D}] & Y \arrow[Rightarrow, from=U, to=D] \end{tikzcd}$$

are given by usual morphisms from A to B in $D_{\text{ét}}(X \times_S Y, \Lambda)$

The composition of one morphisms is given by ”convolution“ in the sense that for a sequence $X \xrightarrow{A} Y \xrightarrow{B} Z$, in the diagram

$$\begin{tikzcd} & & X \times_S Z \arrow[lldd, bend right] \arrow[rrdd, bend left] & & \\ & & X \times_S Y \times_S Z \arrow[u, "\pi_{13}"] \arrow[ld, "\pi_{12}"'] \arrow[rd, "\pi_{23}"] & & \\ & X \times_S Y \arrow[ld] \arrow[rd] & & Y \times_S Z \arrow[ld] \arrow[rd] & \\ X & & Y & & Z \end{tikzcd}$$

we set

$$B \circ A := \pi_{13!}\big(\pi_{12}^* A \otimes_{\Lambda}^{\mathbb{L}} \pi_{23}^* B\big) : X \longrightarrow Z.$$

Here the identity 1-morphism from X to X is given by $\Delta_!\Lambda$ where $\Delta : X \hookrightarrow X \times_S X$.

Remark 9.1.2. — *Any correspondence of locally spatial diamonds*

$$\begin{tikzcd}
& C \arrow[dl] \arrow[dr] & \\
X & & Y
\end{tikzcd}$$

over S (where both morphisms are compactifiable of finite dim.trg.) together with $A \in D_{\text{ét}}(C, \Lambda)$ give rise to a morphism from X to Y in $\mathcal{C}_S$: if $\pi : C \to X \times_S Y$ this is

$$X \xrightarrow{R\pi_! A} Y.$$

We could have defined the 2-category $\mathcal{C}_S$ by fixing a support like C for our correspondences but this is unnecessary and up to replacing (C, A) by $(X \times_S Y, R\pi_! A)$ this does not change anything.

Remark 9.1.3. — *Given $A \in D_{\text{ét}}(X, \Lambda)$, $B \in D_{\text{ét}}(Y, \Lambda)$ and $\pi : C \to X \times_S Y$ as before, a cohomological correspondence from A to B with support in C is nothing else than a 2-morphism*

$$\begin{tikzcd}
X \arrow[rr, "R\pi_!\Lambda"] \arrow[dr, "A"', ""{name=A}] \arrow[dr, bend left=40, "B\circ R\pi_!\Lambda", ""'{name=U}] & & Y \arrow[dl, "B"] \\
& S &
\arrow[Rightarrow, from=U, to=A]
\end{tikzcd}$$

$$\underbrace{B \circ R\pi_!\Lambda \Longrightarrow A}_{\substack{\textit{cohomologically corr. from } B \textit{ to } A \\ \textit{supported on } C}}$$

There is a morphism of 2-categories

$$\mathcal{C}_S \longrightarrow \{\text{2-category of triangulated categories}\} \tag{7}$$

$$X/S \longmapsto D_{\text{ét}}(X, \Lambda) \tag{8}$$

that sends X/S to $D_{\text{ét}}(X, \Lambda)$, the 1-morphism $X \xrightarrow{A} Y$ to the 1-morphism, i.e. the functor, given by the kernel A

$$\begin{array}{rcl} D_{\text{ét}}(X, \Lambda) & \longrightarrow & D_{\text{ét}}(Y, \Lambda) \\ \mathscr{F} & \longmapsto & p_{2!}(A \otimes^{\mathbb{L}}_{\Lambda} p_1^*\mathscr{F}) \end{array}$$

where $p_1 : X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$.

In a 2-category like $\mathcal{C}_S$ there is a notion for a 1-morphism to be a left adjoint (or if you want to have a right adjoint). More precisely, a 1-morphism $X \xrightarrow{A} Y$ is a left adjoint if there exists $Y \xrightarrow{B} X$ and 2-morphisms

$$\begin{cases} \eta : \mathrm{Id} \Rightarrow BA \\ \varepsilon : AB \Rightarrow \mathrm{Id} \end{cases}$$

satisfying

$$\begin{cases} B\varepsilon \circ \eta B = \mathrm{Id}_B \\ \varepsilon A \circ A\eta = \mathrm{Id}_A \, . \end{cases}$$

When this is the case such a B is unique up to a 2-isomorphism. In fact, suppose that $Y \xrightarrow{B'} X$ is equipped with 2-morphisms

$$\begin{cases} \eta' : \mathrm{Id} \Rightarrow B'A \\ \varepsilon' : AB' \Rightarrow \mathrm{Id} \end{cases}$$

satisfying

$$\begin{cases} B'\varepsilon' \circ \eta' B' = \mathrm{Id}_{B'} \\ \varepsilon' A \circ A\eta' = \mathrm{Id}_A \, . \end{cases}$$

Consider

$$u = B'\varepsilon \circ \eta' B : B \Rightarrow B'$$

and

$$v = B\varepsilon' \circ \eta B' : B' \Rightarrow B.$$

One then has

$$\begin{aligned} v \circ u &= B\varepsilon' \circ \underbrace{\eta B' \circ B'\varepsilon}_{BAB'\varepsilon \circ \eta B'AB} \circ \eta' B \\ &= \underbrace{B\varepsilon' \circ BAB'\varepsilon}_{B\varepsilon \circ B\varepsilon' AB} \circ \underbrace{\eta B'AB \circ \eta' B}_{BA\eta' B \circ \eta B} \\ &= B\varepsilon \circ B\varepsilon' AB \circ BA\eta' B \circ \eta B \\ &= B\varepsilon' AB \circ BA\eta' B \\ &= \mathrm{Id} \end{aligned}$$

9.2. Relation to the ULA condition

Theorem 9.2.1. — *The following are equivalent:*

1. *A is f-ULA,*
2. *The natural morphism*
$$\mathbb{D}_{X/S}(A)\boxtimes^{\mathbb{L}}_{\Lambda} A \longrightarrow R\mathscr{H}om_{\Lambda}(p_1^*A, Rp_2^!A)$$
is an isomorphism in $D_{\text{ét}}(X\times_S X,\Lambda)$,
3. *The 1-morphism $X\xrightarrow{A}S$ is a left adjoint in $\mathcal{C}_S$, in which case its right adjoint is given by $S\xrightarrow{\mathbb{D}_{X/S}(A)}X$.*

Proof. — (1)⇒(2) is a consequence of Theorem 8.2.1.

(2)⇒(3): One computes
$$\begin{aligned}\mathbb{D}_{X/S}(A)\circ A &= A\boxtimes^{\mathbb{L}}_{\Lambda}\mathbb{D}_{X/S}(A)\\ A\circ\mathbb{D}_{X/S}(A) &= Rf_!(\mathbb{D}_{X/S}(A)\otimes^{\mathbb{L}}_{\Lambda} A)\end{aligned}$$
Then to give oneself η is the same a morphism
$$\eta:\Delta_!\Lambda\longrightarrow A\boxtimes^{\mathbb{L}}_{\Lambda}\mathbb{D}_{X/S}(A)\xrightarrow{\sim}R\mathscr{H}om_{\Lambda}(p_1^*A, Rp_2^!A)$$
By adjunction this is the same a morphism
$$\Lambda\longrightarrow R\Delta^!R\mathscr{H}om_{\Lambda}(p_1^*A, Rp_2^!A)=R\mathscr{H}om_{\Lambda}(A,A)$$
and we can take the Identity of A.

For ε, this is a morphism
$$\varepsilon: Rf_!(\mathbb{D}_{X/S}(A)\otimes^{\mathbb{L}}_{\Lambda} A)\longrightarrow\Lambda$$
that is to say by adjunction a morphism
$$\mathbb{D}_{X/S}(A)\otimes^{\mathbb{L}}_{\Lambda} A\longrightarrow Rf^!\Lambda$$
and we can take the evident morphism.

One verifies easily those satisfy the adjunction properties.

(3)⇒(1) We use the morphism of 2-categories (7). If $S\xrightarrow{B}X$ is a right adjoint of $X\xrightarrow{A}S$ then by a application of our morphism of 2-categories, then the functor
$$Rf_!(A\otimes^{\mathbb{L}}_{\Lambda}-):D_{\text{ét}}(X,\Lambda)\longrightarrow D_{\text{ét}}(S,\Lambda)$$
has
$$f^*(-)\otimes^{\mathbb{L}}_{\Lambda}B:D_{\text{ét}}(S,\Lambda)\longrightarrow D_{\text{ét}}(X,\Lambda)$$
as a right adjoint. To prove that A is f-ULA we can suppose that S is a strictly totally disconnected perfectoid space. But since the right adjoint commutes with all colimits, it sends compact objects to compact objects and we can conclude that for any $A'\in D_{\text{ét}}(X,\Lambda)_{p.c.}$, $Rf_!(A\otimes^{\mathbb{L}}_{\Lambda}A')$ is perfect constructible.

It remains to prove the overconvergence of A, see the article. □

Let us look at two applications of this that may be very difficult without this 2-categorical point of view.

Proposition 9.2.2. — *1. Any $A \in D_{\text{ét}}(X, \Lambda)$ that is f-ULA is bidual with respect to the Verdier duality,*

$$A \xrightarrow{\sim} \mathbb{D}_{X/S}(\mathbb{D}_{X/S}(A)).$$

2. For $X \xrightarrow{f} S$ and $Y \xrightarrow{g} S$, and $A \in D_{\text{ét}}(X, \Lambda)$ that is f-ULA, and $B \in D_{\text{ét}}(Y, \Lambda)$ that is g-ULA, $A \boxtimes^{\mathbb{L}}_{\Lambda} B$ is ULA relatively to $X \times_S Y \to S$ and

$$\mathbb{D}_{X/S}(A) \boxtimes^{\mathbb{L}}_{\Lambda} \mathbb{D}_{Y/S}(B) \xrightarrow{\sim} \mathbb{D}_{X \times_S Y/S}(A \boxtimes^{\mathbb{L}}_{\Lambda} B).$$

Proof. — Point (1) is a consequence of the isomorphism of 2-categories

$$\mathcal{C}_S \simeq \mathcal{C}_S^{op}.$$

In fact, the 1-morphisms between X and Y are exactly the same as the 1-morphisms between Y and X. Thus, if A is f-ULA then $\mathbb{D}_{X/S}(A)$ is a right adjoint of A and thus A is a right adjoint of $\mathbb{D}_{X/S}(A)$ that is thus f-ULA. From this we deduce point (1) via the identification of the right adjoint of $\mathbb{D}_{X/S}(A)$ with $\mathbb{D}_{X/S}(\mathbb{D}_{X/S}(A))$.

Point (2) is a consequence of the fact that $B \circ A$ is a left adjoint and its right adjoint is the composite of the right adjoint of A with the one of B and is thus $\mathbb{D}_{X/S}(A) \circ \mathbb{D}_{X/S}(B)$. ☐

9.3. An application

Theorem 9.3.1. — *Let $\mathfrak{X} = [*/\underline{H}]$ with H a locally pro-p group. Suppose that $\mathfrak{X}$ is ℓ-cohomologically smooth of dimension 0, for example H is a closed sub-group of $G(E)$ where G is an affine algebraic group over E. Then, via the identification*

$$D(H, \Lambda) \xrightarrow{\sim} D(\mathfrak{X})$$

we have π corresponds to an ULA complex iff for all $K \subset H$ compact open pro-p, π^K is a perfect complex of Λ-modules.

Proof. — We use the identification

$$D(H \times H, \Lambda) \xrightarrow{\sim} D(\mathfrak{X} \times \mathfrak{X}).$$

If π corresponds to an ULA complex then for any π',

$$\widetilde{\pi} \boxtimes^{\mathbb{L}}_{\Lambda} \pi' \xrightarrow{\sim} R\mathscr{H}om_{\Lambda}(\pi \boxtimes 1, 1 \boxtimes \pi')$$

in $D(H \times H, \Lambda)$ (derived smooth dual and derived smooth Hom's). In particular, taking for π' a complex of Λ-modules with trivial H-action, M

$$\widetilde{\pi} \otimes^{\mathbb{L}}_{\Lambda} M \xrightarrow{\sim} R\mathscr{H}om_{\Lambda}(\pi, M).$$

(smooth duals and hom's) Taking the K-invariants with K open pro-p we obtain

$$R\mathrm{Hom}_\Lambda(\pi^K,\Lambda)\otimes^{\mathbb{L}}_\Lambda M \xrightarrow{\sim} R\mathrm{Hom}_\Lambda(\pi^K,M).$$

It is then immediate that π^K is a perfect complex of Λ-modules.

In the other direction, let π be such that for all K open pro-p, π^K is perfect. We have to verify that

$$\widetilde{\pi}\boxtimes^{\mathbb{L}}_\Lambda \pi \xrightarrow{\sim} R\mathscr{H}om_\Lambda(\pi,\pi).$$

It suffices to prove this is the case after applying $D(H\times H,\Lambda)\to D(\Lambda)$. Then the morphism is written as

$$\varinjlim_K R\mathrm{Hom}_\Lambda(\pi^K,\Lambda)\otimes^{\mathbb{L}}_\Lambda \pi^K \longrightarrow \varinjlim_K R\mathrm{Hom}_\Lambda(\pi^K,\pi^K)$$

and the result follows. ☐

9.4. A criterion of smoothness

Proposition 9.4.1. — *Let $f : X\to S$ be a compactifiable of finite dim.trg. morphism of locally spatial diamonds. Then, f is ℓ-cohomologically smooth if and only if $\mathbb{F}_\ell$ is f-ULA and $Rf^!\mathbb{F}_\ell$ is invertible.*

LECTURE 10

FIRST STEP IN THE PROOF OF THE JACOBIAN CRITERION: FORMAL SMOOTHNESS

→ the ULA property is used at two places in the article:

— for the geometric Satake correspondence, in fact the B_{dR}-affine Grassmanian that shows up in the geometric Satake correspondence should be though of as being a Beilinson-Drinfeld type affine Grassmanian i.e. something relative "sitting over the curve" (more precisely over $\mathrm{Div}^1 = \mathrm{Spa}(\breve{E})^\diamond/\varphi^{\mathbb{Z}}$), and we thus need to speak about "families of perverse sheaves" i.e. ULA perverse sheaves,

— for the proof of the Jacobian criterion via Proposition 9.4.1.

We deal here with the proof of the Jacobian criterion.

10.1. A key remark: stability under retracts of the ULA property

Let us begin with a lemma whose proof is a simple computation.

Lemma 10.1.1. — *1. Let*

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

be a diagram of compactifiable of finite dim. trg. morphisms of locally spatial diamonds. Let $\Gamma_h : X \to X \times_S Y$ be the graph of h. Let $A \in D_{\text{ét}}(Y, \Lambda)$. The following diagram in $\mathcal{C}_S$

$$\begin{tikzcd} X \arrow[rr, "(\Gamma_h)_!\Lambda"] \arrow[dr, "h^*A"'] & & Y \arrow[dl, "A"] \\ & S & \end{tikzcd}$$

commutes i.e. there is a canonical (in A) isomorphism

$$A \circ (\Gamma_h)_!\Lambda \xrightarrow{\sim} h^*A.$$

2. *Let* $\mathcal{D}_S$ *be the category of locally spatial diamonds compactifiable of finite dim. trg. over* S*. We upgrade it to a* 2*-category by setting* $2 - Hom(f,g) = \{\mathrm{Id}\}$ *if* $f = g$ *and* $\emptyset$ *if* $f \neq g$*. Then the correspondence*

$$\mathcal{D}_S \longrightarrow \mathcal{C}_S$$

that sends X/S *to* X/S *and* $f : X \to Y$ *an* S*-morphism to* $X \xrightarrow{R(\Gamma_f)_!\Lambda} Y$ *is a morphism of* 2*-categories.*

Proposition 10.1.2. — *Let* $f : X \to S$ *and* $g : Y \to S$ *be morphisms of locally spatial diamonds that are compactifiable of finite dim. trg.. Suppose that* f *is a retract of* g *i.e. there exists morphisms*

$$\begin{tikzcd} X \arrow[rr, shift left, "i"] \arrow[dr, "f"'] & & Y \arrow[ll, shift left, "r"] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

satisfying

$$r \circ i = \mathrm{Id}_X \, .$$

Then, if $A \in D_{\text{ét}}(Y, \Lambda)$ *is* g*-ULA,* i^*A *is* f*-ULA.*

Proof. — Point (1) of the preceding lemma says that is suffices to verify that $X \xrightarrow{\Gamma_{i,!}\Gamma} Y$ is a left adjoint. Point (2) shows that the composite $X \xrightarrow{\Gamma_{i,!}\Gamma} Y \xrightarrow{R(\Gamma_r)_!\Lambda} X$ is canonically isomorphic to Id_X. This gives us a unit

$$\eta : \mathrm{Id}_X \overset{\sim}{\Longrightarrow} R(\Gamma_r)_!\Lambda \circ \Gamma_{i,!}\Lambda.$$

We moreover have

$$R(\Gamma_r)_!\Lambda \circ \Gamma_{i,!}\Lambda = R(\Gamma_{r \circ i})_!\Lambda.$$

Since f is separated, $\mathrm{Id}_X = \Delta_{X/S!}\Lambda = \Delta_{X/S*}\Lambda$. To give a 2-morphism

$$R(\Gamma_{r \circ i})_!\Lambda \Longrightarrow \mathrm{Id}_X$$

is thus the same as a morphism in $D_{\text{ét}}(X, \Lambda)$

$$\Delta^*_{X/S} R(\Gamma_{r \circ i})_! \longrightarrow \Lambda.$$

Proper base change says that the left term is Λ. We thus have a counit

$$\varepsilon : R(\Gamma_r)_!\Lambda \circ \Gamma_{i,!}\Lambda \Longrightarrow \mathrm{Id}_X \, .$$

One verifies that η and ε define an adjunction. □

Here is the corollary that we will use.

Corollary 10.1.3. — *Let $f : X \to S$ and $g : Y \to S$ be morphisms of locally spatial diamonds that are compactifiable of finite dim. trg.. Suppose that f is a retract of g i.e. there exists morphisms*

$$\begin{tikzcd} X \arrow[rr, shift left, "i"] \arrow[dr, "f"'] & & Y \arrow[ll, shift left, "r"] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

satisfying

$$r \circ i = \mathrm{Id}_X .$$

Then if g is ℓ-cohomologically smooth, $\mathbb{F}_\ell$ is f-ULA.

→ contrary to the cohomological smoothness condition, the ULA property is stable under retracts.

10.2. Formal smoothness

The notion of formal smoothness we introduce is a tool we use to prove that $\mathbb{F}_\ell$ is ULA for $\mathcal{M}_Z^{sm} \to Z$ in the proof of the Jacobian criterion of smoothness. It is complementary to the notion of ℓ-cohomologically sm.. All "natural" morphisms that show up that are ℓ-cohomologically sm. are formally smooth too. If there exists a "good" notion of smooth morphisms in our context it has to imply ℓ-cohomologically sm. for all $\ell \neq p$ and formally smooth.

10.2.1. Background on Zariski closed immersions. — Recall the following basic result.

Proposition 10.2.1. — *Let $S = \mathrm{Spa}(A, A^+)$ be affinoid perfectoid and I an ideal of A. The closed subset $V(I) = \{s \in S \mid \forall f \in I,\ |f(s)| = 0\} \subset |S|$ is representable by an affinoid perfectoid space pro-étale inside S.*

→ $V(I) = \varprojlim_{\substack{n \geq 1 \\ f_1,\dots,f_n \in I}} \{|f_1| \leq 1, \dots, |f_n| \leq 1\}$ that is thus a cofiltered limit of affinoid perfectoid spaces.

Definition 10.2.2. — *For S affinoid perfectoid the immersion $S_0 \hookrightarrow S$ defined by an ideal I of $\mathcal{O}(S)$ as before is called a Zariski closed immersion.*

→ one has to be careful that, contrary to the case of schemes, this is not a local condition: if $S = \cup_i U_i$ is a finite rational cover of S and a closed subset $Z \subset |S|$ is such that for all i, $Z \cap U_i$ is Zariski closed in U_i then Z may not be Zariski closed in S in general).

Recall moreover the following basic result that is easy for $\mathbb{F}_p$-perfectoid spaces but more difficult in general.

Theorem 10.2.3. — *Let $S_0 \subset S$ be a Zariksi closed immersion of affinoid perfectoid spaces.*

1. *$S_0 \subset S$ is strongly Zariski closed in the sense that the morphism*
$$\mathcal{O}(S)^+ \to \mathcal{O}(S_0)^+$$
is almost surjective.
2. *$S_0^\flat \to S^\flat$ is a Zariksi closed immersion and thus if $Z \subset |S|$ is Zariski closed then it is Zariski closed in $|S^\flat|$ via the equality $|S| = |S^\flat|$.*

10.2.2. Formally smooth morphisms. — We can now come to our definition. This is related to the topological notion of retracts in the sense of Borsuk.

Definition 10.2.4. — *A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of v-stacks is formally smooth if for all diagrams*

$$\begin{tikzcd}
S_0 \arrow[r] \arrow[d, hook] & \mathfrak{X} \arrow[d, "f"] \\
S \arrow[r] \arrow[ur, dashed] & \mathfrak{Y}
\end{tikzcd}$$

where $S_0 \hookrightarrow S$ is a Zariski closed immersion of affinoid perfectoid spaces, up to replacing $S_0 \hookrightarrow S$ by $S' \times_S S_0 \hookrightarrow S'$ where $S' \to S$ is an étale neighborhood of S_0, we can complete the diagram with the dashed arrow:

$$\begin{tikzcd}
S' \times_S S_0 \arrow[r] \arrow[d, hook] & S_0 \arrow[r] \arrow[d, hook] & \mathfrak{X} \arrow[d, "f"] \\
S' \arrow[r, "\substack{\text{étale nbd.}\\ \text{of } S_0}"'] \arrow[urr, bend left=15] & S \arrow[r] & \mathfrak{Y}
\end{tikzcd}$$

commutes up to a 2-isomorphism i.e. the composite $S' \to \mathfrak{X} \xrightarrow{f} \mathfrak{Y}$ is isomorphic to $S' \to S \to \mathfrak{Y}$ in the groupoid $\mathfrak{Y}(S')$.

→ some evident examples

Example 10.2.5. — *1. Any separated étale morphism of v-stacks is formally smooth.*

2. *The composite of two f.s. morphisms is f.s..*
3. *The base change of a f.s. morphism is f.s..*
4. *The f.s. property is étale local on the source*

Let us give more non-trivial examples.

Lemma 10.2.6. — *Let $\mathbb{B} \to \mathrm{Spd}(\mathbb{Z}_p)$ be the diamond of the morphism sending a $\mathbb{Z}_p$-perfectoid space S to $\mathcal{O}(S)^+$. This is formally smooth.*

Proof. — Let $S_0 \hookrightarrow S$ be a Zariski closed immersion of affinoid perfectoid spaces and $f \in \mathcal{O}(S_0)^+$. Since $\mathcal{O}(S) \to \mathcal{O}(S_0)$ is surjective we can lift it to some $\tilde{f} \in \mathcal{O}(S)$. Now, if $U = \{|f| \leq 1\}$ then $f_{|U} \in \mathcal{O}(U)^+$ and U is a nbd. of S_0. ☐

Lemma 10.2.7. — *The morphism $\mathrm{Spd}(\mathbb{Z}_p) \to \mathrm{Spd}(\mathbb{F}_p)$ is formally smooth.*

Proof. — Let $\xi = \sum_{n\geq 0}[a_n]p^n$ be a degree one primitive element in $W(R^+)$, (R, R^+) a $\mathbb{F}_p$-affinoid perfectoid algebra. Up to multiplying ξ by a unit in $W(R^+)^\times$ we can suppose that $\xi \in p+[\varpi]W(R^+)$ where ϖ is a p.u. in R. Now, if $(A, A^+) \to (R, R^+)$ is a morphism of affinoid rings such that $A^+ \to R^+$ is almost surjective then $A^{\circ\circ} \to R^{\circ\circ}$ is surjective. We deduce that, up to multiplying ξ by a unit it lifts to a degree one primitive element in $W(A^+)$. ☐

Corollary 10.2.8. — *If $f : X \to Y$ is a smooth morphism of Noetherian analytic adic spaces over $\mathbb{Z}_p$ then $f^\diamond : X^\diamond \to Y^\diamond$ is formally smooth.*

Proof. — Since the notion of f.s. is local for the analytic topology on the source we are reduce to proving that if f is the composite of an étale morphism toward $\mathbb{B}^d_Y$, a ball$/Y$, with the projection to Y then $f^\diamond$ is f.s.. Since the diamond of an étale morphism is étale we are reduced to proving that $(\mathbb{B}^d_Y \to Y)^\diamond$ is f.s.. But $(\mathbb{B}^d_Y)^\diamond = \mathbb{B}^d \times_{\mathrm{Spd}(\mathbb{Z}_p)} Y^\diamond$ and the result is deduced from lemma 10.2.6. ☐

10.3. Formal smoothness and the ULA property

→ the following says that, up to some technical "finiteness property of f",

$$f \text{ formally smooth } \Longrightarrow \mathbb{F}_\ell \text{ is } f\text{-ULA}.$$

Proposition 10.3.1. — *Let $f : X \to S$ be a (compactifiable of finite dim. trg.) morphism of locally spatial diamonds satisfying: there exists a v-cover $(T_i \to X)_i$ such that for all i*

1. *T_i is affinoid perfectoid Zariski closed in an ℓ-cohomologically smooth affinoid perfectoid S-space*
2. *$T_i \to X$ is formally smooth and ℓ-cohomologically smooth*

Then, if f is formally smooth, $\mathbb{F}_\ell$ is f-ULA.

Proof. — Since $T_i \to X$ is ℓ-cohomologically smooth, the ULA notion being ℓ-cohomologically smooth local, it is enough to prove that $\mathbb{F}_\ell$ is ULA relatively to $T_i \to S$. This morphism is formally smooth as a composite of two formally smooth morphisms and thus, up to replacing T_i by an étale cover, $T_i \to S$ is a retract of an ℓ-cohomologically smooth affinoid perfectoid space. We can now apply Corollary 10.1.3. ☐

10.4. The main result about formal smoothness

Theorem 10.4.1. — *The morphism $\mathcal{M}_Z^{sm} \to S$ is formally smooth.*

→ we refer to the article; the proof is technical but natural.

10.5. Application of the formal smoothness: first step in the proof of the Jacobian criterion

Proposition 10.5.1. — *There exists a v-cover $(T_i \to \mathcal{M}_Z)_i$ where for all i,*

1. *T_i is affinoid perfectoid and $T_i \to S$ is Zariski closed in an affinoid perfectoid space that is étale over a perfectoid ball $\mathbb{B}_S^{d_i, 1/p^\infty}$,*
2. *$T_i \to \mathcal{M}_Z$ is ℓ-cohomologically sm. and f.s..*

→ using the "quasi-projectivity assumption" for $Z \to X_S$ this is reduced to proving the result for $Z = \mathbb{P}^n_{X_S}$ in which case this is an exercise about BC spaces using the explicit formula for $\mathcal{M}_{\mathbb{P}^n_{X_S}}$.

We can now prove our Theorem.

Theorem 10.5.2 **(First step in the proof of the Jacobian criterion)**
The étale sheaf $\mathbb{F}_\ell$ on $\mathcal{M}_Z^{sm}$ is ULA relatively to the morphism $\mathcal{M}_Z^{sm} \to S$.

→ Apply Propositions 10.3.1 and 10.5.1 together with, of course, Theorem 10.4.1.

LECTURE 11

SECOND STEP IN THE PROOF OF THE JACOBIAN CRITERION: DEFORMATION TO THE NORMAL CONE

We now switch to the deformation to the normal cone argument to finish the proof of the Jacobian criterion.

11.1. Background on the deformation to the normal cone (see Fulton's book)

11.1.1. Normal cones. — Let

$$i : Y \hookrightarrow X$$

be a closed immersion of schemes defined by the ideal $\mathfrak{I}$. Recall that the normal cone to i is

$$C_Y X = \operatorname{Spec}\Big(\bigoplus_{k \geq 0} \mathfrak{I}^k / \mathfrak{I}^{k+1} \Big) \longrightarrow Y.$$

When i is a regular immersion the associated normal bundle is associated to the vector bundle $(\mathfrak{I}/\mathfrak{I}^2)^\vee$ on Y and one has

$$C_Y X = \mathbb{V}\Big((\mathfrak{I}/\mathfrak{I}^2)^\vee \Big)$$

the geometric realization of this vector bundle.

→ the notion of normal cone generalized thus the notion of the normal vector bundle associated to a regular embedding.

→ it is called a cone because it is the spectrum of a graded quasi-coherent algebra and is thus equipped with a $\mathbb{G}_m$-action

$$\begin{tikzcd} C_Y X \arrow[loop right, "\mathbb{G}_m"] \arrow[d] \\ Y \arrow[u, bend left, "0"] \end{tikzcd}$$

where 0 is the zero section of this cone given by $(C_YX)^{\mathbb{G}_m} \xrightarrow{\sim} Y$.

→ thus although in general, when i is not a regular immersion, $C_YX \to Y$ is not (the geometric realization of) a vector bundle, it is still a cone like any vector bundle.

We will use later the following construction. Let S be a scheme and

$$C = \mathrm{Spec}\big(\mathcal{A}\big) \longrightarrow S$$

be a cone, that is to say $\mathcal{A}$ is a graded quasi-coherent $\mathcal{O}_S$-algebra, $\mathcal{A} = \bigoplus_{k\geq 0} \mathcal{A}_k$ with $\mathcal{A}_0 = \mathcal{O}_S$. We define its projective completion as

$$\begin{aligned}\overline{C} &= \mathrm{Proj}\Big(\mathcal{A}_\bullet \quad \underbrace{\otimes_{\mathcal{O}_S}}_{\substack{\text{graded}\\ \text{tensor product}}} \quad \mathcal{O}_S[T]\Big) \longrightarrow S \\ &= (C \times \mathbb{A}^1) \smallsetminus \{0\}/\mathbb{G}_m\end{aligned}$$

where the action of $\mathbb{G}_m$ is the diagonal one and $\{0\} = (C \times \mathbb{A}^1)^{\mathbb{G}_m}$ is the origin of the cone $C \times \mathbb{A}^1$.

There is then a diagram

$$\begin{tikzcd} C \arrow[r, hook, "\text{open}"] \arrow[dr] & \overline{C} \arrow[d] & \overline{C} \smallsetminus C = (C \smallsetminus \{0\})/\mathbb{G}_m \arrow[l, hook'] \arrow[dl] \\ & S & \end{tikzcd}$$

that is to say $\overline{C}$ is $\mathbb{G}_m$-equivariant a compactification of C obtained by adding

$$(C \smallsetminus \{0\})/\mathbb{G}_m = \mathrm{Proj}(\mathcal{A}_\bullet)$$

at ∞ where $\{0\} \hookrightarrow C$ is the origin of the cone, $\mathrm{Spec}(\mathcal{A}_0) \hookrightarrow \mathrm{Spec}(\mathcal{A})$.

If our cone is the geometric realization of a vector bundle $\mathscr{E}$,

$$C = \mathbb{V}(\mathscr{E}),$$

then

$$\overline{C} = \mathbb{P}(\mathscr{E} \oplus \mathcal{O}_S) \supset \overline{C} \smallsetminus C = \mathbb{P}(\mathscr{E}).$$

Example 11.1.1. — *When we have a closed immersion of smooth S-schemes*

$$\begin{tikzcd} Y \arrow[rr, hook, "i"] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

there is an exact sequence of vector bundles on Y

$$0 \longrightarrow T_{Y/S} \longrightarrow i^*T_{X/S} \longrightarrow (\mathfrak{I}/\mathfrak{I}^2)^\vee \longrightarrow 0.$$

Remark 11.1.2. — *The terminology normal bundle comes from the fact that if* (M, g) *is a Riemannian manifold and* $i : N \hookrightarrow M$ *a closed submanifold then the metric* g *allows us to identify the normal bundle with the orthogonal of* T_N *inside* i^*T_M *with respect to the metric* g*, giving a splitting of the sequence of* $\mathcal{C}^\infty$*-vector bundles* $0 \to T_N \to i^*T_N \to \mathcal{N}_{N/M} \to 0$*. At the end, for* $n \in N$*, the normal bundle at* n *is the set of tangents vectors in* $(TM)_n$ *that are orthogonal to* $(TN)_n$*.*

Remark 11.1.3. — *In the context of the preceding remark, let*

$$\exp : TM \longrightarrow M$$

(that is well defined only in a nbd. of the zero section in general if (M, g) *is not complete) be the map that sends the tangent vector* $X \in (TM)_m$ *to* $\gamma(1)$ *where* γ *is the unique geodesic satisfying* $\gamma(0) = m$ *and* $\gamma'(0) = X$*.*

One can find a neighborhood U *of the zero section of* $\mathcal{N}_{N/M} = (TN)^\perp \subset i^*TM$ *such that the map*

$$\exp_{|U} : U \longrightarrow M$$

is an isomorphism onto an open neighborhood of N *inside* M*. This is what's called a tubular neighborhood of* N *inside* M*. The deformation to the normal cone is an algebraic analog of this construction.*

11.1.2. The deformation to the normal cone. —

11.1.2.1. *What we want.* — The deformation to the normal cone is a $\mathbb{G}_m$-equivariant family of closed immersions parametrized by $\mathbb{A}^1$

$$\begin{tikzcd}
Y \times \mathbb{A}^1 \arrow[loop left, "\mathbb{G}_m"'] \arrow[r, hook] & W \arrow[loop right, "\mathbb{G}_m"] \arrow[d] \\
& X \times \mathbb{A}^1 \arrow[loop right, "\mathbb{G}_m"] \arrow[d] \\
& \mathbb{A}^1 \arrow[loop right, "\mathbb{G}_m"]
\end{tikzcd}$$

satisfying:

1. The following diagram

$$\begin{tikzcd} Y \times \mathbb{A}^1 \arrow[r, hook] \arrow[dr, hook, "i \times \mathrm{Id}_{\mathbb{A}^1}"'] & W \arrow[d] \\ & X \times \mathbb{A}^1 \end{tikzcd}$$

commutes.

2. The morphism

$$\begin{tikzcd} W \arrow[d] \\ \mathbb{A}^1 \end{tikzcd}$$

is flat i.e. we have a flat family parametrized by $\mathbb{A}^1$.

3. Its restriction to $\mathbb{G}_m = \mathbb{A}^1 \smallsetminus \{0\}$ is identified with

$$\begin{tikzcd} Y \times \mathbb{G}_m \arrow[r, hook] & W_{|\mathbb{G}_m} \arrow[d, "\simeq"] \\ & X \times \mathbb{G}_m \end{tikzcd}$$

and thus the family of immersions $Y \hookrightarrow W_t$ for $t \in \mathbb{A}^1 \smallsetminus \{0\}$ is the trivial family associated to $i : Y \hookrightarrow X$.

4. Its fiber at $0 \in \mathbb{A}^1$ is identified with

$$\begin{tikzcd} Y \arrow[r, hook, "\text{zero section}\\ \text{of the normal cone}"] & W_0 = C_Y X \arrow[d, "\text{projection to } Y\\ \text{composed wt. } i"] \\ & X \end{tikzcd}$$

and thus the immersion $Y \hookrightarrow W_0$ is identified with the inclusion of the zero section of the normal cone.

11.1.2.2. *The construction.* — This is defined in the following way. Let Z be the blow-up of $Y \times \{0\}$ inside $X \times \mathbb{A}^1$,

$$\begin{tikzcd} & Z \arrow[d, "\text{blow-up}"] \\ Y \times \{0\} \arrow[r, hook] & X \times \mathbb{A}^1. \end{tikzcd}$$

We have

$$Z = \operatorname{Proj}\Big(\bigoplus_{k\geq 0}\sum_{i=0}^{k}\mathfrak{I}^{k-i}T^{i}\mathcal{O}_X[T]\Big).$$

The closed immersion $Y\times\mathbb{A}^1 \hookrightarrow Z$ is defined by the universal property of the blow-up: via the morphism $Y\times\mathbb{A}^1\to X\times\mathbb{A}^1$ the pull-back of $Y\times\{0\}$ inside $X\times\mathbb{A}^1$ is a Cartier divisor.

It is evident that over $\mathbb{G}_m$, the immersion $Y\times\mathbb{A}^1\hookrightarrow Z$ is identified with $Y\times\mathbb{G}_m\hookrightarrow X\times\mathbb{G}_m$.

The fiber over $\{0\}$ is

$$\operatorname{Proj}\Big(\mathcal{O}_X\bigoplus(\mathfrak{I}\oplus\mathcal{O}_X/\mathfrak{I}.t)\bigoplus\cdots\bigoplus(\mathfrak{I}^k\oplus\mathfrak{I}^{k-1}/\mathfrak{I}^k.t\oplus\cdots\oplus\mathfrak{I}/\mathfrak{I}^2.t^{k-1}\oplus\mathcal{O}_X/\mathfrak{I}.t^k)\bigoplus\cdots\Big).$$

Here we use the bigraded ring

$$\mathcal{A} = \bigoplus_{k,l\in\mathbb{N}}\mathcal{A}_{k,l}$$

where

- $\mathcal{A}_{0,l} = \mathfrak{I}^l$,
- $\mathcal{A}_{k,l} = \mathfrak{I}^l/\mathfrak{I}^{l+1}$ if $k>0$.

Then, if $\operatorname{Tot}\mathcal{A}$ is the graded ring such that

$$(\operatorname{Tot}\mathcal{A})_d = \bigoplus_{k+l=d}\mathcal{A}_{k,l},$$

the fiber over $\{0\}$ is

$$\operatorname{Proj}\big(\operatorname{Tot}\mathcal{A}\big).$$

There are two graded ideals in $\operatorname{Tot}\mathcal{A}$,

$$\mathfrak{a} = \bigoplus_{k>0,l\geq 0}\mathcal{A}_{k,l}$$

and

$$\mathfrak{b} = \bigoplus_{l\geq 0}\mathfrak{I}\mathcal{A}_{0,0}$$

contained in the augmentation ideal $(\operatorname{Tot}\mathcal{A})^+$. One has

$$\mathfrak{a}\mathfrak{b} = (0)$$

and thus

$$\operatorname{Proj}\big(\operatorname{Tot}\mathcal{A}\big) = V^+(\mathfrak{a})\cup V^+(\mathfrak{b}).$$

From this formula we see that the fiber of Z over $\{0\}$ is a union of B_YX the blow-up of Y inside X and $\overline{C_YX}$ the projective completion of the normal cone C_YX. There is an identification

$$\overline{C_YX}\smallsetminus C_YX = (C_YX\smallsetminus\{0\})/\mathbb{G}_m.$$

This is identified with the exceptional Cartier divisor $E_YX\subset B_YX$ of the blow-up of Y inside X and our fiber at $\{0\}$ is then

$$Z_0 = B_Y X \coprod_{E_Y X = \overline{C_Y X} \smallsetminus C_Y X} \overline{C_Y X}.$$

One then sets

$$W = Z \smallsetminus \underbrace{B_Y X}_{\substack{\text{closed subset} \\ \text{of the fiber at } \{0\}}} .$$

as an open subscheme of Z.

11.1.2.3. *An explicit formula.* — One can show the following.

Proposition 11.1.4. — *Let*

$$\mathcal{A} = \bigoplus_{n<0} \mathfrak{I}^{-n} t^n \bigoplus \mathcal{O}_X[t] = \mathcal{O}_X\left[\tfrac{\mathfrak{I}}{t}, t\right] \subset \mathcal{O}_X[t, t^{-1}]$$

as a quasi-coherent $\mathcal{O}_X$-algebra. The morphism $\mathcal{O}_X[t] \to \mathcal{A}$ identifies

$$Spec(\mathcal{A}) \longrightarrow X \times \mathbb{A}^1$$

with

$$W \longrightarrow X \times \mathbb{A}^1.$$

In particular, the morphism $W \to X \times \mathbb{A}^1$ is affine.

11.1.3. Some classical applications. —

11.1.3.1. *Verdier's specialization to the normal cone.* — Let k be an algebraically closed field and

$$i : Y \hookrightarrow X$$

be a closed immersion of finite type k-schemes. Let Λ be a finite ring killed by a power of ℓ invertible in k. Recall that an étale constructible sheaf $\mathscr{F}$ of Λ-modules on the cone $C_Y X$ is said to be *monodromic* if for all $\lambda \in \mathbb{G}_m(k)$, if $m_\lambda : C_Y X \to X_Y X$ is the action of λ, then

$$m_\lambda^* \mathscr{F} \simeq \mathscr{F}.$$

This is equivalent to say that for all $x \in C_Y X$, if $w_x : \mathbb{G}_m \to C_Y X$ is the morphism $\lambda \mapsto \lambda.x$ then $w_x^* \mathscr{F}$ is a moderate étale local system of Λ-modules.

Let

$$D_c^b(C_Y X, \Lambda)^{mon}$$

be the category of bounded étale complexes of Λ-modules on $C_Y X$ with constructible monodromic cohomology.

Verdier defines a factorization

$$\begin{tikzcd}
D_c^b(X,\Lambda) \arrow[rr, "i^*"] \arrow[dr, "\mathrm{Sp}_Y X"'] & & D_c^b(Y,\Lambda) \\
& D_c^b(C_Y X,\Lambda)^{mon} \arrow[ur, "0^*"'] &
\end{tikzcd}$$

where $0 : Y \hookrightarrow C_Y X$ is the zero section of the cone. This is called the specialization to the normal cone.

This is defined in the following way. If $\mathrm{pr} : X \times \mathbb{G}_m \longrightarrow X$ is the projection we use the diagram

$$\begin{tikzcd}
C_Y X \arrow[r, hook] \arrow[d] & W \arrow[d] & X \times \mathbb{G}_m \arrow[l, hook'] \arrow[d] \\
\{0\} \arrow[r, hook] & \mathbb{A}_k^1 & \mathbb{G}_{m,k} \arrow[l, hook']
\end{tikzcd}$$

Then, for $A \in D_c^b(X,\Lambda)$,

$$\mathrm{Sp}_Y X(A) := R\Psi_{\overline{\eta}}\big(\,\mathrm{pr}^* A\big)$$

where $\overline{\eta}$ is any geometric point over the generic point of $\mathbb{A}_k^1$.

The fact that $0^* \,\mathrm{Sp}_Y X(A) = i^* A$ and that $\mathrm{Sp}_Y X(A)$ is monodromic is a theorem of Verdier.

11.1.3.2. *Microlocalization.* — When $i : Y \hookrightarrow X$ is a regular immersion, the cone $C_Y X$ is a vector bundle

$$N_Y X = \mathbb{V}\big((\mathfrak{I}/\mathfrak{I}^2)^\vee\big) \longrightarrow Y,$$

the normal bundle. We can look at the dual vector bundle

$$N_Y^* X = \mathbb{V}\big(\mathfrak{I}/\mathfrak{I}^2\big) \longrightarrow Y.$$

There is then a microlocalization functor

$$\mu = \underbrace{\mathscr{F}}_{\substack{\text{Fourier}\\ \text{transform}}} \circ \,\mathrm{Sp}_Y X : D_c^b(X,\Lambda) \longrightarrow D_c^b(N_Y^* X,\Lambda).$$

For example, if X is smooth over $\mathrm{Spec}(k)$, applying this to Y the diagonal of $X \times X$, we can define

$$\mu\mathscr{H}om(A,B) = \mu(R\mathscr{H}om(p_1^* A, p_2^! B)) \in D_c^b(T^* X,\Lambda).$$

11.1.3.3. *Gysin maps.* — See Fulton's book.

11.2. Proof of the Jacobian criterion of smoothness

Recall: we have proven that if

$$\begin{tikzcd}\mathcal{M}^{sm}_Z \arrow[d, "f"] \\ S\end{tikzcd}$$

then $\mathbb{F}_\ell$ is f-ULA. It remains to prove that

$$Rf^!\mathbb{F}_\ell$$

is invertible.

11.2.1. First reduction. — It suffices to prove that for any morphism

$$g : S' \longrightarrow \mathcal{M}^{sm}_Z$$

with S' a strictly totally disconnected perfectoid space,

$$g^*Rf^!\mathbb{F}_\ell$$

is invertible. For this, recall that $\mathbb{F}_\ell$ f-ULA implies that the formation of $Rf^!\mathbb{F}_\ell$ is compatible with base change$/S$. In particular, in the diagram

$$\begin{tikzcd}\mathcal{M}^{sm}_{Z\times_{X_S}X_{S'}} \arrow[r, equal] & \mathcal{M}^{sm}_Z \times_S S' \arrow[r] \arrow[d, "f'"] & \mathcal{M}^{sm}_Z \arrow[d, "f"] \\ & S' \arrow[u, bend left, "s"] \arrow[r, "f\circ g"] & S,\end{tikzcd}$$

where the section s is $(g, \mathrm{Id}_{S'})$, we have

$$g^*Rf^!\mathbb{F}_\ell = s^*Rf'^!\mathbb{F}_\ell.$$

We thus have to prove that

$$s^*Rf'^!\mathbb{F}_\ell$$

is invertible. Up to replacing S by S' we are thus reduced to proving that for any section s of $f : \mathcal{M}^{sm}_Z \to S$,

$$s^*Rf^!\mathbb{F}_\ell$$

is invertible when S is a strictly totally disconnected perfectoid space.

11.2.2. Construction of the deformation: 1st step. — The section s corresponds to a section

$$\begin{tikzcd}Z \arrow[d] \\ X_S \arrow[u, bend right]\end{tikzcd}$$

Let us suppose, to simplify (and this case is sufficient for the application in the article where $Z = (G/P)^{ad} \times X_S$), that, if $S = \mathrm{Spa}(R, R^+)$, via the GAGA morphism of ringed spaces

$$X_S \longrightarrow \mathfrak{X}_{R,R^+}$$

one has

$$Z = \mathfrak{Z}^{ad}$$

where $\mathfrak{Z} \to \mathfrak{X}_{R,R^+}$ is quasi-projective and smooth as a morphism of schemes. The section $X_S \hookrightarrow Z$ then corresponds to a section of $\mathfrak{Z} \to \mathfrak{X}_{R,R^+}$ and we can perform a deformation to the normal cone for this schematical section and then adify it.

At the end we obtain a $\mathbb{G}_m$-equivariant embedding

$$\begin{tikzcd} X_S \times \mathbb{A}^1 \arrow[r, hook] & W \arrow[d] \\ & Z \times \mathbb{A}^1 \end{tikzcd}$$

where

1. when restricted to $Z \times \mathbb{G}_m$ this is

$$\begin{tikzcd} X_S \times \mathbb{G}_m \arrow[r, hook] & Z \times \mathbb{G}_m \arrow[d, "\mathrm{Id}"] \\ & Z \times \mathbb{G}_m \end{tikzcd}$$

2. The fiber at $\{0\}$ is

$$\begin{tikzcd} X_S \arrow[r, hook] & C_{X_S} Z \arrow[d] \\ & Z \end{tikzcd}$$

Recall that

$$\mathcal{M}_{\mathbb{A}^1 \times X_S} = \underline{E}_S$$

and we have the formula $M_{Z_1 \times Z_2} = \mathcal{M}_{Z_1} \times_S \mathcal{M}_{Z_2}$.

We can now consider the associated diagram

$$\begin{tikzcd} \mathcal{M}_W \arrow[d, "g"'] \\ S \times \underline{E} \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

This is equivariant with respect to the action of $\underline{E}^\times$.

At the end one has an $\underline{E}^\times$-equivariant diagram with cartesian squares

$$\begin{tikzcd}
BC(s^*T_{Z/X_S}) \arrow[r, hook] \arrow[d] & \mathcal{M}_W^{sm} \arrow[d, "g"'] & \mathcal{M}_Z^{sm} \times \underline{E} \setminus \{0\} \arrow[l, hook'] \arrow[d, "f"'] \\
S \arrow[u, bend left, "\text{zero section}"] \arrow[r, hook, "\mathrm{Id}_S \times \{0\}"] & S \times \underline{E} \arrow[u, bend right, "\sigma"'] & S \times \underline{E} \setminus \{0\} \arrow[l, hook'] \arrow[u, bend right, "s\times \mathrm{Id}_{\underline{E}\setminus\{0\}}"']
\end{tikzcd}$$

11.2.3. Construction of the deformation: 2nd step. — Up to replacing W by a quasi-compact open nbd. of the section $X_S \times \mathbb{B}^1 \hookrightarrow W$ we can suppose we have an $\underline{\mathcal{O}_E} \setminus \{0\}$-equivariant diagram

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d] & \mathcal{M}_W \arrow[d, "g"'] & \Delta \arrow[l, hook'] \arrow[d, "f"'] \\
S \arrow[u, bend left, "\text{zero section}"] \arrow[r, hook, "\mathrm{Id}_S \times \{0\}"] & S \times \underline{\mathcal{O}_E} \arrow[u, bend right, "\sigma"'] & S \times \underline{\mathcal{O}_E} \setminus \{0\} \arrow[l, hook'] \arrow[u, bend right, "s\times \mathrm{Id}_{\underline{\mathcal{O}_E}\setminus\{0\}}"']
\end{tikzcd}$$

with $\mathcal{M}_W = \mathcal{M}_W^{sm}$ that is quasi-compact and U an open nbd. of the zero section of $BC(s^*T_{Z/X_S})$. Pulling-back the situation to $\pi^{\mathbb{N}\cup\{+\infty\}} \subset \mathcal{O}_E$ we obtain a $\pi^{\mathbb{N}}$-equivariant diagram

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d] & \mathcal{M}_W \arrow[d, "g"'] & \Delta \arrow[l, hook'] \arrow[d, "f"'] \\
S \arrow[u, bend left, "\text{zero section}"] \arrow[r, hook, "\mathrm{Id}_S \times \{0\}"] & S \times \underline{\pi^{\mathbb{N}\cup\{+\infty\}}} \arrow[u, bend right, "\sigma"'] & S \times \underline{\pi^{\mathbb{N}}} \arrow[l, hook'] \arrow[u, bend right, "s\times \mathrm{Id}_{\underline{\pi^{\mathbb{N}}}}"']
\end{tikzcd}$$

11.2.4. Proof of the Jacobian criterion. — Let

$$A = Rg^!\mathbb{F}_\ell \in D_{\text{ét}}(\mathcal{M}_W, \mathbb{F}_\ell)$$

that is $\pi^{\mathbb{N}}$-equivariant. If $i : U \hookrightarrow \mathcal{M}_W^{sm}$, since $\mathbb{F}_\ell$ is g-ULA,

$$i^*A \in D_{\text{ét}}(U, \mathbb{F}_\ell)$$

is étale locally on S isomorphic to $\mathbb{F}_\ell[2d]$ with the trivial $\pi^{\mathbb{N}}$-equivariant structure, where $d = \deg(s^*T_{Z/X_S})$. Since S is strictly totally disconnected we can fix an $E^\times$-equivariant isomorphism

$$\mathbb{F}_\ell[2d] \xrightarrow{\sim} i^*A.$$

Now, we have

$$\varinjlim_{N\geq 0} H^{-2d}(\underbrace{g^{-1}(S \times \underline{\pi^{\mathbb{N}_{\geq N}\cup\{+\infty\}}})}_{\mathcal{M}_W^{\geq N}}, A) \xrightarrow{\sim} H^{-2d}(U, i^*A).$$

We can thus find an integer $N \geq 0$ and a $\pi^{\mathbb{N}}$-equivariant morphism

$$\mathbb{F}_\ell[2d] \longrightarrow A_{|\mathcal{M}_W^{\geq N}}$$

inducing the given equivariant isomorphism $\mathbb{F}_\ell[2d] \xrightarrow{\sim} i^*A$. Let

$$B \in D_{\text{ét}}(\mathcal{M}_W^{\geq N}, \mathbb{F}_\ell)$$

be a cone of this morphism ans an $\pi^{\mathbb{N}}$-equivariant object

$$\mathbb{F}_\ell[2d] \longrightarrow A_{|\mathcal{M}_W^{\geq N}} \longrightarrow B \xrightarrow{+1} .$$

Since $\mathbb{F}_\ell[2d]$ and A (as the Verdier dual of $\mathbb{F}_\ell$, see the stability property of the ULA property under Verdier duality) are g-ULA, B is g-ULA. In particular its Verdier dual is g-ULA and

$$Rg_!\mathbb{D}(B)$$

is constructible, where $\mathbb{D} = \mathbb{D}_{\mathcal{M}_W^{\geq N}/S\times\underline{\pi^{\mathbb{N}_{\geq N}\cup\{+\infty\}}}}$. Since its fiber at $S = S \times \pi^{+\infty}$ is zero we deduce that, up to replacing N by a bigger integer, we can suppose that

$$Rg_!\mathbb{D}(B) = 0.$$

Applying Verdier duality and using the biduality of B we obtain

$$Rg_*B = 0.$$

We not note $\Delta^{(n)}$ for $n \in \mathbb{N}_{\geq N}$ the fiber of g over $S \times \pi^n$. The $\pi^{\mathbb{N}}$-equivariance property implies that we have

$$\Delta^{(n+1)} \subset \Delta^{(n)}$$

with

$$\bigcap_{n\geq N} \Delta^{(n)} = s(S).$$

We thus have if $f^{(n)} : \Delta^{(n)} \to S$, and C is the fiber of B at $S \times \pi^{\mathbb{N}}$: for all $n \geq N$

$$Rf_*^{(n)} C_{|\mathcal{M}_Z^{(n)}} = 0.$$

We deduce that

$$s^*C = \varinjlim_{n\geq N} Rf_*^{(n)} C = 0$$

and thus

$$\mathbb{F}_\ell[2d] \xrightarrow{\sim} s^*Rf^!\mathbb{F}_\ell.$$

This finishes the proof.

LECTURE 12

AN APPLICATION OF THE JACOBIAN CRITERION: COMPACT GENERATION AND FINITENESS OF THE COHOMOLOGY OF LOCAL SHIMURA VARIETIES

12.1. Local Shimura varieties

12.1.1. Definition. —

12.1.1.1. *The tower over the reflex field.* — Let G over E be as before. Let $[b] \in B(G)$ and $\{\mu\}$ be a conjugacy class cocharacters of $G_{\overline{E}}$ where we have fixed $\overline{E}$ an algebraic closure of E. Let $E_\mu|E$ be the field of definition of $\{\mu\}$, $E_\mu \subset \overline{E}$, and $\breve{E}_\mu$ the completion of the maximal unramified extension of E_μ. For $K \subset G(\mathbb{Q}_p)$ compact open let

$$\mathcal{M}_K(G,b,\mu) \longrightarrow \mathrm{Spa}(\breve{E}_\mu)^\diamond$$

be the associated local Shimura variety. If

$$\mathcal{M}(G,b,\mu) = \varprojlim_{K} \mathcal{M}_K(G,b,\mu),$$

The v-sheaf $\mathcal{M}(G,b,\mu)$ sends S an $\overline{\mathbb{F}}_q$-perf. space to an untilt $S^\sharp$ over $\breve{E}_\mu$ and an isomorphism

$$\mathscr{E}_{1|X_S \smallsetminus S^\sharp} \xrightarrow{\sim} \mathscr{E}_{b|X_S \smallsetminus S^\sharp}$$

that is meromorphic along the Cartier divisor $S^\sharp \hookrightarrow X_S$ and is of type $\leq \mu$ geometrically fiberwise over S.

This is representable by a locally spatial diamond compactifiable of finite dim. trg. over $\mathrm{Spa}(\breve{E}_\mu)$. There are two morphisms

$$\begin{tikzcd}
 & \mathcal{M}(G,b,\mu) \arrow[loop above, "\underline{G(E)}\times\widetilde{G}_b"] \arrow[dl, "\pi_{dR}"] \arrow[dr, "\pi_{HT}"'] \arrow[dl, dash, bend right, "\underline{G(E)}"'] \arrow[dr, dash, bend left, "\widetilde{G}_b"] & \\
\mathrm{Gr}_G^{\leq\mu,a} & & \mathrm{Gr}_G^{\leq\mu^{-1},a}
\end{tikzcd}$$

where

1. $\mathcal{M}_K(G,b,\mu) = \underline{K}\backslash\mathcal{M}(G,b,\mu)$,
2. Gr_G is the B_{dR}-affine Grassmanian and $\mathrm{Gr}^{\leq\mu}$ the closed Schubert cell defined by μ, a spatial diamond proper over $\mathrm{Spa}(\breve{E}_\mu)^\diamond$,
3. $\widetilde{G}_b \to *$ is the group of automorphisms of $\mathscr{E}_b$ that sends $S \to *$ to $\mathrm{Aut}(\mathscr{E}_{b|X_S})$, this is a semi-direct product
$$\widetilde{G}_b = \underline{G_b(E)} \ltimes \widetilde{G}_b^\circ,$$
4. π_{dR} is $\widetilde{G}_b$-equivariant and a $\underline{G(E)}$-torsor onto its image, an open subset, the so-called admissible open subset
5. π_{HT} is $\underline{G(E)}$-equivariant and a $\widetilde{G}_b$-torsor over its image, the sub-locally spatial diamond of $\mathrm{Gr}_G^{\leq\mu^{-1}}$ defined by a locally closed generalizing subset of $|\mathrm{Gr}_G^{\leq\mu^{-1}}|$. We still call it the admissible subset.

12.1.1.2. *The Frobenius action.* — The preceding picture descends from $\mathrm{Spa}(\breve{E}_\mu)$ to

$$\mathrm{Div}^1_{E_\mu} = \mathrm{Spa}(\breve{E}_\mu)^\diamond/\varphi^{\mathbb{Z}}_{E_\mu}.$$

In fact, given any degree 1 effective divisor D on X_{S,E_μ} its norm $D' = N_{E_\mu/E}$ is a degree 1 Cartier divisor on $X_S := X_{S,E}$ and we can speak about modifications

$$\mathscr{E}_{1|X_S\smallsetminus D'} \xrightarrow{\sim} \mathscr{E}_{b|X_S\smallsetminus D'}.$$

The moduli space $\mathcal{M}_K(G,b,\mu)$ thus descends via $\mathrm{Spa}(\breve{E}_\mu)^\diamond \to \mathrm{Div}^1_{E_\mu}$ to a moduli space

$$\mathrm{Sht}_K(G,b,\mu) \longrightarrow \mathrm{Div}^1_{E_\mu}$$

and more generally

$$\mathrm{Sht}(G,b,\mu) \longrightarrow \mathrm{Div}^1_{E_\mu}.$$

12.1.1.3. *Coefficients.* — The composite

$$\mathcal{M}(G,b,\mu) \xrightarrow{\pi_{dR}} \mathrm{Gr}_G^{\leq\mu} \longrightarrow [L^+G\backslash \mathrm{Gr}_G^{\leq\mu}] \hookrightarrow [L^+G\backslash \mathrm{Gr}_G^{\leq\mu}] = [L^+G\backslash LG/L^+G]$$

and

$$\begin{tikzcd}
\mathcal{M}(G,b,\mu) \arrow[r, "\pi_{HT}"] & \mathrm{Gr}_G^{\leq\mu^{-1}} \arrow[r] & {[L^+G\backslash \mathrm{Gr}_G^{\leq\mu^{-1}}]} \arrow[r, hook] & {[L^+G\backslash \mathrm{Gr}_G]} \arrow[d, equal] \\
 & & & {[L^+G\backslash LG/L^+G]} \arrow[d, "\simeq"', "g\mapsto g^{-1}"] \\
 & & & {[L^+G\backslash LG/L^+G]}
\end{tikzcd}$$

coincide. This descends to a morphism over $\mathrm{Div}^1_{E_\mu}$

$$\boxed{\mathrm{Sht}(G,b,\mu) \longrightarrow Hck_G^{\leq\mu}}$$

where $Hck_G^{\leq\mu} \to \mathrm{Div}^1_{E_\mu}$ is the closed Schubert strata in the local Hecke stack. This descends to a morphism

$$\big[\underline{G(E)} \times \widetilde{G}_b\backslash \mathrm{Sht}(G,b,\mu)\big] \longrightarrow Hck_G^{\leq\mu}.$$

The geometric Satake correspondence allows us to define a perverse ULA (relative to the morphism toward $\mathrm{Div}^1_{E_\mu}$) sheaf

$$j_{!*}\Lambda \in D_{\text{ét}}(Hck^{\leq\mu},\Lambda)$$

where j is the inclusion of the open Schubert cell. This corresponds to the representation of $\widehat{G} \rtimes W_{E_\mu}$ with weight μ.

By pull-back we obtain a on object

$$\boxed{S_\mu \in D_{\text{ét}}\Big(\big[\underline{G(E)} \times \widetilde{G}_b\backslash \mathrm{Sht}(G,b,\mu)\big],\Lambda\Big)}$$

12.1.2. Cohomology. —

12.1.2.1. *The equivariant cohomology complex.* — One has

$$\mathrm{Div}^1_{E_\mu} = \mathrm{Spa}(\breve{E}_\mu)^\diamond/\varphi_{E_\mu}^{\mathbb{Z}} = [\mathrm{Spa}(\mathbb{C}_p^\flat)/\underline{W_{E_\mu}}].$$

There is thus a morphism

$$\mathrm{Div}^1_{E_\mu} \longrightarrow [*/\underline{W_{E_\mu}}].$$

***Lemma 12.1.1* (Drinfeld lemma; particular case)**
Pull-back induces an equivalence

$$D(G_b(E)\times W_{E_\mu},\Lambda) \xrightarrow{\sim} D_{\text{ét}}([*/\widetilde{G}_b]\times[*/W_{E_\mu}],\Lambda) \xrightarrow{\sim} D_{\text{ét}}([*/\widetilde{G}_b]\times \mathrm{Div}^1_{E_\mu}],\Lambda).$$

12.1.2.2. *The theorem.* — Here is the theorem we want to prove. Let

$$f_K : \big[\widetilde{G}_b\backslash \mathrm{Sht}_K(G,b,\mu)\big] \longrightarrow [*/\widetilde{G}_b]\times \mathrm{Div}^1_{E_\mu}$$

Theorem 12.1.2. — *For all compact open subset K of $G(E)$*

$$Rf_{K!}S_\mu \in D(G_b(E),\Lambda)^{BW_{E_\mu}}$$

is a compact object in $D\big(G_b(E),\Lambda\big)$ i.e. an object in the thick triangulated subcategory of $D\big(G_b(E),\Lambda\big)$ generated by the c-$\mathrm{Ind}_{K'}^{G_b(E)}\Lambda$ when K' goes through the set of compact open pro-p subgroups of $G_b(E)$.

12.2. Local charts using the Jacobian criterion

12.2.1. Construction of the local charts. — Let $[b]\in B(G)$. Suppose, to simplify that G is quasi-split (if not everytuing works since $G\times X_S$ is a quasi-split group scheme over X_S). Let $[\nu_b]\in X_*(A)^+_{\mathbb{Q}}$, M_b its centralizer, a standard Levi, and P_b the standard parabolic subgroup associated. We note b_{M_b} for the canonical reduction of b to M_b.

Definition 12.2.1. — *We note $\mathcal{M}_b$ the small v-stack associating to S a P_b-bundle $\mathscr{E}$ on X_S such that geometrically fiberwise on S, $\mathscr{E}\overset{P_b}{\times} M_b$ is isomorphic to $\mathscr{E}_{b_{M_b}}$.*

There is a cartesian square

$$\begin{tikzcd}
\mathcal{M}_b \arrow[r, hook] \arrow[d] & \mathrm{Bun}_{P_b} \arrow[d] \\
{[\,*\,/\underline{G_b(E)}]} \arrow[r, hook] & \mathrm{Bun}_{M_b}
\end{tikzcd}$$

where the right down map sends a P_b-torsor $\mathscr{E}$ on X_S to $\mathscr{E}\overset{P_b}{\times} M_b$. Let

$$\mathrm{Bun}^{\circ}_{P_b}\subset \mathrm{Bun}_{P_b}$$

be the open sub-stack such that $\mathrm{Bun}^{\circ}_{P_b}(S)$ is the groupoid of P_b-bundles $\mathscr{E}$ on X_S such that the vector bundle $\mathscr{E} \overset{P_b,\mathrm{Ad}}{\times} \mathrm{Lie}\,G/\mathrm{Lie}\,P_b$ has geometrically fiberwise on $S > 0$ HN slopes. The weights of $[\nu_b] \in X_*a(A)^+_{\mathbb{Q}}$ on $\mathrm{Lie}\,gG/\mathrm{Lie}\,P_b$ are < 0. FRom this we deduce that

$$\mathcal{M}_b \subset \mathrm{Bun}^{\circ}_{P_b}.$$

Now, the Jacobian criterion of smoothness implies that the morphism

$$\mathrm{Bun}^{\circ}_{P_b} \longrightarrow \mathrm{Bun}_G$$

is cohomologically smooth. We deduce a diagram

$$\begin{tikzcd}
\widetilde{\mathcal{M}}_b \arrow[r] \arrow[r, bend left, no head, "\underline{G_b(E)}"] \arrow[d] & \mathcal{M}_b \arrow[r, "\substack{\ell\text{-cohomologically}\\ \text{smooth}}"] \arrow[d] & \mathrm{Bun}_G \\
* \arrow[r] \arrow[u, bend left] & {\big[* / \underline{G_b(E)} \big]} \arrow[u, bend right]
\end{tikzcd}$$

where the square is cartesian and defines $\widetilde{\mathcal{M}}_b$. The left vertical section if given by the inclusion $M_p \subset P_b$. Let $K \subset G_b(E)$ be compact open pro-p. We obtain an ℓ-cohomologically sm. morphism

$$f^b_K : \underline{K}\backslash\widetilde{\mathcal{M}}_b \xrightarrow{\substack{\text{separated}\\ \text{étale}}} \mathcal{M}_b \xrightarrow{\substack{\ell\text{-cohomologically}\\ \text{smooth}}} \mathrm{Bun}_G$$

12.2.2. Properties of $\widetilde{M}_b$. — The following are two key points of the local charts constructed.

Theorem 12.2.2. — *The v-sheaf $\widetilde{\mathcal{M}}_b$ satisfies*

1. *$\widetilde{\mathcal{M}}_b \smallsetminus \{*\}$ is a spatial diamond*
2. *If $i : \{*\} \hookrightarrow \widetilde{\mathcal{M}}_b$, for any $A \in D_{\text{ét}}(\widetilde{\mathcal{M}}_b, \Lambda)$ one has*

$$R\Gamma(\widetilde{\mathcal{M}}_b, A) \xrightarrow{\sim} i^*A.$$

12.3. Some compact generators of $D_{\text{ét}}(\text{Bun}_G, \Lambda)$

Definition 12.3.1. — *For $[b] \in B(G)$ and K a compact open pro-p subgroup of $G_b(E)$ define*

$$A_K^b = Rf_{K!}^b \, Rf_K^{b!} \Lambda \in D_{\text{ét}}(\text{Bun}_G, \Lambda).$$

Example 12.3.2. — *If b is basic then*

$$A_K^b = i_!^b \Big(c - \text{Ind}_K^{G_b(E)} \Lambda \Big)$$

and thus A_K^b corresponds in this case to the standard generator $c - \text{Ind}_K^{G_b(E)} \Lambda$ of $D(G_b(E), \Lambda)$.

Proposition 12.3.3. — *The following is satisfied*

1. *For any $B \in D_{\text{ét}}(\text{Bun}_G, \Lambda)$ one has*
$$Hom(A_K^b, B) = (i^b)^* B.$$
2. *The collection $\left(A_K^b\right)_{[b],K}$ is a set of compact generators of $D_{\text{ét}}(\text{Bun}_G, \Lambda)$.*

→ In particular,

$$D_{\text{ét}}(\text{Bun}_G, \Lambda)^{\omega} = \text{thick triangulated sub-cat. generated by } \left(A_K^b\right)_{[b],K}.$$

12.4. The compactness criterion

Here is the main theorem.

Theorem 12.4.1 **(Compacity criterion).** — *An object $A \in D_{\text{ét}}(\text{Bun}_G, \Lambda)$ is compact iff $\{|b| \mid (i^b)^* A \neq 0\}$ is finite (i.e. A is supported on a qc. open subset of Bun_G) and for all $[b] \in B(G)$,*

$$(i^b)^* A \in D(G_b(E), \Lambda)$$

is compact (i.e. lies in the thick triangulated sub-category generated by the collection $\left(c - \text{Ind}_K^{G_b(E)} \Lambda\right)_{\substack{K \subset G_b(E) \\ \text{open pro-}p}}$).

Proof. — Its is evident that A compact implies it is supported on a qc open subset. Let $U \subset \mathrm{Bun}_G$ be such a qc open subset and $A \in D_{\text{ét}}(U, \Lambda)$. Choose $[b] \in |U|$ a closed point. Let $j : U \smallsetminus \{[b]\} \hookrightarrow U$. There is an exact triangle

$$i^b_* i^{b*} A \longrightarrow A \longrightarrow j_! j^* A \xrightarrow{+1} .$$

The functors i^b_* and $j_!$ have right adjoints that commute with arbitrary direct sums (in fact $R(i^b)^!$ is isomorphic to a shift of $(i^b)^*$ since U and $[*/\widetilde{G}_b]$ are ℓ-cohomologically smooth). Thus, at the end we just need to prove, by induction on the cardinality of $|U|$, that if A is compact then j^*A is compact in $D_{\text{ét}}(U \smallsetminus \{[b]\}, \Lambda)$. Since $D_{\text{ét}}(U, \Lambda)^\omega$ is the thick triangulated sub-category generated by the $A^{b'}_{K'}$ with $[b'] \in |U|$ is suffices to prove that for $[b'] \in |U|$ and $K' \subset G_{b'}(E)$ open pro-p,

$$j^* A^{b'}_{K'}$$

is compact. If $[b'] \neq [b]$ one has $j^* A^{[b']}_{K'} = A^{[b']}_{K'}$ and the result is evident. If $[b'] = [b]$ one has

$$j^* A^{[b]}_K = R f^{b,\circ}_{K!}\, R(f^{b,\circ}_K)^! \Lambda$$

that is compact since $\widetilde{\mathcal{M}}^\circ_b := \widetilde{\mathcal{M}}_b \smallsetminus \{*\}$ is spatial and thus quasi-compact. ☐

→ the key point of this proof is to prove that j^* sends compact objects to compact objects. This is absolutely not evident since Rj_* does not commute with arbitrary direct sums in general since, as said before, Bun_G being not quasi-separated, j is not quasi-compact in general.

12.5. Stability of compact objects under the action of Hecke correspondences

12.5.1. Hecke correspondences. — Let us consider the 2-category with one object whose 1-morphisms are $\mathrm{Rep}_\Lambda(\widehat{G})$ with composition given by $\otimes_\Lambda$ and whose 2-morphisms are the usual morphisms in $\mathrm{Rep}_\Lambda(\widehat{G})$. There is a morphism of 2-categories from this 2-category to $\mathcal{C}_*$ the category of correspondences over $*$. This is given by the geometric Satake correspondence.

12.5.2. Stability of compact objects. — The following result is formal.

Proposition 12.5.1. — *For any* $W \in \mathrm{Rep}(\widehat{G})$, $T_W \in \mathit{End}(D_{\text{ét}}(\mathrm{Bun}_G, \Lambda))$ *sends compact objects to compact objects;*

$$T_W : D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)^\omega \longrightarrow D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)^\omega.$$

Proof. — This is a consequence of the fact that

$$T : \mathrm{Rep}(\widehat{G}) \longrightarrow \mathrm{End}(D_{\text{ét}}(\mathrm{Bun}_G, \Lambda))$$

is a monoidal functor, $T_{W_1 \otimes W_2} = T_{W_1} \circ T_{W_2}$. Thus, by application of T to $\mathbf{1} \to \check{W} \otimes W$ and $\check{W} \otimes W \to \mathbf{1}$ on obtains that $T_{\check{W}}$ is a right adjoint to T_W. Since $T_{\check{W}}$ commutes with arbitrary direct sums we deduce the result. ☐

12.6. Proof of the finitess result

It suffices to verify that

$$Rf_{K!} S_\mu \in D(G_b(E), \Lambda)$$

is identified with

$$\boxed{(i^b)^* T_\mu \Big(i^1_! \, c\text{-Ind}_K^{G(E)} \Lambda \Big).}$$

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