φ-Modules and modifications of vector bundles (1)

Recall: $E|\mathbb{Q}_p$, π, $\mathbb{F}_q$

$F|\mathbb{F}_q$ perfectoid

$I \subset]0,1[\rightsquigarrow B_I =$ holo. functions of the variable π with coefficients in F on the annulus defined by I

I compact $\Rightarrow B_I =$ Banach alg.

$Y_I = \mathrm{Spa}(B_I, B_I^{\circ})$ adic space $/E$

$$Y = \varinjlim_{I \subset]0,1[} Y_I$$

φ bijective $\qquad \varphi(\text{Radius } \rho) = \rho^{1/q}$.

$X^{ad} := Y/\varphi^{\mathbb{Z}}$, qc. $/E$.

* $B := B_{]0,1[} = H^0(Y, \mathcal{O}_Y)$

$$X = \mathrm{Proj}\Big(\bigoplus_{d \geq 0} B^{\varphi = \pi^d}\Big), \quad B^{\varphi=\pi^d} = H^0(X^{ad}, \mathcal{O}_{X^{ad}}(d))$$

$$\mathbb{V}(\mathcal{O}_{X^{ad}}(1)) = (Y \times \mathbb{A}^1)/\mathbb{Z}, \quad n \mapsto \varphi^n, \pi^{-n}$$

$$\mathcal{O} = W_{\mathcal{O}_E}(\mathcal{O}_F) = H^0(Y, \mathcal{O}_Y^+) = \Big\{ \sum_{n \geq 0} [a_n]\pi^n \,/\, a_n \in \mathcal{O}_F \Big\}.$$

* primitive elements $= \Big\{ \sum_{n \geq 0} [a_n]\pi^n \in \mathcal{O} \,/\, a_0 \neq 0 \text{ and } \exists d,\ a_d \in \mathcal{O}_F^\times \Big\}$

$|Y| \supset |Y|^{cl} = \{$closed max. ideals of $B\}$

perfectoid residue field.

$= \{$ideals generated by ~~deg 1~~ irreducible primitive elts$\}$.

$$|Y|^{cl}/\varphi^{\mathbb{Z}} \xrightarrow{\sim} |X| \subset |X^{ad}|$$

φ-Modules over $\mathfrak{S}$

$\mathcal{I} = \{\text{primitive elements}\} \underset{\text{multiplicative}}{\subset} \mathfrak{S} \setminus \{0\}$

$x \in \mathcal{I}$ $\quad \mathfrak{S}\left[\frac{1}{u}\right]/(x) \simeq B/(x) = \prod_{y \in |Y|^{cl}} \underbrace{B^+_{dR,y}}_{\widehat{\mathcal{O}}_{Y,y}} / \mathrm{Fil}^{\mathrm{ord}_y(x)} B^+_{dR,y}$

Def. $\varphi\text{-}\mathrm{Mod}_{\mathfrak{S}} = \left\{ (M,\varphi) \;\middle|\; \begin{array}{l} M = \text{free } \mathfrak{S}\text{-module of finite rk.} \\ \varphi : M \to M \ \varphi\text{-linear} \\ \mathrm{Coker}\,\varphi \text{ killed by a primitive elt.} \end{array} \right\}$

$\mathrm{Coker}\,\varphi\left[\frac{1}{u}\right]$ = finite lenght B-module
= Coherent torsion sheaf with finite support on Y.

Ex. $E = \mathbb{Q}_p$. $K|K_0|\mathbb{Q}_p$. $k = \mathcal{O}_K/\pi_K\mathcal{O}_K$. $E(u) \in \mathcal{O}_{K_0}[u]$ Eisenstein minimal of π_K

$K_\infty = K(\pi_K^{1/p^\infty})$ $\quad F = \widehat{K_\infty}^\flat = \widehat{k((\underline{\pi}))}^{perf}$ $\quad \underline{\pi} = (\pi_K^{1/p^n})_{n \geq 0}$

$E([\underline{\pi}])$ primitive of degree $\underline{1}$ and $\mathfrak{S}$: $\mathfrak{S}\left[\frac{1}{p}\right]/(E[\underline{\pi}]) \xrightarrow{\sim} \widehat{K}_\infty$.

$$\mathfrak{S}_{k((\underline{\pi}))} := W(k)[[u]] \hookrightarrow W(\mathcal{O}_F) =: \mathfrak{S}_F$$
$$u \longmapsto [\underline{\pi}]$$

$\rightsquigarrow$ functor $\underbrace{\varphi\text{-}\mathrm{Mod}_{\mathfrak{S}_{k((\underline{\pi}))}}}_{\text{Kisin } \varphi\text{-modules}} \longrightarrow \varphi\text{-}\mathrm{Mod}_{\mathfrak{S}_F}$

(essential image $\subset$ (M,φ) / $\mathrm{Coker}\,\varphi$ killed by $E([\underline{\pi}])$)

Localization

Contrary to the "classical theory" $\mathrm{Coker}\,\varphi[\frac{1}{u}]$ not supported at only one point of Y and φ bijective

$\rightsquigarrow$ needs a localization of $\varphi\text{-}\mathrm{Mod}_\sigma$.

Def: * $\varphi\text{-}\mathrm{Mod}_\sigma[\xi^{-1}]$ = Cat. objects $(M,\varphi)\in\varphi\text{-}\mathrm{Mod}_\sigma$

morphisms: $\mathrm{Hom}((M_1,\varphi_1),(M_2,\varphi_2)) = \mathrm{Hom}_{\xi^{-1}\sigma,\varphi}(\xi^{-1}M_1,\xi^{-1}M_2)$

$\|$

$\varphi\text{-}\mathrm{Mod}_\sigma$ in which we inverted maps whose cokernel is killed by a primitive element.

* $\rho\in]0,1[$, $\varphi\text{-}\mathrm{Mod}^\rho_\sigma = \{(M,\varphi)\in\varphi\text{-}\mathrm{Mod}_\sigma \,/\, \mathrm{supp}(\mathrm{Coker}\,\varphi[\frac{1}{u}])\subset |Y_{[\rho,\rho^q[}|\}$

fundamental domain for projection $Y\to Y/\varphi^{\mathbb{Z}}$.

Ex: In $\varphi\text{-}\mathrm{Mod}_\sigma[\xi^{-1}]$, $(M,\varphi)\sim(\varphi(M),\varphi_{|\varphi(M)})$.

Th: $\forall\rho\in]0,1[$, $\varphi\text{-}\mathrm{Mod}^\rho_\sigma \xrightarrow{\sim} \varphi\text{-}\mathrm{Mod}_\sigma[\xi^{-1}]$

$\rightsquigarrow$ after choice of a fundamental domain, no need to localize.

Modifications of vector bundles

X curve$/E$

$\mathrm{Modif}_X^{\geq 0, ad} = \{ (\mathcal{E}_1, \mathcal{E}_2, u) \mid \mathcal{E}_1, \mathcal{E}_2 \in \mathrm{Bun}_X$ and $u : \mathcal{E}_{1,\eta} \xrightarrow{\sim} \mathcal{E}_{2,\eta} \}$

$u(\mathcal{E}_1) \subset \mathcal{E}_2$ i.e. $u : \mathcal{E}_1 \hookrightarrow \mathcal{E}_2$ torsion cokernel

$\mathcal{E}_1$ semi-stable of slope 0.

i.e. iso. outside a finite number of points.

<u>F alg. closed:</u> $\mathrm{Bun}_X^{ss,0} \xrightarrow{\sim} \mathrm{Vect}_E$

$\mathcal{E} \longmapsto H^0(X, \mathcal{E})$

$V \underset{E}{\otimes} \mathcal{O}_X \longleftarrow V$

$B^+ = \{ x \in B \mid \mathrm{Newt}(x) \geq 0$ i.e. $\lim_{\rho \to 1} |x|_\rho \leq 1 \} = E$-Frechet algebra

$\varphi\text{-}\mathrm{Mod}_{B^+} \xrightarrow{\sim} \varphi\text{-}\mathrm{Mod}_B \xrightarrow{\sim} \mathrm{Bun}_X$

$(D, \varphi) \longmapsto \left(\bigoplus_{d \geq 0} (D \otimes B)^{\varphi = \pi^d} \right)^{\sim}$ on $X = \mathrm{Proj}(P)$

φ bijective

$D \underset{B}{\otimes} \mathcal{O}_Y$ as an φ-equivariant v. b. on Y.

$\circlearrowleft \varphi \otimes \varphi$

<u>2 descriptions of $\mathrm{Modif}_X^{\geq 0, ad}$:</u>

Groth.-Messing periods : ("filtered φ-modules")

$$\mathrm{Modif}_X^{\geq 0, ad} \simeq \{ (M, \varphi, (\Lambda_y)_{y \in |Y|^{cl}} \} \qquad \mathcal{E}_1 \hookrightarrow \mathcal{E}_2$$

$(M, \varphi) \in \varphi\text{-}\mathrm{Mod}_{B^+}$

$\Lambda_y \subset M \underset{B^+}{\otimes} B^+_{dR,y}$ lattice satisfying $\varphi(\Lambda_y) = \Lambda_{\varphi(y)}$

(φ-eq. mod. on Y of the v.b. $M \underset{B^+}{\otimes} \mathcal{O}_Y$)

and $\Lambda_y = M \underset{B^+}{\otimes} B^+_{dR,y}$ for almost all y mod $\varphi^{\mathbb{Z}}$.

$$\dim_E \bigcap_{y \in |Y|^{cl}} \left(M^{\varphi = Id} \cap \Lambda_y \right) = \mathrm{rk}_{B^+} M \Leftrightarrow \dim H^0(X, \mathcal{E}_1) = \mathrm{rk}\, \mathcal{E}_1.$$

$H^0(X, \mathcal{E}_2)$

Hodge-Tate periods: $\left(V, (\Lambda_x)_{x \in |X|} \right)$ $\qquad \mathcal{E}_1 \hookrightarrow \mathcal{E}_2$

finite dim. E-v.s. $\quad \mathcal{E}_1 = V \otimes \mathcal{O}_X$

$V \otimes \widehat{\mathcal{O}}_{X,x} \subset \Lambda_x \subset V \otimes \widehat{\mathcal{O}}_{X,x}\left[\frac{1}{t_x}\right]$ lattice equal to $V \otimes \widehat{\mathcal{O}}_{X,x}$ for almost all $x \in |X|$.

Statement of the theorem

F alg. closed

Th: $\exists$ antiequivalence

$$\varphi\text{-}\mathrm{Mod}_{\mathfrak{S}}[\mathfrak{J}^{-1}]\otimes\mathbb{Q}_p \xrightarrow{\sim} \mathrm{Modif}_X^{\geq 0,\mathrm{ad}}$$

satisfying if $(M,\varphi)\mapsto[\mathcal{E}_1\hookrightarrow\mathcal{E}_2]$:

$*$ if $(M,\varphi)\in\varphi\text{-}\mathrm{Mod}^{\ell}_{\mathfrak{S}}$ then $(\mathcal{E}_2/\mathcal{E}_1)^{\vee}\in\mathrm{Coh}_X^{\mathrm{tor}}$

$$\updownarrow \text{GAGA}$$

$$\underbrace{\bigoplus_{n\in\mathbb{Z}}(\varphi^n)^*\underbrace{\mathrm{Coker}\,\varphi[\tfrac{1}{u}]}_{\text{Coh. torsion finite support on } Y.}}_{\varphi\text{-eq. on } Y}$$

$*$ $H^0(X,\mathcal{E}_1)=\mathrm{Hom}_{\mathfrak{S},\varphi}(M,\mathfrak{S})$.

Construction of the functor $\varphi\text{-}\mathrm{Mod}_{\mathfrak{S}}\longrightarrow\mathrm{Modif}_X^{\geq 0,\mathrm{ad}}$

2- Constructions

Shtukas on Y: **Def:** $\mathrm{Sht}_Y^{\geq 0}=\{(\mathcal{E},u)\mid \mathcal{E}\in\mathrm{Bun}_Y,\ u:\varphi^*\mathcal{E}\hookrightarrow\mathcal{E}$ modification with finite support $\}$

$H^0(Y,-) : \mathrm{Bun}_Y \xrightarrow{\sim} \mathrm{Proj}_B$ = Projective B-modules of finite type

$\mathrm{Sht}_Y^{\geq 0} \simeq \{(M,\varphi) \,/\, M \in \mathrm{Proj}_B,\ \varphi : M \to M$ Cokernel killed by a primitive element $\}$

In particular functor $\varphi\text{-}\mathrm{Mod}_{\mathcal{O}} \longrightarrow \mathrm{Sht}_Y$

* Set $\widetilde{\mathrm{Sht}}_Y^{\geq 0}$ = Category of Shtukas up to isogenies

isogeny = morphism of Shtukas that is a modification with finite support at the level of v.b.

Th: $\widetilde{\mathrm{Sht}}_Y^{\geq 0} \xrightarrow{\sim} \mathrm{Modif}_X^{\geq 0}$

$(\mathcal{E}, u) \in \mathrm{Sht}_Y^{\geq 0}$ $\qquad u : \varphi^* \mathcal{E} \hookrightarrow \mathcal{E}$

$\rightsquigarrow$ System of modifications, $n \in \mathbb{Z}$

$$\cdots \hookrightarrow \varphi^{n+1*}\mathcal{E} \xhookrightarrow{\varphi^{n*}u} \varphi^{n*}\mathcal{E} \xhookrightarrow{\varphi^{n-1*}u} \varphi^{n-1*}\mathcal{E} \hookrightarrow \cdots$$

Fundamental remarks:

If $\mathrm{Supp}(\mathrm{Coker}\, u) \subset Y_I$ $\qquad I$ Compact

then $\mathrm{Supp}(\mathrm{Coker}\, \varphi^{n*}u) \subset Y_{\varphi^n(I)}$ $\qquad \varphi([\rho_1,\rho_2]) = [\rho_1^q, \rho_2^q]$

$\Rightarrow \forall I$ compact $\exists N \in \mathbb{N}$ s.t. $|n| \geq N \Rightarrow (\varphi^{n*}u)_{|Y_I}$ is an iso.

(ind./proj. system essentially constant when restricted to each compact)

$$\Rightarrow \text{ if } \mathcal{E}^{\infty} = \varinjlim_{n \leq 0} \varphi^{n*}\mathcal{E} \in \mathrm{Bun}_{X^{ad}}$$

$$\mathcal{E}_{\infty} = \varprojlim_{n \geq 0} \varphi^{n*}\mathcal{E} \in \mathrm{Bun}_{X^{ad}}$$

$$(\mathcal{E}, u) \longmapsto [\mathcal{E}_{\infty} \hookrightarrow \mathcal{E}^{\infty}] \in \mathrm{Modif}^{\geq 0}_{X^{ad}}$$

$$\downarrow$$

$$[(\mathcal{E}^{\infty})^{\vee} \hookrightarrow \mathcal{E}_{\infty}^{\vee}]$$

* <u>Construction of the functor</u>

If $(\mathcal{E}, \varphi) \rightsquigarrow [\mathcal{E}_{\infty} \hookrightarrow \mathcal{E}^{\infty}]$

$$\mathcal{E}^{\infty} \text{ s.s. of slope } 0 \overset{\text{Kedlaya}}{\Longleftrightarrow} \mathcal{E}^{\infty} \underset{\mathcal{O}_Y}{\otimes} \mathcal{R} = \underbrace{(D, \varphi)}_{\text{isoclinic of slope } 0} \otimes_{\mathcal{R}^b} \mathcal{R}$$

where $\mathcal{R} = \varinjlim_{\rho \to 0, \rho > 0} B_{]0, \rho]}$ = Bezout ring

$\mathcal{R}^b = \{$ elements of $\mathcal{R}$ meromorphic at $0\} =$ Henselian valued field
$= \mathcal{E}^\dagger$

valuation = order at 0

$$\widehat{\mathcal{R}^b} = W_{\mathcal{O}_E}(\mathbb{F})_{\mathbb{Q}} \text{ as usual Dieudonné-Manin.}$$

(D,φ) isoclinic of slope $0 :\Leftrightarrow \exists \Lambda \subset D$ lattice s.t. $\varphi(\Lambda) = \Lambda$
$\widehat{\mathcal{O}_{\mathcal{R}^b}}$

<u>But</u>: If $u : \varphi^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ iso. on $Y_{]0,\rho]}$ then
for $n \leq 0$, $\varphi^{n*}u$ is too. (support of modification thrown to the exterior ($\rho=1$) by φ^n, $n \leq 0$)

$$\Rightarrow \left[\mathcal{E} \underset{\mathcal{O}_Y}{\otimes} \mathcal{R} = \varinjlim_{\rho \to 0} H^0\left(Y_{]0,\rho]}, \mathcal{E}\right) \right].$$

<u>But</u>: via $\mathfrak{S} \hookrightarrow \mathcal{O}_{\mathcal{R}^b}$, $\mathfrak{I} \subset \mathcal{O}_{\mathcal{R}^b}^\times$ (divisor on Y of a primitive element is finite
$\hookrightarrow$ shrinks near 0.)

$\Rightarrow$ If $(M,\varphi) \in \varphi\text{-Mod}_{\mathfrak{S}}$ then
$(M,\varphi) \underset{\mathfrak{S}}{\otimes} \mathcal{R}^b$ is isoclinic of slope 0.

<u>Thus</u>: $\varphi\text{-Mod}_{\mathfrak{S}} \xrightarrow{-\underset{\mathfrak{S}}{\otimes}\mathcal{O}_Y} \mathrm{Sht}_Y^{\geq 0} \longrightarrow \mathrm{Modif}_{X^{ad}}^{\geq 0}$

takes values in admissible modifications.

Application to p-div. groups

$C|\mathbb{Q}_p$ alg. closed, $F = C^\flat$, $\theta : B_F \twoheadrightarrow C$

$$\ker(\theta) = ([\varpi]-p), \quad \varpi^{(0)} = p.$$

$$\wr$$

$$y \in |Y_F|^{cl}.$$

$$\varphi\text{-}\mathrm{Mod}_\sigma^{\leq y} = \{(M,\varphi) \in \varphi\text{-}\mathrm{Mod}_\sigma \;/\; \operatorname{coker}\varphi \text{ killed by } [\varpi]-p\}.$$

Th: $\cancel{BT_{\mathcal{O}_C} \simeq \varphi\text{-}\mathrm{Mod}}$ $BT_{\mathcal{O}_C} \otimes \mathbb{Q}_p \simeq \varphi\text{-}\mathrm{Mod}_\sigma^{\leq y}$.

anti equivalence of categories

$$\simeq \Big\downarrow \text{H.T.} \qquad \swarrow \simeq \qquad \text{killed by } t_\infty$$

$$\{(\mathcal{E}_1 \hookrightarrow \mathcal{E}_2) \in \mathrm{Modif}_X^{\geq 0, ad} \;/\; \mathrm{Supp}(\mathcal{E}_2/\mathcal{E}_1) \subset \{\infty\}$$ and minuscule

$$|Y|^{cl} \twoheadrightarrow |Y|^{cl}/\varphi^{\mathbb{Z}} = |X|$$

$$y \longmapsto \infty$$

$\to$ find back Scholze-Weinstein.