REDUCTION THEORY FOR p-DIVISIBLE GROUPS

LAURENT FARGUES

$K|\mathbb{Q}_p$ complete $v : K \to \mathbb{R} \cup \{+\infty\}$, $v(p) = 1$.

1. Renormalized HN polygon of a p-divisible group

$H/\mathcal{O}_K$ a p-divisible group

$$\forall n \geq 1, \qquad \left.\begin{aligned} \deg(H[p^n]) &= n \dim H \\ \mathrm{ht}(H[p^n]) &= n\mathrm{ht}(H) \end{aligned}\right\} \Rightarrow \boxed{\mu(H[p^n]) = \frac{\dim H}{\mathrm{ht}\, H} =: \mu_H}$$

Definition 1.

$$\begin{aligned} \mathrm{HN}(H) : [0, \mathrm{ht}(H)] &\longrightarrow [0, \dim H] \\ x &\longmapsto \lim_{n \to +\infty} \frac{1}{n} \mathrm{HN}(H[p^n])(nx) \end{aligned}$$

One proves the limit exists is uniform and $\lim\limits_{n\to+\infty} = \inf_{n \geq 1}$. In particular $\mathrm{HN}(H) =$ continuous concave function.

Proposition 1. *H' isogenous to $H \Longrightarrow \mathrm{HN}(H') = \mathrm{HN}(H)$.*

2. Semi-stable p-divisible groups

Definition 2. *H is semi-stable if it satisfies the following equivalent conditions :*
- *$H[p]$ is semi-stable*
- *$\forall n,\ H[p^n]$ is semi-stable*
- *$\forall G \subset H$ finite flat group scheme, $\mu(G) \leq \mu_H$.*

3. A Dieudonné-Manin like theorem

Theorem 1. *Suppose the valuation of K is discrete. Let H be a p-divisible group over $\mathcal{O}_K$. Then $\exists H'$ isogenous to H equiped with a filtration*

$$0 = H'_0 \subsetneq H'_1 \subsetneq \cdots \subsetneq H'_r = H'$$

by p-divisible groups s.t. :
- *$\forall i,\ 1 \leq i \leq r,\ H'_i/H'_{i-1}$ is semi-stable*
- *$\big(\mu_{H'_i/H'_{i-1}}\big)_{1\leq i\leq r}$ is strictly decreasing*
- *$HN(H) =$ concave polygon with slopes $\big(\mu_{H'_i/H'_{i-1}}\big)_{1\leq i \leq r}$ and multiplicities $\big(\mathrm{ht}(H'_i/H'_{i-1})\big)_{1\leq i\leq r}$ $\Rightarrow$ has integral coordinates breakpoints*
- $$\mathrm{HN}(H) \leq \mathrm{Newt}(H \otimes k_K)^{\mathrm{reversed}}$$
  (if $\lambda_1 \geq \cdots \geq \lambda_n$ are the Dieudonné-Manin slopes of $H \otimes k_K$, the reversed Newton polygon is the concave polygon with those slopes)
- $$\dim, \mathrm{ht} : \underbrace{\mathrm{pdiv}_{\mathcal{O}_K} \otimes \mathbb{Q}}_{\textit{abelian category (Tate)}} \longrightarrow \mathbb{N} \quad \textit{additive functions}$$
  The preceding filtration of H' induces the HN filtration of H in $\mathrm{pdiv}_{\mathcal{O}_K} \otimes \mathbb{Q}$ for the slope function $\Gamma \mapsto \mu_\Gamma$

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Corollary 1. *If $H \otimes k_K$ is isoclinic then H is isogenous to a semi-stable p-divisible group.*

Algorithm : $H \longmapsto H'$.

Definition 3. $\mu_{max}(H) = \sup\{\mu(G) \mid \underbrace{G}_{\text{finite flat}} \subset H\}$

One has $\mu_{max}(H) \geq \mu_H = \mu(H[p])$ with equality $\Leftrightarrow$ H semi-stable

<u>*Purpose of the game*</u> : lower $\mu_{max}(H)$

Definition 4. $\forall n \geq 1,\ \underbrace{C_n}_{\text{semi-stable}}$ = *first break of the HN filtration of* $H[p^n]$

Proposition 2. • $\forall n \geq m \geq 1$, $C_m = C_n[p^m]$.
• $\forall n,\ \mu(C_n) = \mu_{max}(H)$ *and* $C_n \in \mathcal{C}^{ss}_{\mu_{max}(H)}$ = *abelian category.*

Corollary 2.

$$\mu_{max}(H) = \mu(C_1) \in \underbrace{\left\{\frac{d}{h} \mid d \in \frac{1}{e_{K/\mathbb{Q}_p}}\mathbb{Z},\ 1 \leq h \leq \text{ht } H\right\} \cap [0,1]}_{X = \text{finite set depending on } e_{K/\mathbb{Q}_p} \text{ and ht } H}$$

Definition 5. • $\mathcal{F}_H = \varinjlim_{n \geq 1} C_n$ *(as an fppf sheaf)*
• $\forall n \geq 1,\ \ a_n = \text{ht}(C_n/C_{n-1})$ *(set $C_0 = 0$).*

Let's remark $\forall n \geq 1,\ \ C_{n+1}/C_n \xhookrightarrow{\times p} C_n/C_{n-1}$ is a monomorphism $\Rightarrow$ the sequence $(a_n)_{n\geq 1}$ is descreasing.
From this we deduce there exists N such that $\forall n \geq N,\ a_n = a_N$.

Then, there are 2 possibilities :
• $a_N = 0 \Rightarrow \mathcal{F}_H = C_N$ = finite flat group scheme. We then have an isogeny

$$H \xrightarrow{\text{isogeny}} H/\mathcal{F}_H \ \text{ with } \underbrace{\mu_{max}(H/\mathcal{F}_H)}_{\in X} < \mu_{max}(H)$$

• $a_N \neq 0$. We then have an isogeny $H \longrightarrow H/C_N$ and an exact sequence of p-divisible groups

$$0 \longrightarrow \underbrace{\mathcal{F}_H/C_N}_{H'} \longrightarrow H/C_N \longrightarrow \underbrace{H/\mathcal{F}_H}_{H''} \longrightarrow 0$$

with
– H' = semi-stable p-divisible group with $\mu_{H'} = \mu_{max}(H) > \mu_H$
– H'' = p-divisible group with $\mu_{H''} < \mu_H$ and $\underbrace{\mu_{max}(H'')}_{\in X} < \mu_{max}(H)$.

Then, X **is a finite set** $\Rightarrow$ **we win the game in finite time**.

Remark 1. *If $\exists n \geq 1,\ a_n = a_{n+1} \Rightarrow C_{n+1}/C_{n-1}$ is a BT_2*

$$\Rightarrow \mu_{max}(H) \in \left\{\frac{d}{h} \mid d \in \mathbb{N}, 0 < h \leq \text{ht}H\right\} \cap [0,1] = \textit{finite set not depending on } e_{K/\mathbb{Q}_p}$$

Outside this finite set, we have a bound for N : $N \leq \text{ht}(H)$ and thus can bound the degre od the isogeny $H \longrightarrow H/C_N$.

4. The general case

Any K. Set $C = \widehat{\overline{K}}$.

Definition 6.

$$\underbrace{\mathrm{VectFil}_{C/\mathbb{Q}_p}}_{\textit{exact category}} := \textit{Category of couples } \Big(\underbrace{V}_{\substack{\mathbb{Q}_p\text{-v.s. of finite}\\ \text{dimension}}} , \underbrace{\mathrm{Fil}^\bullet V_C}_{\substack{\text{finite filtration}\\ \text{of } V\otimes_{\mathbb{Q}_p} C}} \Big)$$

$$\mathrm{rk}, \deg : \mathrm{VectFil}_{C/\mathbb{Q}_p} \longrightarrow \mathbb{Z} \quad \text{additive functions}$$

where

$$\begin{aligned} \mathrm{rk} &= \dim_{\mathbb{Q}_p} V \\ \deg &= \sum_{i\in\mathbb{Z}} i.\dim_C \mathrm{gr}^i V_C. \end{aligned}$$

Have HN filtrations for the slope function $\frac{\deg}{\mathrm{rk}}$.

Let H be a p-divisible group over $\mathcal{O}_K$ and $\alpha_H : V_p(H) \to \omega_{H^D} \otimes C$ be its Hodge-Tate map.
We have an Hodge-Tate exact sequence

$$0 \longrightarrow \omega_H^* \otimes C(1) \xrightarrow{{}^t(\alpha_H\otimes 1)(1)} V_p(H)\otimes C \xrightarrow{\alpha_H\otimes 1} \omega_{H^D}\otimes C \longrightarrow 0.$$

Set

$$\mathrm{HT}(H) = \big(V_p(H), \mathrm{Fil}^\bullet V_p(H)_C\big) \in \mathrm{VectFil}_{C/\mathbb{Q}_p}$$

where $\mathrm{Fil}^0 = V_p(H),\ \mathrm{Fil}^1 = \omega_H^*\otimes C(1)$ and $\mathrm{Fil}^2 = 0$.

Theorem 2. *We have an equality*

$$\mathrm{HN}(H) = \mathrm{HN}\big(\mathrm{HT}(H)\big).$$

In particular $\mathrm{HN}(H)$ *is a polygon with integral coordinates breakpoints !*

When K is discrete with perfect residue field this is "easy" for the following reason. Set $G_K = \mathrm{Gal}(\overline{K}|K)$.

$$\mathrm{pdiv}_{\mathcal{O}_K}\otimes\mathbb{Q} \hookrightarrow \overbrace{\mathrm{Rep}^{HT} G_K}^{\substack{\text{Hodge-Tate}\\ \text{representations}}} \xrightarrow{\deg,\mathrm{ht}} \mathbb{Z}$$

where $\mathrm{pdiv}_{\mathcal{O}_K}\otimes\mathbb{Q}$ and $\mathrm{Rep}^{HT}G_K$ are abelian categories and deg, ht are additive functions with

$$\mathrm{ht} V = \dim_{\mathbb{Q}_p} V$$

$$\deg V = d \text{ if } \det V_C \simeq C(d).$$

HN filtration of H in $\mathrm{pdiv}_{\mathcal{O}_K}\otimes\mathbb{Q}$ = HN filtration of $V_p(H)$ in $\mathrm{Rep}^{HT}G_K$.
We have a functor

$$\begin{aligned} \mathrm{Rep}^{HT} G_K &\longrightarrow \mathrm{VectFil}_{C/\mathbb{Q}_p} \\ V &\longmapsto (V, \mathrm{Fil}^\bullet V_C) \end{aligned}$$

where if $V_C \xrightarrow{\sim} \bigoplus_{i\in Z} C(i)^{a_i}$ then

$$\mathrm{Fil}^k V_C \xrightarrow{\sim} \bigoplus_{i\le k} C(i)^{a_i}.$$

Principle : forget de Galois action (arithmetic), replace by the filtration (geometric)...when $K = C$ for example we don't have any Galois action anymore, replace it by the filtration.

After forgeting the action of Galois, HN filtration of V in $\mathrm{Rep}^{HT}G_K$ = HN filtration of $(V, \mathrm{Fil}^\bullet V_C)$ in $\mathrm{VectFil}_{C/\mathbb{Q}_p}$. *Then, the preceding theorem says this is still true even if the valuation of K is not discrete !.*

Remark 2. *Suppose the valuation of K is discrete and the residue field of K is perfect. Then, we have embedings*

$$\text{pdiv}_{\mathcal{O}_K} \otimes \mathbb{Q} \hookrightarrow \text{Rep}^{cris}(G_K) \xrightarrow[D_{cris}]{\sim} FM_\varphi^{ad} \subset FM_\varphi$$

where FM_φ = exact category with $\boxed{3}$ additive functions : t_H, t_N and ht. *This implies one can cook up a HN* **bifiltration** *(the degre function takes values in $\mathbb{Z}^2$ eqiped with the lexicographic order). FM_φ^{ad} = semi-stable objects with respect to the first index, my filtration is with respect to the second index.*

5. Application to moduli spaces

$F|\mathbb{Q}_p$ discrete.
$\mathfrak{X}/\text{Spf}(\mathcal{O}_F)$ formal scheme locally formally of finite type (locally $\text{Spf}\big(\mathcal{O}_F[[x_1,\ldots,x_n]]\langle y_1,\ldots,y_m\rangle/\text{Ideal}\big)$).
H is a p-divisible group over $\mathfrak{X}$. $h = \text{ht}H, d = \dim H$.
$X = \mathfrak{X}^{an}$ = Berkovich analytic space.

Suppose you are in a "modular situation" (p-adic Shimura varieties, Rapoport-Zink spaces). You have a tower of finite étale coverings of $X = X_{\text{GL}_h(\mathbb{Z}_p)}$

$$(X_K)_{K\subset GL_h(\mathbb{Z}_p)}$$

equiped with a Hecke action of $\text{GL}_h(\mathbb{Q}_p)$.

5.1. HN stratification.

Definition 7. *Set $Poly$ = {concave polygons with integral coordinates breakpoints starting at $(0,0)$ finishing at (d,h)}.*

For $\mathcal{P} \in Poly$ we have a **Hecke invariant** subset

$$X^{HN=\mathcal{P}} = \{x \in X \mid \text{HN}(H_x) = \mathcal{P}\} \subset |X|$$

$$|X| = \bigcup_{\mathcal{P}\in Poly} X^{HN=\mathcal{P}}$$

and

$$X^{HN\geq\mathcal{P}} = \text{ closed subset of } |X|$$

$\Rightarrow$ if $\mathcal{P}_{ss}$ = line with slope $\frac{d}{h}$,

$$X^{HN=\mathcal{P}_{ss}} = \text{ open subset of } X.$$

Gr = Grassmanian of d-dimensional spaces in $\mathbb{Q}_p^h$. Hodge-Tate map between towers

$$HT : \big(|X_K|\big)_{K\subset GL_h(\mathbb{Z}_p)} \longrightarrow \big(K\backslash|Gr^{an}|\big)_{K\subset GL_h(\mathbb{Z}_p)}$$

It is continuous (difficult) and $\text{GL}_h(\mathbb{Q}_p)$-equivariant.

We have a $\text{GL}_h(\mathbb{Q}_p)$-invariant stratification

$$|Gr^{an}| = \bigcup_{\mathcal{P}\in Poly} Gr^{an,\mathcal{P}}$$

defined via the HN filtration of an element of $\text{VectFil}_{K/\mathbb{Q}_p}$ for $K|\mathbb{Q}_p$ complete. Then the preceding stratification of X is pullebacked :

$$\boxed{X^{HN=\mathcal{P}} = HT^{-1}\big(Gr^{an,\mathcal{P}}\big)}.$$

Speculation : Secrets of the p-adic geometry of Shimura varieties/R.Z. spaces lie in $|Gr^{an}|$.

For example, for $d = 1$ there is a retraction (Berkovich) $|Gr^{an}| \longrightarrow$ Compactification of the Brihat-Tits building of PGL_n.

5.2. **Newton stratification.** pulled back from the special fiber via

$$sp : |\mathfrak{X}^{an}| \longrightarrow |\mathfrak{X}_{red}|.$$

One has

$$|X| = \bigcup_{\mathcal{P} \in Poly} X^{Newt=\mathcal{P}}$$

and

$$X^{Newt \leq \mathcal{P}} = \text{open subset}$$

In particular the basic locus is open :

$$X^{Newt=\mathcal{P}_{ss}} = \text{open subset.}$$

5.3. **Relation between both stratifications :** HN $\leq$ $Newt$ $\Rightarrow$ relations between both stratifications. In particular

$$\underbrace{X^{Newt=\mathcal{P}_{ss}}}_{\text{basic locus}} \subset \underbrace{X^{HN=\mathcal{P}_{ss}}}_{\text{iso-semi-stable locus}}$$

5.4. **Main theorem.** Set

$$\mathcal{D} \subset X^{HN=\mathcal{P}_{ss}}$$

to be the semi-stable locus (a closed analytic domain)(compact in the Shimura varieties case).

Theorem 3. *If* $(d, h) = 1$ *then*

$$X^{HN=\mathcal{P}_{ss}} = \bigcup_{T \in GL_h(\mathbb{Z}_p) \backslash GL_h(\mathbb{Q}_p)/GL_h(\mathbb{Z}_p)} T.\mathcal{D}$$

a locally finite covering.

5.5. **Application.** $X = \mathcal{M}$ = generic fiber of R.Z. space of deformations of a p-divisible group over $\overline{\mathbb{F}}_p$ simple up to isogeny. Set $\mathcal{D} \subset \mathcal{M}$, the semi-stable locus. Then preceding theorem says

$$\mathcal{M} = \bigcup_{T \in GL_h(\mathbb{Z}_p) \backslash GL_h(\mathbb{Q}_p)/GL_h(\mathbb{Z}_p)} T.\mathcal{D}$$

a locally fintie covering (induces an admissible open covering of the classical rigi space).

Let

$$\pi : \mathcal{M} \longrightarrow \mathcal{F}$$

be the period morphism and $\mathring{\mathcal{D}} \subset \mathcal{D}$ the stable locus inside the semi-stable one (an open subset).

Theorem 4.

$$\pi_{|\mathring{\mathcal{D}}} : \mathring{\mathcal{D}} \xrightarrow{\sim} \pi(\mathring{\mathcal{D}})$$

is an isomorphisme.

$\pi(\mathcal{D})$ = coarse moduli space of semi-stable p-divisible groups with fixed isogeny class in the special fiber.

$\pi(\mathring{\mathcal{D}})$ = fine moduli space of stable objects.

Question : Is-it possible to define a fine moduli space of stable p-divisible groups as a rigid analytic space (idem for coarser with semi-stable) ?

CNRS–université Paris-Sud Orsay
E-mail address: laurent.fargues@math.u-psud.fr