Factorization of analytic functions in mixed characteristic

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Abstract. Let F be a perfect field complete with respect to a non trivial absolute value $|\ |$ and let $I \subset [0,1[$ an interval. Let

$$Y = \{a \in F \mid |a| \in I\}$$

(a disk or an annulus). From Weierstrass (archimedean case) to Lazard (ultrametric case, [**La62**]), a lot of work has been done about the zeros of functions $f \in B_I$, the ring of *rigid* analytic functions on Y.

This note is a very brief report about a work whose goal is to precise Lazard's results and to extend it to the mixed characteristic case. This work is linked to our joint work [**FF**] on p-adic Hodge theory.

We introduce (§1) the analogue of the B_I's in the mixed characteristic case. We study (§2) a special class of maximal ideals of the B_I's and their residue fields. In §3, we review some of our main results on factorization in the B_I's. We conclude (§4) with a few words about some applications of this work. We refer to [**FF11**] for more details. Proofs will be published elsewhere.

1. The rings of analytic functions

If L is any field equipped with a non archimedean absolute value, or with a valuation, we denote $\mathcal{O}_L$ the valuation ring which is the unit ball and $\mathfrak{m}_L$ the maximal ideal of $\mathcal{O}_L$.

1.1. The setting. We choose:

(1) A field F which is a perfect field of characteristic $p > 0$, complete with respect to a non trivial absolute value $|\ |$.

(2) A field E which is a non archimedean locally compact field, whose residue field k_E (a finite field with q elements, sometimes denoted $\mathbb{F}_q$) is contained in F.

(3) A *uniformizing parameter* π of E.

(If $\mathrm{char}(E) = p$, then $E = \mathbb{F}_q((\pi))$. Otherwise, E is a finite extension of $\mathbb{Q}_p$).

Let $\mathcal{E}$ be the unique field containing E which is complete with respect to a discrete valuation extending the valuation on E, such that π is also a uniformizing parameter of $\mathcal{E}$ and that the residue field of $\mathcal{E}$ is F. There is a unique section of the projection $\mathcal{O}_{\mathcal{E}} \to F$ which is multiplicative and we denote it $a \mapsto [a]$. Any element of $f \in \mathcal{E}$ may be written uniquely

$$f = \sum_{n>>-\infty} [a_n]\pi^n \quad \text{with } a_n \in F \ .$$

– If $\mathrm{char}(E) = p$, the map $a \mapsto [a]$ is an homomorphism of rings. If we use it to identify F to a subfield of $\mathcal{E}$, we have $\mathcal{E} = F((\pi))$.

– Otherwise E is a finite extension of $\mathbb{Q}_p$, we have $\mathcal{E} = E \otimes_{W(k_E)} W(F)$ where, for any ring R, $W(R)$ is the ring of Witt vectors with coefficients in R and, for any $a \in F$, we have $[a] = 1 \otimes (a, 0, 0, \ldots, 0, \ldots)$.

We consider the following two subrings of $\mathcal{E}$:

$$A = \Big\{ \sum_{n=0}^{+\infty} [a_n]\pi^n \ \Big| \ a_n \in \mathcal{O}_F \Big\} \text{ and, with } a \in \mathfrak{m}_F,\ a \neq 0,$$

$$B^b = A\Big[\frac{1}{\pi}, \frac{1}{[a]}\Big] = \Big\{ \sum_{n>>-\infty} [a_n]\pi^n \in \mathcal{E} \ \Big| \ \exists C \text{ such that } |a_n| \leq C,\ \forall n \Big\} \ .$$

1.2. Construction of the B_I's. For $f = \sum_{n>>-\infty} [a_n]\pi^n \in B^b$ and $\rho \in [0, 1]$, set

$$|f|_\rho = \sup_{n\in\mathbb{Z}} \{|a_n|\rho^n\} \ \text{ if } \ \rho \neq 0 \ \text{ and } \ |f|_0 = q^{-r} \ \text{ if } a_n = 0 \text{ for } n < r \text{ and } a_r \neq 0 \ .$$

For $0 < \rho < 1$, we also have $|f|_\rho = \max_{n\in\mathbb{Z}}\{|a_n|\rho^n\}$. In all cases, $|\ |_\rho$ is a multiplicative norm, i.e. is the restriction to the domain B^b of an absolute value on the fraction field of B^b.

If $I \subset [0,1]$ is a non empty interval, we set

$$B_I = \text{ completion of } B^b \text{ for the } |\ |_\rho\text{'s with } \rho \in I\ .$$

In down-to-earth terms – if we define an I *Cauchy sequence* as a sequence $(f_n)_{n\in\mathbb{N}}$ of elements of B^b such that, for all $\rho \in I$ and all $\varepsilon > 0$, there exists $n(\rho,\varepsilon) \in \mathbb{N}$ such that $|f_{n+1} - f_n|_\rho < \varepsilon$ for $n \geq n(\rho,\varepsilon)$ – then an element of B_I may be viewed as an equivalence class of I Cauchy sequences (with an obvious definition for the equivalence).

If $\mathrm{char}(E) = p$ and $I \subset [0,1[$ is $\neq \{0\}$, then $B_{F,I}$ is the ring of rigid analytic functions on

$$\{z \in F \mid |z| \in I \text{ and } z \neq 0\}$$

meromorphic in 0 if $0 \in I$. One has to be careful that $B_{[0,1]}$ is not the ring of rigid analytic functions on $\{|z| \leq 1,\ z \neq 0\}$ meromorphic at 0 but the subring of $B_{[0,1[}$ formed by functions that are bounded near $\{|z| = 1\}$ that is to say, for some (or any) ε satisfying $0 < \varepsilon < 1$, the function is bounded on $\{\varepsilon \leq |z| < 1\}$.

In all cases, the natural map $B^b \to B_I$ is injective and we use it to identify B^b to a subring of B_I. We have $B^b = B_{[0,1]}$. One can show, that, for any $\rho \in I$, the map $B_I \to \mathbb{R}$ deduced by continuity from $|\ |_\rho$ (and that we still denote $|\ |_\rho$) is still a multiplicative norm (i.e. we have $|f|_\rho \neq 0$, if $f \in B_I$ is different from 0). This implies that, if $I \subset J \subset [0,1]$ are non empty intervals, the natural map $B_J \to B_I$ is injective and we use it to identify B_J to a subring of B_I.

If $I = [\rho_1, \rho_2] \subset [0,1[$ is closed, B_I is a Banach E-algebra: we have $B_I = E \otimes_{\mathcal{O}_E} A_I$ where A_I is the π-adic completion of the subring of B^b consisting of those f's such that $|f|_{\rho_1} \leq 1$ and $|f|_{\rho_2} \leq 1$. If $I \subset [0,1[$ and $\mathcal{I}(I)$ is the set of closed intervals contained in I, then

$$B_I = \varprojlim_{J \in \mathcal{I}(I)} B_J$$

is a Fréchet E-algebra (as a projective limit of Banach algebras).

If $0 \in I$,

$$B_I = \Big\{ \sum_{n \gg -\infty} [a_n]\pi^n \ \Big|\ a_n \in F \text{ and } \forall\ \rho \in I\ ,\ |a_n|\rho^n \mapsto 0 \text{ for } n \mapsto +\infty \Big\}\ .$$

– If $0 \notin I$, let $(a_n)_{n\in\mathbb{Z}}$ elements of F such that, for all $\rho \in I$, the sequences $|a_n|\rho^n$ and $a_{-n}\rho^{-n}$ go to 0 when $n \mapsto +\infty$. Then,

$$\sum_{n\in\mathbb{Z}}[a_n]\pi^n$$

is a well defined element of B_I. If $\mathrm{char}(E) = p$, any element of B_I can be written uniquely like that. If $\mathrm{char}(E) = 0$, we don't know if the existence of such a writing is still true. There are elements which can be written like that for which we know that the writing is unique, but it doesn't seem very likely to us that this result extends to any element which has such a writing.

2. Primitive prime ideals

2.1. Primitive elements. A *π-primitive element of degree d* is a $b \in A$ such that $0 < |b - \pi^d|_1 < 1$. The set $Prim(A)$ of π-primitive elements is a sub-monoïd, containing 1, of the multiplicative monoïd of non zero elements of A. The subset $Prim^0(A)$ of π-primitive elements of degree 0 is the subgroup of invertible elements of $Prim(A)$. We denote $\widetilde{Prim}(A)$ the monoïd quotient of $Prim(A)$ by $Prim^0(A)$.

A π-primitive element is *irreducible* if it is of degree > 0 and can't be written $b = b_1b_2$ with b_1 and b_2 π-primitive of degree > 0. We denote $|Y|$ the set of irreducible π-primitive elements in $\widetilde{Prim}(A)$.

If $y \in |Y|$ is the image of a π-primitive element $b = \sum_{n=0}^{+\infty}[b_n]\pi^n$ of degree d, we set

$$||y|| = |b_0|^{1/d}$$

(independent of the choice of b). If $I \subset [0, 1[$ is an interval, we set

$$|Y_I| = \left\{ y \in |Y| \;\middle|\; ||y|| \in I \right\} .$$

2.2. The characteristic p case. Assume $\mathrm{char}(E) = p$.

Let $\mathcal{P}$ the set of irreducible monic polynomials $P \in F[X]$ such that $P(0) \in \mathfrak{m}_F$ and $P(0) \neq 0$. Then, for all $y \in |Y|$:

– There exists a unique $P_y \in \mathcal{P}$ such that y is the class of $P_y(\pi)$. The map

$$|Y| \to \mathcal{P} \text{ sending } y \text{ to } P_y$$

is a bijection.

– The ideal $\mathfrak{m}_y$ of B^b generated by $P_y(\pi)$ is maximal and $B^b/\mathfrak{m}_y = F[X]/(P_y)$ is a finite extension L_y of F.

– Let $I \subset [0,1]$ be a non empty interval. If $y \in |Y_I|$, the ideal $\mathfrak{m}_{y,I}$ of B_I generated by $P_y(\pi)$ is maximal and $B_I/\mathfrak{m}_{I,y} = L_y$. (If $||y|| \notin I$, then $P_y(\pi)$ is invertible in $B_{F,I}$).

2.3. From characteristic 0 to characteristic p. The proof of the previous results are straightforward, they follow from classical Weierstrass type results. To be able to describe the analogue when $\mathrm{char}(E) = 0$, we need a definition and a construction.

Assume $\mathrm{char}(E) = 0$. A *p-perfect extension of E* is a field L complete with respect to a non discrete absolute value $|\ |_\pi$, containing E as a closed subfield, such that $|\pi|_\pi = 1/q$ and that the map

$$\mathcal{O}_L/pO_L \to \mathcal{O}_L/p\mathcal{O}_L \quad \text{sending} \quad x \text{ to } x^p$$

is surjective.

For any p-perfect extension L of E,we set

$$F(L) = \left\{ x = (x^{(n)})_{n\in\mathbb{N}} \mid x^{(n)} \in L \text{ and } (x^{(n+1)})^p = x^{(n)} \right\} .$$

Set $(x+y)^{(n)} = \lim_{m \mapsto +\infty} (x^{(n+m)} + y^{(n+m)})^{p^m}$, $(xy)^{(n)} = x^{(n)} y^{(n)}$.

Then $F(L)$ is a perfect field of characteristic p.

If for $x \in F(L)$, we set $|x|_\pi = |x^{(0)}|_\pi$, then $|\ |_\pi$ is a non trivial absolute value on $F(L)$ for which $F(L)$ is complete.

2.4. The characteristic 0 case.

PROPOSITION 1. *Assume char$(E) = 0$, let $y \in |Y|$, $\mathfrak{m}_y$ the ideal of B^b generated by any representative of y in Prim(A), $L_y = B^b/\mathfrak{m}_y$ and $\theta_y : B^b \to L_y$ the projection. Then*

i) $\mathfrak{m}_y$ is a maximal ideal and there is a unique absolute value $|\ |$ on L_y such that, for all $a \in F$, $|\theta_y([a])| = |a|$.

ii) Equipped with $|\ |$, L_y is a p-perfect extension of F. The map $\iota_y : F \to F(L_y)$ sending a to $(\theta_y([a^{p^{-n}}]))_{n\in\mathbb{N}}$ is a continuous homomorphism of topological fields and $[F(L_y) : F] = \deg(y)$.

iii) If $Y \subset [0,1]$ is a non empty interval and if $y \in |Y_I|$, $\mathfrak{m}_{I,y} = B_I\mathfrak{m}_y$ is a maximal ideal of B_I and $B_I/\mathfrak{m}_{I,y} = L_y$ (If $||y|| \notin I$, then any π-primitive element in the class of y is invertible in B_I).

Remark: If F is algebraically closed, for any $y \in |Y|$, the field $B^b/\mathfrak{m}_y = L_y$ is algebraically closed and ι_y is an isomorphism. If $\lambda \in F$ is such that $\iota(\lambda)^{(0)} = \pi$, then y is the image of the π-primitive element $\pi - [\lambda]$. But such a λ is not unique !

3. Factorization

3.1. The Newton polygon. Let $I \subset [0,1]$ a non empty interval.

Assume first that $\mathrm{char}(E) = p$ or that $0 \in I$, so that any $f \in B_I$ may be written uniquely

$$f = \sum_{n\in\mathbb{Z}} [a_n]\pi^n$$

(with the $a_n \in F$ subject to suitable conditions). Recall that the *Newton polygon of* f is the convex hull $Newt(f)$ of the set of points $(n, v(a_n))_{n\in\mathbb{Z}}$ (where $|a| = q^{-v(a)}$). For any interval $J \subset I$, we denote

$$Newt_J(f)$$

the sub-polygon obtained by keeping only the segments whose slopes are the -s such that $q^{-s} \in J$.

Assume now that $\mathrm{char}(E) = 0$ and $0 \notin I$. We can always write a non zero $f \in B_I$ as the limit of an I Cauchy sequence of elements $(f_n)_{n\in\mathbb{N}}$ with $f_n \in B^b = B_{[0,1]}$. One can show that, for all closed interval $J \subset I$, the sequence of the $Newt_J(f_n)$'s is stationary and the limit $Newt_J(f)$ is a finite polygon independent of the choice of the Cauchy sequence. Moreover, if $J' \subset J$, $Newt_{J'}(f)$ is the sub-polygon of $Newt_J(f)$ obtained by keeping only the segments whose slopes $-s$ are such that $q^{-s} \in J'$. Therefore, we may define

$$Newt_I(f)$$

as the union of the $Newt_J(f)$ for all closed ideals $J \subset I$. Then the classical result relating the length of the projection on the horizontal axis of the segment of the Newton polygon of slope $-s$ to the multiplicity of the zeros of f with absolute value q^{-s} extends to this context. It can be expressed as follow:

Proposition 2. *Let $I \subset [0,1[$ be a non empty interval, let $\rho = q^{-s} \in I$ and $f \in B_I$, non zero. For each $y \in |Y|_I$, let $v_y(f) \in \mathbb{N}$ the biggest integer such that $f \in \mathfrak{m}_{I,y}^{v_y(f)}$. Then*

$$\sum_{||y||=\rho} \deg(y).v_y(f) = \textit{length of the projection on the horizontal axis}$$
$$\textit{of the part of slope } -s \textit{ in } Newt_I(f)\ .$$

The key point for the proof is to show that one may define a distance on $|Y|$ by setting, if $y_1, y_2 \in |Y|$ and if $b_2 \in A$ is a π-primitive element whose image is y_2,

$$d(y_1, y_2) = |\theta_{y_1}(b_2)|\ .$$

This distance is such that, if $J \subset]0,1[$ is a closed interval, then the subset $|Y_J|$ is a complete metric space.

Corollary 1. *If $J \subset]0,1[$ is closed, then B_J is a principal domain and*

$$|Y_J| = \big\{\textit{closed points of } Y_J = \operatorname{Spec}\, B_J\big\}\ .$$

3.2. Divisors. We set $B = B_{]0,1[}$. For each $y \in |Y|$, we choose $\xi_y \in A$ a π-primitive element which is a representative of y and we set $d_y = \deg(y)$.

Let $\mathcal{I}$ the set of closed intervals contained in $]0,1[$. Set

$$Div(Y) = \Big\{ \sum_{||y|| \in Y} n_y[y] \;\Big|\; n_y \in \mathbb{Z}, \forall\, y \in |Y| \text{ and } \forall J \in \mathcal{I}, \text{we have } n_y = 0 \text{ for almost all } y \in |Y_J| \Big\}\ .$$

A divisor is *effective* if $n_y \geq 0$ for all y's.

If $f \in B$ is non zero,

$$\operatorname{div}(f) = \sum_{y \in |Y|} v_y(f)[y]$$

is an effective divisor.

If $D = \sum_{y \in |Y|} n_y[y]$ is an effective divisor, and $0 < \rho < 1$, the infinite product

$$f_{D,\rho} = \prod_{|y| \leq \rho} \frac{\xi_y}{\pi^{d_y}}$$

converges in B and

$$D = \mathrm{div}(f_{D,\rho}) \;+ \sum_{|y|>\rho} n_y[y]\ .$$

If $\mathfrak{a}$ is a closed non zero ideal,

$$\mathrm{div}(\mathfrak{a}) = \sum_{y\in|Y|} v_y(\mathfrak{a})[y]$$

is a divisor. The map

$$div : \big\{\text{non zero closed ideals of } B\big\} \to Div^+(Y)$$

is an isomorphism of monoïds.

Remark 1. *i) Any closed ideal of finite type of B is principal.*

ii)Assume F algebraically closed.

– If $char(E) = p$, then any closed ideal is principal if and only if F is spherically complete (i.e. the intersection of a decreasing sequence of non empty closed balls is non empty).

– If $char(E) = 0$, the condition is necessary (it seems likely that it is also sufficient).

$\big[$ The completion of an algebraic closure of $\mathbb{F}_p((\pi))$ or of $\mathbb{Q}_p$ is not spherically complete. If Γ is a totally ordered divisible abelian group (e.g. $\Gamma = \mathbb{Q}$ or $\mathbb{R}$) and if k is an algebraically closed field, the set $k((\Gamma))$ of functions $\sum_{\gamma\in\Gamma} a_\gamma\gamma$ whose support is a well ordered set is an algebraically closed field which is spherically complete [**Poo93**] $\big]$.

4. The curve X and some applications

4.1. Construction of X. The ring B^b is equipped with an E-automorphism φ

$$\varphi\big(\sum_{n\gg-\infty}[a_n]\pi^n\big) = \sum_{n\gg-\infty}[a_n^q]\pi^n\ .$$

It extends to a continuous automorphism of $B = B_{]0,1[}$. We may consider the graded ring

$$P = \bigoplus_{d\in\mathbb{N}} P_d \quad\text{with}\quad P_d = \big\{b\in B \;\big|\; \varphi(b) = \pi^d b\big\}$$

and the scheme

$$X = \mathrm{Proj}\ P$$

which is equipped with a line bundle

$$\mathcal{O}_X(1) = \widetilde{\bigoplus_{d \in \mathbb{Z}} P_{d+1}}$$

(with $P_d = 0$ for $d < 0$) whose construction depends on the choices of F, E and π. If we change π, it changes the line bundle $\mathcal{O}_X(1)$ but it doesn't change X. We'll write

$$X = X_{F,E}$$

when we want to make clear who were the fields E and F that we used.

It is easy to see that

$$H^0(X, \mathcal{O}_X) = P_0 = E \ .$$

One can show that X is a *regular curve*, i.e. it is a separated noetherian regular scheme of dimension 1. The main lines of the proof are as follow:

One uses some Galois descent to reduce the proof to the case where F is algebraically closed, an assumption that we suppose to be fulfilled until the end of this section.

The cyclic group $\varphi^{\mathbb{Z}}$ acts on B, $|Y|$ and $Div(Y)$. Let $\mathfrak{a}$ be a non zero closed ideal of B and

$$D = div(\mathfrak{a}) = \sum_{y \in |Y|} n_y [y] \ .$$

Then $\mathfrak{a}$ is stable under $\varphi^{\mathbb{Z}}$ if and only if $n_y = n_{\varphi(y)}$ for all y. Assume this is the case and $\mathfrak{a} \neq B$. Let y_0 such that $n_{y_0} \neq 0$. Then $D \geq \sum_{n \in \mathbb{Z}} [\varphi^n(y_0)]$. Choose $\lambda \in F$ such that $\pi - [\lambda]$ is a representative of y_0. Set

$$t_- = \prod_{n \in \mathbb{N}} (1 - \frac{[\lambda^{q^n}]}{\pi}) \in B \ .$$

It is easy to show that there exists $t_+ \in A$, not divisible by π such that

$$\varphi(t_+) = (\pi - [\lambda]) t_+ \ .$$

Morally one has "$t_+ = \prod_{n<0} (\pi - [\lambda^{q^n}])$" although this infinite product does not converge.

One can check that, if $t = t_+ t_-$, then

$$t \neq 0 \ , \ t \in P_1 \ \text{ and } \ \mathrm{div}(t) = \sum_{n \in \mathbb{Z}} [\varphi^n(y_0)] \ .$$

Hence $\mathfrak{a} = (t)\mathfrak{b}$ with $\mathfrak{b}$ a non zero closed ideal of B stable under $\varphi^{\mathbb{Z}}$.

As any non zero $u \in P_d$ generates a closed ideal of B stable under $\varphi^{\mathbb{Z}}$, we see that u is divisible by some $t \in P_1$. From this fact, one deduces easily that u may be written

$$u = t_1 t_2 \dots t_d$$

with $t_1, t_2, \dots, t_d \in P_1$ (unique up to permutation and multiplication by non zero elements in $P_0 = E$)[1].

From this fact it is easy to deduce not only that X is a regular curve but also that:

i) We have a bijection $L \mapsto x_L$ between the set $\mathcal{L}$ of 1-dimensional sub E vector spaces of P_1 and the set $|X|$ of closed points of X: if $L = E.t \in \mathcal{L}$, the homogeneous ideal of P_1 generated by t is prime and x_L is the corresponding point of the topological space underlying X.

ii) We have

$$Div(X) = (Div(Y))^{\varphi(\mathbb{Z})} .$$

iii) For any closed point x, the scheme $X \backslash \{x\}$ is affine and is the spectrum of a principal domain.

iv) For any closed point, the residue field is an algebraically closed field, complete with respect to an absolute value extending the structural absolute value on E.

4.2. Vector bundles on X. One can associate to any closed point x of X a positive integer which is *the degree of* x (when F is algebraically closed, the degree is always 1). Using this degree, one can define in the usual way the degree of a divisor. One can show that, for any non zero f in the function field of X,

$$\deg(div(f)) = 0 .$$

Therefore one can define the rank $r_{\mathcal{F}}$, the degree $d_{\mathcal{F}}$ and the slope $\mu_{\mathcal{F}} = rk_{\mathcal{F}}/d_{\mathcal{F}}$ of a non zero vector bundle $\mathcal{F}$ by the usual recipes. As usual $\mathcal{F}$ is said to be semistable if $\mu_{\mathcal{F}'} \leq \mu_{\mathcal{F}}$ for any non zero sub vector bundle $\mathcal{F}'$. The Harder-Narasimhan filtration of $\mathcal{F}$, defined as usual [**HN75**], exists and is unique.

The most interesting applications relies on the crucial following result ([**FF11**]:

[1] The above discussion also shows that an ideal $\mathfrak{a}$ of B is a non zero closed ideal stable under $\varphi^{\mathbb{Z}}$ if and only if there exists $d \in \mathbb{N}$ and $u \in P_d$, non zero, such that $\mathfrak{a}$ is generated by u.

PROPOSITION 3. *Assume F algebraically closed. A vector bundle $\mathcal{F}$ over X is semistable of slope 0 if and only if there is $r \in \mathbb{N}$ such that $\mathcal{F} \simeq \mathcal{O}_X^r$.*

This leads to (*loc. cit.*) a complete, simple and explicit classification of vector bundles over X and to two other applications:

1– *p-adic Hodge theory*: Let K be a finite extension of $\mathbb{Q}_p$ and $\overline{K}$ be an algebraic closure of K. The p-adic completion C of $\overline{K}$ is a p-perfect extension of $\mathbb{Q}_p$ and we may consider the curve

$$X = X_{F(C),\mathbb{Q}_p} \ .$$

The group $G_K = \mathrm{Gal}(\overline{K}/K)$ acts on X. The study of G_K-equivariant vector bundles over X gives a new approach to p-adic Hodge theory. The classification of p-adic representations of G_K which are de Rham (i.e. the theorem "weakly admissible $\iff$ admissible"[**CF00**] and the theorem "de Rham $\iff$ potentially semistable"[**Be02**] is the semistable of slope 0 part of the classification of *de Rham* G_K-equivariant vector bundles over X (but one should be aware that the use of the word "semistable" for Galois representations comes from the fact that the étale p-adic cohomology of the geometric general fiber of a proper semistable scheme over $\mathcal{O}_K$ produces semistable Galois representations, and is unrelated to the notion of semistability of vector bundles).

2 – *An analogue of Narasimhan-Seshadri's theorem* [**NS65**] linking stable vector bundles of slope 0 on a compact Riemann surface to unitarian representations of the π_1 : If F is not algebraically closed, let $\overline{F}$ be an algebraic closure of F, let $H = \mathrm{Gal}(\overline{F}/F)$ and H^* the group of continuous homomorphisms $H \to E^*$. Using Galois descent, one can construct canonical isomorphisms

$$Pic(X) \xrightarrow{\sim} \mathbb{Z} \times H^* \ , \ Pic^0(X) \xrightarrow{\sim} H^* \ .$$

The later extends to an *equivalence of categories between semistable vector bundles of slope* 0 *over X and finite dimensional E-vector spaces equipped with a linear and continuous action of H.*

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