An introduction to the geometry of Lubin-Tate spaces

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1 From one-dimensional formal group laws to p-divisible groups

Until now we've been working with formal group laws that are rather concrete objects. To go further in the study of Lubin-Tate spaces, we will switch to a more abstract but more powerful object that are p-divisbile groups. Those objects are better suited for the study of higher dimensional formal groups too. We begin with explaining the bridge between one dimensional group laws and p-divisible groups. We refer to the appendix ? ? ? for facts about finite flat group schemes.

Recall the following form of Weierstrass division/factorization :

Lemme 1. *Let A be a complete local ring with maximal ideal $\mathfrak{m}$ and $f = \sum_{k\geq 0} a_k T^k \in A[[T]]$ s.t $f \bmod \mathfrak{m} \neq 0$. Let $n \in \mathbb{N}$ be such that $a_n \in A^\times$ and $\forall k < n\ a_k \in \mathfrak{m}$. Then for each $g \in A[[T]]$ $\exists! Q, R$ with $Q \in A[[T]]$, $R \in A[T]$, $\deg R < n$ and*

$$g = Qf + R$$

Moerover $\exists g \in A[T]$ a degre n unitary polynomial and $h \in A[[T]]^\times$ s.t.

$$f = gh$$

Démonstration. The division assertion is an easy approximation argument, the case when A is a field being evident.

The factorization assertion is, as usual, deduced from the division one by applying it to T^n, writting $T^n = Qf + R$ and constating $Q \in A[[T]]^\times$. $\square$

Thus for such an f there is an isomorphism of A-algebras

$$A[[T]]/(f(T)) \simeq A[T]/(g(T))$$

that is a finite free A-module.

Lemme 2. *Let A be a complete local ring with a car. p residue field k. Let F be a one-dimensional formal group law over A s.t. $F \otimes_A k$ has finite height h. Then for each $k \in \mathbb{N}^*$*

$$F[p^k] = Spec(A[[T]]/([p^k]_F(T)))$$

is a (commutative) finite localy free $Spec(A)$-group scheme of order p^{kh}.

Démonstration. Let $B = A[[T]]/([p^k]_F(T))$ equiped with the T-adic topology. The formal scheme $\mathrm{Spf}(B)$ with (T) being a definition ideal, has a structure of formal group scheme given by the formal group law $F(X,Y)$. In fact, $[p^k]_F(F(X,Y)) = F([p^k]_F(X), [p^k]_F)$ and thus the composed morphism of A-algebras

$$\begin{array}{rcl} A[[T]] & \longrightarrow & A[[X,Y]] \twoheadrightarrow A[[X,Y]]/([p^k]_F(X), [p^k]_F(Y)) = B\widehat{\otimes}_A B \\ T & \longmapsto & F(X,Y) \end{array}$$

verfies $[p^k]_F(F(X,Y)) \in ([p^k]_F(X), [p^k]_F(Y))$ in $A[[X,Y]]$ and thus factorises as a morphism

$$B \longrightarrow B\widehat{\otimes}_A B$$

Now, from the preceding lemma and the definition of height one deduces B is a finite A-module. From this one deduces easily that $B\widehat{\otimes}_A B = B \otimes_A B$. In fact if $[p^k]_F = gh$ as in the preceding lemma with g a unitary polynomial and h a unit then $B \simeq A[T]/(g)$ and $B\widehat{\otimes}_A B = B \otimes_A B = A[X,Y]/(g(X), g(Y))$. □

Remarque 1. *One has to be careful with the following : in the last lemma, for R an A-algebra in general*

$$F[p^k] \otimes_A R \neq \mathit{Spec}(R[[T]]/([p^k]_{F\otimes_A R}(T)))$$

For example if A is an unequal caracteristic discrete valuation ring then $F[p^k] \otimes_A A[\frac{1}{p}]$ is étale allthough $A[\frac{1}{p}][[T]]/([p^k]_F(T))$ is T-adic. $\mathit{Spf}(A[\frac{1}{p}][[T]]/([p^k]_F(T)))$ is the formal completion of $F[p^k] \otimes_A A[\frac{1}{p}]$ along its zero section and is thus equal to $\mathit{Spec}(A[\frac{1}{p}])$. This phenomenon will become clearer in a few moment when we will study the Newton stratfication.

By definition, when we will say a sequence of group schemes is exact this will mean exact as a sheaf sequence on the fppf site.

Lemme 3. *With the hypothesis of the last lemma for each $1 \leq l \leq k$ the sequence of finite flat group schemes*

$$0 \longrightarrow F[p^l] \longrightarrow F[p^k] \xrightarrow{p^l} F[p^{k-l}] \longrightarrow 0$$

is exact.

Démonstration. The sequence

$$0 \longrightarrow F[p^l] \longrightarrow F[p^k] \xrightarrow{p^l} F[p^{k-l}]$$

is easily seen to be exact (even on the big Zariski site). By couting ranks one concludes (? ? ?). □

Lemme 4. *Suppose moreover A is artinian. Consider the formal scheme $\mathit{Spf}(A[[T]])$ with (T) being a definition ideal. It is a formal group scheme, the group law being given by $F(X,Y)$, and thus an ind-group scheme. Then the natural morphism of ind-group schemes*

$$\varinjlim_k \ F[p^k] \longrightarrow \mathit{Spf}(A[[T]])$$

is an isomorphism where $F[p^k]$ is a radicial group scheme over $\mathit{Spec}(A)$.

Démonstration. Let's first prove $F[p^k]$ is radicial. Since A is artinian and the A-module $A[[T]]/([p^k]_F)$ is of finite type it is of finite length. Thus if $I = T.A[[T]]/([p^k]_F)$ is the augmentation ideal of $A[[T]]/([p^k]_F)$, the sequence $(I^k)_{k\geq 0}$ is stationary for $k >> 0$, but $A[[T]]/([p^k]_F)$ being I-adicaly complete one deduces the augmentation ideal I is nilpotent and thus the morphism $F[p^k] \longrightarrow \mathrm{Spec}(A)$ is radicial.

This proves moreover that for each $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that $T^l \in [p^k]_F(T).A[[T]]$.

Now to finish to verify that our morphism of ind-group schemes is an isomorphism we have to prove that for each $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that $[p^l]_F(T) \in T^k.A[[T]]$. But

$$[p]_F \in T^{p^h} A[[T]] + T\mathfrak{m}[[T]]$$

but

$$f(T) \in T^{p^h} A[[T]] + T\mathfrak{m}[[T]] \Longrightarrow f^{\circ k}(T) \in T^{p^h} A[[T]] + T\mathfrak{m}^k[[T]]$$

(compute in $A[T]/(T^{p^h})$). □

Thus, when A is artinian the formal group scheme $\mathrm{Spf}(A[[T]])$ and the inductive system of finite flat group schemes $(F[p^k])_k$ determine each other.

2 p-divisible groups : basics

2.1 p-divisible groups : definition

Let's fix a base scheme S. For an abelian scheaf G we note $G[p^k]$ the kernel of the multiplication by p^k.

A p-divisible group is a particular type of ind-group scheme.

Définition 1 (Grothendieck)**.** *A p-divisible group over S is an abelian fppf sheaf G over S such that :*

- *$p : G \longrightarrow G$ is an epimorphism*
- *The natural monomorphism $\varinjlim_k G[p^k] \longrightarrow G$ is an isomorphism, that is to say G is p^∞-torsion*
- *For each $k \in \mathbb{N}^*$ $G[p^k]$ is representable by a finite localy free group scheme*

The height of $G[p]$ (as a localy constant function on S) is called the height of G.

Of course, as usual for sheaves, when we say G is p^∞-torsion this means that any section is localy killed by a power of p (here it is sufficient to restrict one-self to Zariski localy, and thus globaly on a quasicompact scheme).

Lemme 5 (Equivalent definition)**.** *A p-divisible group over S "is the same" as an inductive system of finite localy free group schemes $(G_k)_{k\geq 1}$ s.t. for each k there is an exact sequence*

$$0 \longrightarrow G_1 \longrightarrow G_k \xrightarrow{\times p} G_{k-1} \longrightarrow 0$$

where the map $G_k \xrightarrow{\times p} G_{k-1} \hookrightarrow G_k$ is multiplication by p on G_k.

Démonstration. To $(G_k)_{k\geq 1}$ one associates $G = \varinjlim_k G_k$ and to G one associates the system $(G[p^k])_{k\geq 1}$. □

Lemme 6 (Equivalent definition not using fppf topology)**.** *A p-divisible group is an inductive system of finite flat group schemes $(G_k)_{k\geq 1}$ such that for all k G_k is killed by p^k, $G_k = G_{k+1}[p^k]$ and if h is the heigth of G_1 then the heigth of G_k is kh.*

Démonstration. This is a consequence of the fact that in the exact sequence

$$0 \longrightarrow G_1 \longrightarrow G_k \xrightarrow{\times p} G_{k-1}$$

the rank of the middle term is the product of the rank of other terms and thus ? ? ?

$$0 \longrightarrow G_1 \longrightarrow G_k \xrightarrow{\times p} G_{k-1} \longrightarrow 0$$

is exact. □

The preceding definition is the original one given by Tate.

Exemple 1.

- *$\mathbb{Q}_p/\mathbb{Z}_p$ or more generaly $(\mathbb{Q}_p/\mathbb{Z}_p)^h$*
- *μ_{p^∞} which is the inductive system formed by the μ_{p^k} for $k \in \mathbb{N}^*$.*
- *If A is an abelian scheme over S then $A[p^\infty]$ is a p-divisible group over S*
- *It follows from last section that if F is a one dimensional formal group law over a complete local ring R with residue field of car. p then the $(F[p^k])_{k\geq 1}$ form a p-divisible group over $Spec(R)$*

Remarque 2. *The category of p-divisible groups is $\mathbb{Z}_p$-linear and pseudo-abelian (any idempotent has a kernel and cokernel). This allows one to construct more p-divisible groups from abelian varities (the category of ab. var. being only $\mathbb{Z}$-linear) in the following way : take A an abelian variety with an action of an order $\mathcal{O}_D$ in a division algebra D over a number field and suppose $D \otimes \mathbb{Q}_p$ is split and $\mathcal{O}_D \otimes \mathbb{Z}_p$ is a maximal order. Then $A[p^\infty]$ has an action of $\mathcal{O}_D \otimes \mathbb{Z}_p$ that is a product of matrix algebras over the ring of integers of finite extensions of $\mathbb{Q}_p$. Then one can apply idempotents in those matrix algebras as in Morita equivalence and obtain new p-divisible groups.*

2.2 Example : the étale case

Définition 2. *A p-divisible group G is étale if $G[p]$ is étale, or equivalently if the $G[p^k]$ for all k are étale.*

The equivalence results from the short exact sequences

$$O \longrightarrow G[p] \longrightarrow G[p^k] \xrightarrow{p} G[p^{k-1}] \longrightarrow 0$$

the fact that an extension of an étale group by an étale one is still étale and induction.

Exemple 2. *If p is invertible over S then any p-divisible group on S is étale.*

Exemple 3. *For any $h \in \mathbb{N}^*$ $(\mathbb{Q}_p/\mathbb{Z}_p)^h$ is an étale p-divisible group.*

Recall a smooth p-adic sheaf on $S_{\text{ét}}$ is a projective system $(\mathcal{F}_k)_{k\geq 1}$ where $\mathcal{F}_k$ is localy free $\mathbb{Z}/p^k\mathbb{Z}$-sheaf on $S_{\text{ét}}$ and $\mathcal{F}_{k+1} \otimes \mathbb{Z}/p^k\mathbb{Z} \xrightarrow{\sim} \mathcal{F}_k$ is an isomorphism.

Lemme 7. *The category of étale p-divisible groups over S is equivalent to the category of smooth $\mathbb{Z}_p$-adic sheaves over $S_{\acute{e}t}$.*

Démonstration. In one direction the equivalence is given by $G \longmapsto$the projective system given by the $G[p^k]$ and with transition mapping $G[p^{k+1}] \xrightarrow{p} G[p^k]$. This projective system can be non-rigorously written " $\varprojlim_k G[p^k]$".

In the other direction the equivalence is "$\mathcal{F} \longmapsto \mathcal{F} \otimes \mathbb{Q}_p/\mathbb{Z}_p$" which means if $\mathcal{F} = (\mathcal{F}_k)_{k\geq 1}$ then one associates the inductive system $(\mathcal{F}_k)_{k\geq 1}$ with transition mappings $\mathcal{F}_k = \mathcal{F}_{k+1} \otimes \mathbb{Z}/p^k\mathbb{Z} \xrightarrow{Id\otimes p} \mathcal{F}_{k+1}$ where $p : \mathbb{Z}/p^k\mathbb{Z} \longrightarrow \mathbb{Z}/p^{k+1}\mathbb{Z}$. □

Let K be a separably closed field. Suppose S is connected and let $\xi \in S(K)$ be a geometric point. Set $\Gamma = \pi_1(S, \xi)$. Recall there is an equivalence between smooth p-adic sheaves on $S_{\text{ét}}$ and continouous representations of Γ in finite type free $\mathbb{Z}_p$-modules. This equivalence is given by

$$(\mathcal{F}_k)_{k\geq 1} \longmapsto \varprojlim_k \mathcal{F}_{k,\xi}$$

Corollaire 1. *There is an category equivalence between étale p-divisible groups over S and continouous representations of Γ in finite type free $\mathbb{Z}_p$-modules. The rank of the $\mathbb{Z}_p$-module equals the height of the p-divisible group.*

Exemple 4. *The $\mathbb{Z}_p$-modules that are free of finite type and have a trivial action of Γ correspond to p-divisible groups isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^h$.*

Exemple 5. *Suppose p is invertible on the base S. Then the p-divisible group μ_{p^∞} is étale. The corresponding one dimensional representation of Γ is the cyclotomic character $\mathbb{Z}_p(1)$.*

Définition 3. *The smooth $\mathbb{Z}_p[\Gamma]$-module attached to an étale p-divisible group is called its Tate module and is denoted $T_p(G)$.*

The notation $T_p(G)$ is ambigous because there no ξ in it. Concretely

$$\begin{aligned} T_p(G) &= \varprojlim_k G[p^k](K) \\ &= \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G \times_S \mathrm{Spec}(K)) \end{aligned}$$

2.3 Cartier duality

If G is a p-divisible group then one defines its Cartier dual G^D by setting $G^D[p^k] = G[p^k]^D$ and the inclusion $G^D[p^k] \hookrightarrow G^D[p^{k+1}]$ is the Cartier dual of multiplication by p. This is a p-divisible group since cartier duality transforms the exact sequence

$$0 \longrightarrow G[p] \longrightarrow G[p^{k+1}] \xrightarrow{p} G[p^k] \longrightarrow 0$$

in

$$0 \longrightarrow G[p^k]^D \longrightarrow G[p^{k+1}]^D \longrightarrow G[p]^D \longrightarrow 0$$

Exemple 6. *In the same way we defined an étale p-divisible group one can define a p-divisible group of multiplicative type by saying G[p] is of multiplicative type that is to say localy isomorphic to some μ_p^h. Then p-divisible groups of multiplicative type are Cartier duals to the étale one.*

Take the notations of the last section on fundamental groups. Tori T over S split after a finite étale base change are equivalent to discrete representations of $\Gamma = \pi_1(S, \xi)$ on finite type free $\mathbb{Z}$-modules. This equivalence is given by $T \mapsto X^*(T_\xi)$. For such a T $T[p^\infty]$ is of multiplicative type and

$$T_p(T[p^\infty]^D) = X^*(T) \otimes \mathbb{Z}_p$$

Exemple 7. *If A is an S-abelian scheme and $A^\vee = \underline{\mathcal{E}xt}^1(A, \mathbb{G}_m)$ is the dual abelian scheme then*

$$A^\vee[p^\infty] = A[p^\infty]^D$$

In fact from the exact sequence

$$0 \longrightarrow A[p^k] \longrightarrow A \xrightarrow{p^k} A \longrightarrow 0$$

one deduces by applying $\underline{Hom}(-, \mathbb{G}_m)$ the exact sequence

$$Hom(A, \mathbb{G}_m) = 0 \longrightarrow A[p^k]^D \longrightarrow A^\vee \xrightarrow{p^k} A^\vee \longrightarrow 0$$

If $\lambda : A \longrightarrow A^\vee$ is a polarization it is symetric, $\lambda^\vee = \lambda$, and induces a map $\lambda : A[p^\infty] \longrightarrow A[p^\infty]^D$ that verifies $\lambda^\vee = -\lambda$ (when you turn triangles in derived categories you have to change the signs !) which expresses the antisymetry of Weil pairing.

2.4 Isogenies

Définition 4. *An isogeny between two p-divisible groups G_1 and G_2 is an epimorphism $f : G_1 \longrightarrow G_2$ such that $\ker f$ is represented by a finite localy free group scheme. The height of $\ker f$ is called the height of the isogeny.*

For example, $p : G \longrightarrow G$ is an isogeny.

In fact one has the following :

Lemme 8. *Let G be a p-divisible group over S and H a subgroup that is finite localy free over S (thus $H \subset G[p^n]$ for $n >> 0$). Then G/H is a p-divisible group and the morphism $G \twoheadrightarrow G/H$ is an isogeny.*

Démonstration. The only non-trivial thing to prove is that foral all n the $(G/H)[p^n]$ are finite localy free group schemes. But if N is such that $H \subset G[p^N]$ then if H' makes the following square cartesian

$$\begin{tikzcd} H' \arrow[r] \arrow[d, hook] & H \arrow[d, hook] \\ G[p^{N+n}] \arrow[r, "p^n"] & G[p^N] \end{tikzcd}$$

then H' is finite localy free over S because the bottom horizontal morphism is finite localy free since it is an epimorphism of finite localy free group schemes and thus the top horizontal morphism is finite localy free which implies H' is finite localy free over H. Then $(G/H)[p^n] = H'/H$. □

Thus, up to isomorphisms, isogenies are given by finite localy free sub-group schemes of the $G[p^n]$, $n \geq 1$.

If $f : A_1 \longrightarrow A_2$ is an isogeny of abelian schemes then the induced morphisms $A_1[p^\infty] \longrightarrow A_2[p^\infty]$ is an isogeny. And this induces a bijection between isogenies from

Remarque 3. *For any two p-divisible groups G_1, G_2 the group $Hom(G_1, G_2)$ is a $\mathbb{Z}_p$-module without p-torsion.*

Lemme 9. *Suppose S is connected or quasicompact then a morphism bewteen p-divisible groups $f : G_1 \longrightarrow G_2$ is an isogeny iff there exists a morphism $g : G_2 \longrightarrow G_1$ and an integer N such that $g \circ f = p^N Id_{G_2}$ and $f \circ g = p^N Id_{G_1}$.*

Démonstration. Suppose f is an isogeny. Then let $N \in \mathbb{N}$ be such that $\ker f \subset G_1[p^N]$. Then, f being an epimorphism f induces an isomorphism

$$G_1/\ker f \xrightarrow{\sim} G_2$$

But the morphism $p^N : G_1/\ker f \longrightarrow G_1/\ker f$ lifts to a morphism

$$\begin{tikzcd} G_1/\ker f \arrow[r, "p^N"] \arrow[dr, dashed, "p^N"] & G_1/\ker f \\ & G_1 \arrow[u] \end{tikzcd}$$

And thus there is a composed morphism $g : G_2 \simeq G_1/\ker f \xrightarrow{p^N} G_1$ that verifies $g \circ f = p^N$. This implies $(f \circ g) \circ f = (p^N Id) \circ f$, but f being an epimorphism this implies $f \circ g = p^N$.

Reciprocaly, let f and g verifie $f \circ g = p^N$ and $g \circ f = p^N$. Then f is an epimorphism since p^N is. Moreover $\ker f \subset G_1[p^N]$. Then the morphisms

$$\begin{tikzcd} G_1[p^N] \arrow[r, shift left, "f_{|G_1[p^N]}"] & G_2[p^N] \arrow[l, shift left, "g_{|G_1[p^N]}"] \end{tikzcd}$$

verifie

$$\ker f_{|G_1[p^N]} = \operatorname{Im}\, g_{|G_2[p^N]}$$

In fact the inclusion $\operatorname{Im}\, g_{|G_2[p^N]} \subset \ker f_{|G_1[p^N]}$ is immediate and if $x \in G_1[p^N] \cap \ker f$ fppf localy $x = p^N y = g(f(y))$ for somme y. But $p^N f(y) = f(g(f(y))) = f(x) = 0$. Thus $f(y) \in G_1[p^N]$.
Now $\ker f = \ker f_{|G_1[p^N]}$ is representable by a finite S-group scheme of finite presentation. Moreover there is an epimorphism $g_{|G_2[p^N]} : G_2[p^N] \longrightarrow \ker f$. This implies (???) that $\ker f$ is finite flat over S. □

Corollaire 2. *There is an isogeny $G_1 \longrightarrow G_2$ iff there is one $G_2 \longrightarrow G_1$. The relation to be isogenous is reflexive and defines an equivalence relation on the set of isomorphism classes of p-divisible groups over S.*

Démonstration. Apply the preceding lemma to verfiy reflexivity and that the composition of two isogenies is an isogeny. □

Exemple 8. *If G is an étale p-divisible group then all isogenous p-divisible groups are étale. A morphism $f : G_1 \longrightarrow G_2$ between étale p-divisible groups is an isogeny iff the induced morphism $f_* : T_p(G_1) \longrightarrow T_p(G_2)$ is injective and makes $T_p(G_1)$ a lattice inside $T_p(G_2)$ that is to say the cokernel of f_* is torsion.*

If G is étale the isomorphism classes of couples (G', f) where $f : G \longrightarrow G'$ is an isogeny are in bijection with Γ-stable lattices in $T_p(G)[\frac{1}{p}]$ containing the lattice $T_p(G)$ (where Γ is the fundamental group). The kernel of the isogeny $f : G \longrightarrow G'$ is the finite étale group-scheme associated to the torsion Γ-module $T_p(G')/f_*T_p(G)$.

3 Formal Lie groups

3.1 Definition

Définition 5. *A (commutative) formal Lie group over S is a formal group scheme G over S s.t. localy on S there is an isomorphism of pointed S-formal schemes $G \simeq Spf(\mathcal{O}_S[[T_1, \dots, T_d]])$ for some d, where G is pointed by its unit section and $Spf(\mathcal{O}_S[[T_1, \dots, T_d]])$ by the section given by $\forall i\ T_i \mapsto 0$.*

In particular G is an affine formal scheme over S.

Exemple 9. *Let G be a smooth group scheme over S. Then $\widehat{G}$, its formal completion along its zero section, is a formal Lie group. This is for example the case if G is an abelian scheme, a torus or a vector bundle.*

If G is a formal Lie group on S we denote ω_G the localy free $\mathcal{O}_S$-module that is the conormal sheaf of the unit section or if you want $e^*\Omega^1_{G/S}$ with e the unit section. Then if $p : G \longrightarrow S$

$$\Omega^1_{G/S} \simeq p^*\omega_G$$

which expresses the fact that the cotangent bundle is trivial, as it-is on any Lie group. Then

$$p^{-1}\omega_G \subset p^*\omega_G = \Omega^1_{G/S}$$

is the sub-sheaf of translation invariant differential forms that is to say the sub-scheaf of $\Omega^1_{G/S}$ of section α such that

$$m^*\alpha = pr_1^*\alpha + pr_2^*\alpha$$

whrere $m : G \times_S G \longrightarrow G$ is the multiplication and $pr_1, pr_2 : G \times_S G \longrightarrow G$ are the two projections.

We note $\mathrm{Lie}\, G = \omega_G^*$.

If $G = \mathrm{Spf}(\mathcal{A})$ with augmentation ideal $\mathcal{I}$ there is a natural isomorphism of augmented $\mathcal{O}_S$-algebras

$$\widehat{\mathrm{Sym}}_{\mathcal{O}_S}\omega_G \xrightarrow{\sim} \mathrm{Gr}_{\mathcal{I}}\mathcal{A}$$

3.2 Connection with formal group laws

The connection with one-dimensional formal Lie groups is given by the following :

Lemme 10. *There is a category equivalence between one dimensional formal group laws over R and the category whose objects are couples (G, r) where G is a one dimensional formal Lie group,*

$$r : R[[T]] \xrightarrow{\sim} \Gamma(Spec(R), \mathcal{O}_G)$$

is an isomorphism of augmented R-algebras and morphisms are $Hom((G_1, r_1), (G_2, r_2)) = Hom(G_1, G_2)$.

Thus, via the functor $(G, r) \mapsto G$, the category of formal group laws is a fibered groupoïd over the category of formal Lie groups. The fiber category over a formal Lie group G is a groupoïd attached to the group of pointed automorphisms of the R-formal scheme $\mathrm{Spf}(R[[T]])$.

3.3 Formal Lie groups as fppf-sheaves

We view S as a formal scheme, (0) being a definition ideal.

Définition 6. *Let $\mathfrak{X}$ be an S-formal scheme. For U an S-scheme we set $\mathfrak{X}(U) = Hom_S(U, \mathfrak{X})$ where U is viewed as a formal scheme with definition ideal* (0)*.*

This defines an fppf sheaf on S still denoted $\mathfrak{X}$. If $\mathfrak{X} = \varinjlim_k X_k$ is a presentation of $\mathfrak{X}$ as an-ind-scheme then the sheaf $\mathfrak{X}$ is the inductive limit of the sheaves associated to the X_k.

More clearly : if $\mathcal{I}$ is a definition ideal of $\mathfrak{X}$ and U is quasicompact

$$\mathfrak{X}(U) = \varinjlim_k \operatorname{Hom}_S(U, \operatorname{Spec}(\mathcal{O}_{\mathfrak{X}}/\mathcal{I}^k))$$

If G is a formal group-scheme, for example a formal Lie group, we wtill note G for the associated fppf sheaf.

3.4 Formal completion of an abelian sheaf along its zero section

Définition 7. *A closed immersion $T \hookrightarrow U$ is called a nil-immersion of order $\leq k$ if the ideal $\mathcal{I}$ defining it verfies $\mathcal{I}^{k+1} = (0)$.*

Définition 8. *Let $\mathcal{F}$ be an abelian fppf sheaf on S. For all $k \in \mathbb{N}^*$ we note $Inf^k\mathcal{F}$ for the set-sheaf associated to the sub-presheaf of $\mathcal{F}$*

$$U \longrightarrow \{s \in \mathcal{F}(U) \mid \exists \textit{nil-immersion } T \hookrightarrow U \textit{ of order } \leq k \textit{ s.t. } s_{|T} = 0\}$$

Remarque 4. *The given presheaf being a sub-presheaf of $\mathcal{F}$ it is separated. Thus the associated sheaf if easy to explicit.*

Exemple 10. *If G is an S-group scheme with augmentation ideal $\mathcal{I}$ then $Inf^kG = Spec(\mathcal{O}_G/\mathcal{I}^{k+1})$.*

For all k $\operatorname{Inf}^k\mathcal{F}$ is a subsheaf of $\mathcal{F}$. One verifies $\operatorname{Inf}^k\mathcal{F} + \operatorname{Inf}^{k'}\mathcal{F} \subset \operatorname{inf}^{k+k'+1}\mathcal{F}$.

Définition 9. *We note $\widehat{\mathcal{F}} = \varinjlim_k Inf^k\mathcal{F}$ as an abelian sub-sheab of $\mathcal{F}$. It is called the formal completion of $\mathcal{F}$ along its zero section.*

Exemple 11. *If G is a formal group scheme then $G = \widehat{G}$.*

Remarque 5. *The quotient $\mathcal{F}/\widehat{\mathcal{F}}$ is formaly net and $\widehat{\mathcal{F}}$ is the smallest abelian sub-sheaf of $\mathcal{F}$ such that $\mathcal{F}/\widehat{\mathcal{F}}$ is formaly net. Thus we have written canonicaly $\mathcal{F}$ as an extension of a formally net sheaf by an "infinitesimal" one.*

3.5 A caracterisation of formal Lie groups in caracteristic p

Let S be a caracteristic p scheme that is to say s.t. $p.\mathcal{O}_S = 0$.

Définition 10. *For a sheaf $\mathcal{F}$ on S_{fppf} we note*

$$\mathcal{F}^{(p)} = Frob^*\mathcal{F}$$

where $Frob : S \longrightarrow S$ is the Frobenius morphism. We note

$$F : \mathcal{F} \longrightarrow \mathcal{F}^{(p)}$$

for the morphism that associates to any U over S the morphism

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U') \\ s & \longmapsto & Frob^*s \end{array}$$

where if $U \xrightarrow{p} S$ *then* $U' = U$ *is the* S*-scheme defined by* $q = Frob \circ p$ *and*

$$\begin{tikzcd} U' \arrow[rr, "Frob"] \arrow[dr, "q"'] & & U \arrow[dl, "p"] \\ & S & \end{tikzcd}$$

Définition 11. *An fppf abelian sheaf* G *on* S *is said to be* F*-divisible if it satisfies the analogous conditions to the one satisfied by a p-divisible group but with p replaced by* F *:*

- $F : G \longrightarrow G^{(p)}$ *is an epimorphism*
- *If* $\forall n\ G(n) = \ker F^n$ *then* G *is* F^∞*-torsion : the natural morphism* $\varinjlim_n G(n) \longrightarrow G$ *is an isomorphism*
- *For all* n $G(n)$ *is represented by a finite localy free* S*-group scheme*

We now note for $\mathcal{F}$ a sheaf $\mathcal{F}(n)$ for the kernel of $F^n : \mathcal{F} \longrightarrow \mathcal{F}^{(p^n)}$.

Théorème 1 (Messing)**.** *An fppf abelian sheaf* G *is* F*-divisible iff it is represented by a formal Lie group over* S*.*

Démonstration. Of course one direction is easy : if G is a formal Lie group it easily satisfies the given conditions. In fact it sufficies to verify the following for the formal affine space $\widehat{\mathbb{A}}^d = \mathrm{Spf}(R[[R_1, \dots, T_d]])$:

- We have $(\widehat{\mathbb{A}}^d)^{(p)} = \widehat{\mathbb{A}}^p$ and the morphism $F : \widehat{\mathbb{A}}^d \longrightarrow \widehat{\mathbb{A}}^d$ is given by $F^*T_i = T_i^p$
- $F : \widehat{\mathbb{A}}^d \longrightarrow \widehat{\mathbb{A}}^d$ is faithfully flat of finite presentation thus an fppf epimorphism
- Forall n the reciprocial image by F of the zero section in $\widehat{\mathbb{A}}^d$ is $\mathrm{Spec}(R[T_1, \dots, T_d](T_1^p, \dots, T_d^p))$ and is thus a finite localy free scheme
- Any section of $\widehat{\mathbb{A}}^d$ over a quasicompact scheme is clearly killed by a power of F

The other direction is based one the following proposition :

Proposition 1. *Let* G *be a finite localy free* S*-group scheme killed by* F^n*. Then the following are equivalent :*

- *Localy on* S $G \simeq \mathit{Spec}(\mathcal{O}_S[T_1, \dots, T_d]/(T_1^{p^n}, \dots, T_d^{p^n}))$
- *For all* $0 \leq i \leq n$ *the morphism* $F^i : G \longrightarrow G(n-i)^{(p^i)}$ *is an epimorphism*

Démonstration. In the second condition we can replace epimorphism by flat since G is flat over S and F^i is always surjective (use ? ? ?).

Now it is easy to verify that the first condition implies the second one.

In the other direction, if the second condition is verified let's first check the first condition when S is the spectrum of a perfect field k. In this case one knows

$$G \simeq \mathrm{Spec}(k[T_1, \dots, T_d]/(T_1^{p^{a_1}}, \dots, T_d^{p^{a_d}}))$$

for some $a_i \in \mathbb{N}$. But then it is an easy commutative algebra computation to check that if for all $0 \leq i \leq n$ $F^i : G \longrightarrow G(n-i)^{(p^i)}$ is flat then $a_1 = \dots = a_d = n$.

Now for S general, to the local choice of a lifting to $\mathcal{I}$ of a generating system of $\omega_G = \mathcal{I}/\mathcal{I}^2$ is associated a morphism localy on S

$$\mathcal{O}_S[T_1, \dots, T_d]/(T_1^{p^n}, \dots, T_d^{p^n}) \longrightarrow \mathcal{O}_G$$

(use the fact that G is killed by F^n) that is an isomorphism on geometric fibers of S by the field case. This is thus an isomorphism since both sides are finite localy free over $\mathcal{O}_S$. $\square$

The proof that if G is F-divisible then it is a formal Lie group is now easy since one can check easily that for all n the $G(n)$ satisfy the hypothesis of the last proposition. Thus, localy on S, if $\mathcal{I}$ is the augmentation ideal of $G(n)$ (which does not depend on n) then any lift of a base of $\mathcal{I}/\mathcal{I}^2$ to $\mathcal{I}$ gives compatibles isomorphisms for varying n

$$G(n) \xrightarrow{\sim} \mathrm{Spec}(\mathcal{O}_S[T_1, \dots, T_d]/(T_1^{p^n}, \dots, T_d^{p^n}))$$

$\square$

4 p-divisible groups in caracteristic p

For any finite flat group scheme G over a caracteristic p scheme there is a Verschiebung morphism $V : G \longrightarrow G^{(p)}$ verifying $FV = p$ and $VF = p$. This defines for any p-divisible group G over a caracteristic p base a morphism $V : G \longrightarrow G^{(p)}$ verifying $FV = p$ and $VF = p$.

Now as a corolary of lemma 9 we have

Lemme 11. *The morphisms F and V are isogenies.*

Théorème 2 (Messing)**.** *Let G be a p-divisible group on a scheme S s.t. $p.\mathcal{O}_S = 0$. Then if $\forall n\ G(n) = \ker F^n$*

$$\varinjlim_n G(n) = \widehat{G}$$

is a formal Lie group on S.

Démonstration. The equality $\varinjlim_n G(n) = \widehat{G}$ is easily checked once one knows $\varinjlim_n G(n)$ is a formal Lie group. In fact the inclusion $\widehat{G} \subset \varinjlim_n G(n)$ is always true and if $\varinjlim_n G(n)$ is a formal Lie group then it is infinitesimal thus the other inclusion is verified.

Let's prove $\varinjlim_n G(n)$ is a formal Lie group by applying theorem 1. It is clearly of F^∞-torsion. Now since F is an isogeny it is clear $\forall n\ G(n)$ is finite localy free over S and F is an epimorphism. $\square$

5 Differential calculus and deformation theory on syntomic schemes

5.1 Cotangent complex

Let S ba a scheme and $p : X \longrightarrow S$ an S-scheme. If p is smooth then the differential calculus of X/S is governed by $\Omega^1_{X/S}$ a localy free $\mathcal{O}_X$-module of rank the relative dimension of X over S. If X is not smooth over S anymore the $\mathcal{O}_X$-module $\Omega^1_{X/S}$ is not the good object anymore.

Définition 12 (Grothendieck-Illusie)**.** *The cotangent complex $\mathbb{L}^\bullet_{X/S} \in \mathbb{D}^{[-\infty,0]}(X, \mathcal{O}_X)$ is the "derived functor" of Ω^1 :*

$$\mathbb{L}^\bullet_{X/S} = \Omega^1_{\mathcal{A}_\bullet/p^{-1}\mathcal{O}_S} \otimes_{\mathcal{A}_\bullet} \mathcal{O}_X \in \mathbb{D}^{[-\infty,0]}(X, \mathcal{O}_X)$$

where :

— *$\mathcal{A}_\bullet$ is a particular simplicial resolution of $\mathcal{O}_X$ as a $p^{-1}\mathcal{O}_S$-algebra : $\mathcal{A}_\bullet$ is a simplicial $p^{-1}\mathcal{O}_S$-algebra s.t. $\forall i \in \mathbb{N}\ \pi_i(\mathcal{A}_\bullet) = 0$ if $i > 0$ and $\pi_0(\mathcal{A}_\bullet) \xrightarrow{\sim} \mathcal{O}_S$. The π_i here being the "local" homotopy groups (those are sheaves on X).*

— *The simplicial complex $\Omega^1_{\mathcal{A}_\bullet/p^{-1}\mathcal{O}_S} \otimes_{\mathcal{A}_\bullet} \mathcal{O}_X$ is seen as a complexe of $\mathcal{O}_X$-modules in negative degrees via the Dold-Kan equivalence between simplicial objects and complexes in negative degres in abelian categories.*

— *The simplical $p^{-1}\mathcal{O}_S$-algebra $\mathcal{A}_\bullet$ is constructed in the following way : consider the adjoint functors couple*

$$p^{-1}\mathcal{O}_S - algebras \underset{G}{\overset{F}{\rightleftarrows}} p^{-1}\mathcal{O}_S - modules$$

where F is the forget functor and $G(\mathcal{M}) = \mathcal{O}_S[\mathcal{M}]$. Then to this couple is associated a simplicial resolution

$$(\mathcal{A}_n)_{n\geq 0} = ((G \circ F)^{\circ n}(\mathcal{O}_X))_{n\geq 0}$$

the edge and faces map being given by the adjonction morphisms

$$\begin{cases} G \circ F \longrightarrow Id \\ Id \longrightarrow F \circ G \end{cases}$$

Then for all n $\mathcal{A}_n$ is a "formaly smooth" $\mathcal{O}_S$-algebra ($\mathcal{O}_S[\mathcal{M}]$ is "formaly smooth") and $\mathbb{L}^\bullet_{X/S} = \left(\Omega^1_{\mathcal{A}_\bullet/\mathcal{O}_S}\right) \otimes_{\mathcal{A}_\bullet} \mathcal{O}_X$.

This cotangent complex is augmented

$$\mathbb{L}^\bullet_{X/S} \longrightarrow \Omega^1_{X/S}$$

and this augmentation gives an isomorphism

$$H^0(\mathbb{L}^\bullet_{X/S}) \xrightarrow{\sim} \Omega^1_{X/S}$$

Of course $\mathbb{L}^\bullet_{X/S}$ is functorial in $X \longrightarrow S$ as is $\Omega^1_{X/S}$.

One of the first results of the theory is the following :

Proposition 2. *Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

be a diagram of schemes. There is a distinguished triangle

$$\begin{tikzcd} f^*\mathbb{L}^\bullet_{Y/S} \arrow[rr] & & \mathbb{L}^\bullet_{X/S} \arrow[dl] \\ & \mathbb{L}^\bullet_{X/Y} \arrow[ul, "+1"] & \end{tikzcd}$$

This triangle generalises the usual short exact sequence

$$f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

that is obtained by applying H^0 to the preceding triangle.

There are particular cases :

Proposition 3. — *Let X be of finite type over S. The scheme X is smooth over S iff the augmentation morphism $\mathbb{L}^\bullet_{X/S} \longrightarrow \Omega^1_{X/S}$ is an isomorphism in $\mathbb{D}(X, \mathcal{O}_X)$ thath is to say $\forall i < 0\ H^i(\mathbb{L}^\bullet_{X/S}) = 0$*

— *Let $X \hookrightarrow Y$ be a closed immersion defined by the ideal $\mathcal{I}$. Then*

$$\tau_{\geq -1}\mathbb{L}^\bullet_{X/Y} \simeq \mathcal{I}/\mathcal{I}^2[1]$$

— *In the preceding if moreover $X \hookrightarrow Y$ is a regular immersion then*

$$\mathbb{L}^\bullet_{X/Y} \simeq \mathcal{I}/\mathcal{I}^2[1]$$

Exemple 12. *In the triangle of the last proposition if the morphism f is a closed immersion then one finds the usual exact sequence*

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow 0$$

Exemple 13. *If f is smooth then one finds back the fact that the sequence*

$$0 \longrightarrow f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

is exact

The following corollary will be fundamental for the study of group schemes.

Corollaire 3. *If X is a localy complete intersection scheme over S then $\mathbb{L}^\bullet_{X/S}$ is a perfect complex in $\mathbb{D}^{[-10]}(X, \mathcal{O}_X)$.*

5.2 Obstruction to smoothness

We have seen that $H^0(\mathbb{L}^\bullet_{X/S}) \xrightarrow{\sim} \Omega^1_{X/S}$. Thus, for any $\mathcal{O}_X$-module $\mathcal{M}$

$$\mathrm{Ext}^0_{\mathcal{O}_X}(\mathbb{L}^\bullet_{X/S}, \mathcal{M}) \simeq \mathrm{Der}_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{M})$$

Recall (EGA ? ? ?) one can define the $\mathcal{O}_X$-module $\mathrm{Extalg}_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{M})$ classifying isomorphism classes of extensions of $\mathcal{O}_S$-algebras of $\mathcal{O}_X$ by the squared zero ideal $\mathcal{M}$, the identity class being the one of the dual numbers algebra $\mathcal{O}_X \oplus \mathcal{M}$.

We can go a step further now :

Théorème 3. *For all $\mathcal{O}_X$-modules $\mathcal{M}$ there is natural isomorphism*

$$\mathit{Ext}^1_{\mathcal{O}_X}(\mathbb{L}^\bullet_{X/S}, \mathcal{M}) \xrightarrow{\sim} \mathit{Extalg}_{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{M})$$

And we find back the short exact sequence of EGA ? ? ?.

Let $T_0 \hookrightarrow T$ be a closed immersion of S-schemes defined by an ideal $\mathcal{I}$ s.t. $\mathcal{I}^2$. It thus corresponds to a class

$$c \in \mathrm{Ext}^1_{\mathcal{O}_{T_0}}(\mathbb{L}^\bullet_{T_0/S}, \mathcal{I})$$

We want to study the application reduction modulo $\mathcal{I} : X(T) \longrightarrow X(T_0)$.

Suppose thus we are given $f : T_0 \longrightarrow T$ an S-morphism. We want to know if it lifts to an element of $X(T)$:

$$\begin{tikzcd} T_0 \arrow[r, "f"] \arrow[d, hook] & X \arrow[d] \\ T \arrow[r] & S \end{tikzcd}$$

This is equivalent to fill the following diagram of $\mathcal{O}_S$-algebras

$$\begin{tikzcd} & & & f^{-1}\mathcal{O}_X \arrow[dl, dashed] \arrow[d, "f^*"] & \\ 0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_T \arrow[r] & \mathcal{O}_{T_0} \arrow[r] & 0 \end{tikzcd}$$

or equivalently to know wether or not the clas of the extension of $\mathcal{O}_{T_0}$ by $\mathcal{I}$ becomes trivial by pullback from $\mathcal{O}_{T_0}$ to $f^{-1}\mathcal{O}_X$ that is to say if through the morphism

$$\mathrm{Ext}^1_{\mathcal{O}_{T_0}}(\mathbb{L}^\bullet_{T_0/S}, \mathcal{I}) \longrightarrow \mathrm{Ext}^1_{f^{-1}\mathcal{O}_X}(f^{-1}\mathbb{L}^\bullet_{X/S}, \mathcal{I}) \simeq \mathrm{Ext}^1_{\mathcal{O}_X}(\mathbb{L}^\bullet_{X/S}, rf_*\mathcal{I})$$

the class c goes to zero (we suppose Rf_* has finite cohomological dimension on quasi-coherent sheaves, for example quasicompact and quasi-separated).

Moerover if such an extension exists the set of such liftings of our element of $X(T_0)$ to an element of $X(T)$is a torsor under $\mathrm{Ext}^0(\mathbb{L}^\bullet_{X/S}, Rf_*\mathcal{I})$.

In particular if T_0 is affine (and X is separated over S which implies f is affine) this obstruction lies in

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathbb{L}^\bullet_{X/S}, f_*\mathcal{I})$$

If X is localy a complete intersection over S, $\mathbb{L}^\bullet_{X/S}$ is a perfect complex and

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathbb{L}^\bullet_{X/S}, f_*\mathcal{I}) \simeq H^1(X, (\mathbb{L}^\bullet_{X/S})^\vee \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} f_*\mathcal{I})$$

6 p-divisible groups and formal Lie groups

Théorème 4 (Grothendieck-Messing)**.** *Let G be a p-divisible group over a base S such that p is localy nilpotent on S. Then G is fomaly smooth and $\widehat{G}$ is a formal Lie group.*

In particular, one can define for such a G ω_G and $\operatorname{Lie} G$ that are localy free $\mathcal{O}_S$-module of finite rank. Localy on S, we have $\omega_G = \omega_{G[p^n]}$ for $n >> 0$, the conormal sheaf associated to the unit section of $H[p^n]$, and the inclusion $H[p^n] \hookrightarrow H[p^{n+1}]$ induces isomorphisms $\omega_{H[p^{n+1}]} \xrightarrow{\sim} \omega_{H[p^n]}$ for $n >> 0$. In fact one has

$$\forall n\ \omega_{H[p^n]} \simeq \omega_H / p^n \omega_H$$

Théorème 5. *Let S and G be as in the preceding theorem. The following are equivalent :*
- *$G = \widehat{G}$*
- *$G[p]$ is radicial*
- *for all n $G[p^n]$ is radicial*

In this case G is said to be a formal p-divisible group. Moerover the following are equivalent :
- *$\widehat{G}$ is p-divisible that is to say is a formal p-divisible group*
- *$G[p]$ is an extension of an étale finite finite group scheme by a finite localy free radicial one*

$$0 \longrightarrow G[p]^0 \longrightarrow G[p] \longrightarrow G[p]^{ét} \longrightarrow 0$$

- *idem for all $G[p^n]$*
- *The function $s \longmapsto$ the separable rank of the fiber $G[p]_s$ is localy constant*

If this is the case then $\widehat{G} = \varinjlim_n G[p^n]^0$ is a formal p-divisible group, $G^{ét} = \varinjlim_n G[p^n]^{ét}$ is an étale one and G is an extension

$$0 \longrightarrow \widehat{G} \longrightarrow G \longrightarrow G^{ét} \longrightarrow 0$$

7 Classification over caracteristic p perfect fields

7.1 The Dieudonné functor

Let k be a caracteristic p perfect field. Let $W(k)$ be the Witt vectors and σ its Frobenius.

Définition 13. *A cristal over k is a couple (M, V) where M is a finite free $W(k)$-module and $V : M \longrightarrow M$ is a σ^{-1}-linear mapping verifying*

$$pM \subset V(M) \subset M$$

Remarque 6. *An equivalent definition is to take a triple (M, F, V) where F is σ-linear, V is σ^{-1}-linear and $FV = p$.*

Théorème 6 (Dieudonné)**.** *There is a category equivalence between p-divisible groups over Spec(k) and cristals over k.*

In fact, more precisely, there is a covariant functor $\mathbb{D}$ from p-divisible groups over a perfect field k to finite free $W(k)$-modules. This functor is compatible with base change : if $k_1 \longrightarrow k_2$ then $\mathbb{D}(- \otimes_{k_1} k_2) \xrightarrow{\sim} W(k_2) \otimes_{W(k_1)} \mathbb{D}(-)$. The Frobenius morphism $F : G \longrightarrow G^{(p)}$ induces $\mathbb{D}(F) : \mathbb{D}(G) \longrightarrow W(k) \otimes_{W(k),\sigma} \mathbb{D}(G)$ and thus a σ^{-1}-linear mapping $V : \mathbb{D}(G) \longrightarrow \mathbb{D}(G)$. In the same way there is a σ-linear mapping $F : \mathbb{D}(G) \longrightarrow \mathbb{D}(G)$ induced by the Verschiebung. Then since $VF = p$ on the p-divisible group we have $FV = p$ on $\mathbb{D}(G)$. Thus to a G there is associated a cristal $(\mathbb{D}(G), F, V)$.

Moreover the rank of $\mathbb{D}(G)$ is equal to the height of G and $[\mathbb{D}(G) : V\mathbb{D}(G)] = \dim(G)$.

Remarque 7. *We take the covariant definition of the Dieudonné modules. There is a contravariant one. Both are linked through Cartier duality.*

Remarque 8. *The Lie algebra functor can be deduced from $\mathbb{D}$ thanks to the relation*

$$Lie\, G = \mathbb{D}(G)/V\mathbb{D}(G)$$

Moreover

$$Lie\, G^D = V\mathbb{D}(G)/p\mathbb{D}(G)$$

and the extension

$$0 \longrightarrow V\mathbb{D}(G)/p\mathbb{D}(G) \longrightarrow \mathbb{D}(G)/p\mathbb{D}(G) \longrightarrow \mathbb{D}(G)/V\mathbb{D}(G) \longrightarrow 0$$

is the Lie algebra of the universal vector extension of G. We will see in the sequel that in fact $\mathbb{D}(G)$ can be defined as the Lie algebra of the universal extension of a lifting of G to $W(k)$.

One of the corollaries of the construction of the Dieudonné functor is :

Fait 1. *A morphism $f : G_1 \longrightarrow G_2$ is an isogeny iff $\mathbb{D}(f)$ is injective and Im $\mathbb{D}(f)$ ia a lattice in $\mathbb{D}(G_2)$, or equivalently $\mathbb{D}(f)$ induces an isomorphism bewtween $\mathbb{D}(G_1) \otimes \mathbb{Q}$ and $\mathbb{D}(G_2) \otimes \mathbb{Q}$. Moreover the height of the isogeny f is equal to the length of coker$(\mathbb{D}(f))$.*

7.2 Three definitions of the Dieudonné functor

Here we give a brief summary of the different approaches to define the Dieudonné module of a p-divisible group.

7.2.1 Via co-Witt vectors

Let for all $n \in \mathbb{N}^*$ W_n be the group scheme of truncated Witt vectors. One has $W_n = \mathrm{coker} V^n$ where W is the functor of Witt vectors and $V : W \longrightarrow W$ is the Verschiebung. As a scheme one has $W_n \simeq \mathbb{A}^n$. One defines the co-Witt vectors as the fppf sheaf

$$CW(-) = \varinjlim_{n} W_n(-)$$

where the transition morphisms are given by the Verschiebung $V : W_n \longrightarrow W_{n+1}$. The idea is to use an analog of the Pontryagin duality $M \mapsto \mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ where CW play the role of the dualizing object. Then one puts for a p-divisible group over Spec(k)

$$\mathbb{D}(G) = \mathrm{Hom}(G, CW)$$

In fact this is not the good definition. One has to modify it a little bit using unipotent co-Witt vectors CW^u to catch the unipotent part of the p-divisible group.

This Dieudonné module is not the one we spoke before : it is the contravariant one (both are linked through Cartier duality).

7.3 Via Cartier theory

7.4 Via the evaluation of a crystal

7.5 Classification of isocrystals

Définition 14. *An isocristal over k is a couple (N, V) where N is a finite dimensional $W(k)_{\mathbb{Q}}$-vector space and $V : N \xrightarrow{\sim} N$ is a σ^{-1}-linear isomorphism. A morphism of isocrystals is a $W(k)_{\mathbb{Q}}$-linear morphism commuting to the action of V.*

Thus to any crystal over k is associated an isocrystal. And thanks to the given caracterisation of isogenies one has

Corollaire 4. *The functor $G \mapsto \mathbb{D}(G)_{\mathbb{Q}}$ is a fully faithfull functor from the category of p-divisible groups up to isogeny (the cateory of p-divisible groups where we have inverted the class of maps given by isogenies, in the sens of localization of categories) to the category of isocrystals over k.*

We will see in a few moment what is the esential image of this functor.

Définition 15. *Let $\lambda \in \mathbb{Q}$. An isocrystal (N, V) is isoclinic with slope λ if there exists a lattice M in N such that $V^s M = p^r M$ where $\lambda = r/s$.*

Théorème 7 (Dieudonné-Manin)**.** *The category of isocrystals over k is a direct orthogonal sum indexed by $\mathbb{Q}$ of the categories of isoclinic isocrystals with some given slope in $\mathbb{Q}$. In down to earth terms : for any (N, V) there is a V-invariant decomposition*

$$N = \bigoplus_{\lambda \in \mathbb{Q}} N_\lambda$$

were (N_λ, V) is isoclinic (zero for only a finite number of $\lambda \in \mathbb{Q}$) and if $\lambda_1 \neq \lambda_2$ then

$$Hom((N_{\lambda_1}, V), (N_{\lambda_2}, V)) = 0$$

If k is algebraically closed then this category is semi-simple, the simple objects being indexed by $\mathbb{Q}$: for exach slople λ there exists a unique simple isocrystal isoclinic with slope λ.

The sets of λ s.t. $N_\lambda \neq 0$ are called the slopes of (N, V).

Corollaire 5. *The functor $G \longmapsto \mathbb{D}(G)_{\mathbb{Q}}$ induces an equivalence between p-divisible groups over k up to isogeny and isocrystals whose slope lie bewteen 0 and 1.*

One know G is étale iff $F : G \longrightarrow G$ is an isomorphism. Thus :

Fait 2. *G is étale iff $\mathbb{D}(G)_{\mathbb{Q}}$ is isoclinic with slope 0. G is a formal p-divisible group iff $\mathbb{D}(G)_{\mathbb{Q}}$ doesn't have the zero slope.*

Thus, in the extension

$$0 \longrightarrow \widehat{G} \longrightarrow G \longrightarrow G^{\text{ét}} \longrightarrow 0$$

$\mathbb{D}(\widehat{G})_{\mathbb{Q}} \subset \mathbb{D}(G)_{\mathbb{Q}}$ is the direct sum of the isoclinic parts with slope > 0.

Remarque 9. *This is very particular to the zero slope case that the zero slope part of $\mathbb{D}(G)_{\mathbb{Q}}$ splits without inverting p at the level of $\mathbb{D}(G)$.*

If $\lambda_1, \dots, \lambda_r$ are the slopes of $\mathbb{D}(G)$ with multiplicity $(a_1, \dots, a_r)$ then if $\forall i \ \lambda_i = \frac{d_i}{h_i}$ with $d_i \wedge h_i = 1$ then

$$\mathrm{ht}(G) = \sum_i a_i h_i \quad \dim(G) = \sum_i a_i d_i$$

Sometimes the number $a_i h_i$ is called the multiplicity of λ_i.

Définition 16. *The Newton polygon attached to the data (λ_i, a_i) is the unique convex polygon whose breakpoints are in $\mathbb{Z}^2$ that begins at $(0,0)$ and whose slpopes are the λ_i each with multiplicity $a_i h_i$.*

Thus this Newton polygon, if attached to (N, V), ends at $(\mathrm{ht}(N, V), \dim(N, V))$.

Corollaire 6. *Let k be algebraically closed. The isogeny classes of height h dimension d p-divisible groups over k are in bijection with convex polygons whose breakpoints are in $\mathbb{N}^2$, whose slopes are in $[01]$, that begin at $(0,0)$ and end at (h, d).*

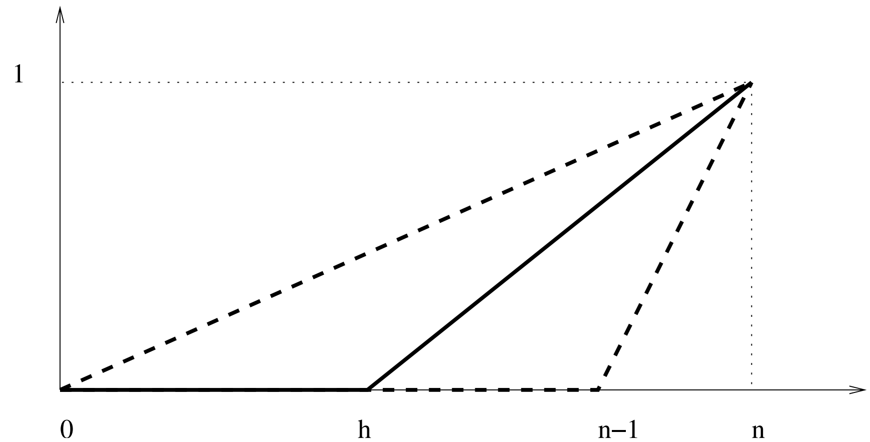

FIGURE 1 – The possible Newton polygons of a one dimensional p-divisible group

7.6 Example in the case of Lubin-Tate groups

Using the preceding results one can find back the classification of one dimensional formal p-divisible groups over $\overline{\mathbb{F}}_p$ by their height.

In fact the possible Newton polygons of a one dimensional p-divisible group over $\overline{\mathbb{F}}_p$ are shown in the following figure

One sees they are parametrized by an integer $h \in \{0, \dots, n-1\}$ where a p-divisible group such that its Newton polygon is attached to the integer h is an extension

$$0 \longrightarrow H^0 \longrightarrow H \longrightarrow H^{\text{ét}} \longrightarrow 0$$

where $H^{\text{ét}}$ is étale of height h and H^0 is a formal one dimensional p-divisibile group with height $n-h$.

Moerover one verifies there is a unique isomorphism class of formal p-divisible groups H of height n. Their crystal is given in the following way. The $W(k)$-module $\mathbb{D}(H)$ has a base $(e_1, \dots, e_n)$ such that

$$\forall 1 \leq i \leq n-1 \;\; V.e_i = e_{i+1} \text{ and } V.e_n = pe_1$$

that is to say the matrix of V is

$$\begin{pmatrix} 0 & \dots & 0 & p \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{pmatrix}$$

Then $\text{End}(H) = \text{End}_{(W(k),V)}(\mathbb{D}(H)) = \mathcal{O}_D$ where D is a division algebra with invariant $1/n$. If $\mathbb{Q}_{p^n} = W(\mathbb{F}_{p^n})[\frac{1}{p}]$ is the degre n unramified extension of $\mathbb{Q}_p$ thent

$$D = \mathbb{Q}_{p^n}[\Pi] \quad \mathcal{O}_D = \mathbb{Z}_{p^n}[\pi]$$

where $\Pi^n = p$ and $\forall x \in \mathbb{Q}_{p^n}$ $\Pi x = x^\sigma \Pi$. Concretely,

$$\forall 1 \leq i \leq n-1 \;\; \Pi.e_i = e_{i+1} \text{ and } \Pi.e_n = pe_1$$

and $\forall x \in \mathbb{Z}_{p^n}$ its action on $\mathbb{D}(H)$ is $x.e_i = \sigma^{-i}(x)e_i$.

8 Algebraization theorem

Let A be an I-adic ring.

Recall (EGA III) the following corollary of Grothendieck's algebraization theorem :

Théorème 8 (Grothendieck)**.** *Define an abelian scheme over Spf(A) as being a compatible system of abelian schemes over the Spec(A/I^n) for $n \in \mathbb{N}^*$ (a cartesian section of the fibered category of abelian schemes over the Spec(A/I^n), $n \geq 1$). Define a polarization of an abelian scheme over Spf(A) as being a compatible system of polarizations.*

Then the functor "formal completion along $V(I)$" induces a category equivalence between polarizable abelian schemes over Spec(A) and polarizable abelian schemes over Spf(A).

We're going to see the same property is satisfied by p-divisible groups.

Définition 17. *Let $\mathfrak{X}$ be a formal scheme with ideal of definition $\mathcal{I}$. A p-divisible group over $\mathfrak{X}$ is a compatible system of p-divisible groups over the Spec($\mathcal{O}_X/\mathcal{I}^n$), $n \geq 1$.*

This means we have forall $n \geq 1$ a p-divisible group G_n over $\mathrm{Spec}(\mathcal{O}_X/\mathcal{I}^n)$ and isomorphisms $G_{n+1} \bmod \mathcal{I}^n \xrightarrow{\sim} G_n$ satisfying the usual cocyle relation.

Proposition 4 (Messing)**.** *The functor $G \longmapsto (G \bmod \mathcal{I}^n)_{n\geq 1}$ induces a category equivalence between p-divisible groups over Spec(A) and over Spf(A).*

9 Rigidity of quasi-isogenies

Let's remark for two p-disvisible groups G_1, G_2 $\mathrm{Hom}(G_1, G_2)$ is a $\mathbb{Z}_p$-module without torsion.

Proposition 5. *Let $S_0 \hookrightarrow S$ be an immersion defined by a localy nilpotent ideal. Suppose p is localy nilpotent on S. Let G, H be two p-divisible groups over S and G_0, S_0 their reduction to S_0. Then the reduction to S_0 induces an injection*

$$Hom_S(G,H) \hookrightarrow Hom_S(G_0,H_0)$$

and if moreover S is quasi-compact there exists $N \in \mathbb{N}$ s.t.

$$p^N Hom_{S_0}(G_0,H_0) \subset Hom_S(G,H)$$

Démonstration. By devissage one is reduced to the case when the immersion $i : S_0 \hookrightarrow S$ is defined by a squared zero ideal $\mathcal{I}$.

We have now

$$\mathrm{Hom}_{S_0}(G_0,H_0) = \mathrm{Hom}_{S_0}(i^*G, i^*H) = \mathrm{Hom}_S(G, i_*i^*H)$$

There is an adjonction morphism $H \longrightarrow i_*i^*H$ and the morphism $\mathrm{Hom}_S(G,H) \longrightarrow \mathrm{Hom}_S(G_0,H_0)$ is nothing else than the morphism induced by applying $\mathrm{Hom}_S(G,-)$ to this adjunction morphism. The formal smoohtness of H implies $H \longrightarrow i_*i^*H$ is an epimorphism. Thus we have an exact sequence

$$0 \longrightarrow \ker(H \longrightarrow i_*i^*H) \longrightarrow H \longrightarrow i_*i^*H \longrightarrow 0$$

But from the following lemma we have

$$\ker(H \longrightarrow i_*i^*H) \simeq i_*\underline{Hom}_{\mathcal{O}_{S_0}}(\omega_{H_0}, \mathcal{I})$$

This gives the exact sequence

$$\mathrm{Hom}_S(G, i_*\underline{Hom}_{\mathcal{O}_{S_0}}(\omega_{H_0},\mathcal{I})) \longrightarrow \mathrm{Hom}_S(G,H) \longrightarrow \mathrm{Hom}_S(G_0,H_0) \longrightarrow \mathrm{Ext}^1(G, i_*\underline{Hom}_{\mathcal{O}_{S_0}}(\omega_{H_0},\mathcal{I}))$$

But p is an epimorphism on G and localy nilpotent on $i_*\underline{Hom}_{\mathcal{O}_{S_0}}(\omega_{H_0},\mathcal{I})$ thus the left term in this sequence is zero. Now if p^N is zero on S then the right term is killed by p^N since it kills $i_*\underline{Hom}_{\mathcal{O}_{S_0}}(\omega_{H_0},\mathcal{I})$. $\square$

Lemme 12. *Let $i : S_0 \hookrightarrow S$ be a closed immersion defined by an ideal $\mathcal{I}$ such that $\mathcal{I}^2 = (0)$. Let H be a group-scheme over S and $H_0 = i^*H$ its reduction modulo $\mathcal{I}$. Let $H \longrightarrow i_*i^*H$ be the adjonction morphism that associates to $s \in H(U)$ $s_{|U\times_S S_0} \in H(U \times_S S_0) = H_0(U \times_S S_0)$. Then as fppf abelian sheaves over S*

$$\ker(H \longrightarrow i_*i^*H) \simeq i_*\underline{Hom}_{\mathcal{O}_{S_0}}(\omega_{H_0}, \mathcal{I})$$

where ω_{H_0} is the conormal quasi-coherent sheaf associated to the unit section of H_0.

Démonstration. Easy. □