An introduction to the geometry of Lubin-Tate spaces

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1 Grothendieck Messing deformation theory

1.1 Universal vector extension

1.2 Motivation

We know the Hodge filtration of the H^1 of an abelian variety over $\mathbb{C}$ determines the deformation theory of this abelian variety. For example is S is a smooth analytic space over $\mathbb{C}$ and $A \longrightarrow S$ is a princiaply polarized abelian variety for each $s \in S$ if one trivializes the Betti relative cohomology of $A \longrightarrow S$ in a a neighborhood of s the Hodge filtration the local trivialization of Betti defines an holomorphic map $U \longrightarrow \mathcal{H} \subset \mathrm{Gr}$ where U is a neighborhood of s and $\mathcal{H} \subset \mathrm{Gr}$ is Siegel space in its associated Grassmanian. Then the germs $(A \times_S V)_V$ where V goes through neigborhoods of s ins S is a versal deformation of A_s iff the tangent map to $U \longrightarrow \mathcal{H}$ at s is an isomorphism that is to say it is a local isomorphism at s.

Let A be an abelian variety of $\mathbb{C}$. Consider its Hodge filtration

$$0 \longrightarrow \Gamma(A, \Omega^1_A) \longrightarrow H^1_{dR}(A) \longrightarrow H^1(A, \mathcal{O}_A) \longrightarrow 0$$

and $\Gamma(A, \Omega^1) = \omega_A$ is the vectorspace of translation invariant differential forms, $H^1(A, \mathcal{O}_A) = \omega^*_{A^\vee}$ where $A^\vee$ is the dual abelian variety.

There is a imbedding $H^1_B(A, \mathbb{Z}) \subset H^1_{dR}(A)$ given by the comparison theorem between De Rham and Betti cohomolohy. Moreover this imbeding composed with the projection $H^1_{dR}(A) \longrightarrow H^1(A, \mathcal{O}_A)$ is still an embedding and

$$A^\vee(\mathbb{C}) = H^1(A, \mathcal{O}_A)/H^1_B(A, \mathbb{Z})$$

as can be easily verified by writting $A = V/\Lambda, A^\vee = V^*/\Lambda^\vee, \omega_{A^\vee} \simeq V, H^1_{dR}(A) \simeq V^* \otimes_{\mathbb{R}} \mathbb{C}$.

Now we're looking for a geometric way to find back the Hodge filtration. Let's look at the following extension of holomorphic Lie groups

$$0 \longrightarrow \omega_A \longrightarrow H^1_{dR}(A)/H^1_B(A, \mathbb{Z}) \longrightarrow A^\vee(\mathbb{C}) \longrightarrow 0$$

obtained by taking the quotient of the Hodge filtation by the discrete $H^1_B(A, \mathbb{Z})$. It is an extension of the abelian variety $A^\vee$ by the vector bundle ω_A.

Moreover, since we took the quotient by a discrete subgroup, one can find back the Hodge filtration from this extension by applying the Lie algebra functor to it.

In fact one can prove :

Fait 1. *The preceding extension is algebraic and it is universal among extension of $A^\vee$ by a vector bundle.*

which gives an intrinsec definition of it.

We are going to do the same for p-divisible groups, apart from the fact that the theory we're going to developp is covariant that is to say is expressed in terms of the “De Rham homology”.

1.3 Universal vector extension of a p-divisible groups

Let S be a scheme on which p is localy nilpotent.

Définition 1. *Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_S$-module. We note $\underline{\mathcal{M}}$ for the associated fppf sheaf.*

Remarque 1. *If $\mathcal{M}$ is localy free of finite rank over S then $\underline{\mathcal{M}}$ is represented by a vector bundle but in general it is not representable by a scheme.*

Définition 2. *Let G be a p-divisible group over S. A vector extension of G is an extension of fppf sheaves*

$$0 \longrightarrow \underline{\mathcal{M}} \longrightarrow E \longrightarrow G \longrightarrow 0$$

where $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_S$-module. A morphism of vector extensions is a morphism of digagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\mathcal{M}_1} & \longrightarrow & E_1 & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \underline{\mathcal{M}_2} & \longrightarrow & E_2 & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

such that the left square is co-cartesian that is to say the morphisms induce an isomorphism between the push-out of the upper extension by $\underline{\mathcal{M}_1} \longrightarrow \underline{\mathcal{M}_2}$ and the bottom extension.

The category of vector extensions of G is rigid in the sens that there is at most one morphism from a vector extension to another one. In fact, two morphisms differ from an element of $\mathrm{Hom}(G, \underline{\mathcal{M}})$. But since p is an epimorphism on G and localy niltpotent on $\underline{\mathcal{M}}$ we have $\mathrm{Hom}(G, \underline{\mathcal{M}}) = 0$.

Thus this has a meaning to speak about a universal vector extension that is to say an initial object in the category of vector extensions.

From now we note $\mathcal{M}$ for $\underline{\mathcal{M}}$ since there's no ambiguity.

Proposition 1. *There exists a universal vector extension*

$$0 \longrightarrow V(G) \longrightarrow E(G) \longrightarrow G \longrightarrow 0$$

Moreover $V(G) = \omega_{G^D}$ and is thus a vector bundle.

Démonstration. We have to prove the functor $\mathcal{M} \longmapsto \mathrm{Ext}^1(G, \underline{\mathcal{M}})$ is representable. This is local on S and we thus can suppose $p^N \mathcal{O}_S = 0$ for an $N \in \mathbb{N}$. The exact sequence

$$0 \longrightarrow G[p^N] \longrightarrow G \xrightarrow{p^N} G \longrightarrow 0$$

induces

$$0 \longrightarrow \mathrm{Hom}(G, \underline{\mathcal{M}}) \xrightarrow{p^N} \mathrm{Hom}(G, \underline{\mathcal{M}}) \longrightarrow \mathrm{Hom}(G[p^N], \underline{\mathcal{M}}) \longrightarrow \mathrm{Ext}^1(G, \underline{\mathcal{M}}) \xrightarrow{p^N} \mathrm{Ext}^1(G, \underline{\mathcal{M}})$$

But in the preceding sequence both maps p^N are zero since p^N is zero on $\underline{\mathcal{M}}$. Thus there is an isomorphism

$$\mathrm{Ext}^1(G, \underline{\mathcal{M}}) \simeq \mathrm{Hom}(G[p^N], \underline{\mathcal{M}})$$

but (? ? ?)

$$\mathrm{Hom}(G[p^N], \underline{\mathcal{M}}) \simeq \mathrm{Hom}_{\mathcal{O}_S}(\omega_{G[p^N]^D}, \underline{\mathcal{M}})$$

Thus the extension obtained py push-out in the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G[p^N] & \longrightarrow & G & \xrightarrow{p^N} & G & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & \omega_{G[p^N]^D} & & & & & & \end{array}$$

where $G[p^N] \longrightarrow \omega_{G[p^N]^D}$ is the morphisms that associates to an $x \in G[p^N] = (G[p^N]^D)^D$, $x : G[p^N]^D \longrightarrow \mathbb{G}_m$, the element $x^* \frac{dT}{T} \in \omega_{G[p^N]^D}$, is universal. $\square$

Exemple 1. *The uniersal vector extension of $\mathbb{Q}_p/\mathbb{Z}_p$ is*

$$0 \longrightarrow \mathbb{G}_a \longrightarrow (\mathbb{G}_a \oplus \mathbb{Q}_p)/\mathbb{Z}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0$$

where $\mathbb{Z}_p \hookrightarrow \mathbb{G}_a \oplus \mathbb{Q}_p$ is the diagonal embeding.

Of course the universal vector extension is functorial in G : for all $f : G_1 \longrightarrow G_2$ there is a morphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V(G_1) & \longrightarrow & E(G_1) & \longrightarrow & G_1 & \longrightarrow & 0 \\ & & \downarrow h & & \downarrow & & \downarrow f & & \\ 0 & \longrightarrow & V(G_2) & \longrightarrow & E(G_2) & \longrightarrow & G_2 & \longrightarrow & 0 \end{array}$$

where $h = (f^D)^* : \omega_{G_1^D} \longrightarrow \omega_{G_2^D}$. This is easily checked using the pullback by f of the bottom extension and the universality property of the upper extension.

Proposition 2. *The fppf-sheaf $E(G)$ is formaly smooth and $\widehat{E}(G)$ is a formal Lie group and is an extension of formal Lie groups*

$$0 \longrightarrow \widehat{V}(G) \longrightarrow \widehat{E}(G) \longrightarrow \widehat{G} \longrightarrow 0$$

where $\widehat{V}(G)$, the formal completion of a vector bundle along its zero section, is Zariski localy on S isomorphic to a sum of copies of $\widehat{\mathbb{G}}_a$.

Moreover the following sequence of $\mathcal{O}_S$-modules is exact

$$0 \longrightarrow V(G) \longrightarrow Lie\, E(G) \longrightarrow Lie\, G \longrightarrow 0$$

Démonstration. This is not difficult, essentialy by writting $E(G) = \varinjlim_n E_n$ where E_n is the reciprocal image of $G[p^n]$ and

$$0 \longrightarrow V(G) \longrightarrow E_n \longrightarrow G[p^n] \longrightarrow 0$$

thus, E_n is a fppf-torsor over $G[p^n]$ under a smooth affine scheme and is thus representable by a smooth $G[p^n]$-scheme. □

Remarque 2. *By functoriality of the universal extension, for a p-divisible group over a formal scheme such that p is in its definition ideal there is associated a universal vector extension.*

1.4 Messing's crystal

Let S be a scheme on which p is localy nilpotent. Let $NCRIS(S)$ be the absolute big Zariski nilpotent cristalline site of S whose objects are $(U \hookrightarrow T, \gamma)$ where U is an S-scheme, $U \hookrightarrow T$ is a nil-immersion defined by an ideal $\mathcal{I}$ equiped with nilpotent divided powers γ. Let $\mathcal{O}_S^{CRIS}$ be the structural sheaf of this site.

Théorème 1 (Messing)**.** *There exists a crystal $\mathcal{E}$ in localy free $\mathcal{O}_S^{CRIS}$-modules on $NCRIS(S)$ such that $\forall (U \hookrightarrow T, \gamma) \in NCRIS(S)$ for any lifting $\widetilde{G_U}$ of $G \times_S U$ to a p-divisible group over T there exists a canonical isomorphism*

$$\mathcal{E}_{(U \hookrightarrow T)} \xrightarrow{\sim} Lie\, E(\widetilde{G_U})$$

Here to explain the word canonical it would need to go inside the construction of this crystal. This is done in the following section.

Définition 3. *We will note $\mathbb{D}(G)$ for the crystal of G.*

Remarque 3. *In fact Messing proves $E(\widetilde{G_U})$ defines a crystal in fppf sheaves. The preceding crystal is deduced by applying the Lie algebra functor.*

1.5 Construction of Messing's crystal

1.5.1 The exponential map

Théorème 2 (Messing). *Let $S_0 \hookrightarrow S$ be an immersion defined by an ideal $\mathcal{I}$ equiped with nilpotent divided powers γ. Let H be a formal Lie group over S and $\mathcal{M}$ a localy free of finite rank $\mathcal{O}_S$-module. Let G_0 and $\mathcal{M}_0$ be the reductions moduli $\mathcal{I}$ of G and $\mathcal{M}$. Then there exists a morphism*

$$\exp : Hom_{\mathcal{O}_S}(\mathcal{M}, \mathcal{I}.Lie\, H) \longrightarrow \ker(Hom_S(\mathcal{M}, H) \longrightarrow Hom_{S_0}(\mathcal{M}_0, H_0))$$

natural in H, $\mathcal{M}$ and $(S, \mathcal{I}, \gamma)$ such that if $\mathcal{I}^2 = p\mathcal{I} = 0$ then via the natural identification between $\ker(Hom_S(\mathcal{M}, H) \longrightarrow Hom_{S_0}(\mathcal{M}_0, H_0))$ *and $Hom_{\mathcal{O}_S}(\mathcal{M}, \mathcal{I}.Lie\, H)$ then*

$$\exp f = \sum_{i \geq 0} (-1)^i \Pi^i \circ f$$

Wehre $\Pi : \mathcal{I} \otimes Lie\, H \longrightarrow \mathcal{I} \otimes Lie\, H$ is defined by $\Pi = \gamma_p \otimes \alpha$ where $\gamma_p : \mathcal{I} \longrightarrow \mathcal{I}$ is Frob-linear and α is the Frob-linear morphism induced at the level of the Lie algebra $Lie\, H$ by $V : H^{(p)} \longrightarrow H$ if H is p-divisible and more generaly by the p-exponentiation of invariant derivations on H for general H (the operation defining the restricted Lie algebra structure on $Lie\,(H)$).

We won't give the proof of this theorem. The proof given in ? ? ? is not realy natural. A more natural one is given in ? ? ? using Cartier theory. The points consists in proving that with the hypothesis of the theorem there is an isomorphism

$$\log : H(\mathcal{I}) \xrightarrow{\sim} \mathcal{I}.\mathrm{Lie}\, H$$

(depending on the divided powers on $\mathcal{I}$ allthough it is not in the notations). This can be done for the infinite dimensional formal group given by the formal completion of the Witt vectors $\widehat{W}(-)$. Then Cartier theory gives a resolution of any formal group by Witt vectors

$$\widehat{W}^m \longrightarrow \widehat{W}^n \longrightarrow H \longrightarrow 0$$

and this enables one to construct such a logarithm isomorphism using the one for Witt vectors.

In the case of a one dimension formal group law the concrete statement about the existence of this logarithm is the following.

Théorème 3. *Let R ba a ring and I an ideal in R equiped with nilpotent divided powers $(\gamma_n)_n$. Let $F \in R[[X, Y]]$ be a formal group law over R. Let $\omega = f(T)dT$ be a generator of the invariant differential forms on F. Let $f(T) = \sum_{n \geq 0} a_n T^n$. For $x \in I$ put*

$$\log(x) = \sum_{n \geq 0} a_n (n-1)! \gamma_n(x)$$

Then log *induces an isomorphism of groups*

$$\log : (I, \underset{F}{+}) \xrightarrow{\sim} (I, +)$$

Exemple 2. *If F is $\widehat{\mathbb{G}}_m$ then $\log(x) = \sum_{n \geq 0} (n-1)! \gamma_n(x)$. Where $(n-1)!\gamma_n(x)$ is the analog of "x^n/n".*

Now let's note $\exp : \mathcal{I}.\mathrm{Lie}\, H \longrightarrow H(\mathcal{I})$ for the inverse of log. To construct

$$\exp : \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{M}, \mathcal{I}.\mathrm{Lie}\, H) \longrightarrow \mathrm{Hom}_S(\mathcal{M}, H)$$

one uses the fact for any flat S scheme T the divided powers extend to $\mathcal{I}.\mathcal{O}_T$. Thus since $\mathcal{M}$ is represented by a flat S-scheme one concludes thanks to Yoneda lemma.

1.5.2 The crystal

Here we explain how using the exponential morphism Messing proves the preceding theorem.

Théorème 4 (Messing)**.** *Let $S_0 \hookrightarrow S$ be a closed immersion defined by an ideal equiped with nilpotent divided powers. Suppose p is localy nilpotent on S. Let G, resp. H, be two p-divisible groups over S and G_0, resp. H_0 their reduction to S_0. Suppose given a morphism $f : G_0 \longrightarrow H_0$. Then there exists a unique morphism $g : E(G) \longrightarrow E(H)$ reducing to $E(f)$ on S_0 such that for any linear morphism $u : V(G) \longrightarrow V(H)$ lifting $V(f) : V(G_0) \longrightarrow V(H_0)$ in the following diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V(G) & \xrightarrow{i} & E(G) & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle g} & & & & \\ 0 & \longrightarrow & V(H) & \xrightarrow{j} & E(H) & \longrightarrow & H & \longrightarrow & 0 \end{array}$$

$j \circ u - g \circ i : V(G) \longrightarrow E(H)$ (that reduces to zero on S_0) is an exponential (with values in the formal Lie group $\widehat{E}(H)$).

This theorem proves for any p-divisible group G_0 over S_0, for two lifts G, G' there exists a unique isomorphism $E(G) \xrightarrow{\sim} E(G')$ deforming the identity and satisfying the conditions of the preceding theorem. Moerover, still thanks to the preceding theorem, this construction is functorial in G, proving the fact that this defines a crystal.

The proof of the preceding theorem is by devissage to the case when the ideal $\mathcal{I}$ defininig he immersion $S_0 \hookrightarrow S$ verifies $\mathcal{I}^2 = p\mathcal{I} = 0$. Then one uses the explicit formula given for the exponential map in this case and the universal property of the universal vector extension.

1.6 Deformation theory

Let $S_0 \hookrightarrow S$ be a divided powers immersion as in the preceding theorem.

For each p-divisible group G_0 over S_0 we can consider its crystal $\mathbb{D}(G_0)$. Its evaluation $\mathbb{D}(G_0)_{(S_0 \xrightarrow{Id} S_0)}$ is identified with $\mathrm{Lie}\, E(G_0)$ and is thus filtered by the sub $\mathcal{O}_{S_0}$-module $V(G_0) = \mathrm{Fil}\, \mathbb{D}(G)_{(S_0 \xrightarrow{Id} S_0)}$ that is a localy free, localy direct summand in $\mathbb{D}(G_0)_{(S_0 \xrightarrow{Id} S_0)}$.

To each lifting G of G_0 over S is associated localy free localy direct factor filtration $V(G)$ of the evaluation of the crystal $\mathbb{D}(G_0)_{(S_0 \hookrightarrow S)}$, filtration thtat reduces modulo de divided powers ideal $\mathcal{I}$ to $\mathrm{Fil}\, \mathbb{D}(G)_{(S_0 \xrightarrow{Id} S_0)}$.

Définition 4. *Let $\mathcal{C}$ be the category whose objects are couples (G_0, Fil) where G_0 is a p-divisible group over S_0 and Fil is a localy free localy direct summand $\mathcal{O}_S$-module in $\mathbb{D}(G_0)_{(S_0 \hookrightarrow S)}$ such that its reduction to S_0 is $V(G_0) \subset \mathbb{D}(G_0)_{(S_0 \xrightarrow{Id} S_0)}$; and $Hom_{\mathcal{C}}((G_0, Fil), (G'_0, Fil'))$ consists in morphisms f from G_0 to G'_0 such that the induced crystal morphism $\mathbb{D}(f)$ verifies*

$$\mathbb{D}(f)_{(S_0 \hookrightarrow S)}(Fil) \subset Fil'$$

Théorème 5. *The functor from the category of p-divisible group over S to $\mathcal{C}$ that associates to G the couple (G_0, Fil) where G_0 is the reduction to S_0 of G and Fil is the vector part $V(G)$ is a category equivalence.*

Remarque 4. *The case when $S_0 \hookrightarrow S$ is defined by a squared zero ideal has been obtained by Grothendieck using the theory of the cotangent complex (? ? ?). The more general case with divided powers is due to Messing.*

1.7 The squared zero case

This case had allready been obtained by Grothendieck using the cotangent complex theory.

Corollaire 1. *Let $S_0 \hookrightarrow S$ be an immersion defined by an ideal $\mathcal{I}$ s.t. $\mathcal{I}^2 = (0)$. Let G_0 be a p-divisible group over S_0. Then the set of liftings of G_0 to a p-divisible group over S is a principal homogenous space under*

$$\omega^*_{G^D} \otimes \omega^*_G \otimes \mathcal{I}$$

Démonstration. The tangent space to the deformation functor of p-divisible groups is identified to the one of a Grassmanian thanks to Grothendieck-Messing's deformation theory. □

Here is a more conceptual restatement of the preceding results

Théorème 6. *The stack $\mathfrak{X}$ of p-divisible groups is formaly smooth. Let G be the universal p-divisible group over $\mathfrak{X}$. Then restricted to schemes on which p is localy nilpotent the tangent bundle to this stack is $Lie\, G^D \otimes Lie\, G$.*

Exemple 3. *Let $\mathcal{O}$ be an inequal caracteristic discrete valuation ring with residue field of caracteristic p. Then if $p \neq 2$, $p\mathcal{O}$ has nilpotent divided powers. Thus the category of p-divisible groups over Spf($\mathcal{O}$) being equivalent to the one over Spec($\mathcal{O}$) there is a fully faithfull functor from the category of p-divisible groups over Spec($\mathcal{O}$) to the category of couples (G_0, Fil) where G_0 is a p-divisible group over $\mathcal{O}/p\mathcal{O}$ and Fil is a direct factor filtration of the evaluation of its crystal on $\mathcal{O} \twoheadrightarrow \mathcal{O}/p\mathcal{O}$.*

If $\mathcal{O}$ is unramified over $\mathbb{Z}_p$ one obtains thus a fully faithfull functor from the category of p-divisibile groups over $\mathcal{O}$ to filtered Dieudonné modules. After inverting p one obtains a fully faithfull functor from p-divisible groups over $\mathcal{O}$ to filtered isocrystals. This works the same when the absolute ramification index e verifies $e \leq p-1$ since then the ideal $p\mathcal{O}$ is equiped with divided powers. The determination of the essential image of this functor is Fontaine's theory of filtered admissible modules in a partciular case.

1.8 Application to the deformation spaces of p-divisible groups

Here we are going to refind and generalize theorem ??? on Lubin-Tate deformation spaces.

Théorème 7. *Let k be a caracteristic p perfect field. Let $\mathbb{H}$ be a dimension d and height h p-divisible group over Spec(k) Let $\mathcal{D}ef_{\mathbb{H}}$ be the functor from artinian rings with residue field k to sets that associates to A the isomorphism classes of couples (H, ρ) where H is a p-divisible group over A and $\rho : \mathbb{H} \xrightarrow{\sim} H \otimes_A k$ is an isomorphism.*

Then $\mathcal{D}ef_{\mathbb{H}}$ is pro-representable :

$$\mathcal{D}ef_{\mathbb{H}} \simeq Spf\big(W(k)[[T_1, \ldots, T_{d(h-d)}]]\big)$$

Démonstration. We apply theorem ??. We know the functor F is formaly smooth. The computation of the tangent space has been done through Grothendieck-Messing deformation theory. □