Geometrization of the local Langlands Correspondence

Motivation: * Conjecture : geometrization of local Langlands

At the interface between
- "classical" Langlands program
- geometric "
- p-adic Hodge theory

Drinfeld, Laumon, Gaitsgory

* Conjecture known for tori (and this was a motivation for the conjecture) using local class field theory

* Explain goal: How to prove directely my conjecture for GL_1 using geometric Langlands methods

$\Rightarrow$ new proof of local class field) - purely geometric, no arithmetic

Background on geometric class field theory

k field $\quad X/k$ smooth proper curve
géo. connected . $K = k(X)$ field of rat. functions

$$\mathrm{Pic} = \coprod_{d \in \mathbb{Z}} \mathrm{Pic}^d \quad \text{Picard scheme}$$

$\mathcal{E} = \overline{\mathbb{Q}_\ell}$ - local system rk. 1 / X
= everywhere unramified rep. $\chi : \mathrm{Gal}(\overline{K}|K) \longrightarrow \overline{\mathbb{Q}_\ell}^\times$

$\rightsquigarrow \mathrm{Aut}_{\mathcal{E}}$ = rk. 1 $\overline{\mathbb{Q}_\ell}$ - local system on Pic

satisfying : $*\ m^* \mathcal{E} \simeq \mathcal{E} \boxtimes \mathcal{E}$ if $m : \mathrm{Pic} \times \mathrm{Pic} \to \mathrm{Pic}$

$*$ if $k = \mathbb{F}_q$ $\quad \mathrm{Pic}(\mathbb{F}_q) = K^\times \backslash \mathbb{A}_K^\times / \prod_v \mathcal{O}_{K_v}^\times$

trace of Frob. fct. of $\mathrm{Aut}_{\mathcal{E}}$ = Hecke character associated to χ
via global class field for K

$\hookrightarrow$ Grothendieck dictionnary
sheaf $\leftrightarrow$ functions.

$\rightarrow$ upgrade a function to a sheaf

Construction of Aut-$\mathcal{E}$

$m^* \mathcal{E} \simeq \mathcal{E} \boxtimes \mathcal{E} \Rightarrow$ suffices to construct Aut-$\mathcal{E}|_{\mathrm{Pic}^d}$ for $d \gg 0$

$d \geq 1$ $\quad \mathrm{Sym}^d X = X^d / \mathfrak{S}_d = \mathrm{Div}^d =$ Hilbert Scheme of deg. d effective Cartier divisors on X

Symmetrization morphism

$$\pi^{(d)} : X^d \longrightarrow \mathrm{Div}^d$$
$$(x_1, \dots, x_d) \longmapsto \sum_{i=1}^{d} [x_i]$$

$$\mathcal{F}^{(d)} = \left(\pi^{(d)}_* \, \mathcal{E}^{\boxtimes d} \right)^{\mathfrak{S}_d} = \text{rk. } 1 \ \overline{\mathbb{Q}}_\ell\text{-loc. sys. on } \mathrm{Div}^d$$

In fact $\pi_1 \left(\mathrm{Div}^d \right) = \pi_1 (X)^{ab}$ if $d > 1$

SGA1 - exp. IX Rem. 5.8:

$\pi_1 (Y / \Gamma) =$ biggest quotient of $\pi_1(Y)$ on which $(\Gamma_{\bar{y}})_{\bar{y} \to Y}$ acts trivially.

quasi-proj.

finite

inertia at $\bar{y}$

géo. point

$\Rightarrow \quad \pi_1(X^d/\sigma_d) =$ biggest quotient of $\pi_1(X^d) = \pi_1(X)^d$ on which σ_d acts trivially

inertia along ~~the~~ diagonal $X \overset{\Delta}{\hookrightarrow} X^d$.

<u>Concretely:</u>

$$\begin{tikzcd} U \arrow[d, "G \text{ abelian pms}"] & U^d \arrow[d, "G^d \text{ pms}"] & (U^d \times_{G^d, \Sigma} G)/\sigma_d \arrow[d, "G \text{ étale Galois}"] \\ X & X^d & X^d/\sigma_d \end{tikzcd}$$

$\Sigma : G^d \twoheadrightarrow G$ sum morphism.

$*$ $\underbrace{\mathcal{P}ic^d}_{\text{Picard stack}} \xrightarrow{\quad} Pic^d =$ Coarse moduli

G_m-gerb

(trivial if $X(k) \neq \emptyset$)

$\Downarrow$

$\mathcal{P}ic^d = [Pic^d/G_m]$

$AJ^d : \mathcal{D}iv^d \longrightarrow \mathcal{P}ic^d \longrightarrow \mathrm{Pic}^d$ Abel-Jacobi

$D \longmapsto \mathcal{O}(D)$

$d > 2g-2$ $\quad \mathcal{L}$ line bundle $/X_S$ deg. d $\quad f : X_S \to S$

$f_*\mathcal{L} = \underbrace{Rf_*\mathcal{L}}_{\text{perfect complex}} =$ vector bundle

⇝ defines $\mathcal{V}$ = vector bundle of rk. $d+1-g$ on $\mathcal{P}ic^d$

$\mathcal{D}iv^d = \{\mathcal{L} \in \mathcal{P}ic^d +$ non-zero section of $\mathcal{L}\}/\sim$

$=$ "$\mathbb{P}(\mathcal{V}) = \mathbb{V}(\mathcal{V}) \setminus \{0\} / \mathbb{G}_m$"

$\mathcal{E}nd(\mathcal{V})$ descends to an Azumaya algebra $\mathcal{A}$ on Pic^d

$[\mathcal{A}] \in \mathrm{Br}(\mathrm{Pic}^d)$

class of the $\mathbb{G}_m$-gerbe $\mathcal{P}ic^d \to \mathrm{Pic}^d$

$\mathcal{V}$ doesn't descend in general unless $X(k) \neq \emptyset$ (even in this case the descended $\mathcal{V}$ ~~does depend~~ is not canonical contrary to $\mathrm{End}(\mathcal{V})$)

Then $\boxed{\mathrm{Div}^d = \underbrace{SB(\mathcal{A})}_{\text{Severi-Brauer}}}$

$\Rightarrow \left[AJ^d : \mathrm{Div}^d \longrightarrow \mathrm{Pic}^d \right.$ étale locally trivial fibration in projective spaces $\left. \mathbb{P}^{d+1-g} \text{ for } d > 2g-2 \right]$

Projective space simply connected $\Rightarrow \pi_1(\mathrm{Div}^d) = \pi_1(\mathrm{Pic}^d)$

$\Rightarrow \mathcal{F}^{(d)}$ descends canonically to a $\overline{\mathbb{Q}}_\ell$-local system $\mathrm{Aut}_{E}|\mathrm{Pic}^d$.

The case of the curve

$$E \begin{cases} \nearrow \mathbb{F}_q((\pi)) \\ \searrow [E:\mathbb{Q}_p] < +\infty \end{cases} \qquad \mathbb{F}_q = \mathcal{O}_E/\pi$$

$F|\mathbb{F}_q$ perfect complete valued field $\left(\underline{\underline{\text{Ex}}}: \widehat{\overline{\mathbb{F}_q((T))}}, \mathbb{F}_q((T^{1/p^\infty})) \right)$

w/ Fontaine have defined a "curve" X_F - Has 2 incarnations:

$\nearrow$ "p-adic Riemann surface" compact X_F^{ad} = E-adic space) not loc. of f.t.

$\searrow$ "proper algebraic curve" X_F^{sch} = E-scheme) not of f.t.

The adic version can be constructed in families:

S perfectoid space $/\mathbb{F}_q$ $\rightsquigarrow$ X_S = E-adic space

= "family of curves $(X_{\underbrace{k(s)}_{\text{perf. field}}})_{s \in S}$" (preceding)

$$X_S = Y_S / \varphi_S^{\mathbb{Z}}$$

$* \; E = \mathbb{F}_q((\pi))$ then $Y_S = S \times_{\mathrm{Spa}\,\mathbb{F}_q} \mathrm{Spa}\, E = \mathbb{D}_S^*$

$$\mathbb{D}_S^* = \{0 < |\pi| < 1\} \subset \mathbb{A}_S^1$$

$\circlearrowleft$

φ_S properly discontinuously.

$* \; E|\mathbb{Q}_p \qquad Y_S^{\diamond} = S \times \mathrm{Spa}(E)^{\diamond}$

$$S = \mathrm{Spa}(\underset{\text{perfectoid}}{R}, R^+) \qquad \mathbb{A} = W_{\mathcal{O}_E}(R^\circ) = \Big\{ \sum_{n \geq 0} [x_n]\pi^n \,/\, x_n \in R^\circ \Big\}$$

$$\mathbb{A}\Big[\frac{1}{\pi}, \frac{1}{[\varpi_R]}\Big] = \Big\{ \sum_{n >> -\infty} [x_n]\pi^n \Big/ \sup_n \|x_n\| < +\infty \Big\}$$

$$\mathcal{O}(Y_S) = \text{Completion of } \mathbb{A}\Big[\frac{1}{\pi}, \frac{1}{[\varpi_R]}\Big]$$

w.r.t. Gauss norms $(\|\cdot\|_\rho)_{\rho \in]0,1[}$

$$\Big\| \sum_n [x_n]\pi^n \Big\|_\rho = \sup_n \|x_n\| \rho^n$$