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$$E \begin{cases} \nearrow \mathbb{F}_q((\pi)) \\ \searrow [E:\mathbb{Q}_p] < +\infty \quad \mathbb{F}_q = \mathcal{O}_E/\pi \end{cases}$$

S $\mathbb{F}_q$-perfectoid space $\rightsquigarrow$ X_S = E-adic space

= "family of curves $(X_{S(s)})_{s\in S}$"

$$\boxed{X_S = Y_S/\varphi_S^{\mathbb{Z}}}$$

* $E = \mathbb{F}_q((\pi))$

$$\underset{\substack{\uparrow \\ \text{categorical product in adic spaces}}}{Y_S = S \times \mathrm{Spa}(E)} \qquad \varphi_S = \mathrm{Frob}_S \times \mathrm{Id}$$

$$= \mathbb{D}_S^* = \{0 < |\pi| < 1\} \subset \mathbb{A}_S^1$$

* $E|\mathbb{Q}_p$ $\quad S = \mathrm{Spa}(R, R^+)$ affinoid perfectoid

$$Y_S = \mathrm{Spa}\Big(\underbrace{W_{\mathcal{O}_E}(R^\circ)}_{\text{Fontaine's } A_{\inf}}, \underbrace{W_{\mathcal{O}_E}(R^\circ)^+}_{[R^+] + \pi W_{\mathcal{O}_E}(R^\circ)}\Big) \setminus V(\pi[\varpi_R])$$

$\varpi_R \in R^{\circ\circ} \cap R^\times$ p.u. (pseudo-unif.)

$A_{\inf} = \left\{ \sum_{n\geq 0} [x_n]\pi^n \,/\, x_n \in R^\circ \right\}$

$\varphi_S \curvearrowright Y_S$ induced by $\varphi\left(\sum_n [x_n]\pi^n\right) = \sum_n [x_n^q]\pi^n$

(usual Frob. of Witt vectors)

<u>Formula:</u> $Y_S^\diamond = S \times \mathrm{Spa}(E)^\diamond$

$\Rightarrow X_S^\diamond = \left(S \times \mathrm{Spa}(E)^\diamond\right) / \varphi_S^{\mathbb{Z}}$

<u>What does it mean?</u>

$\mathrm{Perf}_{\mathbb{Z}_p}$ = Category of perfectoid spaces + pro-étale topology

$\mathrm{Perf}_{\mathbb{F}_p}$ = " " over $\mathbb{F}_p$

Define $\boxed{\mathrm{Spa}(\mathbb{Q}_p)^\diamond \in \widetilde{\mathrm{Perf}}_{\mathbb{F}_p}}$

sheaf of untilts

$$\mathrm{Spa}(\mathbb{Q}_p)^\diamond(S) = \{(S^\#, \iota)\}/\sim$$

perfectoid $/\mathbb{Q}_p$ $\qquad \iota : S \xrightarrow{\sim} S^{\#,b}$

quickly said: $S^\#$ is an untilt of S

$\mathcal{F} \in \widetilde{\mathrm{Perf}}_{\mathbb{Q}_p} \rightsquigarrow \mathcal{F}^\diamond \in \widetilde{\mathrm{Perf}}_{\mathbb{F}_p}$

$$\mathcal{F}^\diamond(S) = \{(S^\#, s) \,/\, S^\# \text{ untilt of } S \text{ and } s \in \mathcal{F}(S^\#)\}/\sim$$

Defines an equivalence of topoi

$$\left[\widetilde{\mathrm{Perf}}_{\mathbb{Q}_p} \xrightarrow{\sim} \widetilde{\mathrm{Perf}}_{\mathbb{F}_p} / \mathrm{Spa}(\mathbb{Q}_p)^\diamond\right]$$

$$\mathcal{F} \longmapsto \mathcal{F}^\diamond / \mathrm{Spa}(\mathbb{Q}_p)^\diamond$$

Moreover the functor

$$\text{Normal } \mathbb{Q}_p\text{-rigid analytic spaces} \longrightarrow \widetilde{\mathrm{Perf}}_{\mathbb{Q}_p}$$

$$X \longmapsto \mathrm{Hom}(-, X)$$

is <u>fully faithfull</u>.

For such an X $\quad \exists \underbrace{\widetilde{X}}_{\text{perfectoid}} \longrightarrow X$ $\quad$ locally $\lim$ of finite étale cov.

<u>Ex</u>: $\mathrm{Spa}\left(\mathbb{C}_p\langle T^{\pm 1/p^\infty}\rangle\right) \longrightarrow \mathrm{Spa}\left(\mathbb{Q}_p\langle T^{\pm 1}\rangle\right)$

Then $\mathcal{R} = \widetilde{X} \times_X \widetilde{X}$ exists and is a perfectoid space

$$\begin{tikzcd} & \mathcal{R} \arrow[dl, "\text{pro-étale}"'] \arrow[dr, "\text{pro-étale}"] & \\ \widetilde{X} & & \widetilde{X} \end{tikzcd}$$

Then $\boxed{X^\diamond = \widetilde{X}^\flat / \mathcal{R}^\flat}$ $\quad$ pro-étale eq. relation on $\widetilde{X}^\flat$ = perfectoid $/\mathbb{F}_p$

Ex: $\mathrm{Spa}(\mathbb{Q}_p)^\diamond = \mathrm{Spa}(\mathbb{Q}_p^{cyc,\flat})/\mathbb{Z}_p^\times$

$$\mathbb{Q}_p^{cyc} = \widehat{\mathbb{Q}_p(\zeta_{p^n})_{n\geq 1}}$$

$$\mathbb{Q}_p^{cyc,\flat} \simeq \mathbb{F}_p((T^{1/p^\infty})) \circlearrowleft \mathbb{Z}_p^\times$$

Thus: Normal $\mathbb{Q}_p$-rigid spaces $\hookrightarrow$ $\overbrace{\text{pro-étale alg. spaces}}^{\text{a diamond, by definition}} / \mathrm{Spa}(\mathbb{Q}_p)^\diamond$

$$X \longmapsto X^\diamond / \mathbb{Q}_p^\diamond$$

In fact, this is the same for pré-perfectoid spaces$/\mathbb{Q}_p$.

$\downarrow$

X adic space$/\mathbb{Q}_p$ s.t. $X \hat{\otimes}_{\mathbb{Q}_p} K$ perfectoid if $K|\mathbb{Q}_p$ is perfectoid.

Ex: $\mathrm{Spa}(\mathbb{Q}_p\langle T^{\pm 1/p^\infty}\rangle)$

Pré-perfectoid spaces$/\mathbb{Q}_p$ $\hookrightarrow$ Diamonds$/\mathbb{Q}_p^\diamond$

$$X \longmapsto (X \hat{\otimes} \mathbb{Q}_p^{cyc})^\flat / \mathbb{Z}_p^\times = (X \hat{\otimes} \mathbb{C}_p)^\flat / \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$$

over $\mathbb{Q}_p^\diamond$.

Now: $Y_{S,E}$ and $X_{S,E}$ are perfectoid.

and one can prove $\boxed{Y_{S,E}^{\diamond} = S \times \mathrm{Spa}(E)^{\diamond}}$

<u>Rem:</u> if $E = \mathbb{F}_q((\pi))$ $\quad \mathrm{Spa}(E)^{\diamond} = \mathrm{Spa}(E)$

The Picard Stack

Pic = stack on $\mathrm{Perf}_{\mathbb{F}_q}$ + pro-étale topology

$\mathrm{Pic}(S)$ = Groupoid of line bundles$/X_S$

$d \in \mathbb{Z}$ $\qquad \mathcal{O}_{X_S}(d)$ = line bundle$/X_S$

géo. realization

$$\begin{array}{c} Y_S \times \mathbb{A}^1 \\ \varphi_S^{\mathbb{Z}} \circlearrowright_{\times \pi^{-d}} \\ \big\downarrow \\ X_S \end{array}$$

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If S = géo. point = $\mathrm{Spa}(F)$ F alg. closed this defines an iso. $\mathbb{Z} \xrightarrow{\sim} \mathrm{Pic}(X_F)$

$$d \mapsto [\mathcal{O}(d)]$$

Set deg = inverse of this bijection.

Then if $\mathcal{L}/X_S$ line bundle the function $|S| \longrightarrow \mathbb{Z}$

$$s \mapsto \deg(\mathcal{L}|X_{\bar{s}})$$

géo. point over s

is loc. constant.

Picard stack of deg. d line bundles.

$$\Rightarrow \boxed{\mathrm{Pic} = \coprod_{d \in \mathbb{Z}} \mathrm{Pic}^d}$$

If A is a locally profinite set it defines a pro. étale sheaf $\underline{A}$

$$\underline{A}(S) = \mathcal{C}(|S|, A)$$

s.t. if $A = \bigcup_{i \in I} B_i$ open pro-finite s.t. $B_i = \varprojlim_{j} \underbrace{B_{ij}}_{\text{finite}}$

then $\underline{A} = \varinjlim_i \varprojlim_j \underbrace{\underline{B_{ij}}}_{\text{usual constant sheaf}}$

$$\Gamma(S, \mathcal{O}_{X_S}) = \underline{E}(S).$$

$\Rightarrow$ the sheaf $S \mapsto \mathrm{Aut}(\mathcal{O}_{X_S}(d))$ is $\underline{E}^\times$

$\underline{\text{Th}}$: $\mathrm{Pic}^d \xrightarrow{\sim} [\mathrm{Spa}(\mathbb{F}_q)/\underline{E}^\times]$

$\mathcal{L} \longmapsto \underline{\mathrm{Isom}}(\mathcal{O}(d), \mathcal{L})$

i.e. if $\mathcal{L}/X_S$ is fiberwise of deg. d then $\exists\, \tilde{S} \to S$ pro-étale cover s.t. $\mathcal{L}|X_{\tilde{S}} \simeq \mathcal{O}_{X_{\tilde{S}}}(d)$.

Rem: $S =$ a point $= \mathrm{Spa}(F)$.

Then $\mathcal{O}(1)$ ample.

$$P_F = \bigoplus_{d \geq 0} \underbrace{H^0(X_F, \mathcal{O}(d))}_{\mathcal{O}(Y_F)^{\varphi = \pi^d} = \underset{\uparrow \text{ if } E = \mathbb{Q}_p}{} (B^+_{\mathrm{cris}})^{\varphi = \pi^d}}$$

$$X_F^{\mathrm{sch}} = \mathrm{Proj}(P_F)$$

Th: * X_F^{sch} is a Dedekind Schem

* $\forall x \in |X_F^{\mathrm{sch}}|$ closed point $k(x)|E$ perfectoid $[k(x)^\flat : F] < +\infty$

$$|X_F^{\mathrm{sch}}| \xrightarrow{\sim} \{(K, \iota)\}/\sim \quad K|E \text{ perfectoid}$$
$$\iota : F \hookrightarrow K^\flat \text{ s.t. } [K^\flat : F] < +\infty.$$

* X_F^{sch} is "complete" $\forall f \in E(X_F^{\mathrm{sch}})^\times$ $\deg(\mathrm{div}\, f) = 0$

where $\deg(x) = [k(x)^\flat : F]$

* F alg. closed $\quad \mathrm{Pic}(X_F^{\mathrm{sch}}) = \underset{\uparrow \text{ not}}{\mathrm{Div}/\sim} \xrightarrow{\sim} \mathbb{Z}$

equivalent to $H^0(X_F^{\mathrm{sch}} \setminus \{\infty\}) = B_{\mathrm{cris}}^{\varphi = \mathrm{Id}}$ is a P.I.D.