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The algebraic curve

$S = \mathrm{Spa}\, F$ ← perf field

$X_F = Y_F / \varphi^{\mathbb{Z}}$ = adic space $/E$

+ $\forall d \in \mathbb{Z}$ $\mathcal{O}(d)$ line bundle $/ X_F$

Has a scheme counterpart by declaring $\mathcal{O}(1)$ ample.

Def: * $P_F = \bigoplus_{d \geq 0} \underbrace{H^0(X_F, \mathcal{O}(d))}_{\mathcal{O}(Y_F)^{\varphi = \pi^d}}$

* $X_F^{sch} = \mathrm{Proj}(P_F)$

Ex: $E = \mathbb{F}_q((\pi))$ $\quad Y_F = \mathbb{D}_F^*$

$$\mathcal{O}(Y_F) = \left\{ \sum_{n \in \mathbb{Z}} a_n \pi^n \,\middle/\, a_n \in F \ \ \forall \rho \in]0,1[\ \lim_{|n| \to +\infty} |a_n| \rho^n = 0 \right\}$$

$\circlearrowleft$

φ $\quad \varphi\left(\sum_n a_n \pi^n\right) = \sum_n a_n^q \pi^n$

$$\mathfrak{m}_F^d \xrightarrow{\sim} P_{F,d}$$

$$(x_0, \dots, x_{d-1}) \longmapsto \sum_{i=0}^{d-1} \sum_{b \in \mathbb{Z}} x_i^{q^{-b}} \pi^{bd+i}$$

$d \geq 1$ $\quad \mathcal{G}_d = \widehat{\mathbb{G}}_a^d / \mathbb{F}_q$ formal group

$\circlearrowleft$ $\quad$ where $\pi \cdot (x_0, \dots, x_{d-1}) = (F(x_{d-1}), x_0, \dots, x_{d-2})$

$\mathcal{O}_E = \mathbb{F}_q[[\pi]]$

$$\widetilde{\mathcal{G}}_d = \varprojlim_{\times \pi} \mathcal{G}_d = \mathrm{Spf}\left(\mathbb{F}_q[[x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}]]\right)$$

formal E-v.s. (E-v.s. in the cat. of formal schemes)
as we defined them w/ Fontaine

$\widetilde{\mathcal{G}}_d(\mathcal{O}_F) \xrightarrow{\sim} P_{F,d}$ and $P_{F,d} \times P_{F,d'} \longrightarrow P_{F,d+d'}$ is given by

mor. of E-v.s. $\quad \widetilde{\mathcal{G}}_d \times \widetilde{\mathcal{G}}_{d'} \to \widetilde{\mathcal{G}}_{d+d'}$

$$\bigoplus_{d \geq 0} \widetilde{\mathcal{G}}_d = \text{graded algebra in the cat. of formal } E\text{-v.s.}$$

($\widetilde{\mathcal{G}}_0 = \underline{E}$ not really formal, étale)

<u>Th</u>: $*$ X_F^{sch} is a Dedekind scheme $/E$

$*$ $\forall x \in |X_F^{sch}|$ closed point $k(x)|E$ is perfectoid
and $k(x)^b|F$ with $\underbrace{[k(x)^b : F]}_{=: \deg(x)} < +\infty$

$*$ This defines a bijection

$$|X_F^{sch}| \xrightarrow{\sim} \{(K, \iota)\}/\sim$$

up to powers of $Frob_F$

perf field $K|E$ $\quad$ $\iota : F \hookrightarrow K^b$ s.t. $[K^b : F] < +\infty$

$*$ $\forall \infty \in |X_F^{sch}|$ s.t. $\deg(\infty) = 1$ $\quad \exists t \in P_1 \setminus \{0\} = H^0(X_F^{sch}, \mathcal{O}(1))$
where $\mathcal{O}(1) = \widetilde{P[1]}$

s.t. ~~$D^+(t)$~~ $V^+(t) = \{\infty\}$ and

$$D^+(t) = X_F^{sch} \setminus \{\infty\} = \operatorname{Spec}\Big(\underbrace{\mathcal{O}(Y_F)\left[\tfrac{1}{t}\right]^{\varphi = Id}}_{B_{cris}(k(\infty))^{\varphi = Id} \text{ if } E = \mathbb{Q}_p}\Big)$$

then $\Gamma(X^{sch}_F \setminus \{\infty\}, \mathcal{O}_X)$ is a P.I.D. if F alg. closed

(only a Dedekind domain in general)

$$\Updownarrow$$

$$\mathrm{Pic}^0(X_F^{sch}) = 0 \Leftrightarrow \deg : \mathrm{Pic} \xrightarrow{\sim} \mathbb{Z}.$$

* X_F^{sch} is complete: $\forall f \in E(X_F^{sch})^\times$ $\deg(\mathrm{div}\, f) = 0$

$\Rightarrow$ good degree function on vector bundles $/X_F^{sch}$

$\Rightarrow$ H.N. reduction theory.

(good notion of semi-stability)

* $\exists$ morphism of ringed spaces

$$\alpha : X_F^{ad} \longrightarrow X_F^{sch}$$

that induces a bijection $|X_F^{ad}|^{cl} \xrightarrow{\sim} |X_F^{sch}|$ = closed points

classical Tate points

D.V.R. since X^{sch} is Dedekind

s.t. if $x \in X_F^{ad}$ is a Tate point $\widehat{\mathcal{O}}_{X^{sch}_F, \alpha(x)} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X^{ad}_F, x}$

$$\overset{\shortparallel}{B^+_{dR}(k(x))}$$

* $\underline{\text{GAGA}}$: $\alpha^*: \mathrm{Bun}_{X_F^{alg}} \xrightarrow{\sim} \mathrm{Bun}_{X_F^{ad}}$.

Back to the Picard stack

$$\mathrm{Pic} = \coprod_{d \in \mathbb{Z}} \mathrm{Pic}^d$$

<u>Notation:</u> A locally pro-finite set

$\underline{A}$ = pro-étale sheaf on $\mathrm{Perf}_{\mathbb{F}_q}$ defined by

$\underline{A}(S) = \mathcal{C}(|S|, A)$

If $A = \bigcup_i \underbrace{\varprojlim_j \underbrace{B_{ij}}_{\text{finite}}}_{\text{pro-finite open in } A}$ then $\underline{A} = \varinjlim_i \varprojlim_j \underline{B_{ij}}$ ← usual constant étale sheaf.

Fact: $\Gamma(S, \mathcal{O}_{X_S}) = \underline{E}(S)$

$\Rightarrow \forall d \in \mathbb{Z}$ the sheaf $S \mapsto \mathrm{Aut}(\mathcal{O}_{X_S}(d))$ is $\underline{E}^\times$

Th: $\forall d \in \mathbb{Z}$

$$\begin{aligned} \mathrm{Pic}^d &\xrightarrow{\sim} [\mathrm{Spa}(\mathbb{F}_q)/\underline{E}^\times] \\ \mathcal{L} &\longmapsto \Big[\mathrm{Isom}\underset{\substack{\circlearrowleft \\ \underline{E}^\times}}{\big(\mathcal{O}(d), \mathcal{L}\big)}\Big] \end{aligned}$$

i.e. if $\mathcal{L}/X_S$ s.t. $\forall \bar{s} \to S$ $\mathcal{L}|_{X_{\bar{s}}} \simeq \mathcal{O}(d)$

then $\exists\, \tilde{S} \to S$ pro-étale cover s.t. $\mathcal{L}|_{X_{\tilde{S}}} \simeq \mathcal{O}_{X_{\tilde{S}}}(d)$.

$\rightarrow$ looks like there is no geometry since $\mathrm{Jac} = \{0\}$ but in fact there is geometry in A.J. morphism.

Div^1:

Div^1 = sheaf of degree 1 - Cartier divisors on the curve.

"Classical context": $\mathrm{Div}^1_X = X$. This is not the case here.

Div^1 = some kind of "mirror image of the curve"

$$\begin{array}{c} \mathrm{Div}^1 \text{ pro-étale sheaf} \\ \downarrow \\ \mathrm{Spa}(\mathbb{F}_q) \end{array}$$

$$\mathrm{Div}^1(S) = \left\{ (\mathcal{L}, f) \;\middle/\; \begin{array}{l} \mathcal{L} = \text{line bundle} / X_S \\ f \in H^0(X_S, \mathcal{L}) \text{ s.t. } \forall s \in S \;\; f_{|X_{k(s)}} \neq 0 \end{array} \right\} \Big/ \simeq$$

homotheties.

$$\boxed{\underline{\mathrm{Th}}: \;\; \mathrm{Spa}(E)^{\diamond} / \varphi^{\mathbb{Z}}_{E^{\diamond}} \xrightarrow{\sim} \mathrm{Div}^1 \text{ which is thus a diamond.}}$$

$$X_S^\diamond = \left(S \times \mathrm{Spa}(E)^\diamond\right)/\varphi_S^{\mathbb{Z}} \quad \Big| \quad \mathrm{Div}^1_S = \left(S \times \mathrm{Spa}(E)^\diamond\right)/\varphi_{E^\diamond}^{\mathbb{Z}}$$

$\varphi_{E^\diamond} \circ \varphi_S$ = absolute Frob acts identically on the étale / pro-étale analytic topos

$\Rightarrow$ $X_S^\diamond$ and Div^1_S have the same top. space / étale site but are not isomorphic.

Eg: $E = \mathbb{F}_q((t))$

$$\begin{tikzcd} & Y_S = \mathbb{D}^*_S \arrow[dl] \arrow[dr] & \\ S & & \mathbb{D}^*_{\mathbb{F}_q} = \mathrm{Spa}(E) \end{tikzcd}$$

$$\begin{tikzcd} X_S = \mathbb{D}^*_S/\varphi_S \arrow[d] \\ \mathrm{Spa}(E) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Div}^1_S = \mathbb{D}^*_S/\varphi_E^{\mathbb{Z}} \arrow[d] \\ S \end{tikzcd}$$