$\mathrm{Div}^1/\mathbb{F}_q$ moduli of deg. 1 effective Cartier divisors on the curve

a relative version of my theorem for φ-mod$_{A_{inf}}$ in the rk. 1 case

Th:

$$\begin{tikzcd} \mathrm{Spa}(E)^\diamond/\varphi^{\mathbb{Z}} \arrow[r, "\sim"] & \mathrm{Div}^1 \\ \underbrace{\mathrm{Spa}(E)^\diamond}_{\text{sheaf of untilts}} \arrow[u] \arrow[ur] & \end{tikzcd}$$

what is this morphism?

$S^\# =$ untilt of S over E then $S^\# \hookrightarrow Y_{S'}$ Cartier divisor.

$S = \mathrm{Spa}(R, R^+)$

Untilts of R to $E \simeq \{\underbrace{\text{deg 1 primitive elements in } A_R}\}/A_R^\times$

$\sum_{n\geq 0}' [x_n]\pi^n \in A_R$ s.t. $x_0 \in R^{\circ\circ} \cap R^\times$ and $x_1 \in (R^\circ)^\times$

$R^\# = A[\frac{1}{\pi}]/\xi \longleftarrow\!\mapsto \xi$ primitive deg. 1

$\mathrm{Spa}(R^\#) = V(\xi) \hookrightarrow Y_{R,R^+}$.

Can see untilts as Cartier div. on Y

$$
\begin{tikzcd}
\mathrm{Spa}(E)^\diamond \arrow[r] \arrow[d] \arrow[rr, bend left, "\text{morphism we are looking for}"] & \text{Cartier div. on } Y \arrow[r] & \mathrm{Div}^1 \\
\mathrm{Spa}(E)^\diamond/\varphi^{\mathbb{Z}} \arrow[urr, bend right, "\text{factorization}"'] & &
\end{tikzcd}
$$

$\varphi^{\mathbb{Z}}$-periodization

$$D \mapsto \sum_{n\in\mathbb{Z}} \varphi^n(D)$$

Local systems on $\mathrm{Div}^1_{\overline{\mathbb{F}}_q}$

$\overline{\mathbb{F}}_q | \mathbb{F}_q$

$\breve{E} = \widehat{E^{un}} \circlearrowleft \sigma$

$$\mathrm{Spa}(\breve{E})^\diamond = \mathrm{Spa}(E)^\diamond \times_{\mathrm{Spa}(\mathbb{F}_q)} \mathrm{Spa}(\overline{\mathbb{F}}_q)$$

$\mathrm{Div}^1 = E^\diamond/\varphi^{\mathbb{Z}}$

$\overline{\mathbb{Q}}_\ell$-local systems on $\mathrm{Div}^1_{\overline{\mathbb{F}}_q} = \varphi_{E^\diamond}$-eq. loc. syst. on $\mathrm{Spa}(E)^\diamond \times \mathrm{Spa}(\overline{\mathbb{F}}_q)$

in the pro-étale sense: loc. constant for the pro-ét. top.

$\simeq \mathrm{Frob}_{\overline{\mathbb{F}}_q}$-eq. loc. syst. on $\mathrm{Spa}(E)^\diamond \times \mathrm{Spa}(\overline{\mathbb{F}}_q)$

same as before:

$\varphi_{E^\diamond} \circ \varphi_{\overline{\mathbb{F}}_q} = \text{abs. Frob.}$

$= \sigma$-eq. local systems on $\mathrm{Spa}(\breve{E})^\diamond$

②

$= \text{Weil } \overline{\mathbb{Q}}_\ell\text{- loc. systems on } \mathrm{Spa}(\breve{E})$

de Jong type π_1

$= \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(W_E)$

" $\pi_1(\mathrm{Div}^1_{\overline{\mathbb{F}}_q}) = W_E$ "

$\ast\ AJ^1 : \mathrm{Div}^1_{\overline{\mathbb{F}}_q} \longrightarrow \mathrm{Pic}^1_{\overline{\mathbb{F}}_q} = [\mathrm{Spa}(\overline{\mathbb{F}}_q)/\underline{E}^\times]$

induces $\pi_1(AJ^1) : W_E \longrightarrow E^\times$

$\tau \bmod \pi = \mathrm{Frob}^{v(\tau)}$

<u>Fact</u>: This morphism is given by $\tau \longmapsto \chi_{\mathcal{LT}}(\tau)\, \pi^{-v(\tau)}$

Lubin-Tate character: $W_E^{ab} \to \mathcal{O}_E^\times$

(<u>Side remark</u>: Since the beginning I've fixed π to define $\mathcal{O}(1)$ and thus a Lubin-Tate group.)

<u>Reason is the following</u>:

$$\begin{tikzcd} \mathrm{Spa}(E)^\diamond \arrow[r] \arrow[rr, bend right] & \mathrm{Div}^1 \arrow[r, "AJ^1"] & {[\cdot/\underline{E}^\times]} \end{tikzcd}$$

pullback of univ. torsor via on $E^\diamond = \varphi_{E^\diamond}$- eq. $\underline{E}^\times$- torsor on $\mathrm{Spa}(E)^\diamond$

$$\begin{array}{l} \mathcal{T}_{LT} \\ \downarrow \; \circlearrowright \underline{\mathcal{O}}_E^\times \\ \mathrm{Spa}(E)^\diamond \end{array}$$

Lubin-Tate torsor

has a canonical $\varphi_{E^\diamond}$-eq. structure

$$\mathrm{Can}: \varphi^* \mathcal{T}_{LT} \xrightarrow{\sim} \mathcal{T}_{LT}$$

(the L.T. group is defined over $\mathcal{O}_E$)

Then the preceding φ-eq. torsor is given by

$$\left(\mathcal{T}_{LT} \underset{\underline{\mathcal{O}}_E^\times}{\times} \underline{E}^\times , \; \pi^{-1} \cdot \mathrm{Can} \right)$$

* Thus local class field (Artin reciprocity is an iso.)

is equivalent to:

[Th: $\chi : W_E \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ ~~,~~ $\mathcal{L}_\chi$ $\overline{\mathbb{Q}}_\ell$-loc. sys on $\mathrm{Div}^1_{\overline{\mathbb{F}}_q}$

Then $\mathcal{L}_\chi$ descends to a local system on $\mathrm{Pic}^1_{\overline{\mathbb{F}}_q}$ via AJ^1.]

$\forall d \geq 1 \quad \mathrm{Div}^d / \mathrm{Spa}(\mathbb{F}_q)$

$$\mathrm{Div}^d(S) = \left\{ (\mathcal{L}, f) \;\middle/\; \begin{array}{l} \mathcal{L} = \text{deg. } d \text{ line bundle} / X_S \\ f \in H^0(X_S, \mathcal{L}) \text{ non zero fiberwise}/S \end{array} \right\} \Big/ \sim$$

Consequence of $\forall \xi \in A_{\inf, F}$ primitive deg. d, F alg. closed $\xi = \xi_1 \dots \xi_d$

$$AJ^d : \mathrm{Div}^d \longrightarrow \mathrm{Pic}^d$$
$$[(\mathcal{L}, f)] \longmapsto \mathcal{L}$$

Th: $(\mathrm{Div}^1 \times \dots \times \mathrm{Div}^1)/\mathfrak{S}_d \xrightarrow{\sim} \mathrm{Div}^d$

pro-étale quotient

$$(\mathcal{L}_1, f_1), \dots, (\mathcal{L}_d, f_d) \longmapsto (\mathcal{L}_1 \otimes \dots \otimes \mathcal{L}_d, f_1 \otimes \dots \otimes f_d)$$

$\Rightarrow \mathrm{Div}^d$ is a diamond.

$$\mathrm{Div}^d = \left(\mathrm{Spa}(E)^\diamond \times \dots \times \mathrm{Spa}(E)^\diamond\right) \Big/ \left(\varphi_{E^\diamond}^{\mathbb{Z}} \times \dots \times \varphi_{E^\diamond}^{\mathbb{Z}}\right) \rtimes \mathfrak{S}_d$$

$$\pi^{(d)} : \mathrm{Div}^1 \times \dots \times \mathrm{Div}^1 \longrightarrow \mathrm{Div}^d$$ Symmetrization morphism

$$\mathcal{L}_X^{(d)} = \left(\pi^{(d)}_* \mathcal{L}_X^{\boxtimes d}\right)^{\mathfrak{S}_d}$$

$= $ rk. 1 $\overline{\mathbb{Q}}_\ell$ - loc. system / $\mathrm{Div}^d_{\overline{\mathbb{F}}_q}$

Then the preceding statement is equivalent to:

should be true for $d=2$ – Technical problem.

Th: For $d \geq 2$ (resp. $d \geq 3$ if $E|\mathbb{Q}_p$) any $\overline{\mathbb{Q}}_\ell$-local system on $\mathrm{Div}^d_{\overline{\mathbb{F}}_q}$ descends along $AJ^d : \mathrm{Div}^d_{\overline{\mathbb{F}}_q} \to \mathrm{Pic}^d_{\overline{\mathbb{F}}_q}$.

Def: $\mathbb{B}$ = sheaf on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ defined by $\mathbb{B}(S) = \Gamma(Y_S, \mathcal{O}_{Y_S})$.

Thus $\forall d \geq 1$, $\mathbb{B}^{\varphi = \pi^d}$ = sheaf of global sections of $\mathcal{O}(d)$.

= absolute Banach Colmez space

Using $\mathrm{Pic}^d = [\,\cdot\,/\underline{E}^\times]$

→ not a diamond, becomes one

$\mathcal{L} \mapsto$ torsor of iso. between $\mathcal{O}(d)$ and $\mathcal{L}$.

$$\boxed{\mathrm{Div}^d = \left(\mathbb{B}^{\varphi = \pi^d} \setminus \{0\}\right)/\underline{E}^\times \xrightarrow{AJ^d} \left[\mathrm{Spa}(\overline{\mathbb{F}}_q)/\underline{E}^\times\right]}$$

Thus, everything is a consequence of: (4)

$\underline{\text{Th}}$: $\forall d \geq 2$ (resp. $d \geq 3$ if $E/\mathbb{Q}_p$)

$\underbrace{\mathbb{B}_{\mathbb{F}_q}^{\varphi=\pi^d} \setminus \{0\}}_{\text{analogous to projective space in my situation.}}$ is a simply connected diamond.

$\underline{\text{Le cas } E = \mathbb{F}_q((\pi))}$

$\underline{\text{Recall}}$: $\mathbb{B}_{\mathbb{F}_q}^{\varphi=\pi^d} = \widetilde{\mathcal{G}_d} = \mathrm{Spf}\left(\mathbb{F}_q[[x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}]]\right)$

represented by a perfect formal scheme $/\mathbb{F}_q$

$\varprojlim_{\times\pi} \mathcal{G}_d$ $\qquad$ $\mathcal{G}_d = \widehat{\mathbb{G}}_a^d \circlearrowleft \mathcal{O}_E = \mathbb{F}_q[[\pi]]$

$\Rightarrow \mathbb{B}_{\mathbb{F}_q}^{\varphi=\pi^d} \setminus \{0\} = \mathrm{Spa}\left(\mathbb{F}_q[[x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}]]\right) \setminus V(x_0, \dots, x_{d-1})$

$= \mathrm{Spa}(-)_a$ = perfectoid space

analytic points

$$X = \mathrm{Spa}(-)_a = \bigcup_{i=0}^{d-1} \underbrace{D(x_i)}_{\{x_i \neq 0\}} \rightsquigarrow \text{looks like } \mathbb{P}^{d-1}$$

$$D(x_i) = \text{perfectoid open ball} / \overline{\mathbb{F}}_q((x_i^{1/p^\infty}))$$

$$\downarrow$$

$$\mathrm{Spa}(\overline{\mathbb{F}}_q((x_i^{1/p^\infty})))$$

~~Elkik ⇒ finite étale~~ / Spa($\overline{\mathbb{F}}_q[x_i]$

Elkik

$$\Downarrow$$

$$2\text{-}\varinjlim_n \mathrm{F.\acute{e}t}/\mathrm{Spa}(\overline{\mathbb{F}}_q[[x_i^{1/p^n}]]_i)_a \xrightarrow{\sim} \mathrm{F.\acute{e}t}/\mathrm{Spa}(\overline{\mathbb{F}}_q[[x_i^{1/p^\infty}]])_a$$

$$\boxed{\begin{array}{l}\underline{\text{Th.}}\ k \text{ separably closed and } d \geq 2. \text{ Then} \\ \qquad \mathrm{Spa}(k[[x_0,\dots,x_{d-1}]]) \setminus V(x_0,\dots,x_{d-1}) \\ \text{is simply connected.}\end{array}}$$

Consequence of GAGA:

Th (GAGA): A noetherian I-adic.

Vector bundles $/ \operatorname{Spec}(A) \setminus V(I)$ $\xrightarrow{\sim}$ Vector bundles $/ \operatorname{Spf}(A) \setminus V(I)$

$\Rightarrow$ Finite étale $/ (-)$ $\xrightarrow{\sim}$ Finite étale $/ (-)$

$+$ Zariski - Nagata purity $+$ Hensel

$\downarrow$

any finite étale covering of $\operatorname{Spec}(k[[x_0, \dots, x_{d-1}]]) \setminus V(x_0, \dots, x_{d-1})$ extends to $\operatorname{Spec}(k[[x_0, \dots, x_{d-1}]])$.