Recall:

$$\underset{\substack{\cap \\ \mathrm{Perf}_{\mathbb{F}_q}}}{B(S)} = \mathcal{O}(Y_S) \qquad X_S = Y_S/\varphi^{\mathbb{Z}}$$

Th: $\forall d \geq 2$ (resp. $d \geq 3$ if $E|\mathbb{Q}_p$)

$B^{\varphi=\pi^d}_{\overline{\mathbb{F}}_q} \setminus \{0\}$ is a simply connected diamond.

Proof when $E = \mathbb{F}_q((\pi))$: $B^{\varphi=\pi^d}_{\mathbb{F}_q}$ represented by

$$\widetilde{\mathcal{G}}_d = \varprojlim_{\times \pi} \mathcal{G}_d \qquad \mathcal{G}_d = \widehat{\mathbb{G}}_a^d \circlearrowleft \mathcal{O}_E = \mathbb{F}_q[[\pi]]$$

$$\pi.(x_0, \dots, x_{d-1}) = (F(x_{d-1}), x_0, \dots, x_{d-2})$$

$$\simeq \mathrm{Spf}\left(\mathbb{F}_q[[x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}]]\right)$$

$$(R^{\circ\circ})^d \xrightarrow{\sim} \widetilde{\mathcal{G}}_d(R, R^+)$$

$$(x_0, \dots, x_{d-1}) \longmapsto \sum_{i=0}^{d-1} \sum_{b \in \mathbb{Z}} x_i^{q^{-b}} \pi^{bd+i}$$

$$\mathbb{B}^{\varphi=\pi^d} = \mathrm{Spa}\left(\mathbb{F}_q[\![x_0^{1/p^\infty},\dots,x_{d-1}^{1/p^\infty}]\!]\right)$$

perfect adic space not perfectoid (non-analytic)

$\forall S/\mathbb{F}_q$ perfectoid $\mathbb{B}^{\varphi=\pi^d}\times S \simeq \mathring{\mathbb{B}}_S^{d,1/p^\infty}$ = perfectoid open ball

$\longrightarrow$ absolute perfectoid space non-perfectoid

$$\mathbb{B}_{\mathbb{F}_q}^{\varphi=\pi^d}\setminus\{0\} = \mathrm{Spa}\left(\mathbb{F}_q[\![x_0^{1/p^\infty},\dots,x_{d-1}^{1/p^\infty}]\!]\right)\setminus V(x_0,\dots,x_{d-1})$$

$$X = \mathrm{Spa}(-)_a = \text{perfectoid q.c. space}$$

$$X = \bigcup_{i=0}^{d-1}\underbrace{\{x_i\neq 0\}}_{U_i} \qquad U_i \simeq \mathring{\mathbb{B}}^{d-1,1/p^\infty}_{\mathbb{F}_q((x_i^{1/p^\infty}))}$$

= perfectoid open ball radius $1/\mathbb{F}_q((x_i^{1/p^\infty}))$

<u>But</u> no structural morphism $\longrightarrow$ perfectoid field since

$$\mathcal{O}(X) = \mathbb{F}_q[\![x_0^{1/p^\infty},\dots,x_{d-1}^{1/p^\infty}]\!]$$

Other set of charts

$$X = \bigcup_{i=0}^{d-1} \underbrace{\{\forall j \quad |x_j| \le |x_i| \neq 0\}}_{V_i}$$

$$\underbrace{V_i}_{\text{Closed ball in } \mathring{\mathbb{B}}^{d-1}_{\mathbb{F}_q((x_i^{1/p^\infty}))}} \subset\subset U_i \qquad X = \bigcup_i V_i = \bigcup_i U_i$$

$\Rightarrow X$ is proper ..... but not really

since $\forall F|\mathbb{F}_q$ perf. field $X_F \simeq \mathring{\mathbb{B}}^{d,1/p^\infty}_F \setminus \{0\}$ not q.c.

$X \to \mathrm{Spa}(\mathbb{F}_q)$ not adic

Analogy with $\mathbb{P}^{d-1}_F = \{[x_0 : \dots : x_{d-1}]\} = \bigcup_{i=0}^{d-1} \underbrace{\{x_i \neq 0\}}_{U_i \simeq \mathbb{A}^{d-1}_F}$

$$= \bigcup_{i=0}^{d-1} \underbrace{\{\forall j \quad |x_j| \le |x_i| \neq 0\}}_{V_i \simeq \mathbb{B}^{d-1}_F}$$

$$V_i \subset\subset U_i$$
closed ball.

Th.: $\forall d \geq 2$ $\mathrm{Spa}\left(\overline{\mathbb{F}}_q[[x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}]]\right) \setminus V(x_0, \dots, x_{d-1})$ is simply connected.

Rem.: $C|\overline{\mathbb{F}}_q$ $\quad X_C \simeq \mathring{\mathbb{B}}^d_C \setminus \{0\}$ not simply connected

$\Rightarrow$ we really nee to work absolutely $/\overline{\mathbb{F}}_q$

Proof: Elkik + our space is q.c.

$\hookrightarrow$ a Scholze purity (trivial in equal char.)

$$\Rightarrow 2\text{-}\varinjlim_n \mathrm{F\acute{e}t.}/\mathrm{Spa}(\overline{\mathbb{F}}_q[[x_i^{1/p^n}]]_i)_a \xrightarrow{\sim} \mathrm{F\acute{e}t.}/\mathrm{Spa}(\overline{\mathbb{F}}_q[[x_i^{1/p^\infty}]])_a$$

Th.: k sep. closed $\mathrm{Spa}\left(k[[x_0, \dots, x_{d-1}]]\right) \setminus V(x_0, \dots, x_{d-1})$ is simply connected.

Th (local GAGA): A noetherian I-adic

$$\text{Vector bundles}/\mathrm{Spec}(A)\setminus V(I) \xrightarrow{\sim} \text{Vector bundles}/\mathrm{Spa}(A)\setminus V(I)$$

$\Downarrow$

$$\text{F. ét}/\_\_ \xrightarrow{\sim} \text{F. ét}/\_\_$$

$\longrightarrow$ Raynaud's flatification + GAGF (EGA III)

$X = \mathrm{Spec}(A) \qquad \mathfrak{X} = \mathrm{Spf}(A)$

$$\mathrm{Spa}(A)_a = \varprojlim_{\tilde{\mathfrak{X}} \to \mathfrak{X}} \tilde{\mathfrak{X}}$$

I-admissible formal blow-up

$$= \varprojlim_{\tilde{X} \to X} \widehat{\tilde{X}}$$

I-adic completion along exceptional divisor.

I-admissible blowup

Any v.b. $/\mathrm{Spa}(A)_a$ has a model $/\widehat{\tilde{X}}$ for some $\tilde{X} \to X$

GAGF $\Rightarrow$ Any v.b. $/\widehat{\tilde{X}}$ comes from a v.b. $/\tilde{X}$

restrict it to $\tilde{X}\setminus V(I) = \mathrm{Spec}(A)\setminus V(I)$ $\square$

Conclusion of the proof:

$\boxed{\text{Zariski - Nagata purity}}$ + Hensel

$$\forall d \geq 2 \qquad \mathrm{F.\acute{e}t}/\mathrm{Spec}(R) \setminus V(m) \qquad \left(R = k[[x_0, \dots, x_{d-1}]] \right)$$

$$\downarrow \simeq$$

$$\mathrm{F.\acute{e}t}/\mathrm{Spec}(R) \Big] \text{ trivial by Hensel}$$

The case $E|\mathbb{Q}_p$

Problems: * $d > 1$ $\quad \overbrace{\mathbb{B}^{\varphi = \pi^d}_{\overline{\mathbb{F}}_q}}^{\text{absolute diamond}} \setminus \{0\}$ diamond not representable by a perfectoid space

* $C|\overline{\mathbb{F}}_q$ $\quad \underbrace{\mathbb{B}^{\varphi = \pi^d}_{C}}_{\text{diamond}} \setminus \{0\}$ not simply connected

→ Can't do an extension to a perfectoid field. Have to work with absolute diamonds that are not diamonds

Guided by the equal char. case where at the end the result is a consequence of Zariski - Nagata purity.

→ Proves that : $U / \mathbb{B}^{\varphi = \pi^d}_{\overline{\mathbb{F}}_q} \setminus \{0\}$ finite étale extends to $\mathbb{B}^{\varphi = \pi^d}_{\overline{\mathbb{F}}_q}$ then it is trivial.

⇒ one has to prove purity for the inclusion

$$\mathbb{B}^{\varphi = \pi^d}_{\overline{\mathbb{F}}_q} \setminus \{0\} \hookrightarrow \mathbb{B}^{\varphi = \pi^d}_{\overline{\mathbb{F}}_q}$$

Consequence of the following purity theorem in rigid analytic geometry:

<u>Th</u>: $d \geq 3$. $\rho \in]0,1[$. Then

Finite Etale covers $/ \mathbb{B}^d \setminus \mathbb{B}^d(\rho) \xrightarrow{\sim}$ F. ét. covers $/ \mathbb{B}^d$.

→ Lütkebohmert's adaptation of Siu's thesis:

Any v.b. $/\mathbb{B}^d \setminus \mathbb{B}^d(\rho)$ extends canonically to a $\boxed{\text{Coherent sheaf}}$ $/\mathbb{B}^d$

$\Rightarrow$ any finite étale $/\mathbb{B}^d \setminus \mathbb{B}^d(\rho)$ extends to a finite $/\mathbb{B}^d$

+ **Th.** $d \geq 2$. U normal s.t. $U_{|\mathbb{B}^d \setminus \mathbb{B}^d(\rho)}$

$$\begin{tikzcd} U \arrow[d, "\text{finite}"] & U_{|\mathbb{B}^d \setminus \mathbb{B}^d(\rho)} \arrow[d, "\text{étale}"] \\ \mathbb{B}^d & \mathbb{B}^d \setminus \mathbb{B}^d(\rho) \end{tikzcd}$$

Then $U/\mathbb{B}^d$ is étale.

false for $d=2$

No Zariski's main theorem in rigid geometry

$$\begin{tikzcd} U \arrow[r, dashed, "?"] \arrow[d] & \cdot \arrow[d, "\text{finite}"] \\ \mathbb{B}^d \setminus \mathbb{B}^d(\rho) \arrow[r, hook] & \mathbb{B}^d \end{tikzcd}$$

quasi-finite separated