p-adic Twistors

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$$\begin{cases} [E:\mathbb{Q}_p] < +\infty & \mathbb{F}_q = \mathcal{O}_E/\pi \\ F|\mathbb{F}_q \text{ perfectoid} & \underline{\text{Ex}}: \mathbb{F}_q((T^{1/p^\infty})),\ \widehat{\overline{\mathbb{F}_q((T))}} \end{cases}$$

$\rightsquigarrow$ a Curve $\to$ "p-adic Riemann surface" X^{ad} = E-adic space

Joint work with Fontaine

a Curve $\to$ "algebraic curve" X = one dim. noetherian regular scheme$/E$ (not of finite type)

$\ast$ $X^{ad} = Y/\varphi^{\mathbb{Z}}$

$$Y = E\text{-adic Stein space} = \mathrm{Spa}\left(W_{\mathcal{O}_E}(\mathcal{O}_F)\right) \setminus V(p[\varpi_F]), \quad 0 < |\varpi_F| < 1$$

φ acts properly discontinuously

$\hookrightarrow$ induced by $\varphi\left(\sum_n [x_n]\pi^n\right) = \sum_n [x_n^q]\pi^n$

Line bundle $\underline{\mathcal{O}(1)}$ on X^{ad}

$$\begin{array}{c} Y \times_{\varphi^{\mathbb{Z}}} \mathbb{A}^1 \circlearrowleft \varphi \text{ via } \times \pi^{-1} \\ \downarrow \\ Y/\varphi^{\mathbb{Z}} \end{array}$$

Declare $\mathcal{O}(1)$ ample:

$$\ast \quad X = \mathrm{Proj}\Big(\bigoplus_{d \geq 0} \underbrace{H^0(X^{ad}, \mathcal{O}(d))}_{\mathcal{O}(Y)^{\varphi = \pi^d}}\Big) = \text{Dedekind scheme}$$

· equipped with a morphism $\boxed{X^{ad} \longrightarrow X}$ that induces

a bijection $\underbrace{|X^{ad}|^{cl}}_{\text{classical Tate points}} \xrightarrow{\sim} \underbrace{|X|}_{\text{closed points}}$

$$|X^{ad}|^{cl} = |Y|^{cl}$$

$|Y|^{cl} \simeq$ untilts of finite deg. extensions of F.

if $x^{ad} \mapsto x$ then $\left[\widehat{\mathcal{O}_{X,x}} \xrightarrow{\sim} \widehat{\mathcal{O}_{X^{ad},x}} = B^+_{dR}(k(x))\right]$

x^{ad} classical

$k(x) = k(x^{ad})|E$ perfectoid $[k(x)^\flat : F] < +\infty$

X^{ad} is "proper" and X is complete: $\forall f \in E(X)^\times$, $\deg(\mathrm{div} f)$

although not t.f. of finite type

where $\deg(x) = [k(x)^\flat : F]$

$= 1$ if F alg. closed

* GAGA $\mathrm{Bun}_X \xrightarrow{\sim} \mathrm{Bun}_{X^{ad}}$

* X has a canonical $\widehat{\mathbb{Z}}$-pro-Galois cover $X_h^{ad} := Y/\varphi^{h\mathbb{Z}}$, $h \geq$

$$X_h^{ad} = X^{ad} \otimes_E E_h, \quad E_h|E \text{ unram. deg. } h$$

$$\left.\begin{matrix}(X_h)_{h\geq 1}\\ \downarrow\\ X_1 = X\end{matrix}\right)\widehat{\mathbb{Z}} \text{ unfolding cover.}$$

F alg. closed from now on:

* $\overline{\mathbb{F}_q} \subset F \qquad L = W_{O_E}(\overline{\mathbb{F}_q})_{\mathbb{Q}} = \widehat{E^{un}} \circlearrowleft \sigma = \varphi$

$$\varphi\text{-}\mathrm{Mod}_L = \{(\underbrace{D}_{\text{finite dim. } L\text{-v.s.}},\underbrace{\varphi}_{\text{semi-linear iso.}})\} = \text{Isocrystals}$$

$$\begin{array}{ccccc} \varphi\text{-}\mathrm{Mod}_L & \longrightarrow & \mathrm{Bun}_{X^{ad}} & \xleftarrow[\sim]{GAGA} & \mathrm{Bun}_X \\ (D,\varphi) & \longmapsto & \mathcal{E}(D,\varphi)^{ad} & \longleftarrow\!\shortmid & \mathcal{E}(D,\varphi) \end{array}$$

Th (F.-Fontaine): This is essentialy surjective.

→ Dieudonné-Manin $\quad \forall \mathcal{E} \in \mathrm{Bun}_X \qquad \mathcal{E} \simeq \bigoplus_i \underbrace{\mathcal{O}(\lambda_i)}_{\text{stable of slope } \lambda_i}, \quad \lambda_i \in \mathbb{Q}$

$\lambda = \frac{d}{h} \qquad \mathcal{O}(\lambda) :=$ pushforward of $\mathcal{O}(d)$ via $X_h \xrightarrow[\mathbb{Z}/h\mathbb{Z}\text{-Galois}]{} X$

* G reductive gp. $/E \qquad B(G) = G(L)/G\text{-conj.}$ (Kottwitz)

$$b \in G(L) \rightsquigarrow \quad \begin{array}{ccccc} \mathrm{Rep}\, G & \longrightarrow & \varphi\text{-}\mathrm{Mod}_L & \xrightarrow{\mathcal{E}(-)} & \mathrm{Bun}_X \\ (V,\rho) & \longmapsto & (V_L, \rho(b)\sigma) & & \end{array}$$

$$\mathcal{E}_b = G\text{-torsor}$$

$$\boxed{\underline{\text{Th}}: \; B(G) \xrightarrow{\sim} H^1_{et}(X, G) \\ [b] \longmapsto [\mathcal{E}_b]}$$

<u>Nice features:</u> dictionnary Kottwitz $\longleftrightarrow$ Atiyah-Bott reduction theory

$\underline{\text{Ex}}$: b basic $\iff$ $\mathcal{E}_b$ semi-stable.

<u>Archimedean picture:</u> $E = \mathbb{C}$ $\quad X = \mathbb{P}^1_{\mathbb{C}}$

$E = \mathbb{R}$ $\quad X = \widetilde{\mathbb{P}}^1_{\mathbb{R}} = \mathbb{P}^1_{\mathbb{C}} / z \sim -\frac{1}{\bar{z}}$

(smooth quadric without real points) $\quad$ = twisted projective line
= Severi-Brauer attached to $\mathbb{H}$

$$\begin{array}{l} \mathbb{P}^1_{\mathbb{C}} \\ \downarrow u \Big) \mathbb{Z}/2\mathbb{Z}\text{-cover analogous to the } \hat{\mathbb{Z}}\text{-cover } (X_h)_{h \geq 1} \\ \widetilde{\mathbb{P}}^1_{\mathbb{R}} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad X_1 = X \end{array}$$

For $\lambda \in \frac{1}{2}\mathbb{Z}$ define $\mathcal{O}_{\widetilde{\mathbb{P}}^1_{\mathbb{R}}}(\lambda) = \begin{cases} \text{line bundle s.t. } u^*\mathcal{O}(\lambda) = \mathcal{O}(2\lambda) \text{ if } \lambda \in \mathbb{Z} \\ \underbrace{u_*\mathcal{O}(2\lambda)}_{\text{rk. 2 v.b.}} \text{ if } \lambda \notin \mathbb{Z} \end{cases}$

$\mathrm{End}(\mathcal{O}(\frac{1}{2})) = \mathbb{H}$

Prop: $\forall \mathcal{E} \in \mathrm{Bun}_{\widetilde{\mathbb{P}}^1_{\mathbb{R}}}$ $\quad \mathcal{E} \simeq \bigoplus_i \mathcal{O}(\lambda_i)$, $\lambda_i \in \frac{1}{2}\mathbb{Z}$

Can go further: $G/\mathbb{R}$ reductive

Kottwitz: $B(G) := H^1_{alg}(W_{\mathbb{R}}, G(\mathbb{C}))$

acts via $W_{\mathbb{R}} \twoheadrightarrow \mathrm{Gal}(\mathbb{C}|\mathbb{R})$

Cocycles $b : W_{\mathbb{R}} \to G(\mathbb{C})$
whose restriction to $\mathbb{C}^\times$ is algebraic

$$1 \to \mathbb{C}^\times \to W_{\mathbb{R}} \to \mathrm{Gal}(\mathbb{C}|\mathbb{R}) \to 1$$

Then: $\left[\textbf{Prop:}\ B(G) \xrightarrow{\sim} H^1(\widetilde{\mathbb{P}}^1_{\mathbb{R}}, G) \right]$

Modifications: $\infty \in \widetilde{\mathbb{P}}^1_{\mathbb{R}}$ s.t. $u^{-1}(\infty) = \{0, \infty\}$

$t = \frac{1}{z}$ local parameter at ∞

$$\begin{array}{ccc} \mathbb{P}^1_{\mathbb{C}} & \circlearrowleft & \mathbb{C}^\times \\ u \downarrow & & \cup \\ \widetilde{\mathbb{P}}^1_{\mathbb{R}} & \circlearrowleft & U(1) \end{array}$$

acts via $\lambda \cdot t = \lambda t$, $\lambda \in \mathbb{C}^\times$

Gr = affine Grassmanian of $G_{\mathbb{C}}$ at ∞

$\mathrm{Gr}(\mathbb{C}) = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ $\qquad \mathrm{Gr} \circlearrowleft \mathbb{C}^\times$ via $t \mapsto \lambda t$.

$\forall \mu \in X_*(T)_+$ $\qquad$ Gr_μ = Schubert-cell

$$\mathrm{Gr}_\mu \xrightarrow{\mathbb{C}^\times\text{-invariant affine bundle}} G/P_\mu$$

$\rightsquigarrow$ $\mathrm{Gr}_\mu^{\mathbb{C}^\times} \xrightarrow{\sim} G/P_\mu$ with $\mathrm{Gr}_\mu = \mathrm{Gr}_\mu^{\mathbb{C}^\times}$ if μ minuscule.

For GL_n: $V = \mathbb{C}$-v.s. Filtrations of $V \xrightarrow{\sim}$ $\mathbb{C}^\times$-invariant lattices in $V((t))$

$$\mathrm{Fil}^\bullet V \longmapsto \sum_{b\in\mathbb{Z}} t^{-b}\,\mathrm{Fil}^b V[[t]]$$

Thus for $\mu \in X_*(T)_+$, $U(1)$-equivariant ~~affine~~ modifications of $\mathcal{E}_1$ of type μ

$$\simeq (G/P_\mu)(\mathbb{C})$$

Moreover for $V \in \mathrm{Vect}_{\mathbb{R}}$, $\mathrm{Fil}^\bullet V_{\mathbb{C}}$ = filtration of $V_{\mathbb{C}}$

$\rightsquigarrow$ $\underbrace{\mathcal{E}(V, \mathrm{Fil}^\bullet V_{\mathbb{C}})}_{U(1)\text{-equivariant}}$ modification of $V \otimes_{\mathbb{R}} \mathcal{O}_{\widetilde{\mathbb{P}}^1_{\mathbb{R}}}$

Then $\mathcal{E}(V, \mathrm{Fil}^\bullet V_{\mathbb{C}})$ semi-stable of slope $\frac{w}{2}$

$$\Updownarrow$$

$(V, \mathrm{Fil}^\bullet V_{\mathbb{C}})$ = pure Hodge structure of weight w.

<u>Thus:</u> $\mathcal{X}/\mathbb{C}$ proper smooth, $d \in \mathbb{N}$

$\exists\, U(1)$-equivariant modification

$$H^d_{\mathrm{Betti}}(\mathcal{X}, \mathbb{R}) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}^1_{\mathbb{R}}} \dashrightarrow \text{slope } \tfrac{d}{2} \text{ semi-stable vector bundle linked to de Rham coho.}$$

$\uparrow$ Hodge filtration

$\text{slope } \tfrac{d}{2} \text{ semi-stable vector bundle} \simeq \mathcal{O}\left(\tfrac{d}{2}\right)^n, \ n \in \mathbb{N}.$

* <u>p-adic analog</u>: $E = \mathbb{Q}_p$. K discrete valuation perfect residue field.

$$K | \underbrace{K_0 | \mathbb{Q}_p}_{\text{max. unramified}}$$

$$C = \widehat{\overline{K}}, \quad F = C^\flat, \quad G_K = \mathrm{Gal}(\overline{K}|K)$$

$X_F \ni \infty$ closed point w.t. residue field C

$\circlearrowright$ $G_K = U(1)$ stabilizes ∞

$$\widehat{\mathcal{O}}_{X,\infty} = B^+_{dR} \circlearrowleft G_K \text{ via } \sigma(t) = \chi_{cyc}(\sigma)\, t$$

<u>Same principle</u>: $(D, \varphi) \in \varphi\text{-}\mathrm{Mod}_{K_0}$

$$\text{Filtrations of } D_K \xrightarrow{\sim} \{ G_K\text{-invariant lattices in } D \otimes_{K_0} B_{dR} \}$$

$$\mathrm{Fil}^\bullet D_K \longmapsto \sum_{b \in \mathbb{Z}} \mathrm{Fil}^b D_K \otimes_K t^{-b} B_{dR}^+$$

$\uparrow$ via
$$\begin{tikzcd} B_{dR}^+ \arrow[r, two heads, "\theta"] & C \\ & K \arrow[u, hook] \arrow[ul, bend left] \end{tikzcd}$$

$$B_{dR}^{+\,G_K} = K.$$

$$\varphi\text{-ModFil}_{K/K_0} \xrightarrow{\sim} G_K\text{-eq. modifications of } \mathcal{E}(D,\varphi)$$

$$(D, \varphi, \mathrm{Fil}^\bullet D_K) \longmapsto \mathcal{E}(D, \varphi, \mathrm{Fil}^\bullet D_K)$$

Fontaine's filtered φ-modules

Ex. (Reformulation of Tsuji's Comparison théo.)

$\mathfrak{X}/\mathcal{O}_K$ proper smooth, $d \in \mathbb{N}$.

$$(D, \varphi) = \left(H^d_{cris}\left(\mathfrak{X}_{k_K}/W(k_K)\right)\left[\tfrac{1}{p}\right], \text{ crystalline Frob.} \right)$$

$$\mathrm{Fil}^\bullet D_K = \text{Hodge filtration of } H^1_{cris} \otimes_{\mathcal{O}_{K_0}} K = H^1_{dR}(\mathfrak{X}_K/K)$$

Then: $\exists$ G_K-equivariant modification

$$H^d_{et}(\mathfrak{X}_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathcal{O}_X = \mathcal{E}(D, \varphi, \mathrm{Fil}^\bullet D_K) \dashrightarrow \mathcal{E}(D, \varphi).$$

What to do with this?

Transfert of p-adic ideas to archimedean ones

All constructions linked to basic R.Z. spaces $\mathcal{M}(b,\mu)$ and more generally Shtuka $\mathrm{Sht}(b,\mu)$ transfert to the archimedean world:

$$\mathcal{M}(\underset{\text{basic}}{b},\underset{\text{minuscule}}{\mu}) \qquad \mathrm{Sht}(\underset{\text{basic}}{b},\underset{\text{any}}{\mu})$$

* Hodge-de Rham periods: $\pi_{dR}: \mathcal{M}(b,\mu) \to \mathcal{F}_{\mu}$
* Hodge-Tate period map: $\pi_{HT}: \mathcal{M}(b,\mu)_{\infty} \to \mathcal{F}_{-\mu}$
* Twin tower isomorphism. $\mathcal{M}(b,\mu) \simeq \mathcal{M}(b',\mu')$

  $[b]\in B(G)$ basic $\leadsto$ $[b']\in B(J_b)$

Ex: $\mathbb{C}\setminus\mathbb{R} \subset \mathbb{P}^1_{\mathbb{C}} = \mathcal{F}_{\mu}$ $\qquad G = GL_{2,\mathbb{R}}$, $\mu=(1,0)$

$\mathbb{C}\setminus\mathbb{R}$ = locus where the modification of $\mathcal{O}_{\widetilde{\mathbb{P}}^1_{\mathbb{R}}}$ is semi-stable (of slope $\frac{1}{2}$) i.e. $\simeq \mathcal{O}(\frac{1}{2})$

$\mathrm{End}(\mathcal{O}(\frac{1}{2})) = \mathbb{H}$

$$T \circlearrowleft GL_2(\mathbb{R})\times \mathbb{H}^{\times}$$

$$\begin{tikzcd}
 & T \arrow[dl, "\pi_{HT}"', "\mathbb{H}^{\times}\text{-torsor}"] \arrow[dr] & \\
\text{Drinfeld space} = \mathbb{C}\setminus\mathbb{R} \subset \mathbb{P}^1_{\mathbb{C}} \circlearrowleft GL_2(\mathbb{R}) & & T/SO_2(\mathbb{R}) = \text{Lubin-Tate} \arrow[d, "\pi_{dR}"] \\
 & & T/GL_2(\mathbb{R}) = \mathbb{P}^1_{\mathbb{C}} \circlearrowleft \mathbb{H}^{\times}
\end{tikzcd}$$

In general for (G, X) = Shimura datum

$\underbrace{h : \mathbb{S}}_{\text{Deligne torus}} \to G \qquad \exists\, [b_{can}] \in B(\mathbb{S}) = H^1(\widetilde{\mathbb{P}}^1_{\mathbb{R}}, \mathbb{S})$

$b := h_* b_{can} \qquad \underbrace{[b]}_{\text{basic class attached to } (G, X)} \in B(G)$.

$\mathcal{E}_b$ = semi-stable G-torsor on $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$.

$G(\mathbb{R})/K = X \subset \mathcal{F}_\mu$

locus where the modification of $\mathcal{E}_1$ is isomorphic to $\mathcal{E}_b$.

G' = Compact inner form of G

$G'(\mathbb{R}) = \{ g \in G(\mathbb{C}) \,/\, h(i)\,\overline{g}\,h(i)^{-1} = g \}$

$G'(\mathbb{R}) \cap G(\mathbb{R}) = K$.

$\mathrm{Sht}(G,b,\mu)$ = moduli space of modifications
$\mathcal{E}_1 \dashrightarrow \mathcal{E}_b$ $\boxed{J_b = G'}$

$$\begin{tikzcd}
& G(\mathbb{R}) \times_K G'(\mathbb{R}) = \mathrm{Sht}(G,b,\mu) \circlearrowleft G(\mathbb{R}) \times G'(\mathbb{R}) \arrow[dl, "\pi_{HT}"'] \arrow[dr] & \\
X = G(\mathbb{R})/K \subset \mathcal{F}_\mu & & \mathrm{Sht}/K \arrow[d, "\pi_{dR}"] \\
& & G'(\mathbb{R})/K = \mathcal{F}_{-\mu}
\end{tikzcd}$$

$X \circlearrowleft G(\mathbb{R})$

+ twin tower $(G,G') \longleftrightarrow (G',G)$.

via $H^1(\widetilde{\mathbb{P}}^1_{\mathbb{R}}, G) = H^1(\widetilde{\mathbb{P}}^1_{\mathbb{R}}, G')$

$\uparrow \; G'_{\widetilde{\mathbb{P}}^1_{\mathbb{R}}} = G^{\mathcal{E}_b}_{\widetilde{\mathbb{P}}^1_{\mathbb{R}}} \leftarrow$ inner twist.

* In general ~~Sht~~ if μ non minuscule or b not associated to a Shimura datum Griffiths-Schmidt symmetric domains appear.

via $\mathrm{Sht}(G,b,\mu)^{U(1)} \leftarrow$ $U(1)$-equivariant modifications

<u>Not the good object when μ is non-minuscule</u>

In general: should look at $\quad Gr_{\mu}$

$$\downarrow$$

$$\cancel{G/P_\mu} = \cancel{G_{/P_0}^{U(1)}} \Big] \text{ Not the good thing.}$$

$\longrightarrow$ Griffiths-Schmidt domains appearing in G/P with $P \neq P_\mu$ with μ minuscule are not the good objects.

Archimedean vs p-adic: What is the analog of $\mathbb{H}$ in the p-adic case?

$(\mathcal{A}, \deg, \mathrm{rk})$ = abelian Harder-Narasimhan category

$$\left\{\begin{array}{l} \deg : \mathcal{A} \to \mathbb{Z} \\ \mathrm{rk} : \mathcal{A} \to \mathbb{N} \end{array}\right. \qquad \mu = \frac{\deg}{\mathrm{rk}}$$

<u>Ex</u>: Coh_X, X curve

$\mathcal{A}^{\geq 0}$ = objects with ≥ 0 H.N. slopes
$\mathcal{A}^{< 0}$ = " " < 0 " "
} exact subcategories.

$(\mathcal{A}^{\geq 0}, \mathcal{A}^{<0})$ = torsion structure on $\mathcal{A}$
$\rightsquigarrow$ t-structure on $\mathbb{D}^b(\mathcal{A})$ (see Bridgeland's work on semi-stability conditions in derived categories of coherent sheaves)

heart $\underbrace{\widehat{\mathcal{A}}}_{\text{abelian cat.}} = \left\{ \mathcal{E} \in \mathbb{D}^{[-1,0]}(\mathcal{A}) \;\middle|\; H^{-1}(\mathcal{E}) \in \mathcal{A}^{<0},\ H^0(\mathcal{E}) \in \mathcal{A}^{\geq 0} \right\}$

$\deg, \mathrm{rk} : \mathbb{D}^b(\mathcal{A}) \to \mathbb{Z}$, $\deg(C^\cdot) = \sum_i (-1)^i \deg H^i(C^\cdot)$

Set $\widehat{\deg} = -\mathrm{rk}|_{\widehat{\mathcal{A}}} : \widehat{\mathcal{A}} \to \mathbb{Z}$

$\widehat{\mathrm{rk}} = \deg|_{\widehat{\mathcal{A}}} : \widehat{\mathcal{A}} \to \mathbb{N}$

<u>Prop</u>: If $\forall A \in \mathcal{A}^{<0}$ $\forall B \in \mathcal{A}^{\geq 0}$ $\mathrm{Ext}^1(A,B) = 0$ then

$\widehat{\widehat{\mathcal{A}}} \xrightarrow{\sim} \mathcal{A}$ $\quad \widehat{\widehat{\deg}} = \deg$, $\widehat{\widehat{\mathrm{rk}}} = \mathrm{rk}$

applies to curves X, $\mathbb{P}^1_{\mathbb{R}}$.

<u>Ex.</u> If $X =$ elliptic curve, $\mathcal{A} = \mathrm{Coh}_X$

$\widehat{\mathcal{A}} = \mathcal{F}^{-1}(\mathrm{Coh}_X[1])$ via Fourier-Mukai

$\mathcal{F} : D^b(\cancel{}\mathrm{Coh}_X) \xrightarrow{\sim} D^b(\mathrm{Coh}_X)$.

If X is a curve $\quad \mathrm{Coh}_X / \mathrm{Coh}_X^{\mathrm{rk}=0} \xrightarrow{\sim} \mathrm{Vect}_{\eta}(X)$

Serre subcategory of torsion coherent sheaves.

$\mathcal{F} \mapsto \mathcal{F}_\eta$

<u>What is $\widehat{\mathrm{Coh}_X} / \widehat{\mathrm{Coh}_X}^{\mathrm{rk}=0}$?</u> $\to$ function field of $\widehat{X}$ even if $\widehat{X}$ does not exist

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Fact: $X = \widetilde{\mathbb{P}}^1_{\mathbb{R}}$ (or any Severi-Brauer attached to a quaternion algebra/field)

$\widehat{Coh}_X / \widehat{Coh}_X^{\widehat{rk}=0}$ = semi-simple ab. cat. with only one simple object $\mathcal{O}(\frac{1}{2})$

$$\mathrm{End}_{\widehat{Coh}/\widehat{Coh}^{\widehat{rk}=0}}\left(\mathcal{O}(\tfrac{1}{2})\right) = \mathbb{H}$$

$$\Rightarrow \widehat{Coh}_X / \widehat{Coh}_X^{\widehat{rk}=0} \xrightarrow{\sim} \mathrm{Vect}_{\mathbb{H}}.$$

: X/E, $[E:\mathbb{Q}_p] < +\infty$ the p-adic curve.

$\mathcal{BC}$ = abelian category of finite dimensional Banach spaces in Colmez sense (Banach-Colmez spaces)

= smallest abelian subcategory of the ab. cat. of E-vector spaces on $\mathrm{Spa}(C)_{\text{pro-ét}}$ stable under extension and containing Vect_E and Vect_C. (here $\infty \in |X|$, C = residue field)

$$\mathrm{Vect}_E \longrightarrow \mathcal{BC} \qquad \mathrm{Vect}_C \to \mathcal{BC}$$
$$V \longmapsto \underline{V} \qquad W \mapsto W \otimes_C \mathcal{O}_a$$

If $S = \mathbb{F}_q$ perfectoid space can define X_S^{ad} = "family of curves $(X^{ad}_{k(s)})_{s \in |S|}$"

$\rightsquigarrow$ $\widetilde{(X^{ad}_{F,E})}_{\text{pro-ét}}$

$$\downarrow f$$

$$\widetilde{\mathrm{Spa}(F)}_{\text{pro-ét}} = \widetilde{\mathrm{Spa}(C)}_{\text{pro-ét}}$$

[Th: $Rf_* : \widehat{\mathrm{Coh}_X} \xrightarrow{\sim} \mathcal{BC}$]

Now: Colmez $\mathcal{BC}/\mathrm{Vect}_E$ = semi-simple with one simple object $\rightsquigarrow$ division algebra $\mathcal{D}/E$ (infinite dimensional)

$$\| $$

$$\widehat{\mathrm{Coh}_X} \Big/ \widehat{\mathrm{Coh}_X}^{\mathrm{rk}=0}$$

What can we do with $\mathcal{D}$?

[Can we reconstruct X from $\mathcal{D}$?]